SEMICLASSICAL ESTIMATES IN ASYMPTOTICALLY EUCLIDEAN SCATTERING

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1. Introduction

The purpose of this note is to obtain semiclassical resolvent estimates for long range perturbations of the Laplacian on asymptotically Euclidean manifolds.

For an estimate which is uniform in the Planck constant h we need to assume that the energy level is *non-trapping*. In the high energy limit (that is, when we consider $\Delta - \lambda^2$, as $\lambda \to \infty$, which is equivalent to $h^2\Delta - 1$, $h \to 0$), this corresponds to the global assumption that the geodesic flow is non-trapping. We note here that a sufficiently small neighbourhood of infinity is always non-trapping.

The resolvent estimates in the classical ($h = 1$) and semi-classical cases have a long tradition going back to the *limiting absorption principle* – see [1] and references given there. Various variants of the theorem we present were proved in Euclidean potential scattering by Jensen-Mourre-Perry [9], Robert-Tamura [14], Gérard-Martinez [5], Gérard [4] and Wang [15], and for more general elliptic operators by Robert [12],[13]. The proofs were based on *Mourre theory* whose underlying feature is the positive commutator method accompanied by functional analytic techniques for obtaining a resolvent estimate. While the work of Gérard-Martinez [5] explains the role of geometry in the positive commutator estimate itself, it refers to Mourre's work for the functional analytic argument. We adopt a completely geometric approach based on direct microlocal ideas.

The classical version of the estimate on asymptotically Euclidean manifolds ($h = 1$ in which case there is no need for the non-trapping assumption) is essentially in Melrose's original paper on the subject [10] in which he introduced a fully microlocal point of view to scattering. However, the proof presented here is somewhat different in spirit: a global positive commutator argument is used to derive an estimate on the resolvent directly.

Referring to (2.3) below for the definition of a scattering metric, to (2.4),(2.5) for the definition of a long range semi-classical perturbation, and (2.7) for the definition of a non-trapping energy, we state our main

Theorem. *Let X be a manifold with boundary and let Δ be the Laplacian of a scattering metric on X. If $P = h^2\Delta + V$ is a semi-classical long range perturbation of $h^2\Delta$, and $R(\lambda) = (P - \lambda)^{-1}$ its resolvent, then for all $m \in \mathbb{R}$,*

$$\|R(\lambda + it)f\|_{H_{sc}^{m,-1/2-\epsilon}(X)} \leq C_0 h^{-1} \|f\|_{H_{sc}^{m-2,1/2+\epsilon}(X)}, \quad \epsilon > 0, \tag{1.1}$$

with C_0 independent of $t \neq 0$ real and $\lambda \in I$, $I \subset (0, +\infty)$ a compact interval in the set of non-trapping energies for P.

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Here $H^{m,k}_{\mathrm{sc}}(X)$ denote Sobolev spaces adapted to the scattering calculus, that is to asymptotically Euclidean structures. The index m indicates smoothness and k the rate of decay at infinity: the larger the better in both cases.

To indicate the main idea of the proof let $p - \lambda$ be the principal symbol of $P - \lambda$. Here the principal symbol is meant in both the semi-classical sense and the scattering sense – see Sect.2. Near the characteristic variety of $P - \lambda$, we construct a function $q \geq 0$ such that q is decreasing along the Hamilton vector field H_p. This gives the required estimate for the resolvent when we apply a variant of the well known commutator method – see [7] for the now standard application to the propagation of singularities for operators of principal type.

On the quantum level, propagation of singularities corresponds to the propagation by the classical flow. The use of commutators is natural as their symbols are given by the classical Poisson brackets. The microlocal approach is thus motivated by the quantum-classical correspondence. In a scattering problem, estimates on the resolvent are closely related to quantum propagation estimates. Hence we can apply the same strategy for directly relating analysis and geometry.

We stress that to prove, say, the outgoing resolvent estimate, one needs to keep the signs of both q and $H_p q$ fixed throughout phase space, and in case of the outgoing estimate, these signs must be opposite. Indeed, it is the fixed sign of q that makes it possible to eliminate the machinery of Mourre's method. The positivity of q shows that in the outgoing region, where bicharacteristics tend as $t \to +\infty$, q must be of the form $x^r a$, $a \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*X)$, $r > 0$, (here x is a boundary defining function), and in the incoming region, where bicharacteristics tend as $t \to -\infty$, it must be of the form $x^{-s} b$, $b \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*X)$, $s > 0$. The difference between these two weights, which can be made arbitrarily small, but is never 0, plus the improvement by 1 in the order when calculating a commutator, explains how the weighting of the Sobolev spaces works.

For applications of the non-trapping estimates to more general operators we refer to a recent paper by Bruneau-Petkov [2]. It is clear that the "black box" set-up discussed there can be easily adapted to the manifold situation.

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2. Preliminaries

Let X be a $\mathcal{C}^\infty$ manifold with boundary, ∂X and let x be a boundary defining function. Thus, in a small collar neighborhood $[0, \epsilon_0) \times \partial X$ of the boundary ∂X, we have 'semi-global coordinates' (x, y, ξ, η) on $T^*_{[0,\epsilon_0)\times\partial X}X$.

Microlocal techniques adapted to asymptotically Euclidean structure near ∂X (see (2.3)) were introduced by Melrose [10]. We start by recalling the scattering cotangent bundle ${}^{\mathrm{sc}}T^*X$ which is the natural *phase space*. It is defined as the dual of the scattering tangent bundle ${}^{\mathrm{sc}}TX$, which in turn is defined so that the space of vector fields $x\mathcal{V}_b(X)$, where $\mathcal{V}_b(X)$ are vector fields tangent to ∂X, is given by sections

$$x\mathcal{V}_b(X) = \mathcal{C}^\infty(X; {}^{\mathrm{sc}}TX)\,,$$

see [10] for a thorough discussion. Since ${}^{\mathrm{sc}}TX \hookrightarrow TX$ we have a natural map $T^*X \to {}^{\mathrm{sc}}T^*X$. In 'semi-global coordinates' (x, y, ξ, η) on ${}^{\mathrm{sc}}T^*_{[0,\epsilon_0)\times\partial X}X$ it is given by

$$(x, y, \tau, \mu) = (x, y, x^2\xi, x\eta)\,,$$

and this identification is worth keeping in mind since the symplectic and contact structures are inherited from T^*X, that is, from the (x, ξ) coordinates. In particular, when we speak of the Hamilton vector fields on ${}^{\mathrm{sc}}T^*X$, we mean the natural extention of the usual Hamilton vector field on ${}^{\mathrm{sc}}T^*X^o \simeq T^*X^o$, to ${}^{\mathrm{sc}}T^*X$ – see [10] and [11]. We also note that the variable μ is naturally identified with $\mu \in T^*_y\partial X$.

