DIFFRACTION BY EDGES

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Abstract. In these expository notes we explain the role of geometric optics in wave propagation on domains or manifolds with corners or edges. Both the propagation of singularities, which describes where solutions of the wave equation may be singular, and the diffractive improvement under non-focusing hypotheses, which states that in certain places the diffracted wave is more regular than a priori expected, is described. In addition, the wave equation on differential forms with natural boundary conditions, which in particular includes a formulation of Maxwell's equations, is studied.

1. Introduction

The aim of the present notes is twofold. On the one hand, these are expository notes intended to explain certain aspects of diffraction by edges and corners. On the other hand, they contain the announcement of three new results. The first new result is joint with Richard Melrose and Jared Wunsch, namely diffractive improvements for the scalar wave equation on $X = M \times \mathbb{R}$ where M is a manifold with corners equipped with a smooth Riemannian metric; this is explained in the last section. The second new result is propagation of singularities for the scalar wave equation on Lorentzian manifolds with corners (with time-like boundary faces), i.e. where X is not a metric product like above. Furthermore, we explain the more general setting of Maxwell's equations *with natural boundary conditions*, and discuss microlocal elliptic estimates for these. The proofs of the last two results turn out to be a rather simple modification of the proofs for the scalar wave equation on product spaces, in the sense that the same method works, but some additional care needs to be taken in constructions, so we will not need to describe the technicalities that do not need any change in too great detail. The proof of the propagation of singularities results for natural boundary conditions are more technical, so it will be discussed elsewhere.

The full details of the proof of the main propagation of singularities theorem, Theorem 13, in the scalar metric product setting (also valid directly for the wave equation on forms with Dirichlet or Neumann boundary conditions, which are however not the interesting ones), are written up in [27]. The diffractive improvement in a model case, namely edge manifolds, defined in the last section, is proved in [15], and its extension to manifolds with corners is currently being written up in [14]. Moreover, [28] contains an expository description of the propagation results for the scalar equation, while [26] provides an exposition at an intermediate level: the main technical points are explained there.

Date: September 2008.
This work is partially supported by NSF grant DMS-0733485.

Light is described by Maxwell's equations, which in turn imply that in free space each component u of the electromagnetic field satisfies the wave equation,

$$Pu = 0, \quad Pu = D_t^2 u - \Delta_g u,$$

Δ_g is the Laplacian, so it is $c^2 \sum_{j=1}^n D_{x_j}^2$ in $\mathbb{R}^n$, where c is the speed of light (this corresponds to a Riemannian metric $g = c^{-2} \sum dx_j^2$), $D_{x_j} = \frac{1}{i}\partial_{x_j}$. If light propagates in regions with boundaries, one also needs suitable boundary conditions. A typical condition, if the boundary is a perfect conductor, is that the tangential component of the electric field and the normal component of the magnetic field vanish at the boundary hypersurfaces. This is an example of a *natural boundary condition*, as we shall soon see.

As PDE are relatively complicated, it is natural to ask whether one can find important qualitative information about solutions of the wave equation without actually solving the equation. A step in this direction is given by *geometric optics*.

According to geometric optics, light propagates in straight lines (in homogeneous media), reflects/refracts from surfaces according to Snell's law: energy and tangential momentum are conserved. Thus, when reflecting from a hypersurface (which has codimension one) one gets the usual law of incident and reflected rays enclosing an equal angle to the normal to the surface. Indeed, conservation of tangential momentum and kinetic energy implies that of the *magnitude* of the normal component. When reflecting from a higher codimension (≥ 2) corner, the law is unchanged (momentum tangential to the corner and energy are conserved) – but now this allows each incident ray to generate a whole cone of reflected rays, see Figures 1-2. In addition, even the local geometry of the rays can be very complicated because of rays tangential to a boundary face: one can even have an accummulation of reflection points, as shown by an example of Taylor [22].

It is natural to ask how these points of view are related. One way of discussing the relationship between these is that singularities (lack of smoothness) of solutions of $Pu = 0$ follow geometric optics rays. Due to its relevance, this problem has a long history, and has been studied extensively by Keller and others in the 1940s and 1950s in various special settings, see e.g. [1, 9]. The present work (and ongoing projects continuing it, especially joint work with Melrose and Wunsch [15], see also [2, 16]) can be considered a justification of Keller's work in the general geometric setting (curved edges, variable coefficient metrics, etc). In order to describe this relationship precisely, I discuss an even more general setting.

The first main result discussed here is a precise statement of this result for domains with corners in a general (Riemannian or Lorentzian) geometric setting, including for the wave equation on differential forms with certain boundary conditions. In the analytic setting for scalar equations this result is due to Lebeau.

The second result discussed here, which is joint work with R. Melrose and J. Wunsch [15], is that while the preceeding result is optimal, for a rather large class of solutions of the wave equation, namely those 'not focusing' on the corner, it can be improved. As an illustration, consider spherical waves emanating from a source near the boundary or corner: on the one hand, most of the spherical wave misses the corner, i.e. only a lower dimensional part hits it, but on the other hand, a full dimensional part of the spherical wave hits the boundary hypersurfaces (or smooth boundary).

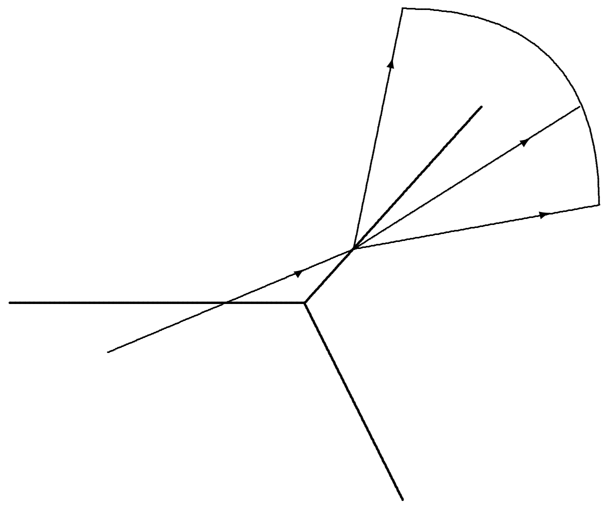

FIGURE 1. Geometric optics rays hitting a surface at a codimension 2, dimension 1, corner (which may be called an edge). The momentum component parallel to the edge is preserved when the edge is hit, as is the magnitude of the normal component, so a single incident ray generates a cone with apex at the point where the edge is hit, axis given by the edge, and angle at the apex given by the angle between the incident ray and the edge. On the picture only the projection of the rays to the spatial factor, M, is shown; time can be thought of as the arclength parameter along the rays.

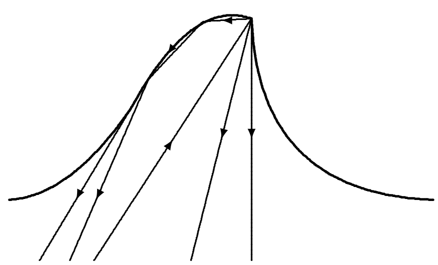

FIGURE 2. Geometric optic rays hitting a corner. Even if a ray hits the corner non-tangentially to any boundary hypersurface, the reflected rays may be tangential to one of these, hence their geometry may be complicated.

Informally stated, this second result is that under a non-focusing assumption, which holds for instance for spherical waves emanating from a source near the edge, the diffracted wave is $1/2-\epsilon$ order more regular (in a Sobolev sense, for all $\epsilon>0$) than either the incident or the reflected wave. This result is expected to be useful in inverse problems, e.g. when studying the reflection of seismic waves from cracks in the Earth. In 2 dimensions, in the analytic category, there is a corresponding result due to Gérard and Lebeau [4] for conormal incident waves. There is also a long history of the subject in applied mathematics, especially in the work of Keller.

The original version of these notes were based on my transparencies and lecture notes at the inverse quantum scattering conference at Siófok, Hungary, in August 2007, with additional material included later on. I am very grateful to the conference organizers for the invitation and for hosting the meeting so well, as well as for their patience as I overran many deadlines while preparing these notes.

2. The wave equation

In this section we briefly discuss the wave equation on Lorentz manifolds. A more thorough description can be found in Taylor's book [24, Sections 2.10-2.11]. Below, if X is an n-dimensional C^∞ manifold, ΩX denotes the space of *densities* on X, i.e. for $z \in X$, $\Omega_z X$ consists of maps $\omega : \wedge^n T_z X \to \mathbb{R}$ (where $\wedge^n T_z X$ can be identified with completely antisymmetric n-linear maps on $T_z^* X$) satisfying

$$\omega(tV_1 \wedge \ldots \wedge V_n) = |t|\omega(V_1 \wedge \ldots \wedge V_n).$$

On $\mathbb{R}^n$, $\Omega\mathbb{R}^n$ is trivialized by $\omega_0 = |dz_1 \wedge \ldots \wedge dz_n|$ which satisfies $\omega_0(\partial_{z_1}, \ldots, \partial_{z_n}) = 1$. One can naturally integrate densities, and on oriented manifolds they can be identified with n-forms; see below for more. Here we will usually not differentiate between real vector spaces and their complexification, so e.g. we write both the real and complex tangent spaces at z as $T_z X$, rather than say $T_z^{\mathbb{C}} X$ for the complex case.

On $\mathbb{R}^n$, each element f of $C(\mathbb{R}^n)$ defines a continuous linear functional on $C_c^\infty(\mathbb{R}^n)$ (still denoted by f), where the subscript c denotes compact support, by

$$f : C_c^\infty(\mathbb{R}^n) \ni \phi \mapsto \int_{\mathbb{R}^n} f\phi \in \mathbb{C}.$$

While $C_c^\infty(\mathbb{R}^n)$ can be thought of as the space of 'extremely nice' functions, $\mathcal{D}'(\mathbb{R}^n)$ stands for the space of distributions on X (i.e. continuous linear functionals on $C_c^\infty(\mathbb{R}^n)$), which, by the above observation, are 'generalized functions'. On a manifold X the same argument goes through except we can only integrate densities, so $\mathcal{D}'(X)$ is the dual of $C_c^\infty(X; \Omega X)$. (If we fix a non-vanishing density ω, e.g. arising from a Riemannian or Lorentzian metric, as discussed below, we can trivialize the density bundle, and identify $C_c^\infty(X; \Omega X)$ with $C_c^\infty(X)$.)

Suppose that X is a manifold without boundary of dimension n, and let h be a Lorentz metric on X, i.e. h is a real non-degenerate symmetric 2-cotensor of signature $(1, n-1)$. (Some people prefer signature $(n-1, 1)$, which would amount to switching some signs below.) Thus, for each $z \in X$, $h(z)$ is a symmetric bilinear map $T_z X \times T_z X \to \mathbb{R}$, $h(z)(V, W) = 0$ for all $W \in T_z X$ implies $V = 0$, and the maximal dimension of a subspace of $T_z X$ to which the restriction of h is positive definite is 1. In local coordinates, $h = \sum_{ij} h_{ij}(z)\, dz_i \otimes dz_j$, with (h_{ij}) symmetric, and having one positive and $n-1$ negative eigenvalues as an endomorphism of $\mathbb{R}^n$. With our signature convention, vectors $V \in T_z X$ are called *time-like* if $h(V, V) > 0$, *space-like* if $h(V, V) < 0$, and *light-like* or *characteristic* if $h(V, V) = 0$. The metric h gives rise to a smooth measure, or density, in local coordinates $d\mathrm{vol}_h = |\det h|\, |dz|$, with $\det h = \det(h_{ij})$, i.e. for $f \in C_c^\infty(X)$ supported in the coordinate chart, $\int f\, d\mathrm{vol}_h = \int f(z) |\det h(z)|\, dz$. As h is non-degenerate, the determinant never vanishes, so in particular we get a positive definite inner product on $C_c^\infty(X)$, and on $L^2(X)$.

A special case of such (X,h) is products $X = M\times\mathbb{R}$, where (M,g) is a Riemannian manifold, and $\mathbb{R}$ is 'time', with the Lorentz metric on X being $h = dt^2 - g$. Thus, for a vector $V = (V_M, V_T)$, where $V_M \in T_m M$, $V_T \in \mathbb{R} = T_t\mathbb{R}$, $z = (m,t) \in X$, $h(V,V) = |V_T|^2 - g(V_M,V_M)$. In particular, $(V_M, 0)$ is space-like (for $V_M \neq 0$), $(0, V_T)$ is time-like (for $V_T \neq 0$), while (V_M, V_T) is characteristic if $|V_T|^2 = g(V_M, V_M)$.

The Lorentz metric also gives rise to a dual metric, which is a non-degenerate symmetric bilinear form on $T^*_z X \times T^*_z X$. Indeed, non-degeneracy implies that the map $\hat h : T_z X \ni V \mapsto h(V,.) \in T^*_z X$ is injective, hence an isomorphism as $\dim T_z X = \dim T^*_z X$, and then we can define the dual metric H by

$$H(\alpha,\beta) = h(\hat h^{-1}\alpha, \hat h^{-1}\beta),\ \alpha,\beta\in T^*_z X.$$

We call a covector $\alpha \in T_z X$ time-like, space-like or characteristic if $H(\alpha,\alpha) > 0$, $H(\alpha,\alpha) < 0$ or $H(\alpha,\alpha) = 0$. We recall that if S is a submanifold of X then N^*S is the *conormal bundle* of S; at a point $p \in S$, the fiber $N^*_p S$ consists of all covectors $\alpha \in T^*_p S$ such that $\alpha(V) = 0$ for all $V \in T_p S$. Another way of looking at N^*S is that the space of its smooth sections is spanned (over $C^\infty(S)$) by da, as a ranges over all elements of $C^\infty(X)$ that vanish on S. If S is a *hypersurface*, i.e. has codimension 1, we call S space-like, time-like resp. characteristic, if non-zero elements of its conormal bundle are time-like, space-like (note the reversal!), resp. characteristic. In particular, in the product case, $X = M \times \mathbb{R}$, dt is time-like while if f is a function on M pulled back to X, then df is space-like (whenever it is non-zero). Correspondingly, $M \times \{t_0\}$ is space-like (with conormal dt), while if S_0 is a hypersurface in M, then $S_0 \times \mathbb{R}$ is time-like.

This bilinear form H on $T^*_z X$ then extends to the differential form bundle, ΛX, i.e. for each z, one has a non-degenerate symmetric bilinear form on the 2^n-dimensional vector space $\Lambda_z X$, with respect to which the grading of forms by degree is an orthogonal decomposition. Namely, on k-forms,

$$H(dz_{i_1}\wedge\ldots\wedge dz_{i_k}, dz_{j_1}\wedge\ldots\wedge dz_{j_k}) = \sum_{\pi\in S_k} (\operatorname{sgn}\pi) H(dz_{i_1}, dz_{j_{\pi(1)}})\ldots H(dz_{i_k}, dz_{j_{\pi(k)}}),$$

where the sum is over all permutations π of $(1,\ldots,k)$, extended linearly to $\Lambda^k_z X$. Thus, one has a (non-positive but non-degenerate) inner product on sections of ΛX, namely

$$\langle\alpha,\beta\rangle = \int_X (\alpha,\beta)_H \,\mathrm{dvol}_h,\ (\alpha,\beta)_H = H(\alpha,\beta).$$

In particular, we can define formal adjoints for differential operators

$$P : C^\infty(X;\Lambda X) \to C^\infty(X;\Lambda X)$$

by

$$\langle P^* u, v\rangle = \langle u, Pv\rangle,\ u,v \in C^\infty(X;\Lambda X),$$

$P^* : \mathcal{D}'(X;\Lambda X) \to \mathcal{D}'(X;\Lambda X)$, and it is then straightforward to check that P^* itself is a differential operator. Then the d'Alembertian $\Box$ on differential forms is defined by

$$\Box = (d + d^*)^2 = dd^* + d^*d$$

in analogy with the Laplace-Beltrami operator in the Riemannian setting. Thus, for $u, f \in C^\infty(X;\Lambda X)$, $\Box u = f$ if and only if $\langle \Box u, v\rangle = \langle f, v\rangle$ for all $v \in C^\infty_c(X;\Lambda X)$, i.e. in the symmetric quadratic form formulation, if and only if

$$\langle du, dv\rangle + \langle d^*u, d^*v\rangle = \langle f, v\rangle$$

for all $v \in C_c^\infty(X; \Lambda X)$; indeed this holds even if $u, f \in \mathcal{D}'(X; \Lambda X)$.

A useful way of relating d and d^* is given by the Hodge star operator $*$, which requires that X be orientable (i.e. the existence of a global non-vanishing section ω_0 of $\Lambda^n X$), and a choice of orientation. For such an X, there is a unique n-form ω with the correct orientation (i.e. being a *positive* multiple of a preferred section) and with $|H(\omega,\omega)| = 1$; in fact, the choice of an orientation gives an isomorphism between densities and n-forms. If X is oriented (which always holds locally), then the Hodge star operator $* : \Lambda^k X \to \Lambda^{n-k} X$ is characterized by $u \wedge *v = H(u, v)\omega$. Then $d^* = (\operatorname{sgn} h)(-1)^{(k+1)n+1} * d *$ on $C^\infty(X, \Lambda^k X)$, where $\operatorname{sgn} h$ is the signature of h (1 in the Riemannian setting, $(-1)^{n-1}$ in the Lorentz setting with our signs) and $*\Box = \Box *$, so u solves $\Box u = 0$ if and only if $*u$ solve $\Box * u = 0$.

We can now turn to boundaries and corners. First, we define C^∞ manifolds with corners. These are topological manifolds with boundary with a C^∞ structure with corners, which means that each point p in X has a neighborhood $O = O_p$ diffeomorphic to an open subset U of $[0,\infty)^k \times \mathbb{R}^{n-k}$; we denote the corresponding coordinates by (x, y), so $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_{n-k})$, k depends on O, with the transition maps between the coordinate charts C^∞. The *tangent* and *cotangent bundles* on X can be either defined the usual way on X, or by embedding X in a manifold without boundary (by 'doubling' it locally over each boundary hypersurface), and restricting the the resulting bundles to X. Thus, covectors have the form

$$\alpha = \sum_{i=1}^{k} \xi_i \, dx_i + \sum_{i=1}^{n-k} \zeta_i \, dy_i, \tag{1}$$

and (x, y, ξ, ζ) give local coordinates on T^*X. (Actually, they are global on the fibres of $T^*X \to X$.)

If X is a manifold with C^∞ boundary, $d, d^*, \Box$ are differential operators on X with smooth coefficients, defined at first by the above formulae for $C_c^\infty(X^\circ; \Lambda X)$, then noting that by the smoothness of their coefficients, they act on $C^\infty(X; \Lambda X)$. We assume that ∂X is time-like. This is the case for instance if $X = M \times \mathbb{R}$, where now M is a manifold with boundary. As usual, one needs *boundary conditions* so that $\Box$ is symmetric. For instance, one could take Dirichlet boundary conditions, which amounts to requiring that $u \in C^\infty(X; \Lambda X)$ vanishes at ∂X, and

$$\langle du, dv\rangle + \langle d^*u, d^*v\rangle = \langle f, v\rangle \tag{2}$$

for all $v \in C_c^\infty(X; \Lambda X)$ that vanishes at ∂X. Or, we could take Neumann boundary conditions, which amounts to requiring $u \in C^\infty(X; \Lambda X)$ and (2) holds for all $v \in C_c^\infty(X; \Lambda X)$. However, for either of these boundary conditions, $d + d^*$ itself is not symmetric.

Natural boundary conditions for forms are $\nu \wedge u = 0$ at ∂X if ν is a non-vanishing conormal to $H = \partial X$, called the *relative* boundary condition, and its dual $\iota_\nu u = 0$, called the *absolute* boundary condition (with these conditions being independent of the choice of ν). $\nu \wedge u = 0$ is interpreted as u is *normal* to H, while $\iota_\nu u = 0$ as u being *tangential* to H. We write $\Lambda_R(H)$ for the subbundle of $\Lambda X|_H$ consisting of normal forms; it is thus the kernel of the endomorphism $\alpha \mapsto \nu \wedge \alpha$. We also write

$$C^\infty_R(X;\Lambda X) = \{u \in C^\infty(X;\Lambda X):\ u|_H \in C^\infty(H;\Lambda_R H)\}.$$

Note that the *vanishing* of $\nu \wedge u$ is well-defined, independent of the choice of h, while the vanishing of ι_ν depends on h: one needs the non-degenerate h to identify ν with a vector $H(\nu,.) = \hat{h}^{-1}(\nu)$, and then $\iota_\nu = \iota_{\hat{h}^{-1}(\nu)}$, with ι_V being the evaluation of forms on a vector field V. However, these conditions are *dual* to each other in that $\iota_\nu v = 0$ if and only if $\nu \wedge *v = 0$, i.e. v satisfies absolute boundary conditions if and only if $*v$ satisfies relative boundary conditions. In local coordinates $(x, y_1, \dots, y_{n-1})$ near ∂X, $x = x_1$ (recall ∂X is C^∞), one can take $\nu = dx$. For a form u, $dx \wedge u = 0$ states that u is a linear combination of $dx \wedge dy_{j_1} \wedge \dots \wedge dy_{j_m}$, explaining why u is normal. On the other hand, *if* $\nu = dx$ *is orthogonal to* $dy_1, \dots, dy_{n-1}$ *with respect to* H, $\iota_{dx} u = 0$ states that u is a linear combination of $dy_{j_1} \wedge \dots \wedge dy_{j_m}$, explaining why u is tangential. In the product case, $X = M \times \mathbb{R}_t$, it is natural to keep t as one of the y variables, i.e. $y_{n-1} = t$.

If X is a manifold with corners, $d, d^*, \Box$ are still differential operators with smooth coefficients, and the boundary conditions are required for all codimension 1 boundary faces, i.e. for all *boundary hypersurfaces*. The local form of the relative boundary conditions is as follows: at a codimension k corner, given by $x_1 = \dots = x_k = 0$, the condition on $u \in C^\infty(X, \Lambda X)$ is that $dx_j \wedge u = 0$ at $H_j = \{x_j = 0\}$ for all j. In order to make these compatible for our analysis, we need a local trivialization of $\Lambda^p X$ for all p, i.e. a map $\Lambda^p O \to O \times \mathbb{R}^N$, $N = \dim \Lambda^k$ being given by the binomial coefficient, and an index set $J_j \subset \{1, \dots, N\}$ for $j = 1, \dots, k$, such that for each j and at each $q \in O \cap H_j$, for a form α to satisfy $dx_j \wedge u = 0$ requires that $\alpha_m = 0$ for $m \in J_j$, where $\alpha = (\alpha_1, \dots, \alpha_N)$ with repect to the trivialization. This is straightforward, however, using

$$(3) \qquad dx_{i_1} \wedge \dots \wedge dx_{i_s} \wedge dy_{\ell_1} \wedge \dots \wedge dy_{\ell_{p-s}},\ i_1 < \dots < i_s,\ \ell_1 < \dots < \ell_{p-s},$$

as the basis of $\Lambda^p_q X$, $dx_j \wedge u = 0$ amounts to saying that all components of α in which j is not one the i_r's vanish. Similarly, using the Hodge star operator, there is such a good trivialization for the absolute boundary condition as well, namely $*$ applied to the basis of (3).

Now recall that if X is a smooth manifold with corners, $H^k(X)$ can be defined as the completion of $C^\infty_c(X)$ in the H^k norm (here $k \geq 0$ integer), or equivalently as the space of restrictions of H^k functions from the 'double of X' in which X has been extended across all boundary hypersurfaces. In the Riemannian setting there is a natural H^k norm given by ∇ and the metric, but over compact set all choices of metrics give rise to equivalent norms, so in fact $H^k_{\mathrm{loc}}(X)$ and $H^k_c(X)$ are defined independently of such choices. This immediately extends to sections of vector bundles: we again need a metric on the fibers of the bundle for a global definition, but the local definition is independent of any such choices. Now, the restriction to boundary hypersurfaces for $C^\infty(X;E)$ induces a restriction map $H^1(X;E) \to H^{1/2}(H_j;E)$, as usual, and in view of the trivialization (3), $C^\infty_R(X;\Lambda X)$, resp. $C^\infty_{R,c}(X;\Lambda X)$ are dense in $H^1_{R,\mathrm{loc}}(X;\Lambda X)$, resp. $H^1_{R,c}(X;\Lambda X)$, where $H^1_{R,\mathrm{loc}}(X,\Lambda X)$ and $H^1_{R,c}(X,\Lambda X)$ are defined analogously to $C^\infty_R(X;\Lambda X)$.

For products $X = M \times \mathbb{R}$, $h = dt^2 - g$, one also has a functional analytic picture. Namely, with Dirichlet, Neumann, or natural boundary conditions, Δ_g is self-adjoint on $L^2(M;\Lambda M)$ (with respect to the induced inner product). Moreover, as shown by Mitrea, Taylor and the author in [23], the quadratic form domain $\mathcal{D} =$

$\mathcal{D}_1$ of Δ_g with respect to natural boundary conditions is the subspace $H^1(M,\Lambda M)$ given $\nu\wedge u=0$, resp. $\iota_\nu u=0$, at all boundary hypersurfaces, where ν denotes a conormal. In addition, there is an orthogonal decomposition of the form bundle, $\Lambda X=\Lambda M\oplus(dt\wedge\Lambda M)$, and $C^\infty(X;\Lambda M)$, resp. $C^\infty(X;dt\wedge\Lambda M)$ are preserved by Δ_g and D_t^2, so the d'Alembertian on X gives rise to the wave equation on $M\times\mathbb{R}$ with values in ΛM. Thus, the solutions of the wave equation in this functional analytic sense, i.e. the solutions of $D_t^2u=\Delta_g u$ with $u\in C(\mathbb{R};\mathcal{D})\cap C^1(\mathbb{R},L^2(M,\Lambda M))$ are exactly the $u\in H^1_{\mathrm{loc}}(X;\Lambda M)$ satisfying the boundary condition (say, $\nu\wedge u=0$ at ∂X), such that (2) holds for all $v\in H^1_c(X;\Lambda M)$ satisfying the boundary condition, with $f=0$.

We can now explain Maxwell's equations in units in which the speed of light is 1. In $\mathbb{R}^4=\mathbb{R}^3\times\mathbb{R}_t$, writing the coordinates on $\mathbb{R}^3$ as (z_1,z_2,z_3), we can identify the electric field $E:\mathbb{R}^4\to\mathbb{R}^3$ and magnetic field $B:\mathbb{R}^4\to\mathbb{R}^3$ with the 2-form on $\mathbb{R}^4$ given by

$$\begin{aligned}-\mathcal{F}=&B_1\,dz_2\wedge dz_3+B_2\,dz_3\wedge dz_1+B_3\,dz_1\wedge dz_2\\&+E_1\,dz_1\wedge dt+E_2\,dz_2\wedge dt+E_3\,dz_3\wedge dt.\end{aligned}$$

Maxwell's equations in free space (without charges and currents) are then equivalent to $\Box\mathcal{F}=0$; the general version is $\Box\mathcal{F}=f$, with f given by the charges and currents. If we impose Maxwell's equations in $z_3\geq 0$, and write $x_1=z_3$, $y_j=z_j$, $j=1,2$, $\mathcal{F}$ being normal means that $B_3=0$ and $E_1=E_2=0$, i.e. the electric field is normal to the boundary while the magnetic field is tangential to it. This then generalizes to other regions with smooth boundaries and also to other Lorentz metrics (i.e. a background from general relativity) to fit into the framework described above, with X a 4-manifold, and u being a 2-form. We refer to [24, Section 2.11] for a more detailed discussion.

It is often useful to choose local coordinates with somewhat more care. First, in the product setting, we always use local coordinates $(x,\tilde{y})$ arising from an open set U in M and t as local coordinates on X, i.e. on X we have local coordinates $w=(x,y)$ where $y=(\tilde{y},t)$. In such local coordinates the dual metric G on M is

$$G(x,y)=\sum_{i,j}\tilde{A}_{ij}(x,\tilde{y})\,\partial_{x_i}\,\partial_{x_j}+\sum_{i,j}2\tilde{C}_{ij}(x,\tilde{y})\,\partial_{x_i}\,\partial_{\tilde{y}_j}+\sum_{i,j}\tilde{B}_{ij}(x,\tilde{y})\,\partial_{\tilde{y}_i}\,\partial_{\tilde{y}_j}\tag{4}$$

with A,B,C smooth. Moreover, the coordinates on M can be chosen (i.e. the $\tilde{y}_j$ can be adjusted) so that $C(0,\tilde{y})=0$. Then on $U\times\mathbb{R}$,

$$H|_{x=0}=\partial_t^2-\sum_{ij}\tilde{A}_{ij}(\tilde{y})\,\partial_{x_i}\,\partial_{x_j}-\sum_{ij}\tilde{B}_{ij}(\tilde{y})\,\partial_{\tilde{y}_i}\,\partial_{\tilde{y}_j},\tag{5}$$

with $\tilde{A}$, $\tilde{B}$ positive definite matrices depending smoothly on $\tilde{y}$.

In the more general Lorentzian setting, the analogue of (4) on X is

$$H(x,y)=\sum_{i,j}A_{ij}(x,y)\,\partial_{x_i}\,\partial_{x_j}+\sum_{i,j}2C_{ij}(x,y)\,\partial_{x_i}\,\partial_{y_j}+\sum_{i,j}B_{ij}(x,y)\,\partial_{y_i}\,\partial_{y_j}\tag{6}$$

with A,B,C smooth. In this paper *we assume that every boundary face F is time-like in the sense that the restriction of H to N^*F is negative definite*, so A is negative definite (for the conormal bundle N^*F is given by $\zeta=0$ at $x=0$). Then H is Lorentzian on the H-orthocomplement $(N^*F)^\perp$ of N^*F. In fact, note that for $p_0\in F$,

$$T^*_{p_0}X=N^*_{p_0}X\oplus(N^*_{p_0}X)^\perp,\tag{7}$$

for if V is in the intersection of the two summands, then $H(V,V)=0$ and $V\in N^*_{p_0}F$, so the definiteness of the inner product on N^*F shows that $V=0$, hence (7) follows as the dimension of the summands sums up to the dimension of $T^*_{p_0}X$. Choosing an orthogonal basis of $(N^*F)^\perp$ at a given point $p_0\in F^\circ$, and then coordinates y_j with differentials equal to these basis vectors, we have in the new basis that $C_{ij}(0,0)=0$ and

$$\sum B_{ij}(0,0)\partial_{y_i}\partial_{y_j}=\partial^2_{y_{n-k}}-\sum_{i<n-k}\partial^2_{y_i},$$

and we write coordinates on T^*X as

$$x,\ t=y_{n-k},\ \tilde y=(y_1,\ldots,y_{n-k-1}),\ \xi,\ \tau=\zeta_{n-k},\ \tilde\zeta=(\zeta_1,\ldots,\zeta_{n-k-1}),$$

cf. (1). Thus B is non-degenerate, Lorentzian, near p_0, so a simple calculation shows that and the coordinates on X can be chosen (i.e. the y_j can be adjusted) so that $C(0,y)=0$. Then

$$H|_{x=0}=\sum_{i,j}A_{ij}(0,y)\,\partial_{x_i}\,\partial_{x_j}+\sum_{i,j}B_{ij}(0,y)\,\partial_{y_i}\,\partial_{y_j} \tag{8}$$

Note that in the product setting (with $t=y_{n-k}$) $A_{ij}=-\tilde A_{ij}$, $C_{ij}=-\tilde C_{ij}$, $j<n-k$, $C_{i,n-k}=0$, $B_{ij}=\tilde B_{ij}$, $i,j<n-k$, $B_{n-k,n-k}=1$, $B_{n-k,j}=B_{j,n-k}=0$ if $j<n-k$.

It is also useful to have a positive definite inner product on ΛX. Thus, in addition to the given Lorentzian metric h we often also consider a Riemannian metric $\tilde h$. Let H, resp. $\tilde H$ denoting the dual metrics, as well as the induced metrics on forms; these can be thought of as maps $\Lambda_pX\to(\Lambda_pX)^*$, hence

$$J=H^{-1}\tilde H \tag{9}$$

is an isomorphism of Λ_pX. Note that the form inner products then satisfy

$$\langle u,Jv\rangle_H=\langle u,v\rangle_{\tilde H}, \tag{10}$$

and the inner product on the right hand side is positive definite.

One nice feature of the product case is that there is a natural Riemannian metric on X as well, namely $\tilde h=dt^2+g$. Then J commutes with Δ_g and D_t. Moreover, it preserves $C^\infty_R(X;\Lambda X)$ as well as $C^\infty_A(X;\Lambda X)$. In fact, with $(x,\tilde y)$ coordinates on M, $y=(\tilde y,t)$, it maps $\alpha=dx_{i_1}\wedge\ldots\wedge dx_{i_s}\wedge dy_{\ell_1}\wedge\ldots\wedge dy_{\ell_{p-s}}$ to $J\alpha=(-1)^p\alpha$ if $\ell_{p-s}\neq n-k$, and to $J\alpha=(-1)^{p-1}\alpha$ if $\ell_{p-s}=n-k$.

In the general case we cannot pick $\tilde h$ arbitrarily because we need to preserve boundary conditions. However, with coordinates as in (8), taking $\tilde h=-\xi\cdot A\xi+|\zeta|^2$ does the job, for with respect to this metric the summands in (7) are orthogonal, $J=H^{-1}\tilde H=-1$ on the span of the dx_j, so in particular J maps normal forms to normal forms (it affects the dy terms in (3), but not the dx terms, up to an overall negative sign.)

In this paper we will be concerned with the general Lorentzian setting, assuming that *every boundary face F of X is time-like* in the sense that H restricts to be negative definite on N^*F. We will fully treat the scalar equation, as well as the equation on forms, with Dirichlet or Neumann boundary conditions. However, for natural boundary conditions we only deal with the elliptic regions due to some issues that are explained in the penultimate section. The propagation results will be taken up elsewhere.

3. Microlocal analysis on manifolds without boundary

Suppose X is a manifold without boundary of dimension n. As outlined in the introduction, we want to connect analytic objects (such as the wave operator) with geometric objects (such as certain curves related to the light rays). This is accomplished by the so-called microlocal, or phase space, analysis. The standard setting for microlocal analysis is the cotangent bundle – T^*X *is* the phase space. If z_j are local coordinates on X, and we write one-forms as $\sum \zeta_j\, dz_j$, then (z_j, ζ_j), $j = 1, \dots, n$, are local coordinates on T^*X.

For our purposes there are two important structures on T^*X. First, being a vector bundle, T^*X is *equipped with an* $\mathbb{R}^+$*-action* (dilation in the fibers): $\mathbb{R}^+_s \times T^*X \ni (s, z, \zeta) \mapsto (z, s\zeta)$. In particular, homogeneous degree m functions with respect to the $\mathbb{R}^+$-action, also called *positively homogeneous functions* on $T^*X \setminus o$ (o denoting the zero section) are those functions p for which $p(z, s\zeta) = s^m p(z, \zeta)$ for $s > 0$. (There are no smooth functions p which are homogeneous of order $m \in \mathbb{R} \setminus \mathbb{N}$, with the problem being smoothness at the zero section, which explains why we disregard the latter.) T^*X is also a *symplectic manifold*, equipped with a canonical symplectic form ω, $\omega = \sum d\zeta_j \wedge dz_j$ in local coordinates.