The fiber radial compactification of ${}^{\mathrm{sc}}T^*X$ is denoted by ${}^{\mathrm{sc}}\bar{T}^*X$; ${}^{\mathrm{sc}}\bar{T}^*X$ is thus a ball bundle over X. Classical symbols, $a \in S^{m,l}_{cl}(X)$, are functions $a \in x^l\rho_\infty^{-m}\mathcal{C}^\infty({}^{\mathrm{sc}}\bar{T}^*X)$. By $a \in S^{m,l}(X)$ it is meant that $a \in \mathcal{C}^\infty(T^*X)$, $x^{-l}\rho^m_\infty a \in L^\infty(T^*X)$, and the same estimate holds after the application (to a) of any b-differential operator on ${}^{\mathrm{sc}}\bar{T}^*X$, that is, an operator in the algebra generated by $\mathcal{V}_b({}^{\mathrm{sc}}\bar{T}^*X)$, vector fields tangent to $\partial{}^{\mathrm{sc}}\bar{T}^*X$.

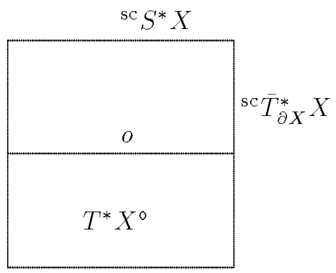

FIGURE 1. The fiber compactification ${}^{\mathrm{sc}}\bar{T}^*X$ of ${}^{\mathrm{sc}}T^*X$ is a manifold with corners. Its boundary hyperfaces are ${}^{\mathrm{sc}}\bar{T}^*_{\partial X}X$, which is a ball bundle over ∂X, and the cosphere bundle ${}^{\mathrm{sc}}S^*X$, which is a sphere bundle over X. The zero section is denoted by o.

The semiclassical calculus for the Weyl metric on T^*X.

$$\frac{dz^2}{1+|z|^2} + \frac{d\zeta^2}{1+|\zeta|^2}$$

is well known and, for instance, it is discussed in great generality in [6]. The natural generalization to manifolds with asymptotically Euclidean structure near infinity is given in the Appendix to [16]. We will review and slightly extend it below.

The semi-classical symbols are defined as follows: $a \in S^{m,l,k}(X)$ means that $a \in \mathcal{C}^\infty((0,1) \times T^*X)$, $h^k x^{-l}\rho^m_\infty a \in L^\infty((0,1)\times T^*X)$, and the same estimate holds after the application of any b-differential operator ${}^{\mathrm{sc}}\bar{T}^*X$. Thus, $a(h,.) \in S^{m,l}(X)$ for $h \in (0,1)$, and the symbol estimates are uniform in h. The corresponding class of classical symbols, $a \in S^{m,l,k}_{cl}(X)$ are functions with $h^k x^{-l}\rho^m_\infty a \in \mathcal{C}^\infty([0,1) \times {}^{\mathrm{sc}}\bar{T}^*X)$.

For $a \in S^{m,l,k}(X)$ we define a semiclassical operator $\mathrm{Op}(a) \in \Psi^{m,l,k}_{\mathrm{sc}}(X)$ as in Appendix to [16]: we first use local Euclidean coordinates in a cone near infinity, identified with a neighbourhood of a boundary point (see Figure 2) to define

$$Au(z) = \left(\frac{1}{2\pi h}\right)^n \int e^{i(z-w)\cdot\xi/h}a(h,z,\xi)u(w)dwd\xi.$$

with $a \in S^{m,l,k}(\mathbb{R}^n)$. Invariance under local changes of coordinates then gives $\mathrm{Op}(a)$ and leads to the definition of the class $\Psi^{m,l,k}_{\mathrm{sc}}(X)$.

We then have the symbol map $\sigma_{\mathrm{sc},h} : \Psi^{m,l,k}_{h,\mathrm{sc}}(X) \to S^{m,l,k}(X)$ with the usual properties, and in particular with the short exact sequence

$$0 \to \Psi^{m-1,l+1,k-1}_{h,\mathrm{sc}}(X) \to \Psi^{m,l,k}_{\mathrm{sc}}(X) \xrightarrow{\sigma^{m,l,k}_{h,sc}} S^{m,l,k}(X)/S^{m-1,l+1,k-1}(X) \longrightarrow 0\,.$$

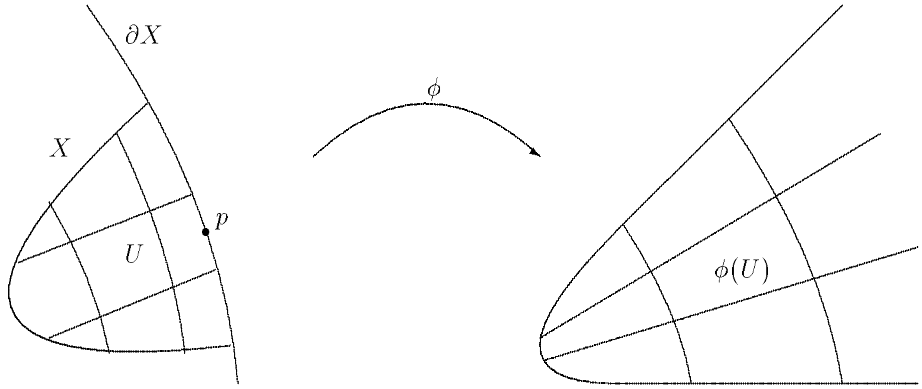

FIGURE 2. Local Euclidean coordinates near $p \in \partial X$ identify a neighborhood U of p in X with a conic neighborhood $\phi(U)$ of infinity in $\mathbb{R}^n$.

Another important property of $\Psi_{\mathrm{scc},h}(X)$ is that it is commutative to top order, and the principal symbol of a commutator is given by the Poisson bracket of the principal symbols of the commutants. That is, if $A \in \Psi^{m,l,k}_{\mathrm{scc},h}(X)$, $B \in \Psi^{m',l',k'}_{\mathrm{scc},h}(X)$, then $[A,B] \in \Psi^{m+m'-1,l+l'+1,k+k'-1}_{\mathrm{scc},h}(X)$ and

$$\sigma^{m+m'-1,l+l'+1,k+k'-1}_{h,\mathrm{sc}}([A,B]) = \frac{h}{i} H_a b, \tag{2.1}$$

where a, b are the principal symbols of A and B, and H_a denotes the Hamilton vector field of a.

We will also make use of the sharp Gårding estimate:

Lemma 2.1. *Suppose $A \in \Psi^{0,0,0}_{scc,h}(X)$ is self-adjoint, and its (joint semiclassical) principal symbol is $a \geq 0$. Then there exists $C > 0$ such that*

$$\langle u, Au\rangle \geq -Ch\|u\|_{H^{-1/2,1/2}_{sc}(X)}. \tag{2.2}$$

In particular, if $A \in \Psi^{2m,-2l,0}_{scc,h}(X)$ has principal symbol $a \geq 0$, then $A \geq hR$ for some $R \in \Psi^{2m-1,-2l+1,0}_{scc,h}(X)$.

Proof. The inequality is well known in the case of $\mathbb{R}^n$ – see Sect.18.4 of [8] (easily adapted to the semi-classical setting), [3] and [6]. The localization argument presented in the Appendix of [16] then gives the lemma. $\square$

Now, let g be a scattering metric on X, that is, a metric which near ∂X takes the form

$$\frac{dx^2}{x^4} + \frac{h'}{x^2}, \quad h'|_{\partial X} = h \text{ is a metric on } \partial X\,. \tag{2.3}$$

This defines an asymptotically Euclidean structure near ∂X: a neighbourhood of ∂X is isometric to a perturbation of the large end of the cone $\mathbb{R}_+ \times \partial X$ with the metric $dr^2 + r^2 h$.

We will consider the following self-adjoint, classically elliptic operators in $\mathrm{Diff}_{h,\mathrm{sc}}{}^{2,0,0}(X) \subset \Psi^{2,0,0}_{h,\mathrm{sc}}$:

$$P = h^2\Delta_g + V \tag{2.4}$$

where in any compact set, V is a second order semiclassical operator ($V = \sum_{|\alpha|\leq 2} v_\alpha(z,h)(hD_z)^\alpha$ in local coordinates) and near the boundary ∂X, in local coordinates $y \in \partial X$,

$$
\begin{aligned}
&V = x^\gamma V_0\,,\ \ V_0 = \sum_{|\alpha|+k\leq 2} v_{k\,\alpha}(x,y,h)(hx^2D_x)^k(hD_y)^\alpha\,,\\
&\qquad v_{k\,\alpha} - v^0_{k\,\alpha} \in hS^{0,0,0}(X)\,,\ \ v^0_{k\,\alpha} \in S^{0,0}(X)\ \ \gamma > 0\,.
\end{aligned}
\tag{2.5}
$$

The condition that the coefficients are symbols independent of the fiber variables means that $|(x\partial_x)^l\partial_y^\beta v_{k,\alpha}| \leq C_{l\beta}$. In the Euclidean setting it corresponds to assuming that the coefficients are symbols in the Euclidean base variables. Due to the vanishing of $v_{k\,\alpha} - v^0_{k\,\alpha}$ in $S^{0,0,0}(X)$ when $h = 0$, the semiclassical principal symbol of P is

$$
p = g + x^\gamma \sum_{|\alpha|+k\leq 2} v^0_{k\,\alpha}(x,y)\tau^k\mu^\alpha,
\tag{2.6}
$$

where g also denotes the (dual) metric function of the metric g. Thus, p can be represented by an h-independent function, which will be convenient for the construction in the last section of this paper. Note, however, that in (2.5), $v_{k\,\alpha} - v^0_{k\,\alpha} \in hS^{0,0,0}(X)$ could be replaced by $v_{k\,\alpha} - v^0_{k\,\alpha} \in h^\rho S^{0,0,0}(X)$, $\rho > 0$, or indeed by the assumption that $v_{k\,\alpha}$ is continuous on $[0,1)_h$ with values in $S^{0,0}(X)$, at the expense of minor changes in the next section.

For obtaining the uniform resolvant estimates in h for $R(\lambda \pm i0)$, we make the assumption that the Hamiltonian is non-trapping at energy λ

$$
\begin{aligned}
&\text{for any } \xi \in T^*X^\circ \text{ satisfying } p(\xi) = \lambda,\\
&\qquad \lim_{t\to\pm\infty} x(\exp(tH_p)(\xi)) = 0.
\end{aligned}
\tag{2.7}
$$

As discussed in [5], this implies that an interval of energies around λ is non-trapping:

$$
\begin{aligned}
&\exists\delta_0 > 0 \text{ such that for any } \xi \in T^*X^\circ \text{ satisfying } p(\xi) \in (\lambda - \delta_0, \lambda + \delta_0),\\
&\qquad \lim_{t\to\pm\infty} x(\exp(tH_p)(\xi)) = 0.
\end{aligned}
\tag{2.8}
$$

The symbolic functional calculus applies in the semiclassical setting as well – see [3] and references given there. Here, we will restrict the discussion to the operator P given by (2.4). The formula

$$
f(P) = \frac{1}{2\pi i}\int_{\mathbb{C}} \bar\partial_z \tilde f(z)(P-z)^{-1} d\bar z \wedge dz\,,\quad \tilde f \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{C})\,,\ \ \tilde f|_{\mathbb{R}} = f\,,\ \ \bar\partial \tilde f = \mathcal{O}(|\operatorname{Im} z|^\infty)\,,
$$

($\tilde f$ is an almost analytic extensions of f) shows that for $f \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$, $f(P) \in \Psi^{-\infty,0,0}_{\mathrm{sc},h}(X)$. Also $\sigma^{\star,0,0}_{h,\mathrm{sc}}(f(P)) = f(p)$.