If F is a vector bundle over X, $\pi : T^*X \to X$ the bundle projection, then π^*F is a vector bundle over T^*X whose fiber over (z, ζ) is F_z, the fiber of F over z. If p is a section of π^*F, then for fixed z, but different ζ's, $p(z, \zeta)$ lies in the same vector space, F_z, so one can talk about positively homogeneous sections of degree m of π^*F, namely the ones for which $p(z, s\zeta) = s^m p(z, \zeta)$ for $s > 0$.

We can now turn to differential operators. It is useful to recall the multiindex notation: if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, then $D_z^\alpha = D_{z_1}^{\alpha_1} \dots D_{z_n}^{\alpha_n}$, with $D_j = D_{z_j} = \frac{1}{i}\partial_{z_j}$ (and $\mathbb{N}$ is the set of non-negative integers). (The appearance of the factor of $\frac{1}{i}$ is explained by the intertwining relation given by the Fourier transform.)

If P is a scalar differential operator on X, say $P = \sum_{|\alpha| \leq m} a_\alpha(z) D_z^\alpha$ in some local coordinates, one can associate a *principal symbol*

$$p(z, \zeta) = \sigma_m(P)(z, \zeta) = \sum_{|\alpha| = m} a_\alpha(z) \zeta^\alpha$$

to P; this is a positively homogeneous degree m function on $T^*X \setminus o$.

If P is an mth order differential operator acting on sections of a rank ℓ vector bundle E over X (the set of which is denoted by $\mathrm{Diff}(X, E)$), then in local coordinate charts in which E is trivial, P is given by a ℓ-by-ℓ matrix $P = (P_{jk})$, where each $P_{jk} = \sum_{|\alpha| \leq m} a_{jk,\alpha}(z) D_z^\alpha$ is a scalar differential operator, hence has its principal symbol $p_{jk} = \sum_{|\alpha| = m} a_{jk,\alpha}(z)\zeta^\alpha$ as above, and the principal symbol of P itself is the ℓ-by-ℓ matrix of the p_{jk}. Invariantly, $\sigma_m(P)$ is a positively homogeneus degree m function on $T^*X \setminus o$ valued in endomorphisms of π^*E (which is $\pi^*\mathrm{Hom}(E, E)$, i.e. the pull back of a bundle from X itself). We say that P has a *scalar principal symbol* if for all (z, ζ), $\sigma_m(P)$ is a multiple of the identity operator on E_z.

As an example, $d \in \mathrm{Diff}^1(X, \Lambda X)$ has principal symbol $i\zeta\wedge$, i.e. at $(z, \zeta) \in T^*X$, $\sigma_1(d)(z, \zeta)u = i\zeta \wedge u$, $u \in \pi^*_{(z,\zeta)}\Lambda X$, while $\sigma_1(d^*) = -i\iota_\zeta$, where $\iota_\zeta u$ is the evaluation of the form u on the tangent vector at z associated to ζ by the dual metric H, i.e. $H(\zeta)$ – see [24, Section 2.10] for details. As

$$\iota_\zeta(\zeta \wedge .) + \zeta \wedge \iota_\zeta. = H(\zeta, \zeta)., \tag{11}$$

it follows that

$$\sigma_2(\Box_h) = \sigma_2(d^*d + dd^*) = H\,\mathrm{Id},$$

where H on the right hand side is understood as the metric function (i.e. $H(z,\zeta) = H_z(\zeta,\zeta)$ is the squared length of a covector $\zeta \in T_zX$). In particular, $\Box_h$ has scalar principal symbol.

In fact, the same works for a more general class of operators, called *pseudodifferential operators*, or ps.d.o.'s for short. I will give a concrete description of what these are, but one may learn more by listing their properties first. For $P \in \Psi^m_{cl}(X,E)$, i.e. P is a *classical* pseudodifferential operator of order m acting on sections of E, $p = \sigma_m(P)$ is homogeneous degree m function with values in endomorphisms of π^*E on $T^*X \setminus o$, o denoting the zero section. There is also a slightly larger class consisting of all pseudodifferential operators, $\Psi^m(X,E)$, whose principal symbols are merely (equivalence classes of) *symbols* in the sense discussed below, see (12). We will always work with *properly supported* ps.d.o's, i.e. such that either projection $X \times X \to X$ is proper (compact sets have compact pre-image) when restricted to the support of the Schwartz kernel of the operators – this ensures that the operators can be composed, etc., and as the Schwartz kernels of these operators are non-smooth only at the diagonal, this is not a serious restriction.

From an algebraic point of view, some of the most important properties are that $\Psi^\infty(X,E) = \cup_m \Psi^m(X,E)$ is an order-filtered ring, the space $\Psi^m(X,E)$ increasing with m, so

$$A \in \Psi^m(X,E),\ B \in \Psi^{m'}(X,E) \Rightarrow AB \in \Psi^{m+m'}(X,E),$$

that the principal symbol is a ring homomorphism, that $\Psi^0(X,E)$ is bounded on $L^2(X)$, $\Psi^m(X,E)$ (m arbitrary) maps $C^\infty(X,E)$ (and distributional sections of E, $\mathcal{D}'(X,E)$) to itself, and that there is a short exact sequence

$$0 \to \Psi^{m-1}_{cl}(X,E) \to \Psi^m_{cl}(X,E) \to S^m_{\mathrm{hom}}(T^*X \setminus o, \pi^*\mathrm{Hom}(E,E)) \to 0;$$

where S^m_{hom} stands for C^∞ homogeneous functions of degree m.

On the other hand, for $X = \mathbb{R}^n$, there are explicit maps, called *quantizations*, sending appropriate classes of functions on T^*X to pseudodifferential operators on X. The standard class of such functions to consider is that of symbols: a *symbol* of order m on T^*X ($X = \mathbb{R}^n$) is a C^∞ function with specified behavior as $\xi \to \infty$ (and uniform control as $x \to \infty$, although this is much less relevant here): for all $\alpha, \beta \in \mathbb{N}^n$ there is $C_{\alpha,\beta} > 0$ such that for all $(x,\xi) \in T^*X$,

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}(1 + |\xi|)^{m-|\beta|}. \tag{12}$$

The set of these symbols is denoted by $S^m(T^*X)$ or $S^m(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$ if one wants emphasize explicit coordinates (hence product structure on T^*X). This generalizes polynomials in ξ (recall that symbols of differential operators are polynomials): the order of a polynomial decreases each time one differentiates it. Note that a smooth homogeneous function of degree m on $T^*X \setminus o$ is in fact a symbol of order m in $|\xi| > 1$ over bounded regions in x, i.e. it satisfies the symbol estimates (12) there – we need to work away from the zero section, $\xi = 0$, for any smooth homogeneous function on all of T^*X is in fact a polynomial. A *one-step polyhomogeneous symbol* a of order m is a symbol of order m for which there exist smooth homogeneous degree $m - j$ functions a_j ($j \in \mathbb{N}$) on $T^*X \setminus o$ such that, for all k, $a - \sum_{j=0}^{k-1} a_j$ is a symbol of order $m-k$ in $|\xi|>1$. For quantization, for instance, one can take the 'left quantization'

$$(q_L(a)u)(x) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a(x,\xi)u(y)\,dy\,d\xi, \tag{13}$$

$q_L(a)$ is (by definition) a ps.d.o. of order m if a is a symbol of order m. Note that if a is a polynomial in ξ depending smoothly on x, i.e. $a(x,\xi)=\sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$, then $q_L(a)=\sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$, explaining the connection to differential operators. Vector bundles E (of rank ℓ) over $\mathbb{R}^n$ are trivial, and one can use a given trivialization to quantize $a\in S^m_{\mathrm{cl}}(T^*X;\pi^*\mathrm{Hom}(E,E))$ by identifying a with an ℓ-by-ℓ matrix of functions a_{jk}, and letting $q_L(a)=(q_L(a_{jk}))$ be the matrix of the quantizations. For general manifolds one can transfer this definition by localization. These quantizations q have the property that $\sigma_m(q(a))-a$ is a symbol of order $m-1$ – so to leading order $q(a)$ is independent of the choice of q, but there are still many choices.

It should be emphasized that, in the present setting, the relevant region for microlocal analysis is the asymptotic regime as $\xi\to\infty$. Making various objects homogeneous, or conic, is a way of 'bringing infinity to a finite region'. Another way of accomplishing this is to compactifying the fibers of the cotangent bundle – this is the approach taken by Melrose, e.g. in [11].

The symplectic form ω turns scalar valued functions p, or rather the differential dp, into a vector field H_p (called the *Hamilton vector field* of p) on T^*X via demanding that $\omega(V,H_p)=Vp$ for all vector fields V. Thus,

$$H_p=\sum_j \frac{\partial p}{\partial\zeta_j}\frac{\partial}{\partial z_j}-\frac{\partial p}{\partial z_j}\frac{\partial}{\partial\zeta_j}.$$

Note that H_p is homogeneous of degree $m-1$. If $p\in S^m_{\mathrm{hom}}(T^*X\setminus o;\pi^*\mathrm{Hom}(E,E))$ is a scalar multiple of identity, $p=\tilde p\,\mathrm{Id}$, then we write $H_p=H_{\tilde p}$. Note that we only define H_p is p is scalar valued. As mentioned above, an example with $p=h$, a Riemannian or Lorentzian metric, with E being either scalars or differential forms, is $P=\Delta_h$, the Laplace-Beltrami operator (in the Riemannian case) and $P=\Box_h$, the d'Alembertian or wave operator (in the Lorentzian case).

Definition 1. Suppose that p is homogeneous degree m on $T^*X\setminus o$ and scalar valued. The *characteristic set* of p is $\Sigma=p^{-1}(\{0\})$. *Bicharacteristics* are integral curves of H_p inside Σ.

The role that H_p plays in analysis becomes apparent upon noticing that if $P\in\Psi^m_{cl}(X)$, $Q\in\Psi^{m'}_{cl}(X)$ then $[P,Q]=PQ-QP\in\Psi^{m+m'-1}_{cl}(X)$, and

$$\sigma_{m+m'-1}(i[P,Q])=H_pq. \tag{14}$$

If instead $P\in\Psi^m_{\mathrm{cl}}(X,E)$, $Q\in\Psi^{m'}_{\mathrm{cl}}(X,E)$ and P has scalar principal symbol, then $[P,Q]\in\Psi^{m+m'-1}_{cl}(X,E)$ still since

$$\sigma_{m+m'}(PQ)=\sigma_m(P)\sigma_{m'}(Q)=\sigma_{m'}(Q)\sigma_m(P)=\sigma_{m+m'}(QP),$$

so $\sigma_{m+m'}([P,Q])=0$, but the principal symbol is given by a more complicated expression, for it depends on P modulo $\Psi^{m-2}_{cl}(X,E)$. However, if *both* P and Q have scalar principal symbols, then (14) holds as can be seen by locally trivializing E and computing the commutator.

To do analysis, we also need a notion of singularity of a function or distribution u. The roughest notion is that of the *wave front set* $\mathrm{WF}(u)$, which locates at which points and in which direction a function u is not smooth, here meaning C^∞. Immediately from the definition, given below, this is a closed conic subset of $T^*X \setminus o$; u is C^∞ if and only if $\mathrm{WF}(u) = \emptyset$. In fact, for any point $z_0 \in X$, z_0 has a neighborhood in X on which u is C^∞ if and only if $\mathrm{WF}(u) \cap (T^*_{z_0}X \setminus o) = \emptyset$.

One way of defining $\mathrm{WF}(u)$ for distibutions u is the following:

Definition 2. Suppose that $u \in \mathcal{D}'(X, E)$. We say that $q \in T^*X \setminus o$ is *not* in $\mathrm{WF}(u)$ if there exists $A \in \Psi^0(X, E)$ such that $\sigma_0(A)(q)$ invertible (i.e. A is *elliptic* at q; in the scalar case this just means $\sigma_0(A)(q) \neq 0$) and $Au \in C^\infty(X, E)$.

To get a feeling for this, one should think of A as the quantization of a scalar symbol a which is supported in a cone around q, identically 1 on the $\mathbb{R}^+$-orbit through q (at least outside some compact subset of T^*X).

For example, if δ_0 is the delta distribution at the origin, then

$$\mathrm{WF}(\delta_0) = \{(0, \zeta) : \zeta \neq 0\} = N^*\{0\} \setminus o,$$

i.e. δ_0 is singular only at the origin, and it is singular there in every direction – which is quite reasonable. As an aside, conormal bundles are *Lagrangian submanifolds* of T^*X, i.e. the symplectic form vanishes when restricted to their tangent space, and are maximal dimensional (i.e. n-dimensional) with this property. Conic Lagrangian submanifolds of $T^*X \setminus o$ play an important role in many parts of microlocal analysis.

A more interesting example is that of a domain Ω with a C^∞ boundary, and χ_Ω the characteristic function of Ω. If locally $\partial\Omega$ is defined by f, i.e. over some open set $O \subset X$, $\partial\Omega \cap O = \{z \in O : \ f(z) = 0\}$ and df never vanishes on $\partial\Omega \cap O$, then, over O, the space of sections of $N^*\partial\Omega$ is spanned by df, so any covector in $N^*\partial\Omega$ has the form $\alpha\, df$, $\alpha \in C^\infty(\partial\Omega)$. In this case,

$$\mathrm{WF}(\chi_\Omega) = N^*\partial\Omega \setminus o.$$

That is, χ_Ω is smooth both in Ω and in the complement of its closure (after all, it is constant there!), and it is singular at $\partial\Omega$, but it is only singular in the conormal directions: it is smooth when one moves along $\partial\Omega$. (This can be seen directly from the definition of WF: consider differentiating χ_Ω along a vector field tangential to the boundary, and note that the principal symbol of such a vector field vanishes on the conormal bundle!)

One can measure singularities with respect to other spaces: e.g. the Sobolev spaces $H^s_{\mathrm{loc}}(X, E)$, where we would write $\mathrm{WF}^s(u)$, or with respect to real analytic functions, where we would write $\mathrm{WF}_A(u)$. Indeed, $\mathrm{WF}^s(u)$ plays a role in the proofs of various results stated below; one often proves in an inductive manner that u is microlocally in H^s for every s (hence is C^∞ microlocally), rather than proving directly that u is C^∞ microlocally. We can define $\mathrm{WF}^s(u)$ for $u \in \mathcal{D}'(X, E)$ by saying that $q \in T^*X \setminus o$ is *not* in $\mathrm{WF}^s(u)$ if there exists $A \in \Psi^0(X, E)$ such that $\sigma_0(A)(q) \neq 0$ and $Au \in H^s_{\mathrm{loc}}(X, E)$. Equivalently, one can shift the weight to the ps.d.o. from the function space:

Definition 3. Suppose that $u \in \mathcal{D}'(X, E)$. We say that $q \in T^*X \setminus o$ is *not* in $\mathrm{WF}^s(u)$ if there exists $A \in \Psi^s(X, E)$ with $\sigma_0(A)(q) \neq 0$ and $Au \in L^2_{\mathrm{loc}}(X, E)$.

The main facts about the analysis of P, which in this generality are due to Hörmander and Duistermaat-Hörmander [6, 3, 8] are:

(1) Microlocal elliptic regularity. Let $\Sigma(P)$ be the characteristic set of P, i.e. the set of points in $T^*X \setminus o$ at which $\sigma(P)$ is not invertible. (Thus, if

$\sigma(P) = p\,\mathrm{Id}$ is scalar, this is just $p^{-1}(\{0\})$.) If $u \in \mathcal{D}'(X, E)$ then $\mathrm{WF}(u) \subset \mathrm{WF}(Pu) \cup \Sigma(P)$. In particular, if $Pu \in C^\infty(X, E)$ then $\mathrm{WF}(u) \subset \Sigma(P)$.

(2) Propagation of singularities. Suppose that $\sigma(P) = p\,\mathrm{Id}$ is scalar, p is real, $Pu \in C^\infty(X, E)$. Then $\mathrm{WF}(u)$ is a union of maximally extended bicharacteristics in $\Sigma(P)$. That is, if $q \in \mathrm{WF}(u)$ (hence in $\Sigma(P)$) then so is the whole bicharacteristic through q.

For analogy with the manifolds with corners setting, we restate part of these conclusions in a special case:

Theorem 4. *(See Hörmander and Duistermaat-Hörmander* [6, 3, 8]*.) Suppose* $P \in \Psi^m(X, E)$*,* $\sigma_m(P) = p\,\mathrm{Id}$ *is scalar, real,* $Pu = 0$*,* $u \in \mathcal{D}'(X, E)$*. Then* $\mathrm{WF}(u) \subset \Sigma = \Sigma(P)$*, and it is a union of maximally extended bicharacteristics of* P*.*

Note that (2) may be vacuous; indeed, if H_p is *radial*, i.e. tangent to the orbits of the $\mathbb{R}^+$-action, then it does not give any information on $\mathrm{WF}(u)$, as the latter is already known to be conic. Such points are called radial points, and in recent work with Hassell and Melrose [5], they have been extensively analyzed under non-degeneracy assumptions. If P is the wave operator, there are no radial points in $\Sigma = \Sigma(P)$, but such points are very important in scattering theory (where the $\mathbb{R}^+$-action, or its remnants, are in the base variables z).