If $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$, $\psi \equiv 1$ near λ, then for $t \in \mathbb{R}$, $1 - \psi(\sigma) = \tilde\psi_t(\sigma)(\sigma - (\lambda + it))$, $\tilde\psi \in S^{-1}_{cl}(\mathbb{R})$ satisfying uniform symbol estimates as t varies over compact sets, so $\tilde\psi(P) \in \Psi^{-2,0,0}_{\mathrm{sc},h}(X)$, and we have proved the following lemma.

Lemma 2.2. *Let P be as in (2.4). Suppose that $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$, $\psi \equiv 1$ near λ, and suppose that $r, s \in \mathbb{R}$. Then there exists $C > 0$, independent of t as long as t varies in compact sets, such that for all $u \in \mathcal{C}^{-\infty}(X)$ with $(P - (\lambda + it))u \in H^{r-2,s}_{sc}(X)$, the following estimate holds:*

$$
\|(\mathrm{Id} - \psi(P))u\|_{H^{r,s}_{sc}(X)} \leq C\|(P - (\lambda + it))u\|_{H^{r-2,s}_{sc}(X)}.
\tag{2.9}
$$

3. Semiclassical estimates

In this section we will prove the semi-classical resolvent estimates under the assumption that there exists $q \in S^{0,-\epsilon,0}(X)$, $\epsilon \in (0, \frac{1}{4})$, such that

$$\begin{gathered} 2qH_p q = -b\psi(p)^2 \,, \\ b \in S^{0,1-2\epsilon,0}(X)\,, \quad \psi \in \mathcal{C}_c^\infty(\mathbb{R};[0,1])\,, \ \psi \equiv 1 \text{ near } \lambda, \\ b \geq c_0 x^{1+2\epsilon} > 0\,. \end{gathered} \tag{3.1}$$

The existence of q under global non-trapping assumptions will be established in Sect.4.

If we write $Q = \mathrm{Op}(q)$ and $B = (\mathrm{Op}(b) + \mathrm{Op}(b)^*)/2$ then, as reviewed in Sect.2,

$$i[Q^*Q, P] = h\psi(P)B\psi(P) + h^2 R \tag{3.2}$$

with $R \in \Psi_{\mathrm{scc},h}^{0,2-2\epsilon,0}(X)$. Note that $\psi(P) \in \Psi_{\mathrm{scc},h}^{-\infty,0,0}(X)$, i.e. it is smoothing, so the differentiability order r in the weighted Sobolev spaces $H_{\mathrm{sc}}^{r,s}(X)$ is mostly irrelevant below. Suppose that $u \in H_{\mathrm{sc}}^{0,\frac{1}{2}+\epsilon}(X)$. Then for $t > 0$,

$$\langle u, i[Q^*Q, P]u\rangle = -2\,\mathrm{Im}\langle u, Q^*Q(P - (\lambda + it))u\rangle - 2t\|Qu\|^2. \tag{3.3}$$

(Note that Q^*QP and PQ^*Q are in $\Psi_{\mathrm{scc},h}^{0,-2\epsilon,0}(X)$, so the expressions of the form $\langle u, Q^*QPu\rangle$, make sense.) Thus, taking into account that $2t\|Qu\|^2 \geq 0$,

$$h\langle u, \psi(P)B\psi(P)u\rangle \leq 2|\langle u, Q^*Q(P - (\lambda + it))u\rangle| + h^2|\langle u, Ru\rangle|. \tag{3.4}$$

By the Cauchy-Schwartz inequality we have, for any $\delta > 0$,

$$\begin{aligned} |\langle u, Q^*Q(P - (\lambda + it))u\rangle| &\leq \|x^{\frac{1}{2}+\epsilon}u\|\|x^{-\frac{1}{2}-\epsilon}Q^*Q(P - (\lambda + it))u\| \\ &\leq \delta h\|x^{\frac{1}{2}+\epsilon}u\|^2 + \delta^{-1}h^{-1}\|x^{-\frac{1}{2}-\epsilon}Q^*Q(P - (\lambda + it))u\|^2. \end{aligned} \tag{3.5}$$

Note that $x^{-\frac{1}{2}-\epsilon}Rx^{-\frac{1}{2}-\epsilon} \in \Psi_{\mathrm{scc},h}^{0,1-4\epsilon,0}(X)$, hence bounded on $L^2_{\mathrm{sc}}(X)$ since $\epsilon \in (0, \frac{1}{4})$. Similarly, $x^{-\frac{1}{2}-\epsilon}Q^*Qx^{\frac{1}{2}+3\epsilon} \in \Psi_{\mathrm{scc},h}^{0,0,0}(X)$ is also bounded on the L^2 space. Thus,

$$\begin{aligned} &h\langle u,\psi(P)B\psi(P)u\rangle - (\delta h + h^2\|x^{-\frac{1}{2}-\epsilon}Rx^{-\frac{1}{2}-\epsilon}\|_{\mathcal{B}(L^2_{\mathrm{sc}}(X))})\|x^{\frac{1}{2}+\epsilon}u\|^2 \\ &\qquad \leq \delta^{-1}h^{-1}\|x^{-\frac{1}{2}-\epsilon}Q^*Qx^{\frac{1}{2}+3\epsilon}\|^2_{\mathcal{B}(L^2_{\mathrm{sc}}(X))}\|x^{-\frac{1}{2}-3\epsilon}(P - (\lambda + it))u\|^2. \end{aligned} \tag{3.6}$$

We will now use the last assumption in (3.1): $x^{-1+2\epsilon}b\psi(p) \geq c_0x^{2\epsilon}\psi(p)$. Hence by the sharp Gårding estimate,

$$x^{-\frac{1}{2}+\epsilon}\psi(P)B\psi(P)x^{-\frac{1}{2}+\epsilon} \geq c_0^2x^{2\epsilon}\psi(P)^2x^{2\epsilon} + hR_1, \quad R_1 \in \Psi_{\mathrm{sc},h}^{-\infty,1,0}(X). \tag{3.7}$$

Adding $c_0^2x^{2\epsilon}(\mathrm{Id} - \psi(P)^2)x^{2\epsilon}$ to both sides gives

$$x^{-\frac{1}{2}+\epsilon}\psi(P)B\psi(P)x^{-\frac{1}{2}+\epsilon} + c_0^2x^{2\epsilon}(\mathrm{Id} - \psi(P)^2)x^{2\epsilon} \geq c_0^2x^{4\epsilon} + hR_1. \tag{3.8}$$

We also note that $|\langle x^{\frac{1}{2}-\epsilon}u, R_1x^{\frac{1}{2}-\epsilon}u\rangle| \leq C'\|x^{1-\epsilon}u\|^2$. Thus, applying both sides of (3.8) to $x^{\frac{1}{2}-\epsilon}u$, and pairing with $x^{\frac{1}{2}-\epsilon}u$ afterwards yields

$$\begin{aligned} c_0^2\|x^{\frac{1}{2}+\epsilon}u\|^2 &\leq \langle u, \psi(P)B\psi(P)u\rangle + c_0^2|\langle(\mathrm{Id} + \psi(P))x^{\frac{1}{2}+\epsilon}u, (\mathrm{Id} - \psi(P))x^{\frac{1}{2}+\epsilon}u\rangle| + C'h\|x^{1-\epsilon}u\|^2 \\ &\leq \langle u, \psi(P)B\psi(P)u\rangle + 2c_0^2\delta\|x^{\frac{1}{2}+\epsilon}u\|^2 + \delta^{-1}\|(\mathrm{Id} - \psi(P))x^{\frac{1}{2}+\epsilon}u\|^2 + C'h\|x^{1-\epsilon}u\|^2, \end{aligned} \tag{3.9}$$

The last term is clearly bounded by $C'h\|x^{\frac{1}{2}+\epsilon}u\|^2$ and the second to last term can be estimated using (2.9). Choosing $\delta < 1/4$, $h_1 = c_0^2/4C'$ gives that for $h \in (0, h_1)$,

$$\|x^{\frac{1}{2}+\epsilon}u\|^2 \leq C_1\langle u, \psi(P)B\psi(P)u\rangle + C_2\|(P-(\lambda+it))u\|^2_{H_{\mathrm{sc}}^{-2,-\frac{1}{2}-\epsilon}(X)}. \tag{3.10}$$

The norm in the second term on the right hand side can be replaced by the $H_{\mathrm{sc}}^{-2,\frac{1}{2}+\epsilon}(X)$ norm.

Combining (3.6) and (3.10), we thus conclude that there exists $h_0 > 0$ such that for $h \in (0, h_0)$,

$$\langle u, \psi(P)B\psi(P)u\rangle \leq Ch^{-2}\|x^{-\frac{1}{2}-3\epsilon}(P-(\lambda+it))u\|^2. \tag{3.11}$$

Again using (3.10), we conclude that for all $\epsilon > 0$,

$$\|u\|_{H_{\mathrm{sc}}^{0,-1/2-\epsilon}(X)} \leq Ch^{-1}\|(P-(\lambda+it))u\|_{H_{\mathrm{sc}}^{0,1/2+3\epsilon}(X)}, \quad h \in (0, h_0). \tag{3.12}$$

We can modify this argument slightly by inserting $(P+i)(P+i)^{-1}$ in (3.5) between Q and $P-(\lambda+it)$, to see that the last factor in (3.6) can be replaced by $\|(P+i)^{-1}x^{-\frac{1}{2}-3\epsilon}(P-(\lambda+it))u\|^2$, and correspondingly the norm on the right hand side of (3.12) can be replaced by $\|(P-(\lambda+it))u\|_{H_{\mathrm{sc}}^{-2,1/2+3\epsilon}(X)}$. A further slight modification in the same spirit allows us to conclude that the smoothness order r in $H_{\mathrm{sc}}^{r,s}(X)$ can be shifted by the same amount on both sides of (3.12):

$$\|u\|_{H_{\mathrm{sc}}^{r,-1/2-\epsilon}(X)} \leq Ch^{-1}\|(P-(\lambda+it))u\|_{H_{\mathrm{sc}}^{r-2,1/2+3\epsilon}(X)}, \quad h \in (0, h_0). \tag{3.13}$$

Now let $u = u_t = R(\lambda+it)f$, $f \in H_{\mathrm{sc}}^{r,1/2+3\epsilon}(X)$. Since $R(\lambda+it) = (P-(\lambda+it))^{-1} \in \Psi_{\mathrm{scc},h}^{-2,0,0}(X)$ for $t > 0$, we see that $u_t \in H_{\mathrm{sc}}^{r+2,1/2+3\epsilon}(X)$ for $t > 0$. Thus, the above estimate is applicable and we conclude that

$$\|R(\lambda+it)f\|_{H_{\mathrm{sc}}^{r+2,-1/2-\epsilon}(X)} \leq Ch^{-1}\|f\|_{H_{\mathrm{sc}}^{r,1/2+3\epsilon}(X)}, \quad h \in (0, h_0). \tag{3.14}$$

Note that for a fixed ψ, we can let λ be arbitrary inside the region where $\psi \equiv 1$, so a compactness argument gives the uniform estimate in λ as stated in our main Theorem.

Remark 3.1. As in Melrose's paper [10], using these estimates one can show that for fixed $h > 0$ the limits $R(\lambda \pm i0)f$ exist in $H_{\mathrm{sc}}^{r+2,-1/2-\epsilon}(X)$ for $f \in H_{\mathrm{sc}}^{r,1/2+\epsilon}(X)$, $\epsilon > 0$. Hence, (3.14) yields

$$\|R(\lambda+i0)f\|_{H_{\mathrm{sc}}^{r+2,-1/2-\epsilon}(X)} \leq Ch^{-1}\|f\|_{H_{\mathrm{sc}}^{r,1/2+\epsilon}(X)}, \quad h \in (0, h_0), \tag{3.15}$$

as well.