As an example, consider the wave operator $P = D_t^2 - \Delta_g$, $X = M \times \mathbb{R}$, M a manifold without boundary. Then $p = \sigma_2(P) = \tau^2 - |\xi|_g^2$, where (x, t, ξ, τ) are coordinates on T^*X (so ξ is dual to x, and τ is dual to t), and the projection of bicharacteristics to M are geodesics. If $M \subset \mathbb{R}^n$ and g is the Euclidean metric, then $H_p = 2\tau\partial_t - 2\xi \cdot \partial_x$, and bicharacteristics inside $p = 0$, i.e. $|\tau| = |\xi|$, are straight lines

$$s \mapsto (x_0 - 2s\xi_0, t_0 + 2\tau_0 s, \xi_0, \tau_0),$$

which explains geometric optics in the absence of boundaries.

4. Propagation of singularities on manifolds with corners: the phase space

On manifolds with corners, roughly, the results have the same form as in the boundaryless case, but the definitions of wave front set and the bicharacteristics change significantly. In particular, the relevant wave front set is $\mathrm{WF}_b(u)$, introduced by Melrose (see [17], [7, Section 18.2] for the setting of smooth boundaries, [19] for manifolds with corners). Both $\mathrm{WF}_b(u)$ and the image of the (generalized broken) bicharacteristics are subsets of a new phase space, the *b-cotangent bundle* ${}^bT^*X$.

The reason for this is that one cannot microlocalize in T^*X: naively defined ps.d.o's do not act on functions on X in general, and even when they do, they do not preserve boundary conditions. This causes technical complications, for we are interested in the wave operator, $P = D_t^2 - \Delta$, whose principal symbol is a C^∞ function on T^*X, *not* on ${}^bT^*X$ where we microlocalize. In fact, from a PDE point of view, this discrepancy is what causes the diffractive phenomena.

Rather than defining ${}^bT^*X$ directly, I describe its main properties: these can be easily made into a definition as we shortly see. Being a vector bundle, locally in X it is trivial, and in the local coordinate product decomposition above, it will take the form $U_{x,y} \times \mathbb{R}^k_\sigma \times \mathbb{R}^{n-k}_\zeta$, with $U \subset [0, \infty)^k_x \times \mathbb{R}^{n-k}_y$, where σ is the 'b-dual' variable of x and ζ is the b-dual variable of y.

There is a natural map $\pi : T^*X \to {}^bT^*X$, which in these local coordinates takes the form

(15)
$$\pi(x, y, \xi, \zeta) = (x, y, x\xi, \zeta), \quad \text{with } x\xi = (x_1\xi_1, \ldots, x_k\xi_k).$$

(That is, $\sigma_j = x_j\xi_j$.) Thus, π is a C^∞ map, but at ∂X, it is not a diffeomorphism.

Over the *interior* X° of X, ${}^bT^*X$ and T^*X are naturally identified via π, and

$$\mathrm{WF}_b(u) \cap {}^bT^*_{X^\circ}X = \pi(\mathrm{WF}(u) \cap T^*_{X^\circ}X).$$

Note that if q is a linear function on each fiber of ${}^bT^*X$, then it has the form

$$q = \sum a_j(x, y)\sigma_j + \sum b_j(x, y)\zeta_j,$$

so

$$\pi^*q = \sum a_j(x, y)x_j\xi_j + \sum b_j(x, y)\zeta_j,$$

which is the principal symbol of

(16)
$$Q = \sum a_j(x, y)x_jD_{x_j} + \sum b_j(x, y)D_{y_j}.$$

Vector fields of this form are exactly the vector fields tangent to all boundary faces of X; we denote their space by $\mathcal{V}_b(X)$.

In fact, this indicates how ${}^bTX$ can be defined intrinsically: the set of all smooth vector fields tangent to all boundary faces is the set of all smooth sections of a vector bundle; indeed, x, y, a_j, b_j above give local coordinates on ${}^bTX$. Then ${}^bT^*X$ can be defined as the dual vector bundle. However, as long as all considerations are local, and they are mostly such here, it is safe to consider ${}^bT^*X$ a space arising from a singular change of variables on T^*X (given by (15)) – it is for this reason that it is sometimes called the compressed cotangent bundle.

There are two closely related pseudodifferential algebras microlocalizing $\mathcal{V}_b(X)$ and the induced algebra of differential operators $\mathrm{Diff}_b(X)$, corresponding to $\Psi_{cl}(X)$ and $\Psi(X)$ in the boundaryless case. These are denoted by $\Psi_b(X)$ and $\Psi_{bc}(X)$, respectively. There is also a principal symbol on $\Psi_b^m(X)$; this is now a homogeneous degree m function on ${}^bT^*X \setminus o$. $\Psi_b(X)$ has the algebraic properties analogous to $\Psi(X)$ on manifolds without boundary. $\Psi_b(X)$ can be described quite explicitly; this was done for instance in [19, 27] in the corners setting, and in [7, Section 18.3] for smooth boundaries. In particular, a *subset* of $\Psi_b(X)$ (which would morally suffice for our purposes here) consists of operators with Schwartz kernels supported in $U \times U$, $U \subset X$ a coordinate chart with coordinates x, y as above, with Schwartz kernels of the form

(17)
$$q(a)u(x, y) = (2\pi)^{-n} \int e^{i((x-x')\cdot\xi + (y-y')\cdot\zeta)} \phi\Big(\frac{x-x'}{x}\Big) a(x, y, x\xi, \zeta) u(x', y')\, dx'\, dy'\, d\xi\, d\zeta,$$

understood as an oscillatory integral, where $a \in S^m(\mathbb{R}^n_{x,y}; \mathbb{R}^n_{\sigma,\zeta})$ (with $\sigma = x\xi$, cf. (15)), $\phi \in C_c^\infty((-1/2, 1/2)^k)$ is identically 1 near 0, $\frac{x-x'}{x} = (\frac{x_1-x_1'}{x_1}, \ldots, \frac{x_k-x_k'}{x_k})$, and the integral in x' is over $[0, \infty)^k$. This formula is similar to (13), but ξ is replaced by $x\xi$ here. Thus, if a is a polynomial in its third and fourth slots, i.e. in

$x\xi$ and ζ, depending smoothly on x, y, i.e.

$$a(x, y, \xi, \zeta) = \sum_{|\alpha|+|\beta| \leq m} a_{\alpha\beta}(x, y)(x\xi)^\alpha \zeta^\beta,$$

then

$$q(a) = \sum_{|\alpha|+|\beta| \leq m} a_{\alpha\beta}(x, y)(xD_x)^\alpha D_y^\beta,$$

thus connecting $\mathcal{V}_b(X)$ and $\mathrm{Diff}_b(X)$ to $\Psi_b(X)$ in view of (16). One can also construct $\Psi_b(X; E, F)$ – acting between distributional sections of vector bundles E and F over X. Elements of $\Psi^m_{bc}(X)$ have the important property that they map $C^\infty(X) \to C^\infty(X)$, and more generally they map $x_j C^\infty(X) \to x_j C^\infty(X)$, so if $A \in \Psi^m_{bc}(X)$, then $(Au)|_{H_j}$ depends only on $u|_{H_j}$ for $u \in C^\infty(X)$. In particular, Dirichlet boundary conditions are automatically preserved by such A, which makes $\Psi_b(X)$ easy to use in the analysis of the Dirichlet problem in [27]. We will need more care for natural boundary conditions, which is a point we address in the next section.

Now $\mathrm{WF}_b(u)$ can be defined analogously to $\mathrm{WF}(u)$. For simplicity we state this here for $u \in L^2_{\mathrm{loc}}(X; E)$ where E is a vector bundle; this is how the main theorem is stated below, but see [27, Section 3] for the more general setting. (Here we put an arbitrary Riemannian metric on X and an arbitrary fiber metric on E; the resulting L^2-norms are equivalent over compact sets.) The space of 'very nice' functions corresponding to $\mathcal{V}_b(X)$ and $\mathrm{Diff}_b(X)$, replacing $C^\infty(X)$, is the space of L^2 conormal functions to the boundary, i.e. functions $v \in L^2_{\mathrm{loc}}(X; E)$ such that $Qv \in L^2_{\mathrm{loc}}(X; E)$ for every $Q \in \mathrm{Diff}_b(X; E)$ (of any order). Then $q \in {}^bT^*X \setminus o$ is *not* in $\mathrm{WF}_b(u)$ if there is an $A \in \Psi^0(X; E)$ such that $\sigma_{b,0}(A)(q)$ is invertible and Au is L^2-conormal to the boundary. Spelling out the latter explicitly:

Definition 5. (See [27, Section 3] for the more general setting.) Suppose $u \in L^2_{\mathrm{loc}}(X; E)$. Then $q \in {}^bT^*X \setminus o$ is *not* in $\mathrm{WF}_b(u)$ if there is an $A \in \Psi^0(X; E)$ such that $\sigma_{b,0}(A)(q)$ is invertible and $QAu \in L^2_{\mathrm{loc}}(X; E)$ for all $Q \in \mathrm{Diff}_b(X; E)$.

Note that the definition of WF could be stated in a completely parallel manner: we would require (for X without boundary) $QAu \in L^2(X)$ for all $Q \in \mathrm{Diff}(X)$ – this is equivalent to $Au \in C^\infty(X)$ by the Sobolev embedding theorem.

Moreover, the wave front set is microlocal, i.e. $\mathrm{WF}_b(Bu) \subset \mathrm{WF}'_b(B) \cap \mathrm{WF}_b(u)$, so the standard characterization applies: $q \notin \mathrm{WF}_b(u)$ if there is an open set O containing q such that for every $B \in \Psi^0(X; E)$ with $\mathrm{WF}'_b(B) \subset O$, Bu is L^2-conormal to the boundary.

In fact, technically it is useful to work with the space of functions conormal relative to $H^1_{\mathrm{loc}}(X; E)$, as the latter is almost the quadratic form domain when $E = \Lambda X$. Moreover, we also need spaces of distributions possessing finite regularity.

Definition 6. Suppose $u \in H^1_{\mathrm{loc}}(X; E)$. Then $q \in {}^bT^*X \setminus o$ is *not* in $\mathrm{WF}^{1,\infty}_b(u)$ if there is an $A \in \Psi^0(X; E)$ such that $\sigma_{b,0}(A)(q)$ is invertible and $QAu \in H^1_{\mathrm{loc}}(X; E)$ for all $Q \in \mathrm{Diff}_b(X; E)$.

Moreover, $q \in {}^bT^*X \setminus o$ is *not* in $\mathrm{WF}^{1,m}_b(u)$ if there is an $A \in \Psi^m(X; E)$ such that $\sigma_{b,0}(A)(q)$ is invertible and $Au \in H^1_{\mathrm{loc}}(X; E)$.

The key observation in making the definition useful is that any $A \in \Psi^0_{bc}(X; E)$ with compact support defines a continuous linear maps $A : H^1(X; E) \to H^1(X; E)$ with norms bounded by a seminorm of A in $\Psi^0_{bc}(X; E)$. This follows from the analogous statement for scalar operators, proved in [27, Lemma 3.2], by using local trivializations. However, we recall the essence of the argument as it involves some important concepts.

The *indicial operators* $\hat{N}_j(A)(\sigma_j)$, introduced by Melrose, see [27, Section 2] for a discussion in the present context, play an important role below. These capture the behavior of A at a boundary hypersurface H_j. For a differential operator,

$$A = \sum_{|\alpha|+|\beta|\leq m} a_{\alpha\beta}(x,y)(xD_x)^\alpha D_y^\beta \Rightarrow$$
$$\hat{N}_j(A) = \sum_{|\alpha|+|\beta|\leq m} a_{\alpha\beta}((\hat{x},0),y)\sigma_j^{\alpha_j}(\hat{x}D_{\hat{x}})^{\hat{\alpha}} D_y^\beta$$

where we write $\hat{x}$ for x with the coordinate x_j dropped, similarly for $\hat{\alpha}$, and $(\hat{x},0)$ is x with the jth coordinate replaced by 0. Thus, $\hat{N}_j(A)$ captures A fully at H_j, i.e. not merely its principal symbol (which only captures the highest derivatives), in the sense that $\hat{N}_j(A)(\sigma_j) = 0$ for all $\sigma_j \in \mathbb{R}$ implies $A \in x_j \mathrm{Diff}_b^m(X)$ (this also holds for ps.d.o's). More invariantly, also for $A \in \Psi_{bc}^m(X)$,

$$\hat{N}_j(A)(\sigma)\tilde{u}(\hat{x},y) = (x_j^{-i\sigma_j} A x_j^{i\sigma_j} u)|_{H_j},\ u|_{H_j} = \tilde{u},$$

which follows from $x_j^{-i\sigma_j} A x_j^{i\sigma_j} \in \Psi_{bc}^m(X)$, which hence maps $C^\infty(X) \to C^\infty(X)$ and $x_j C^\infty(X) \to x_j C^\infty(X)$ (so the left hand side is independent of the particular choice of u). One application of indicial operators is to note that

$$A \in \Psi_{bc}^m(X) \Rightarrow [x_j D_{x_j}, A] \in x_j \Psi_{bc}^m(X)$$

(rather than merely in $\Psi_{bc}^m(X)$), since its indicial operator is $[\hat{N}(x_j D_{x_j}), \hat{N}(A)] = 0$, for $\hat{N}_j(x_j D_{x_j}) = \sigma_j$ is a constant (rather than a general differential operator). This implies that for $A \in \Psi_{bc}^0(X)$, $A : H^1(X) \to H^1(X)$, for A is bounded on $L^2(X)$, and

$$D_{x_j} A = x_j^{-1}(x_j D_{x_j})A = x_j^{-1}[x_j D_{x_j}, A] + (x_j^{-1} A x_j) D_{x_j}, \tag{18}$$

with the first term on the right hand side in $\Psi_{bc}^0(X)$, hence bounded on $L^2(X)$, and $x_j^{-1} A x_j \in \Psi_{bc}^0(X)$, hence bounded on $L^2(X)$ as well. In view of $x_j^{-1} A x_j = A + x_j^{-1}[A, x_j]$, an immediate consequence of (18) is the following

Lemma 7. *(Lemma 2.8 of [27]) Let ∂_{x_j}, ∂_{σ_j} denote local coordinate vector fields on ${}^bT^*X$ in the coordinates (x,y,σ,ζ). For $A \in \Psi_b^m(X)$ with Schwartz kernel supported in the coordinate patch, $a = \sigma_{b,m}(A) \in C^\infty({}^bT^*X \setminus o)$, we have $[D_{x_j}, A] = A_1 D_{x_j} + A_0 \in \mathrm{Diff}^1 \Psi_b^{m-1}(X)$ with $A_0 \in \Psi_b^m(X)$, $A_1 \in \Psi_b^{m-1}(X)$ and*

$$\sigma_{b,m-1}(A_1) = \frac{1}{i}\partial_{\sigma_j} a,\ \sigma_{b,m}(A_0) = \frac{1}{i}\partial_{x_j} a. \tag{19}$$

This result also holds with $\Psi_b(X)$ replaced by $\Psi_{bc}(X)$ everywhere.

Here we introduced some notation for operators of the form

$$\sum_j Q_j A_j,\ Q_j \in \mathrm{Diff}^k(X),\ A_j \in \Psi_b^m(X), \tag{20}$$

where the sum is locally finite; we write $\mathrm{Diff}^k\Psi_b^m(X)$ for their set, and analogously for $\mathrm{Diff}^k\Psi_{bc}^m(X)$. A calculation analogous to (18), see [27, Lemma 2.5] shows that

$$B_j \in \mathrm{Diff}^{k_j}\Psi_b^{m_j}(X)\ (j=1,2) \Rightarrow$$
$$B_1B_2 \in \mathrm{Diff}^{k_1+k_2}\Psi_b^{m_1+m_2}(X),\ [B_1,B_2] \in \mathrm{Diff}^{k_1+k_2}\Psi_b^{m_1+m_2-1}(X),$$

i.e. the b-ps.d.o order of the commutator is one order lower than that of the products.

Again, one has microlocality, see [27, Lemma 3.9]: for $B \in \Psi_b^k(X;E)$, $u \in H^1_{\mathrm{loc}}(X;E)$,

$$\mathrm{WF}_b^{1,m-k}(Bu) \subset \mathrm{WF}_b'(B) \cap \mathrm{WF}_b^{1,m}(u).$$

One can also work relative to the dual space, $\dot H^{-1}(X)$, of $H^1(X)$, and define $\mathrm{WF}_b^{-1,*}(f)$ for $f \in \dot H^{-1}(X)$; we simply refer to [27, Section 3] here.

5. Elliptic estimates

If $P \in \mathrm{Diff}^m(X)$, with $\sigma(P) = p\,\mathrm{Id}$ scalar, the characteristic set $\Sigma(P) = p^{-1}(\{0\})$ is a subset of T^*X. Let $\dot\Sigma = \pi(\Sigma(P)) \subset {}^bT^*X$ be the *compressed characteristic set.* If P is elliptic, then $\dot\Sigma$ is empty, but even if P is not elliptic, $\dot\Sigma$ is often a proper subset of ${}^bT^*X$, outside which P behaves as if it were elliptic. In particular, as we prove below, if P is the wave operator on forms, $Pu = 0$ with Dirichlet, Neumann or natural boundary conditions, then $\mathrm{WF}_b^{1,\infty}(u) \subset \dot\Sigma$. If $P = \Delta - \lambda$, Δ the Laplacian on forms, then $Pu = 0$ with Dirichlet, Neumann or natural boundary conditions implies that u is H^1-conormal, which is the statement of *elliptic regularity.*

We make this more concrete for the wave operator $P = D_t^2 - \Delta_g$ on $X = M \times \mathbb{R}$. Using coordinates from Section 2, namely such that (5) holds, on $T^*_{U\times\mathbb{R}}X \setminus o$,

$$p|_{x=0} = \tau^2 - \xi \cdot \tilde A(\tilde y)\xi - \tilde\zeta \cdot \tilde B(\tilde y)\tilde\zeta, \tag{21}$$

with $\tilde A$, $\tilde B$ positive definite matrices depending smoothly on $\tilde y$. Thus, with $\mathcal{U} = \{x = 0\} \cap {}^bT^*_{U\times\mathbb{R}}X \setminus o$, writing local coordinates on ${}^bT^*X$ as $(x, \tilde y, t, \sigma, \tilde\zeta, \tau)$,

$$\dot\Sigma \cap \mathcal{U} = \{(0, \tilde y, t, 0, \tilde\zeta, \tau) :\ \tau^2 \ge \tilde\zeta \cdot \tilde B(\tilde y)\tilde\zeta,\ (\tilde\zeta, \tau) \ne 0\}. \tag{22}$$

Note that $\dot\Sigma = \pi(\Sigma(P))$ is disjoint from all points $(x, \tilde y, t, \sigma, \zeta, \tau)$ with $x = 0$ at which either $\sigma \ne 0$ (for $\sigma_j = x_j\xi_j = 0$ for all j) or $\tau^2 < \zeta \cdot \tilde B(\tilde y)\zeta$.