4. Symbol construction

Let p be the principal symbol of P. Thus, near ∂X,

$$p = \tau^2 + g_\partial(y,\mu) + x^\gamma r, \quad r \in S^{2,0}(X), \tag{4.1}$$

where g_∂ is the metric on the boundary, and we denote the metric function on the cotangent bundle the same way. Its Hamilton vector field H_p is of the form

$$x(2\tau(x\partial_x + \mu\cdot\partial_\mu) - 2g_\partial\partial_\tau + H_{g_\partial}) + x^{1+\gamma}W, \quad W \in \mathcal{V}_{\mathrm{b}}({}^{\mathrm{sc}}T^*X) \otimes S^{1,0}(X)\,; \tag{4.2}$$

see [10, Equation (8.17)] for a detailed calculation. Here we will be mainly concerned with the (x,τ) variables, so we rewrite this as

$$H_p = x(2\tau + x^\gamma a)(x\partial_x) - x(2g_\partial + x^\gamma b)\partial_\tau + 2x\tau\mu\cdot\partial_\mu + xH_{g_\partial} + x^{1+\gamma}W', \tag{4.3}$$

where $a, b \in S^{1,0}(X)$, and W' is now a vector field tangent to the ∂X fibers, i.e. it is a vector field in ∂_y and ∂_μ with coefficients in $S^{1,0}(X)$. In this section we take λ^2, not λ, as the spectral parameter.

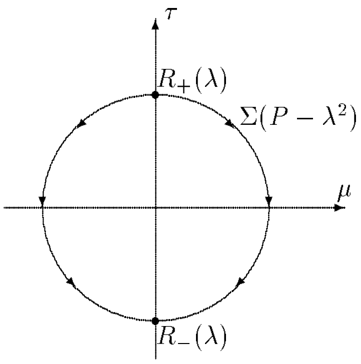

FIGURE 3. The projection of the characteristic variety $\Sigma(P-\lambda^2)$ and the bicharacteristics of H_p inside it to the (τ,μ)-plane.

As indicated, we make the assumption that a small interval of energies around λ^2 is non-trapping, i.e.

$$\begin{gathered}\exists\delta_0>0 \text{ such that for any } \xi\in T^*X^\circ \text{ satisfying } p(\xi)\in(\lambda^2-\delta_0,\lambda^2+\delta_0),\\ \lim_{t\to\pm\infty} x(\exp(tH_p)(\xi))=0.\end{gathered} \tag{4.4}$$

Now,

$$H_p(x^{-1}\tau)=-2(\tau^2+g_\partial)+x^\gamma f,\ f\in S^{1,0}(X)\,, \tag{4.5}$$

so there exists $\epsilon_1>0$ such that for $\xi\in{}^{\mathrm{sc}}T^*X$ satisfying $p(\xi)\in(\lambda^2/2,2\lambda^2)$, $x<\epsilon_1$, $-(H_p(x^{-1}\tau))(\xi)\geq c_0>0$. Since p is constant along integral curves of H_p, we see that if $\exp(-tH_p)(\xi)$, $t\geq T$, stays in $x<\epsilon_1$ (which holds under our non-trapping assumption for sufficiently large T), then $x^{-1}\tau$ tends to $+\infty$; in particular τ is non-negative for all large t. By reducing $\epsilon_1>0$ if necessary, we also see that there exist $\delta_1>0$, $\epsilon_1>0$ such that for $\xi\in{}^{\mathrm{sc}}T^*X$,

$$|p(\xi)-\lambda^2|<\delta_1,\ x(\xi)<\epsilon_1,\ |\tau|<7\lambda/8\Rightarrow g_\partial(\xi)\geq c_1>0. \tag{4.6}$$

Reducing $\epsilon_1>0$ further if necessary, we can thus arrange that

$$|p(\xi)-\lambda^2|<\delta_1,\ x<\epsilon_1,\ |\tau|<7\lambda/8\Rightarrow -x^{-1}H_p\tau(\xi)\geq c_1>0. \tag{4.7}$$

Thus, we see that given any $x_0>0$, $\xi\in T^*X^\circ$ with $|p(\xi)-\lambda^2|<\delta_1$, there exists $T>0$ such that

$$t\geq T\Rightarrow \tau(\exp(-tH_p)(\xi))>2\lambda/3,\ x(\exp(-tH_p)(\xi))<x_0/2. \tag{4.8}$$

We now define a symbol $q\in S^{-\infty,0}({}^{\mathrm{sc}}T^*X)$ whose most important properties are that

$$q\geq 0 \text{ and } x^{-1}H_pq\leq 0. \tag{4.9}$$

We will always use a localization in the energy via a factor $\psi(p)$ where $\psi\in\mathcal{C}_c^\infty(\mathbb{R})$ is supported in $(\lambda^2-\delta,\lambda^2+\delta)$, where $\delta\in(0,\lambda^2)$ is a fixed small constant with $\delta<\delta_1$, δ_1 as above. Let

$$M=\sup\{|a(\xi)|+|b(\xi)|:\ p(\xi)\leq 2\lambda^2\}<+\infty; \tag{4.10}$$

here we used that $p^{-1}((-\infty,2\lambda^2])$ is a compact subset of ${}^{\mathrm{sc}}T^*X$. Also, let

$$x_0=\min\{(\lambda/6(M+1))^{1/\gamma},(c_1/2(M+1))^{1/\gamma},\epsilon_1\}. \tag{4.11}$$

Let $\chi_- \in \mathcal{C}^\infty(\mathbb{R})$ be supported in $(\lambda/3, +\infty)$, identically 1 on $(2\lambda/3, +\infty)$, with $\chi_-' \geq 0$, and similarly let $\chi_+ \in \mathcal{C}^\infty(\mathbb{R})$ be supported in $(-\infty, -\lambda/3)$, identically 1 on $(-\infty, -2\lambda/3)$, with $\chi_+' \leq 0$. Also, let $\chi_\partial \in \mathcal{C}_c^\infty(\mathbb{R})$ be supported in $(-7\lambda/8, 7\lambda/8)$, with $\chi_\partial' \geq (6\lambda/c_1)\chi_\partial \geq 0$ on $(-7\lambda/8, 3\lambda/4)$, and $\chi_\partial(-3\lambda/4) > 0$. Let $\rho \in \mathcal{C}_c^\infty([0, +\infty))$ be identically 1 on $[0, 1/2]$, supported in $[0, 1)$, $\rho' \leq 0$ on $[0, +\infty)$. In the incoming region we will take the symbol

$$q_- = x^{-\epsilon}\chi_-(\tau)\psi(p)\rho(x/x_0), \tag{4.12}$$

in the outgoing one the symbol

$$q_+ = x^{\epsilon}\chi_+(\tau)\psi(p)\rho(x/x_0), \tag{4.13}$$

with $\epsilon \in (0, \frac{1}{4})$. In the intermediate region we take

$$q_\partial = x^{-\epsilon}\chi_\partial(\tau)\psi(p)\rho(x/x_0). \tag{4.14}$$

Note that for any $\alpha \in \mathbb{R}$, $\chi, \rho \in \mathcal{C}^\infty(\mathbb{R})$,

$$\begin{aligned} &x^{-\alpha-1}H_p(x^\alpha\chi(\tau)\rho(x/r)) \\ &= (2\tau + x^\gamma a)(\alpha\rho(x/r) + r^{-1}\rho'(x/r))\chi(\tau) - (2g_\partial(\xi) + x^\gamma b)\rho(x/r)\chi'(\tau). \end{aligned} \tag{4.15}$$

Note that in the definition of q_-, $\alpha = -\epsilon < 0$, so $\alpha\rho(x/r) + r^{-1}\rho'(x/r) \leq 0$ everywhere. Moreover, on $\operatorname{supp}\chi_-$, $\tau > \lambda/3 > 0$, so for $x(\xi) \leq x_0$, $\xi \in \operatorname{supp}\psi(p)$, $\tau(\xi) \in \operatorname{supp}\chi_-$, $2\tau + x^\gamma a \geq \lambda/3 > 0$. In addition, $\tau \leq 2\lambda/3$ on $\operatorname{supp}\chi_-'$, so if $\xi \in \operatorname{supp}(\rho(x/x_0)\chi'(\tau)\psi(p))$ then $g_\partial \geq c_1 > 0$, hence $2g_\partial + x^\gamma b \geq c_1 > 0$ there. Thus,

$$x^{-1+\epsilon}H_p q_- \leq 0. \tag{4.16}$$

Moreover, $x \leq x_0/2$ implies $\rho'(x/x_0) = 0$, and $\tau \geq 2\lambda/3$ implies $\chi_-'(\tau) = 0$, so

$$x \leq x_0/2,\ \tau \geq 2\lambda/3 \Rightarrow -x^{-1+\epsilon}H_p q_- \geq c_2\psi(p),\ c_2 > 0. \tag{4.17}$$

The difference between q_- and q_+ is that $\tau\rho'$ is positive on $\operatorname{supp}\chi_+$, and $-\chi_+'$ is also positive, so the negativity estimate only holds away from $\operatorname{supp}\rho'$ and $\operatorname{supp}\chi_+'$. Thus, there is no analogue of (4.16), but the following analogue of (4.17) still holds:

$$x \leq x_0/2,\ \tau \leq -2\lambda/3 \Rightarrow -x^{-1-\epsilon}H_p q_+ \geq c_3\psi(p),\ c_3 > 0. \tag{4.18}$$

Next, q_∂ provides the connection between the incoming and outgoing regions. Since χ_∂' can be used to estimate χ_∂ on $(-7\lambda/8, 3\lambda/4)$, we see that

$$\tau(\xi) \in (-7\lambda/8, 3\lambda/4),\ x(\xi) \leq x_0/2,\ \xi \in \operatorname{supp}\psi(p) \Rightarrow |(2\tau + x^\gamma a)\chi_\partial(\tau)| \leq c_1\chi_\partial'(\tau)/2. \tag{4.19}$$

Since $\alpha = -\epsilon$, $|\alpha| < 1$, so we conclude that

$$\tau(\xi) \in (-7\lambda/8, 3\lambda/4),\ x(\xi) \leq x_0/2,\ \xi \in \operatorname{supp}\psi(p) \Rightarrow -x^{-1+\epsilon}H_p q_\partial \geq c_1\chi_\partial'(\tau)\psi(p) \geq 0. \tag{4.20}$$

Note that on $(-3\lambda/4, 3\lambda/4)$, $\chi_\partial' \geq C > 0$, so

$$\tau(\xi) \in (-3\lambda/4, 3\lambda/4),\ x(\xi) \leq x_0/2,\ \xi \in \operatorname{supp}\psi(p) \Rightarrow -x^{-1+\epsilon}H_p q_\partial \geq c_4\psi(p),\ c_4 > 0. \tag{4.21}$$

For $\xi \in T^*X^\circ$ with $p(\xi) \in (\lambda^2 - \delta_0, \lambda^2 + \delta_0)$, take $T = T_\xi > 0$ as in (4.8), so for $t \geq T$ we have $\tau(\exp(-tH_p)(\xi)) > 2\lambda/3$, $x(\exp(-tH_p)(\xi)) < x_0/2$. We will define a symbol q_ξ which is supported in a neighborhood of the bicharacteristic segment $\{\exp(-tH_p)(\xi) : t \in [0, T+1]\}$, and which satisfies $H_p q \leq 0$ over

$$K' = \{\xi' \in T^*X^\circ : x(\xi') \geq x_0/2 \text{ or } (x(\xi') \leq x_0/2 \text{ and } \tau(\xi') \leq 2\lambda/3\}. \tag{4.22}$$

Namely, let Σ be a hypersurface through ξ which is transversal to H_p. Then there is a neighborhood U_ξ of ξ, such that $V_\xi = \{\exp(-t(U_\xi \cap \Sigma)) : t \in (-1, T+2)\}$ is a neighborhood of the above bicharacteristic segment, which we can think of as a product $(-1, T+2) \times (U_\xi \cap \Sigma)$, and