In the more general Lorentzian setting, by Section 2 we have coordinates, namely such that (8) holds, in which

$$p|_{x=0} = \xi \cdot A(y)\xi + \zeta \cdot B(y)\zeta. \tag{23}$$

This gives that

$$\dot\Sigma \cap \mathcal{U} = \{(0, y, 0, \zeta) :\ 0 \le \zeta \cdot B(y)\zeta,\ \zeta \ne 0\}. \tag{24}$$

We have already seen that any $A \in \Psi^0_{bc}(X)$ preserves Dirichlet boundary conditions. For natural boundary conditions we need to be more careful, which is a topic we now address.

The basic idea of proving microlocal estimates for solutions of the wave equation, i.e. for $u \in H^1_{R,\mathrm{loc}}(X; \Lambda X)$ (so $\nu \wedge u = 0$ at ∂X) satisfying

$$\langle du, dv\rangle + \langle d^*u, d^*v\rangle = \langle f, v\rangle \tag{25}$$

for all $v \in H^1_{R,c}(X; \Lambda X)$, is to use the equation with v replaced by an operator applied to u, and then rewrite it to obtain an estimate for $\|Bu\|^2_{H^1}$ for some B.

More concretely, we would like to use $A \in \Psi^m_{bc}(X;\Lambda X)$ with $v = A^*Au$. In order to do this, we need that $A^*Au \in H^1(X;\Lambda X)$ which is automatically true if $A \in \Psi^0_{bc}(X;\Lambda X)$, *and* $\nu \wedge A^*Au = 0$. This latter part is what requires particular care, in fact does not work in general, but we have the following result:

Lemma 8. *Suppose that $a \in S^m({}^bT^*X)$. Then there exists an $A = q(a) \in \Psi^m_{bc}(X,\Lambda X)$ such that $\sigma_{b,m}(A)(w,\eta) = a(w,\eta)I_{(w,\eta)}$ (I being the identity operator on $\pi^*\Lambda X$) and $\hat{N}_j(A) : C^\infty(H_j;\Lambda_R(H_j)) \to C^\infty(H_j;\Lambda_R(H_j))$ for each boundary hypersurface H_j. Thus, for $m = 0$, $A : H^1_{R,c}(X;\Lambda X) \to H^1_{R,c}(X;\Lambda X)$, and similarly for the local spaces..*

Proof. Using a partition of unity $\{\phi_k\}$ on X, it suffices to prove the result with a replaced by $\phi_k a$, for if A_k denotes the associated operator, $A = \sum A_k$ satisfies the requirements of the theorem. Thus, we may assume that a is supported in a coordinate chart O, in which the trivialization $u = (u_m)$, as in (3), of ΛX can be used (with the notation of Section 2), so $\nu \wedge u|_{H_j} = 0$ means exactly that $u_m|_{H_j} = 0$ for $m \in J_j$. Then let $A_0 \in \Psi^m_{bc}(X)$ be a *scalar* operator with principal symbol a and Schwartz kernel supported in $O \times O$, and let $A \in \Psi^m_{bc}(X;\Lambda X)$ be given, using the trivialization, by the *diagonal* matrix of operators on ΛX with all diagonal entries equal to A_0. Then $\hat{N}_j(A)$ is diagonal with respect to the induced trivialization of $\Lambda X|_{H_j}$ with diagonal entries $\hat{N}_j(A_0)$. If $u \in C^\infty(X;\Lambda X)$ then $(Au)|_{H_j} = \hat{N}_j(A)(u|_{H_j})$, so if $u|_{H_j} \in C^\infty(X;\Lambda_R(H_j))$ then $(\hat{N}_j(A)(u|_{H_j}))_m = \hat{N}_j(A_0)(u_m|_{H_j}) = 0$ for $m \in J_j$ proving the claim.

Due to the density of $C^\infty_{R,c}(X;\Lambda X)$ in $H^1_{R,c}(X;\Lambda X)$, for $a \in S^0({}^bT^*X)$ we deduce that the A given by the lemma maps $H^1_{R,c}(X;\Lambda X) \to H^1_{R,c}(X;\Lambda X)$. □

If A is as in the Lemma, A^* does not map $H^1_{R,c}(X;\Lambda X) \to H^1_{R,c}(X;\Lambda X)$ in general, for that would correspond to $\hat{N}_j(A)$ mapping sections of $(\Lambda_R H_j)^\perp$ to itself. However, we can use as replacement the operator $A^\dagger$ with $A^\dagger = q(\bar{a})$, writing $A_0^\dagger$ be A_0 of the lemma constructed with $\bar{a}$ in place of a, and then $\sigma_{b,m}(A^*) = \bar{a}\, I = \sigma_{b,m}(A^\dagger)$.

We also want to rewrite $\langle du, dv\rangle + \langle d^*u, d^*v\rangle$, $u \in C^\infty(X,\Lambda X)$, $v \in C^\infty_c(X,\Lambda X)$. Here recall that $\langle .,.\rangle$ is the inner product given by h, so is not positive definite if h is Lorentzian, only if h is Riemannian. The following calculation follows [23, Section 4], which in turn was based on the work of Mitrea [21] and Mitrea, Mitrea and Taylor [20]. Let ∇ be any first order differential operator with the same symbol of the Levi-Civita connection, so

$$\sigma_1(\nabla)(w,\tilde{\xi})u = i\tilde{\xi} \otimes u,\ \ u \in \pi^*_{(w,\tilde{\xi})}\Lambda X.$$

Writing $\delta = d^*$, $\sigma_2(\delta d + d\delta) = \sigma_2(\nabla^*\nabla)$ (as both are given by the dual metric function, H), and

$$\delta d + d\delta - \nabla^*\nabla = R \in \mathrm{Diff}^1(X;\Lambda X)$$

is formally self-adjoint. We remark that the Levi-Civita connection itself is special because if ∇ is the Levi-Civita connection then $R \in \mathrm{Diff}^0(X;\Lambda X)$, i.e. is a bundle isomorphism, due to the Weitzenböck formulae (see [24, Chapter 10, Section 4]), but this does not play a role here. Since for any first order differential operator

$P \in \mathrm{Diff}(X; E, F)$ one has (see [24, Chapter 2, Proposition 9.1])

$$\langle Pu, v\rangle_{L^2(X;F)} = \langle u, P^* v\rangle_{L^2(X;E)} + \frac{1}{i}\langle \sigma_1(P)(\nu_w)u, v\rangle_{L^2(\partial X;F)},$$
$$\langle \sigma_1(P)(\nu_w)u, v\rangle_{L^2(\partial X;F)} = \int_{\partial X} (\sigma_1(P)(\nu_w)u, v)_{F_w}\, dS(w),$$

where P^* is the formal adjoint of P, and in the last term one takes the inner product on the fiber F_w of F over $w \in \partial X$ and integrates with respect to the surface measure induced by the restriction of h to ∂X (which is non-degenerate by the time-like assumption), one deduces that for $u \in C^\infty(X; \Lambda X)$, $v \in C^\infty_c(X; \Lambda X)$,

$$\begin{aligned} &\langle (d\delta + \delta d - \nabla^*\nabla)u, v\rangle \\ &= \langle du, dv\rangle + \langle \delta u, \delta v\rangle - \langle \nabla u, \nabla v\rangle + \int_{\partial X} ((\nu \wedge \delta u, v) - (\nu \vee du, v) + (\nabla_\nu u, v))\, dS \\ &= \langle du, dv\rangle + \langle \delta u, \delta v\rangle - \langle \nabla u, \nabla v\rangle + \int_{\partial X} ((\nu \wedge \delta u, v) - (du, \nu \wedge v) + (\nabla_\nu u, v))\, dS. \end{aligned}$$

If $\nu \wedge v = 0$ (i.e. v is normal), the penultimate term on the right can be dropped. Writing the integral over ∂X as a sum of integrals over the boundary hypersurfaces H_j, and extending the conormal ν_j to H_j to a smooth 1-form on X for each j, note that if u is normal, so $\nu_j \wedge u = 0$ at H_j, then $\nu_j \wedge u = x_j \tilde{u}$ for some smooth form $\tilde{u}$, so $\delta(\nu_j \wedge u) = \delta(x_j \tilde{u}) = x_j \delta\tilde{u} - dx_j \vee \tilde{u}$ is equal to $-dx_j \vee \tilde{u}$ at H_j, so $(\delta(\nu_j \wedge u), v) = -(dx_j \vee \tilde{u}, v) = (\tilde{u}, dx_j \wedge v) = 0$ at H_j if v is normal. Thus, the integral over H_j can be rewritten as

$$\int_{H_j} (\delta(\nu_j \wedge u) + \nu_j \wedge \delta u + \nabla_{\nu_j} u, v)\, dS = \int_{H_j} (P_{\nu_j} u, v)\, dS,$$

where $P_{\nu_j} u = \delta(\nu_j \wedge u) + \nu_j \wedge \delta u + \nabla_{\nu_j} u$ is a priori a first order differential operator. However, the principal symbol of iP_{ν_j} is $\tilde{\xi} \vee (\nu_j \wedge .) + \nu_j \wedge (\tilde{\xi} \vee .) - (\nu_j, \tilde{\xi}). = 0$ (this is just the polarized version of (11)), so P_{ν_j} is in fact zeroth order, i.e. a bundle endomorphism. We thus have for $u \in C^\infty_R(X; \Lambda X)$, $v \in C^\infty_{R,c}(X; \Lambda X)$:

$$\langle du, dv\rangle + \langle \delta u, \delta v\rangle = \langle \nabla u, \nabla v\rangle + \langle Ru, v\rangle + \int_{\partial X} (\tilde{R}u, v)\, dS$$

for some smooth bundle endomorphism $\tilde{R}$ and a first order differential operator R; by continuity and density this holds for all $u \in H^1_{R,\mathrm{loc}}(X; \Lambda X)$, $v \in H^1_{R,c}(X; \Lambda X)$. Thus, the inhomogeneous wave equation becomes

$$\langle f, v\rangle = \langle \nabla u, \nabla v\rangle + \langle Ru, v\rangle + \int_{\partial X} (\tilde{R}u, v)\, dS \tag{26}$$

for all $v \in H^1_{R,c}(X; \Lambda X)$. We can think of the last two terms on the right hand side as error terms, for

$$|\langle Ru, v\rangle|, \Big|\int_{\partial X} (\tilde{R}u, v)\, dS\Big| \leq C(\|u\|_{H^1}\|v\|_{L^2} + \|u\|_{L^2}\|v\|_{H^1}), \tag{27}$$

can be estimated using one derivative on u and v altogether, while $\langle \nabla u, \nabla v\rangle$ involves two derivatives altogether. This formula is valid both for Lorentzian and Riemannian metrics g; the difference is that for Lorentz metrics the quadratic form given by ∇ is not positive. However, it is microlocally positive in the elliptic region for scalar operators, and in general can be adjusted in this region by use of a twist J discussed below, giving rise to the elliptic estimates.

If $a \in S^m({}^bT^*X)$ and $A \in \Psi^m_{bc}(X;\Lambda X)$ is given by Lemma 8, then we may take $v = A^\dagger JAu$, where J is the bundle isomorphism given by (9). As our formula is valid for any ∇ with principal symbol $i\tilde{\xi}\otimes$, we may choose to ∇ to be the gradient corresponding to the flat metric in the coordinate chart O (where v is supported), and then (keeping in mind that the inner products still correspond to the actual metric h!)

$$\langle \nabla u, \nabla A^\dagger JAu\rangle = \sum_{ij}\sum_{\alpha,\beta}\langle H_{\alpha\beta}H_{ij}D_{w_i}u_\alpha, D_{w_j}A_0^\dagger JA_0 u_\beta\rangle_{L^2(X)},$$

where the remaining pairing is the L^2-pairing on functions, i.e. is the integral of the product (with a complex conjugation). Thus, commuting $A_0^\dagger$ through D_{w_j}, taking the adjoint, commuting through the metric factors and then D_{w_i}, and also commuting J through D_{w_j}, with each commutator giving an operator of one lower b-differential order,

$$\begin{aligned}\langle \nabla u, \nabla A^\dagger JAu\rangle &= \sum_{ij}\sum_{\alpha,\beta}\langle H_{\alpha\beta}H_{ij}D_{w_i}(A_0^\dagger)^* u_\alpha, JD_{w_j}A_0 u_\beta\rangle\\ &\quad+\sum_{ij}\sum_{\alpha,\beta}(\langle \tilde{B}_{\alpha\beta,i}u_\alpha, D_{w_j}A_0u_\beta\rangle + \langle D_{w_i}u_\alpha, \tilde{C}_{\alpha\beta,j}A_0u_\beta\rangle)\end{aligned}$$

with

$$\tilde{B}_{\alpha\beta,i},\tilde{C}_{\alpha\beta,j}\in \mathrm{Diff}^1\Psi^{m-1}_{bc}(X;\Lambda X),\ \mathrm{WF}_b'(\tilde{B}_{\alpha\beta,i}),\mathrm{WF}_b'(\tilde{C}_{\alpha\beta,j})\subset \mathrm{WF}_b'(A);$$

see (20) for the definition of $\mathrm{Diff}\,\Psi_b(X)$. As $(A_0^\dagger)^* - A_0 \in \Psi^{m-1}_{bc}(X)$, the contribution of this difference can be absorbed into $\tilde{B}_{\alpha\beta,i}$, so one obtains in view of (10),

$$\begin{aligned}\langle \nabla u, \nabla A^\dagger JAu\rangle &= \sum_{ij}\sum_{\alpha,\beta}\langle \tilde{H}_{\alpha\beta}H_{ij}D_{w_i}A_0u_\alpha, D_{w_j}A_0u_\beta\rangle\\ &\quad+\sum_{ij}\sum_{\alpha,\beta}(\langle B_{\alpha\beta,i}u_\alpha, D_{w_j}A_0u_\beta\rangle + \langle D_{w_i}u_\alpha, C_{\alpha\beta,j}A_0u_\beta\rangle,\end{aligned}$$

with $B_{\alpha\beta,i}$ and $C_{\alpha\beta,j}$ having similar properties to $\tilde{B}_{\alpha\beta,i}$ and $\tilde{C}_{\alpha\beta,j}$. The first term on the right hand side is just the *twisted Dirichlet form*,

$$\mathcal{Q}(Au,Au) = \langle \nabla Au, \nabla Au\rangle_{L^2(X,\Lambda X\otimes T^*X,\tilde{H}\otimes H)}, \tag{28}$$

with the inner product on $\Lambda X\otimes T^*X$ induced by $\tilde{H}$ (which is positive definite) on ΛX, and H on T^*X (which is not positive definite). Rewriting

$$\langle Ru, A^\dagger Au\rangle = \langle (A^\dagger)^*Ru, Au\rangle = \langle RAu, Au\rangle + \langle ((A^\dagger)^* - A)Ru, Au\rangle + \langle [A,R]u, Au\rangle,$$

$((A^\dagger)^* - A)R, [A,R] \in \mathrm{Diff}^1\Psi^{m-1}_b(X;\Lambda X)$, so the commutator in $\langle Ru, A^*Au\rangle$ can be treated as the $B_{\alpha\beta,i}$ terms. Moreover,

$$\begin{aligned}\langle \tilde{R}u, A^\dagger Au\rangle &= \langle \tilde{R}u, \hat{N}(A^\dagger)\hat{N}(A)u\rangle\\ &= \langle \tilde{R}\hat{N}(A)u, \hat{N}(A)u\rangle_{\partial X} + \langle (\hat{N}((A^\dagger)^*) - \hat{N}(A))\tilde{R}u, \hat{N}(A)u\rangle_{\partial X}\\ &\quad+\langle [\hat{N}(A),\tilde{R}]u, \hat{N}(A)u\rangle_{\partial X},\end{aligned}$$

and similar estimates apply again.

Now suppose that

$$P = \delta d + d\delta + P_1 = \Box + P_1,\ P_1 \in \mathrm{Diff}^1(X),$$

with natural boundary conditions, that is u solves $Pu = f$, $f \in \dot{H}^{-1}(X, \Lambda X)$ in the sense that $u \in H^1_{\mathrm{loc}}(X; \Lambda X)$ with $\nu \wedge u = 0$ and

$$\langle du, dv \rangle + \langle \delta u, \delta v \rangle + \langle P_1 u, v \rangle = \langle f, v \rangle \tag{29}$$

for all $v \in H^1_c(X; \Lambda X)$ with $\nu \wedge v = 0$. Then in the argument above P_1 can simply be incorporated into R, so the same arguments work.

If $a \in S^m({}^bT^*X)$ is supported near a point $q \in {}^bT^*X \setminus (\mathrm{WF}_b^{-1,m}(f) \cup \dot{\Sigma})$, $a(q) \neq 0$, so that $\mathrm{WF}'_b(A) \cap (\mathrm{WF}_b^{-1,m}(f) \cup \dot{\Sigma}) = \emptyset$, and $\mathrm{WF}_b^{1,m-1/2}(u) \cap \mathrm{WF}'_b(A) = \emptyset$ then $\langle B_{\alpha\beta,i} u_\alpha, D_{w_j} A_0 u_\beta \rangle$ and $\langle D_{w_i} u_\alpha, C_{\alpha\beta,j} A_0 u_\beta \rangle$ are finite, and similarly all the terms arising from R and R', while $\langle f, A^\dagger J A u \rangle$ can be handled by Cauchy-Schwartz. If the calculation we performed were directly valid, this would give that the twisted Dirichlet form $\mathcal{Q}(Au, Au)$ is finite, i.e. $q \notin \mathrm{WF}_b^{1,m}(u)$. As the calculation actually requires more regularity, we need to run an approximation argument, replacing A by a family $A_r \in \Psi_{bc}^{-\infty}(X)$, $r \in (0, 1]$, such that A_r is uniformly bounded in $\Psi_{bc}^m(X)$, and $A_r \to A$ as $r \to 0$ in $\Psi_{bc}^{m+\epsilon}(X)$ for all $\epsilon > 0$ (we cannot do the approximation without losing ϵ). The calculation with A replaced by A_r applies directly, and now gives that the twisted Dirichlet form $\mathcal{Q}(A_r u, A_r u)$ is uniformly bounded. If the metric is Riemannian, the (twisted) Dirichlet form is positive definite (there is of course no need for twisting in this case, but on the other hand it does not hurt either), so this (together with the metrizability and weak-* compactness of the unit ball in $H^1(X, \Lambda X)$) gives directly that for any sequence of r's converging to 0 there is a weak-* convergent subsequence, $A_{r_j} u$, in $H^1(X, \Lambda X)$; as $A_{r_j} u \to Au$ in distributions, we deduce that $Au \in H^1(X, \Lambda X)$, so that $u \notin \mathrm{WF}_b^{1,m}(q)$.

In general, if the metric is not Riemannian, we need to note that away from $\dot{\Sigma}$, the twisted Dirichlet form $\mathcal{Q}(u, u)$ is microlocally positive, $u \notin \mathrm{WF}_b^{1,m}(q)$ still holds. As given the uniform estimate for $A_r u$ this part of the argument is essentially the same as in the scalar product case (except that one specifies the localization in y as well, so that A is supported sufficiently close to $y = 0$), we refer the reader to [27, Section 4] for details, and we simply state the result on microlocal elliptic regularity:

Theorem 9 (Microlocal elliptic regularity for Riemannian and Lorentz metrics.)**.** *Suppose that* $u \in H^1_{R,\mathrm{loc}}(X; \Lambda X)$*,* $Pu = f \in \dot{H}^{-1}_{\mathrm{loc}}(X; \Lambda X)$ *in the sense of* (29)*. Then*

$$\mathrm{WF}_b^{1,m}(u) \setminus \dot{\Sigma} \subset \mathrm{WF}_b^{-1,m}(f).$$

In particular, if $f = 0$*, then* $\mathrm{WF}_b^{1,\infty}(u) \subset \dot{\Sigma}$*, while if* h *is Riemannian (but* $f \neq 0$ *necessarily) then* $\mathrm{WF}_b^{1,m}(u) \subset \mathrm{WF}_b^{-1,m}(f)$*.*

In fact, we can switch to the L^2 (rather than H^1) based b-wave front set: as shown in [27, Lemma 6.1], for solutions of the scalar wave equation $\Box u = 0$, $\mathrm{WF}_b^{1,m}(u) = \mathrm{WF}_b^{m+1}(u)$, and given the elliptic estimates we just sketched, the proof goes through unchanged for the form-valued wave equation. In fact, the stronger statement holds (again, as in [27, Lemma 6.1]):

$$\mathrm{WF}_b^{1,m}(u) \setminus \mathrm{WF}_b^{-1,m}(Pu) = \mathrm{WF}_b^{m+1}(u) \setminus \mathrm{WF}_b^{-1,m}(Pu). \tag{30}$$

6. Propagation of singularities on manifolds with corners: the bicharacteristic geometry

After the elliptic discussion, we turn to propagation and bicharacteristics. So we let

$$P \in \mathrm{Diff}^2(X, E),\ \sigma_2(P) = p\,\mathrm{Id},\ p = h, \tag{31}$$

where h is Lorentzian. Recall that $\dot{\Sigma} = \pi(\Sigma(P)) \subset {}^bT^*X$ is the compressed characteristic set. Generalized broken bicharacteristics are curves inside $\dot{\Sigma}$, satisfying a Hamilton vector field condition, plus an additional requirement where the boundary is smooth. More precisely:

Definition 10. *Generalized broken bicharacteristics* are continuous maps $\gamma : I \to \dot{\Sigma}$, where I is an interval, satisfying

(1) for all $f \in C^\infty({}^bT^*X)$ real valued,
$$\liminf_{s\to s_0} \frac{(f\circ\gamma)(s) - (f\circ\gamma)(s_0)}{s - s_0}$$
$$\geq \inf\{H_p(\pi^* f)(q) :\ q \in \pi^{-1}(\gamma(s_0)) \cap \Sigma(P)\},$$

(2) and if $q_0 = \gamma(s_0) \in {}^bT^*_{p_0}X$, and p_0 lies in the interior of a boundary hypersurface (i.e. a boundary face which has codimension 1, so near p_0, ∂X is smooth), then in a neighborhood of s_0, γ is a generalized broken bicharacteristic in the sense of Melrose-Sjöstrand [12], see also [7, Definition 24.3.7].

(1) is a very natural requirement. In the interior of X, we have defined bicharacteristics as integral curves of the Hamilton vector field of p in the characteristic set. Thus, if γ is an bicharacteristic segment over X°, then for all $f \in C^\infty(T^*X)$, the derivative of f along γ at s_0, i.e. $\lim_{s\to s_0} \frac{(f\circ\gamma)(s)-(f\circ\gamma)(s_0)}{s-s_0}$, is equal to $(H_p f)(\gamma(s_0))$. When we go back to the manifold with corners X, H_p is a vector field on T^*X, while the image of γ lies in ${}^bT^*X$. Moreover, π is not one-to-one, even when restricted to $\Sigma(P)$. Thus, the preimage of $\gamma(s_0)$ under π often contains many points (although it is still compact). Hence we cannot expect that f is differentiable along γ, although we can still expect *bounds* for the lim inf (and lim sup) of the difference quotients by taking the worst case scenario as we evaluate $H_p(\pi^* f)(q)$ over $q \in \pi^{-1}(\gamma(s_0)) \cap \Sigma(P)$, which explains the infimum. Note that replacing f by $-f$, the lim inf estimate for $-f$ gives a lim sup estimate for f, so (1) really gives a two-sided estimate. Thus, it is very natural to demand the estimate in the definition above – and conversely, this gives a useful notion of generalized broken bicharacteristics.

Without (2) the propagation theorem below would still hold, but would be weaker. In fact, our definition, without the strengthening given by (2), is equivalent to Lebeau's [10], see Lemma 11 below. While it is nice to have a stronger result, it is important to note what (2) actually achieves: it rules out certain rays tangent to the boundary hypersurface (where the boundary is smooth): it prevents rays gliding along the boundary to enter the shadow of an obstacle. We remark that this strengthening, which is a result of Melrose and Sjöstrand [12], is special to C^∞ singularities; if we were discussing the analytic wave front set, we could not do so.

Now, if P is a 'perturbed wave operator', as in (31), then Snell's law is encoded in the statement that γ is continuous. Indeed, any (locally defined) smooth functions on ${}^bT^*X$, such as $x, \tilde{y}, t, \sigma, \tilde{\zeta}, \tau$, are continuous along γ, i.e. their composition with

γ is continuous (since γ is a continuous map into ${}^bT^*X$). However, $\xi_j = x_j^{-1}\sigma_j$ is *not* continuous, so the normal momentum may jump.

In order to better understand the generalized broken bicharacteristics for P as in (31), we divide $\dot\Sigma$ into two subsets. We thus define the *glancing set* $\mathcal{G}$ as the set of points in $\dot\Sigma$ whose preimage under $\hat\pi = \pi|_\Sigma$ consists of a single point, and define the *hyperbolic set* $\mathcal{H}$ as its complement in $\dot\Sigma$. Thus, $q \in \dot\Sigma$ lies in $\mathcal{G}$ if and only if on $\hat\pi^{-1}(\{q\})$, $\xi_j = 0$ for all j. More explicitly, with the notation of (24),

$$\begin{aligned} &\mathcal{G}\cap\mathcal{U} = \{(0,y,0,\zeta):\ \zeta\cdot B(y)\zeta = 0,\ \zeta\neq 0\},\\ &\mathcal{H}\cap\mathcal{U} = \{(0,y,0,\zeta):\ \zeta\cdot B(y)\zeta > 0,\ \zeta\neq 0\}. \end{aligned} \tag{32}$$

In particular, for product metrics on $X = M\times\mathbb{R}$,

$$\begin{aligned} &\mathcal{G}\cap\mathcal{U} = \{(0,\tilde y,t,0,\tilde\zeta,\tau):\ \tau^2 = \tilde\zeta\cdot\tilde B(\tilde y)\tilde\zeta,\ (\tilde\zeta,\tau)\neq 0\},\\ &\mathcal{H}\cap\mathcal{U} = \{(0,\tilde y,t,0,\tilde\zeta,\tau):\ \tau^2 > \tilde\zeta\cdot\tilde B(\tilde y)\tilde\zeta,\ (\tilde\zeta,\tau)\neq 0\}. \end{aligned} \tag{33}$$

We can then describe broken bicharacteristics more concretely:

Lemma 11. *(Stated and proved in* [28] *in the product setting, but the same proof works in general.) Suppose γ is a generalized broken bicharacteristic. Then*

(1) *If $\gamma(s_0)\in\mathcal{G}$, let q_0 be the unique point in the preimage of $\gamma(s_0)$ under $\hat\pi = \pi|_\Sigma$. Then for all $f\in C^\infty({}^bT^*X)$ real valued, $f\circ\gamma$ is differentiable at s_0, and*
$$\frac{d(f\circ\gamma)}{ds}|_{s=s_0} = H_p\pi^* f(q_0).$$
(2) *If $\gamma(s_0)\in\mathcal{H}$, lying over a corner given in local coordinates by $x=0$, then exists $\epsilon>0$ such that $x(\gamma(s))=0$ for $s\in(s_0-\epsilon,s_0+\epsilon)$ if and only if $s=s_0$. That is, γ does not meet the corner $\{x=0\}$ in a punctured neighborhood of s_0. (Here, as usual, x is considered as a vector valued function, $x=(x_1,\dots,x_k)$.)*

Part (2) of this lemma indicates the possibility of an iterative description of the bicharacteristics: at $\mathcal{H}$, where we do not know in which direction they will travel, we still know that they will be in a less singular stratum (a lower codimensional corner) in a punctured neighborhood of s_0. Thus, if we already understand bicharacteristics in less singular strata, we can also understand their behavior at the corner under consideration.

In fact, we have an even stronger description of generalized broken bicharacteristics at $\mathcal{H}$, as in Lebeau's paper.

Lemma 12. *(Lebeau,* [10, Proposition 1]*) If γ is a generalized broken bicharacteristic, $s_0\in I$, $q_0=\gamma(s_0)$, then there exist unique $\tilde q_+,\tilde q_-\in\Sigma(P)$ satisfying $\pi(\tilde q_\pm)=q_0$ and having the property that if $f\in C^\infty({}^bT^*X)$ then $f\circ\gamma$ is differentiable both from the left and from the right at s_0 and*
$$\left(\frac{d}{ds}\right)(f\circ\gamma)|_{s_0\pm} = H_p\pi^* f(\tilde q_\pm).$$

Thus, one can associate an incoming and an outgoing point in T^*X, rather than merely in ${}^bT^*X$, into which the curve γ maps – the point being that even incoming and outgoing *normal* momenta are defined, although they can certainly differ. This indicates that, at least away from rays hitting the boundary tangentially, Figure 1 gives an accurate indication of the bicharacteristic geometry.

7. Propagation of singularities on manifolds with corners: the main theorem

We are now ready to state the main propagation theorem for the perturbed wave equation with Dirichlet boundary condition on Lorentzian manifolds – the Neumann case is completely analogous. Thus, consider $u \in H^1_{0,\mathrm{loc}}(X)$ satisfying

$$\langle Pu, v\rangle = \langle \nabla u, \nabla v\rangle_{L^2(X;T^*X)} + \langle Ru, v\rangle_{L^2(X)} = \langle f, v\rangle,\ v \in H^1_{0,c}(X), \tag{34}$$

$R \in \mathrm{Diff}^1(X)$, where the first equality defines $\langle Pu, v\rangle$. In fact, we may consider systems, i.e. allow u to be $\mathbb{C}^r$-valued, with an inner product $(.)_{k(p)}$ on $\mathbb{C}^r$ depending smoothly on $p \in X$, and write

$$\langle \phi, \psi\rangle_{L^2(X;\mathbb{C}^r)} = \int_X (\phi, \psi)_k \, d\mathrm{vol}_h.$$

and demand that $u \in H^1_{0,\mathrm{loc}}(X;\mathbb{C}^r)$ satisfies

$$\langle Pu, v\rangle = \langle \nabla u, \nabla v\rangle_{L^2(X;\mathbb{C}^r\otimes T^*X)} + \langle Ru, v\rangle_{L^2(X;\mathbb{C}^r)} = 0,\ v \in H^1_{0,c}(X;\mathbb{C}^r), \tag{35}$$

$R \in \mathrm{Diff}^1(X;\mathbb{C}^r)$. As propagation results are local, the (globally) more general case of u being a section of a vector bundle with an inner product is an immediate consequence; in particular, we obtain propagation of singularities for the wave equation on differential forms, which is of this form, with Dirichlet or Neumann (but *not* natural) boundary conditions.

Theorem 13 (See [27] for the scalar equation if $X = M \times \mathbb{R}$ with a product metric.)**.** *Suppose $u \in H^1_{0,\mathrm{loc}}(X;\mathbb{C}^r)$ and $Pu = f$ in the sense of (35) holding for all $v \in H^1_{0,c}(X;\mathbb{C}^r)$. Then $(\mathrm{WF}_b^{1,m}(u)\cap\dot\Sigma)\setminus\mathrm{WF}_b^{-1,m+1}(f)$, is a union of maximally extended generalized broken bicharacteristics of P in $\dot\Sigma \setminus \mathrm{WF}_b^{-1,m+1}(f)$.*

In particular, if $Pu = 0$ then $\mathrm{WF}_b^{1,\infty}(u) \subset \dot\Sigma$ is a union of maximally extended generalized broken bicharacteristics of P.

The scalar version of this theorem for $X = M \times \mathbb{R}$ with product metrics was proved in the real analytic setting by Lebeau [10], and in the C^∞ Lorentzian setting with C^∞ boundaries (and no corners) by Melrose, Sjöstrand and Taylor [12, 13, 22]. This result is thus the C^∞ version of Lebeau's theorem: the geometry is similar in the real analytic vs. C^∞ settings, but the analysis is quite different, though the C^∞ proof can be considered as an infinitesimal version of the real analytic argument.