$(T+1/2, T+2) \times (U_\xi \cap \Sigma)$ is disjoint from K'. Now let $\phi_\xi \in \mathcal{C}_c^\infty(U_\xi \cap \Sigma)$ be identically 1 near ξ, and let $\chi_\xi \in \mathcal{C}_c^\infty(\mathbb{R})$ be supported in $(-1, T+2)$, $\chi_\xi \geq 0$, $\chi_\xi' \geq 0$ on $(-1, T+2/3)$. Using the product coordinates, we can think of ϕ_ξ and χ_ξ as functions of ${}^{\mathrm{sc}}T^*X$ with compact support in V_ξ. Let

$$q_\xi = \chi_\xi \phi_\xi \psi(p), \tag{4.23}$$

so

$$H_p q_\xi = -\chi_\xi' \phi_\xi \psi(p). \tag{4.24}$$

Thus, for $\xi' \in K'$, $H_p q_\xi(\xi') \leq 0$.

Now let $K \subset T^*X^\circ$ be the compact set

$$K = \{\xi \in T^*X^\circ : \ \xi \in \operatorname{supp} \psi(p), \ x(\xi) \geq x_0/4\}. \tag{4.25}$$

Since K is compact, applying the previous argument for every $\xi \in K$ gives a U_ξ, and a $U_\xi' \subset U_\xi$ on which $\phi_\xi = 1$. Since $\{U_\xi' : \ \xi \in K\}$ covers K, the compactness of K shows that we can pass to a finite subcover, $\{U_{\xi_j}' : \ j = 1, \ldots, N\}$. We let

$$q_\circ = \sum_{j=1}^N q_{\xi_j}. \tag{4.26}$$

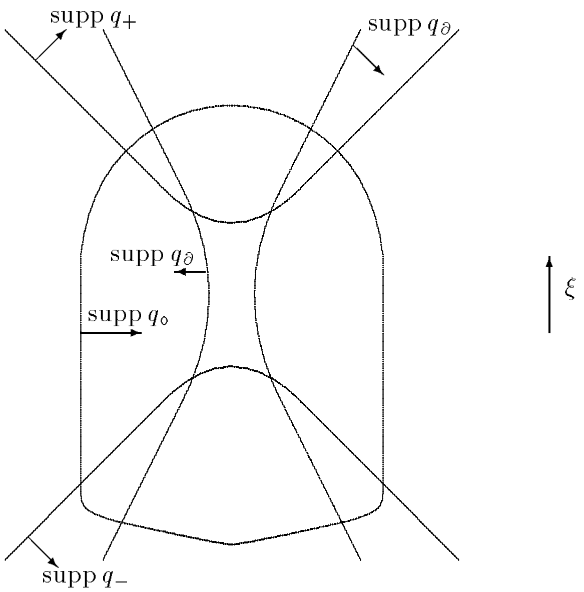

FIGURE 4. Supports of q_+, q_-, $q_\circ$ and q_∂ for $X = \overline{\mathbb{R}^n}$. ${}^{\mathrm{sc}}T^*X$ is identified with $\overline{\mathbb{R}^n} \times \mathbb{R}^n_\xi$, and the covector ξ is fixed on the picture.

The symbol we use in the positive commutator estimate is

$$q = q_- + C'' q_\partial + C q_\circ + C' q_+, \tag{4.27}$$

with $C, C', C'' > 0$ chosen appropriately. Namely note that in the region $x \leq x_0/2$, $\tau \geq 2\lambda/3$, which is the only place where $H_p q_\circ$ is positive, we have the estimate $-x^{-1+\epsilon} H_p q_- \geq c_2 > 0$. Since

$x^{-1+\epsilon}H_p q_\diamond$ is bounded, we can choose $C > 0$ sufficiently small so that $-x^{-1+\epsilon}H_p(q_- + Cq_\diamond)$ is still bounded below by a positive constant in this region. Then $-x^{-1+\epsilon}H_p(q_- + Cq_\diamond)$ is non-negative everywhere, and it is bounded below by a positive constant on $x \geq x_0/2$ as well as on $x \leq x_0/2$, $\tau \geq 2\lambda/3$. But this is the only region where the bounded function $x^{-1+\epsilon}H_p q_\partial$ is positive, so by choosing $C'' > 0$ sufficiently small, we can arrange that $-x^{-1+\epsilon}H_p(q_- + Cq_\diamond + C''q_\partial)$ is non-negative everywhere, and it is bounded below by a positive constant on $x \geq x_0/2$, as well as on $x \leq x_0/2$, $\tau \geq -3\lambda/4$. But this is the only region where $x^{-1-\epsilon}H_p q_+ > 0$. Thus, by choosing $C' > 0$ sufficiently small, and taking into account that $x^{-1+\epsilon}H_p = x^{2\epsilon}x^{-1-\epsilon}H_p q_+$, with $x^{-1-\epsilon}H_p q_+$ as well as $x^{2\epsilon}$ bounded, we can arrange that $-x^{-1+\epsilon}H_p q$ is non-negative everywhere, and $-x^{-1-\epsilon}H_p q$ bounded below by a positive constant everywhere. In summary, we have proved the proposition needed in Sect.3 (see (3.1))

Proposition 4.1. *There exist functions $q \in S^{-\epsilon,\infty}(X)$, $\psi \in \mathcal{C}_c^\infty(\mathbb{R};[0,1])$, $\psi \equiv 1$ near λ^2, and $c', c'' > 0$ such that*

$$q \geq c'x^\epsilon\psi(p), \ -H_p q \geq c''x^{1+\epsilon}\psi(p). \tag{4.28}$$

Thus, the results of the previous section show that there exists $h_0 > 0$ such that for $h \in (0, h_0)$,

$$\|R(\lambda^2 + it)f\|_{H_{sc}^{*,-1/2-\epsilon}(X)} \leq C_0 h^{-1}\|f\|_{H_{sc}^{*,1/2+3\epsilon}(X)}. \tag{4.29}$$

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