The general technique in proving the theorem is to prove positive commutator estimates. We start by discussing the scalar case. In view of (30), it suffices to prove an L^2-based wave front estimate, i.e. to show that $Bu \in L^2$ for certain $B \in \Psi_b^{m+1}(X)$, with invertible principal symbol at a point q in question. In fact, B will have scalar principal symbol b, and will arise as a commutator. Namely, from Section 5 we deduce that for $u \in C^\infty(X)$ vanishing at ∂X, $A \in \Psi_{bc}^m(X)$ with principal symbol $\sigma_{b,m}(A) = a$,

$$\langle f, A^*Au\rangle - \langle A^*Au, f\rangle = \langle \nabla u, \nabla A^*Au\rangle + \langle Ru, A^*Au\rangle - \langle \nabla A^*Au, \nabla u\rangle - \langle A^*Au, Ru\rangle$$

The leading terms are the ones involving ∇, as they have the highest differential order, and expanding ∇ in local coordinates over a coordinate chart U, assuming

A is supported in $U \times U$,

$$
\begin{aligned}
&\langle \nabla u, \nabla A^*Au\rangle_{L^2(X;T^*X)} - \langle \nabla A^*Au, \nabla u\rangle_{L^2(X;T^*X)}\\
&= \sum_{ij} \langle H_{ij} D_{w_i} u, D_{w_j} A^*Au\rangle_{L^2(X)} - \langle H_{ij} D_{w_i} A^*Au, D_{w_j} u\rangle_{L^2(X)}\\
&= \sum_{ij} \langle A^*AH_{ij} D_{w_i} u, D_{w_j} u\rangle + \sum_{ij} \langle H_{ij} D_{w_i} u, [D_{w_j}, A^*A]u\rangle\\
&\qquad - \langle [H_{ij} D_{w_i}, A^*A]u, D_{w_j} u\rangle - \langle A^*AH_{ij} D_{w_i} u, D_{w_j} u\rangle,
\end{aligned}
\tag{36}
$$

and the first and last terms on the right cancel. As

$$
\begin{aligned}
&[D_{x_j}, A^*A] = A_1 D_{x_j} + A_0, \text{ with } A_1 \in \Psi_{bc}^{2m-1}(X),\ A_0 \in \Psi_{bc}^{2m}(X)\\
&\qquad \sigma(A_1) = i^{-1}\partial_{\sigma_j} a^2,\ \sigma(A_0) = i^{-1}\partial_{x_j} a^2
\end{aligned}
$$

and

$$
\begin{aligned}
&[D_{y_j}, A^*A] \in \Psi_{bc}^{2m}(X), \text{ with } \sigma(i[D_{y_j}, A^*A]) = \partial_{y_j} a^2,\\
&[H_{ij}, A^*A] \in \Psi_{bc}^{2m-1}(X),\\
&\qquad \text{with } \sigma(i[H_{ij}, A^*A]) = -\sum_k (x_k \partial_{x_k} H_{ij})\partial_{\sigma_k} a^2 - \sum_k (\partial_{y_k} H_{ij})\partial_{\zeta_k} a^2.
\end{aligned}
$$

Here we use that if $B \in \Psi_{bc}^k(X)$ then $B^* \in \Psi_{bc}^k(X)$ with principal symbol that of the adjoint of B, i.e. if the principal symbol of B is scalar, b, then that of B^* is $\bar{b}$. We deduce that

$$
\begin{aligned}
&\langle \nabla u, \nabla A^*Au\rangle_{L^2(X;T^*X)} - \langle \nabla A^*Au, \nabla u\rangle_{L^2(X;T^*X)}\\
&= \sum_{ij} \langle Q_{ij} D_{x_i} u, D_{x_j} u\rangle_{L^2(X)}\\
&\qquad + \sum_i (\langle Q_i D_{x_i} u, u\rangle_{L^2(X)} + \langle Q_i' u, D_{x_i} u\rangle_{L^2(X)}) + \langle Q_0 u, u\rangle_{L^2(X)}
\end{aligned}
\tag{37}
$$

where (with $H = \sum A_{ij}\xi_i\xi_j + 2\sum C_{ij}\xi_i\zeta_j + \sum B_{ij}\zeta_i\zeta_j$)

$$
Q_{ij} \in \Psi_{bc}^{2m-1}(X),\ Q_i, Q_i' \in \Psi_{bc}^{2m}(X),\ Q_0 \in \Psi_{bc}^{2m+1}(X),
$$

with

$$
\begin{aligned}
&i\sigma(Q_{ij}) = (\partial_{\sigma_i} a^2 + \partial_{\sigma_j} a^2) A_{ij} - \sum_k (x_k \partial_{x_k} A_{ij})\partial_{\sigma_k} a^2 - \sum_k (\partial_{y_k} A_{ij})\partial_{\zeta_k} a^2,\\
&i\sigma(Q_i) = i\sigma(Q_i') = \sum_j C_{ij} \partial_{y_j} a^2 \zeta_j + \sum_j A_{ij} \partial_{x_j} a^2\\
&\qquad\qquad - (\sum_k (x_k \partial_{x_k} C_{ij})\partial_{\sigma_k} a^2 + \sum_k (\partial_{y_k} C_{ij})\partial_{\zeta_k} a^2)\zeta_j
\end{aligned}
$$

and finally

$$
\begin{aligned}
i\sigma(Q_0) &= \sum_j C_{ij}(\partial_{x_i} a^2)\zeta_j + \sum_{ij} B_{ij}(\partial_{y_i} a^2 + \partial_{y_j} a^2)\zeta_i\zeta_j\\
&\qquad - \sum_{ij} (\sum_k (x_k \partial_{x_k} B_{ij})\partial_{\sigma_k} a^2 + \sum_k (\partial_{y_k} B_{ij})\partial_{\zeta_k} a^2)\zeta_i\zeta_j.
\end{aligned}
$$

All but the first term in $i\sigma(Q_0)|_{x=0}$ arise from considering the commutator of A^*A with $\Box_B$, the d'Alembertian on F given by $B|_{x=0}$. Also, at $x = 0$ all terms with C vanish. For the proof of the theorem we choose a appropriately, based on different ideas at $\mathcal{G}$ and $\mathcal{H}$. The basic (rough!) idea is that at $\mathcal{G}$, $D_{x_i}u$ can be estimated by a small multiple of $D_t u$ modulo lower order terms, so all terms but $\langle Q_0 u, u\rangle$ in (37) are negligible. Hence, one can proceed essentially as if one was studying propagation of singularities on the boundaryless manifold F, which explains why at $q_0 \in \mathcal{G}$ singularities move in the direction of $H_p(\hat{\pi}^{-1}(q_0))$, cf. Definition 10. At $\mathcal{H}$ more terms matter, but fortunately we need a weaker estimate – essentially that singularities leave F immediately, for which the basic idea is explained below.

As

$$\langle R, A^*Au\rangle - \langle A^*Au, Ru\rangle = \langle [A^*A, R]u, u\rangle + \langle (R - R^*)A^*Au, u\rangle,$$

and $[A^*A, R] \in \mathrm{Diff}^1\Psi_b^{2m-1}(X)$, this commutator can be absorbed into the Q_i and Q_0 terms above *without affecting the principal symbols*, thus can be neglected. If $R = R^*$, i.e. $P = \nabla^*\nabla + R$ is formally self-adjoint, the last term vanishes, thus can be dropped. In general this is not the case, but

$$R - R^* \in \mathrm{Diff}^1(X),\ R - R^* = \sum R_i D_{x_i} + R_0,\ R_0 \in \mathrm{Diff}_b^1(X),\ R_i \in C^\infty(X),$$

so $\langle (R - R^*)A^*Au, u\rangle$ has the same form as the Q_i and Q_0 terms in (37), with principal symbol a multiple of a^2 (rather than a multiple of a derivative of a^2). Making the derivative of a^2 large compared to the size of a^2 (which is what one usually does in any case to deal with regularization, see the discussion following [27, Equation (6.19)]), this term also becomes negligible as well. Thus, R does not affect the propagation estimates hence can be neglected.

Allowing u to be $\mathbb{C}^r$-valued barely affects the calculations. A key point though is that if $A \in \Psi_b^m(X;\mathbb{C}^r)$ with scalar principal symbol a, then $A^* \in \Psi_b^m(X;\mathbb{C}^r)$ with scalar principal symbol $\bar{a}$, and A^*A preserves Dirichlet or Neumann boundary conditions. Thus, the above calculation goes through unchanged (except that all $\langle ., .\rangle$ now are replaced by $\langle ., .\rangle_{L^2(X;\mathbb{C}^r)}$), thus the proof can be finishes exactly as in the scalar case [27].

This also shows the difficulty with natural boundary conditions on differential forms. In order to preserve the boundary conditions, we need to replace A^*A by $A^\dagger A$, with $A, A^\dagger$ as in Lemma 8. (We remark that there is an additional boundary term, the $\tilde{R}$ term in (26), but as $\tilde{R}$ is zeroth order, one can handle it just as the R term is handled above, using (27).) Then in (36) we have an extra term, $\langle ((A^\dagger A)^* - A^\dagger A)H_{ij}D_{w_i}u, D_{w_j}u\rangle$, and $(A^\dagger A)^* - A^\dagger A \in \Psi_b^{2m-1}(X;\Lambda X)$. Now, modulo commutator terms we can control (because they are lower order), we can shift this to one of the u's, i.e. we need to deal with $\langle H_{ij}D_{w_i}u, D_{w_j}((A^\dagger A)^* - A^\dagger A)u\rangle$. *If* $(A^\dagger A)^* - A^\dagger A$ preserves the natural boundary condition, this vanishes modulo the lower order R term in (34), so the previous argument goes through unchanged. If $(A^\dagger A)^* - A^\dagger A = E + F$, where E preserves boundary conditions and $F \in \Psi_b^{2m-2}(X;\Lambda X)$, the argument still goes through, and indeed it goes through even if $F \in \Psi_b^{2m-1}(X;\Lambda X)$ but $\sigma(F)|_{x=0}$ vanishes. However, this is not necessarily the case, and in general it seems that one needs to blow up the corner $x = 0$ (see the next section for a discussion of blow-ups) to construct a well-behaved A.

To be more precise, the proof of the theorem is thus based on two propositions giving propagation estimates at hyperbolic, resp. glancing points. Given these propositions, an argument of Melrose and Sjöstrand [12, 13], see also [7, Chapter XXIV] and [10], implies the theorem immediately – in particular, the proof from [27, Section 8] applies unchanged.

We only state the following propagation results for propagation in the forward direction along the generalized broken bicharacteristics. A similar result holds in the backward direction. The propagation results are *local*, so we can work in local coordinates $(x, \tilde{y}, t)$ on some open set U, and are *very rough* in the sense that they do not localize sharply along generalized broken bicharacteristics. It is the argument of Melrose and Sjöstrand that gives the precise bicharacteristic propagation then.

For instance, the tangential result states that in order to ensure that a point q_0 in $\mathcal{G}$ is not in the wave front set of u, we only need to ensure that for some sufficiently small $\delta > 0$, an $\mathcal{O}(\delta^2)$-sized ball at distance δ backwards along the $H_p(\hat{\pi}^{-1}(q_0))$ direction from $\hat{\pi}^{-1}(q_0)$ is disjoint from the wave front set of u. (Here recall that $\hat{\pi} = \pi|_\Sigma$. There is also a uniformity statement in the proposition for compact subsets that is used in turning the result into the theorem, i.e. in the Melrose-Sjöstrand argument.) Because of the $\mathcal{O}(\delta^2)$-sized requirement, which is necessary as we are not following a bicharacteristic precisely (we are simply fixing H_p at the point in question and extending it as a constant vector field using the local coordinates), we could use integral curves of any vector field W on T^*X with $W(\hat{\pi}^{-1}(q_0)) = H_p(\hat{\pi}^{-1}(q_0))$, at the cost of changing the constant in $\mathcal{O}(\delta^2)$.

To motivate the normal result, consider the function $\eta = -\sum \sigma_j/|\tau|$ (in the local coordinates) on ${}^bT^*X \setminus o$, so η vanishes on $\dot{\Sigma} \cap {}^bT^*_F X$, $F = \{x = 0\}$. Then $\pi^*\eta = -\frac{\sum x_j \xi_j}{|\tau|}$, and if

$$p_0 = -\sum \tilde{A}_{ij}(y)\xi_i\xi_j + \sum B_{ij}(y)\zeta_i\zeta_j,$$

with $t = y_{n-k}$, $\tau = \zeta_{n-k}$, then

$$\frac{|\tau|}{2} H_{p_0}\pi^*\eta = \sum_{ij} \tilde{A}_{ij}(y)\xi_i\xi_j - (\frac{1}{2}\sum_{ij} \partial_t B_{ij}(y))\eta = \sum B_{ij}(y)\zeta_i\zeta_j - p_0 - r\eta,$$

so in particular, it is positive on $\hat{\pi}^{-1}(\mathcal{H}) \cap {}^bT^*_F X$, for p_0 and η vanish there, and $\sum B_{ij}(y)\zeta_i\zeta_j > 0$ by (32). Thus, η is an *increasing* function along generalized broken bicharacteristics in view of Definition 10. Thus, the normal propagation result states that in order to conclude that q_0 is not in the wave front set in u it suffices to know that q_0 has a neighborhood such that $\mathrm{WF}_b(u)$ is absent from the half of the neighborhood where η is negative – note that η is certainly negative along *backward* generalized broken bicharacteristic segments from q_0 as we just remarked, and $\eta(q) < 0$ implies $q \notin F$.

It is remarkable that the argument of Melrose and Sjöstrand allows one to combine these rather rough results rather simply to get the full precise Theorem; here we merely point out that as $\eta(q) < 0$ implies $q \notin F$ in the normal case, there is a possibility for an induction on the dimension of boundary faces, cf. the remarks following Lemma 11. Below we simply state the results; as we already mentioned, the proofs require only simple modification of the proofs given for the Dirichlet (and Neumann) problems given in [27] and the positive commutator calculation above to compute the principal symbol of the commutator.

Proposition 14. *(Normal propagation.) Let* $q_0 = (0, \tilde{y}_0, t_0, 0, \tilde{\zeta}_0, \tau_0) \in \mathcal{H} \cap {}^bT^*_F X$, $F \cap U = U \cap \{x = 0\}$*, and let* $\eta = -\frac{\sum_j \sigma_j}{|\tau|}$ *be the function defined in the local coordinates discussed above, and suppose that* $u \in H^1_{0,\mathrm{loc}}(X; \mathbb{C}^r)$, $q_0 \notin \mathrm{WF}_b^{-1,\infty}(f)$, $f = Pu$ *in the sense of* (35)*. If there exists a conic neighborhood* U *of* q_0 *in* ${}^bT^*X$

such that

$$q \in U \text{ and } \eta(q) < 0 \Rightarrow q \notin \mathrm{WF}_b^{1,\infty}(u) \tag{38}$$

then $q_0 \notin \mathrm{WF}_b^{1,\infty}(u)$.

In fact, if the wave front set assumptions are relaxed to $q_0 \notin \mathrm{WF}_b^{-1,s+1}(f)$ *(*$f = Pu$*) and the existence of a conic neighborhood* U *of* q_0 *in* ${}^bT^*X$ *such that*

$$q \in U \text{ and } \eta(q) < 0 \Rightarrow q \notin \mathrm{WF}_b^{1,s}(u), \tag{39}$$

then we can still conclude that $q_0 \notin \mathrm{WF}_b^{1,s}(u)$.

Proposition 15. *(Tangential propagation.) Let* $u \in H^1_{0,\mathrm{loc}}(X; \mathbb{C}^r)$*, and let* $\tilde\pi : T^*X \to T^*F$ *be the coordinate projection* $\tilde\pi : (x, \tilde y, t, \xi, \tilde\zeta, \tau) \mapsto (\tilde y, t, \tilde\zeta, \tau)$*. Given* $K \subset {}^bS^*_UX$ *compact with*

$$K \subset (\mathcal{G} \cap T^*_FX) \setminus \mathrm{WF}_b^{-1,\infty}(f),\ f = Pu, \tag{40}$$

in the sense of (34)*, there exist constants* $C_0 > 0$*,* $\delta_0 > 0$ *such that the following holds. If* $q_0 = (\tilde y_0, t_0, \tilde\zeta_0, \tau_0) \in K$*,* $\alpha_0 = \hat\pi^{-1}(q_0)$*,* $W_0 = H_p(\alpha_0)$ *considered as a constant vector field in local coordinates, and for some* $0 < \delta < \delta_0$*,* $C_0\delta \le \epsilon < 1$ *and for all* $\alpha = (x, y, t, \xi, \zeta, \tau) \in \Sigma(P)$

$$\begin{aligned} &\alpha \in T^*X \text{ and } |\tilde\pi(\alpha - \alpha_0 - \delta W_0)| \le \epsilon\delta \text{ and } |x(\alpha)| \le \epsilon\delta \\ &\qquad \Rightarrow \pi(\alpha) \notin \mathrm{WF}_b^{1,\infty}(u), \end{aligned} \tag{41}$$

then $q_0 \notin \mathrm{WF}_b^{1,\infty}(u)$.

8. Geometric improvement

We now discuss a geometric improvement to the propagation theorem *for the scalar wave equation*; this is joint work with Richard Melrose and Jared Wunsch.

Definition 16. Suppose F is a boundary face (boundary hypersurface or edge) of X, and let $q \in \mathcal{H} \cap {}^bT^*_{F^\circ}X$. The *b-flow-out* of q, $\dot{\mathcal{F}}_{O,q}$, is the union of the images of generalized broken bicharacteristics $\gamma : [0, \infty) \to \dot\Sigma$ with $\gamma(0) = q$. The b-flowout of a set $S \subset \mathcal{H} \cap {}^bT^*_{F^\circ}$ is $\dot{\mathcal{F}}_{O,S} = \cup_{q \in S} \dot{\mathcal{F}}_{O,q}$. Also let the time T flow-outs $\dot{\mathcal{F}}_{O,q}(T)$ and $\dot{\mathcal{F}}_{O,S}(T)$ defined similarly, replacing $[0, \infty)$ by $[0, T)$. The b-flow-in $\dot{\mathcal{F}}_{I,q}$ is defined similarly, with the domain of definition of γ replaced by $(-\infty, 0]$.

Requiring $q \in \mathcal{H}$ makes the flow-out better behaved; indeed if $F = \partial X$ is smooth, then the flowout is a smooth manifold; indeed, as long S is an open subset of $\mathcal{H} \cap {}^bT^*_{F^\circ}X$ with compact closure, the flowout is smooth for short times, i.e. for sufficiently small T (depending on S), $\dot{\mathcal{F}}_{O,S}(T)$ is smooth.

If X has corners, even if a bicharacteristic is normal to a corner F (of codimension $k \ge 2$), it may be tangential to one of the boundary hypersurfaces through F, hence all the complications in bicharacteristic geometry that may take place on manifolds with smooth boundaries for bicharacteristics tangent to the boundary are still present, even for small times. We are thus led to distinguish between different kinds of rays in $\dot{\mathcal{F}}_{O,q}$.

Near an interior point p of F, one has local coordinates $(x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$ centered at p such that locally X is given by $x_1 \ge 0, \ldots, x_k \ge 0$, and F is given by $x = 0$ (i.e. $x_1 = \ldots = x_k = 0$). Write H_j for the boundary hypersurface given by $x_j = 0$. Below we also need the *(real) blow-up* of X at F, $[X; F]$, in which

F is replaced by a new boundary hypersurface, ff. Thus, $[X;F]$ is a manifold with corners with a smooth blow-down map $\beta : [X;F] \to X$ that restricts to a diffeomorphism $[X;F] \setminus \mathrm{ff} \to X \setminus F$ away from a boundary hypersurface, called the *front face* and denoted by ff, that is mapped onto F.

In order to describe this, first consider the blow-up $[\mathbb{R}^k;\{0\}]$ of the origin in $\mathbb{R}^k$; this amounts to introducing spherical coordinates around 0, i.e. $[\mathbb{R}^k;\{0\}] = [0,\infty)_r \times \mathbb{S}^{k-1}_\omega$, and the blow-down map β_k maps $\beta_k(r,\omega) = r\omega \in \mathbb{R}^k$, which is thus a diffeomorphism away from the boundary hypersurface $r = 0$, but is degenerate (though smooth) at $r = 0$. We can intersect this with the positive quadrant $[0,\infty)^k$; thus $[[0,\infty)^k;\{0\}] = [0,\infty)_r \times (\mathbb{S}^{k-1}_+)_\omega$, where $\mathbb{S}^{k-1}_+$ is the positive sector in $\mathbb{S}^{k-1}$, i.e. writing $\omega = (\omega_1,\ldots,\omega_k)$, is given by $\omega_j \geq 0$ for all j. Note that, away from $r = 0$, $r = \sqrt{\sum x_j^2}$ and $\omega_j = \frac{x_j}{r}$ – this is slightly cumbersome as one has to use $k-1$ of the ω_j's as local coordinates, and there are no $k-1$ of them that work globally. One can use projective coordinates instead, which can be made global: then with $r = \sum x_j$, $\theta_j = x_j/r$, r together with any $k-1$ of the θ_j's gives global coordinates – taking say $\theta_1,\ldots,\theta_{k-1}$, the cross section (which in this point of view is the standard $k-1$-simplex), $\mathbb{S}^{k-1}_+$, is given by $\theta_j \geq 0$, $j = 1,\ldots,k-1$, $\sum_{j=1}^{k-1}\theta_j \leq 1$.

Now, on a neighborhood of a point $p \in F^\circ$ in X, one can use local coordinates (x,y) as above, so in particular X is locally a product $U \times V$, $U \subset [0,\infty)^k$, $V \subset \mathbb{R}^{n-k}$ open and then $\beta^{-1}(U \times V)$ is a product $\tilde U \times V$, where $\tilde U = [U;\{0\}]$ is an open subset of $[0,\infty) \times \mathbb{S}^{k-1}_+$. It is not hard to check to check that the smooth structure of $[X;F]$ is independent of the choice of local coordinates, etc., so it is a manifold with corners. We refer to [18], the appendix of [11], and [25, Section 2] for more detailed discussions of blow-ups.

Definition 17. A generalized broken bicharacteristic segment γ, defined on $[0,s_0)$ or $(-s_0,0]$, $\gamma(0) = q \in \mathcal{H} \cap {}^bT^*_{F^\circ}X$ is said to approach F *normally* as $s \to 0$ if for all j

$$\lim_{s\to 0\pm} \frac{x_j(\gamma(s))}{s} \neq 0;$$

this limit always exists by Lemma 12.

This definition is independent of the particular choices of the x_j's. Indeed, equivalently, by Lemma 12, γ approaches F normally if $\tilde q_+$ (or $\tilde q_-$), given by the Lemma, satisfies

$$\tilde q_+ \in T^*X \setminus \cup_{j=1}^k T^*H_j,$$

where T^*H_j is the cotangent space given by the metric (i.e. the image of the tangent space under the Riemannian isomorphism), for by the Lemma, the limit above is $-2\sum_i \tilde A_{ij}(\tilde y((\tilde q_+)))\xi_i(\tilde q_+)$, i.e. is the x_j-component of the image of the covector $\tilde q_+$ under the inverse Riemannian isomorphism up to a constant factor.

The limits in Definition 17 are either all nonnegative or nonpositive, depending on the sign of s, and cannot vanish for all j simultaneously as $\gamma(0) = q \in \mathcal{H} \cap {}^bT^*_{F^\circ}X$, so $\sum \tilde A_{ij}(\tilde y((\tilde q_+)))\xi_i(\tilde q_+)\xi_j(\tilde q_+) > 0$. Thus, using $r = \sum x_j$ or $r = (\sum x_j^2)^{1/2}$ as the defining function of ff in $[X;F]$, and considering the positive time case for definiteness, we see from Lebeau's result (using that $r(\gamma(s))$ is comparable to s, so dividing by s in Definition 17 is analogous to dividing by r in constructing coordinates on the blown-up space) that the projection of $\gamma|_{(0,s_0)}$ to X extends to a continuous map $c_+ : [0,s_0) \to [X;F]$, and γ being normally incident means that $c_+(0) \in \mathrm{ff}$ does not lie on the lift of any of the boundary hypersurfaces H_j.

Note also that if γ approaches F normally then for $s_1 > 0$ sufficiently small, $\gamma|_{(0,s_1)}$ (or $\gamma|_{(-s_1,0)}$) lies in ${}^bT^*_{X^\circ}X = T^*X^\circ$, i.e. the restriction of γ_0 to a smaller open interval is actually a null-bicharacteristic, and its projection is a geodesic. In particular, as bicharacteristics through a point $q' \in T^*X^\circ$ are unique until they hit ∂X, we deduce that for $s \in (0, s_1)$ (resp. $s \in (-s_1, 0)$), the only generalized broken bicharacteristics through $\gamma(s)$ are reparameterizations of extensions of γ. Correspondingly, we make the definition:

Definition 18. For $q \in \mathcal{H} \cap {}^bT^*_{F^\circ}X$, the *regular part* $\mathcal{F}_{O,q,\mathrm{reg}}$ *of the flow-out* of q is the union of images $\gamma((0, s_0))$ of normally approaching generalized broken bicharacteristics $\gamma : [0, s_0) \to \dot{\Sigma}$ with $\gamma(0) = q$ and $\gamma(s) \in T^*X^\circ$ for $s \in (0, s_0)$.

The regular part of the flow-out of a set $S \subset \mathcal{H} \cap {}^bT^*_{F^\circ}X$ is

$$\mathcal{F}_{O,S,\mathrm{reg}} = \cup_{q\in S}\mathcal{F}_{O,q,\mathrm{reg}}.$$

In fact, one can easily see that $\mathcal{F}_{O,q,\mathrm{reg}}$ is a smooth embedded submanifold of T^*X° (by using the standard bicharacteristic flow to parameterize it), and if U is an open subset of $[X; F]$, with $\bar{U}$ disjoint from the lifts of the boundary hypersurfaces (but intersecting the front face ff, i.e. the lift of F), then for sufficiently small T (depending on U)

$$\dot{\mathcal{F}}_{O,q}(T) \cap {}^bT^*_{U\setminus \mathrm{ff}}X \subset \mathcal{F}_{O,q,\mathrm{reg}},$$

i.e. the small-time flow-out over U is smooth. Moreover, if S is a conic open subset of $\mathcal{H} \cap {}^bT^*_{F^\circ}X$ with $\bar{S} \subset \mathcal{H} \cap {}^bT^*_{F^\circ}X$, then $\dot{\mathcal{F}}_{O,q}(T) \cap {}^bT^*_{U\setminus \mathrm{ff}}X$ is a smooth conic *coisotropic* submanifold of $T^*X^\circ \setminus o$, i.e. its tangent space contains its symplectic orthocomplement.

Definition 19. We say that a generalized broken bicharacteristic $\gamma : (-s_0, s_0) \to \dot{\Sigma}$ with $\gamma(0) \in \mathcal{H} \cap {}^bT^*_{F^\circ}X$ is *limiting* at F if it is the limit of generalized broken bicharacteristics in $\dot{\Sigma} \setminus {}^bT^*_FX$, i.e. if there exist $\gamma_n : I \to \dot{\Sigma} \setminus {}^bT^*_FX$ such that $\gamma_n \to \gamma$ uniformly.

Thus, limiting generalized broken bicharacteristics are limits of bicharacteristics γ_n that just miss the edge F. As the γ_n hit only lower codimensional boundary faces, e.g. if F has codimension 2, only boundary hypersurfaces are hit, one has a better picture of propagation along the γ_n – e.g. if γ_n only hits boundary hypersurfaces, and only does so normally, singularities of solutions u of the wave equation necessarily propagate along γ_n as generalized broken bicharacteristics are unique through a given point in this region, so if u has a singularity along γ_n for some negative time, say $\gamma(-s_1) \in \mathrm{WF}_b^{1,r}(u)$, then it also has a singularity along γ_n for positive times, i.e. $\gamma(s) \in \mathrm{WF}_b^{1,r}(u)$ for all s. Thus, the limiting process indicates that one can expect that singularities along γ for positive times are as strong as those along γ for negative times.

In order to understand the limiting process better, it is useful to blow up F as above. The front face ff of $[X; F]$ has a fibration arising from $\beta : [X; F] \to X$, namely it is $\phi_0 = \beta|_{\mathrm{ff}}$. As explained above, the fibers $Z_p = \phi_0^{-1}(p)$, $p \in F$, of ϕ_0 are diffeomorphic to $\mathbb{S}^{k-1}_+$, or equivalently the standard simplex. Moreover, the Riemannian metric on X induces a metric on the fibers. Indeed, it is best to consider for each $p \in F$, coordinates (x, y) as in (4) with $C(0, y) = 0$, then on $[0,\infty)^k_x$, $\sum_{ij} a_{ij}(y)\, dx_i\, dx_j$ gives a translation-invariant Riemannian metric (where a_{ij} is the inverse of A_{ij}) which in terms of spherical coordinates corresponding to

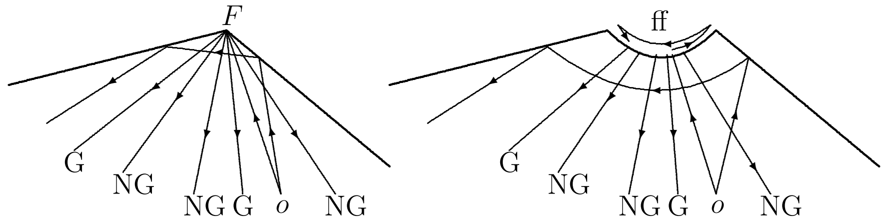

FIGURE 3. Geometric optic rays hitting a corner F, emanating from a point o. The rays labelled G are geometric at F, while those labelled NG are non-geometric at F. The leftmost geometric ray is a limit of rays like the unlabelled one shown on the figure that just miss F. The blown up version of the picture is shown on the right, with the reflecting line indicating the broken geodesic of length π induced on the front face ff (which is one fiber in this case). Thus, the total length of the three segments shown on ff is π; this can be thought of as the sum of three angles on the picture on the left: namely the angles between the incident ray and the right boundary (corresponding to the first segment), the right and left boundaries, finally the left boundary and the emanating reflected ray.

this metric, i.e. letting $\mathbb{S}_+^{k-1}$ be given by $\sum a_{ij}\omega_i\omega_j = 1$, has the form $dr^2 + r^2 k$, $h = \sum a_{ij}\,d\omega_i\,d\omega_j$ a Riemannian metric on $Z_y = \mathbb{S}_+^{k-1}$. Thus, Z_y is a manifold with corners with a Riemannian metric, and correspondingly one can talk about generalized broken geodesics (projections of *unit speed* generalized broken bicharacteristics on $Z_y \times \mathbb{R}_t$ to Z_y, i.e. $\frac{dt}{ds} = \pm 1$ along these) on Z_y. It turns out that families $\gamma_n \to \gamma$ can be rescaled to give rise to such generalized broken geodesics $\tilde{c} : [-\pi/2, \pi/2] \to Z_y$ of length π (this length π corresponds to the antipodal map if there are no breaks), with $\tilde{c}(\pm\pi/2) = \lim_{s\to 0\pm} c_\pm(s)$, with $c_\pm$ the continuous extensions of the projection of γ to X lifted to $[X; F]$, as discussed above. We thus make the following definition:

Definition 20. We say that a generalized broken bicharacteristic $\gamma : (-s_0, s_0) \to \dot{\Sigma}$ with $q = \gamma(0) \in \mathcal{H} \cap {}^bT^*_F X$ is *geometric* at F if there is a generalized broken geodesic $\tilde{c} : [-\pi/2, \pi/2] \to Z_{\pi(q)}$ of length π, where $\tilde{c}(\pm\pi/2) = \lim_{s\to 0\pm} c_\pm(s)$, with $c_\pm$ the continuous extensions of the projection of γ to X lifted to $[X; F]$, as discussed above.

Definition 21. Suppose $q \in \mathcal{H} \cap {}^bT^*_{F^\circ} X$, $\gamma_0 : (-s_0, 0] \to \dot{\Sigma}$ is a generalized broken bicharacteristic with $\gamma_0(0) = q$. The *non-geometric diffracted front* $\dot{\mathcal{F}}_{O,NG,\gamma_0}(T)$ emanating from γ_0 is the union of the images $\gamma|_{[0,T)}$ of *non-geometric* generalized broken bicharacteristics $\gamma : (-s_0, T] \to \dot{\Sigma}$ with $\gamma|_{(-s_0,0)} = \gamma_0$,

The *regular part of the non-geometric diffracted front* emanating from γ_0 is

$$\mathcal{F}_{O,NG,\gamma_0,\mathrm{reg}}(T) = \dot{\mathcal{F}}_{O,NG,\gamma_0}(T) \cap \mathcal{F}_{O,\gamma_0(0),\mathrm{reg}},$$

i.e. the union of the images $\gamma|_{[0,T)}$ of non-geometric generalized broken bicharacteristics $\gamma : (-s_0, T) \to \dot{\Sigma}$ with $\gamma|_{(-s_0,0)} = \gamma_0$, such that $\gamma|_{(0,T)}$ approaches F normally.

We expect that unless one is dealing with a solution that focuses on the corner, on the non-geometric broken bicharacteristics the reflected wave should be less singular than the incident wave. Although the full result (which would also include results along non-normally approaching bicharacteristics emananting from F) is too hard with the current state of technology, we have partial results on manifolds with corners (with corners of arbitrary codimension), and the full result in a model setting (manifolds with so-called edge metrics).

An example is the fundamental solution of wave equation with pole o near the edge, and we state the first version of the theorem in this case in order to make it more concrete. The general version is stated below in Theorem 28. Let $s = -n/2+1$, $n = \dim M = \dim X - 1$, so that the fundamental solution of the wave equation with pole $o \in X^\circ$ lies in $H^{s'}_{\mathrm{loc}}(X^\circ)$ for all $s' < s$ for non-zero times.

Theorem 22 (Melrose, Vasy and Wunsch, [14]). *Let F be a codimension k corner of X.*

Suppose that $\tilde{U}$ is an open subset of the front face ff of $[X;F]$, with $\overline{\tilde{U}} \subset \mathrm{ff}^\circ$, and $S \subset \mathcal{H} \cap {}^bT^*_{F^\circ}X$ is compact. Then there is an open set U in $[X;F]$ with $\tilde{U} \subset U$ and $T > 0$ such that the following holds.

Let $o \in U \cap X^\circ$, and let $\gamma_0 : [-s_0, 0] \to \dot{\Sigma}$, $0 < s_0 < T$, be a bicharacteristic normally approaching F with $\gamma_0(0) \in S$, $\gamma_0(-s_0) \in T^*_oX$. Let u be the forward fundamental solution of the wave equation with pole at o.

Then microlocally *near the regular part of the non-geometric diffractive front emanating from γ_0, u is in $H^{s'+(k-1)/2}_{\mathrm{loc}}(X^\circ)$ for all $s' < -n/2+1$, $n = \dim M$, i.e.*

$$\mathrm{WF}^{s'+(k-1)/2}(u) \cap \mathcal{F}_{O,NG,\gamma_0,\mathrm{reg}} = \emptyset. \tag{42}$$

Remark 23. The role of U, S and T is to ensure that there are no generalized broken bicharacteristics that go through both $\gamma_0(0)$ and N^*o, apart from $\gamma_0(0)$. In particular, there are no non-normally approaching generalized broken bicharacteristics through N^*o and q.

A different way of stating the result would be not to specify S, T, but say that if γ_0 hits F in a sufficiently small time (depending on o), and does so normally, then (42) holds, with the point being that the first rays emanating from o to hit F do so at hyperbolic points. This is how the analogue of this result for edge metrics is stated below in Theorem 24.

In order to explain the more general version, we start by considering a geometrically simpler case. Our model is manifolds with *edge metrics*. These are manifolds with boundary $\tilde{M}$, whose boundary has a fibration, $\phi_0 : \partial\tilde{M} \to Y$ with compact fibers Z (without boundary), and a Riemannian metric g compatible with this fibration.

More precisely, we assume that on a neighborhood U of $\partial\tilde{M}$, in which x is a boundary defining function, g is of the form

$$\begin{aligned} &g = dx^2 + \tilde{\phi}_0^* h + x^2 k, \ \text{where} \\ &h \in C^\infty([0,\epsilon) \times Y; \mathrm{Sym}^2\, T^*([0,\epsilon) \times Y)),\ k \in C^\infty(U; \mathrm{Sym}^2\, T^*\tilde{M}); \end{aligned}$$

we further assume that $h|_{x=0}$ is a nondegenerate metric on Y and $k|_{x=0}$ is a nondegenerate fiber metric. Here we extended the fibration ϕ_0 to a fibration $\tilde{\phi}_0 : U \to [0,\epsilon) \times Y$ on a neighborhood U of $\partial\tilde{M}$, and Sym^2 stands for symmetric 2-cotensors.

As an example, let $\tilde{M}$ be the real blow up of a C^∞ submanifold Y of a manifold without boundary M: $\tilde{M} = [M; Y]$. As explained above this means that we introduce 'spherical coordinates' around Y in M. Then the fibers Z are spheres, and a smooth metric on M would give rise to an edge metric on $\tilde{M}$. A particular example is the z axis in $\mathbb{R}^3$ blown up. This amounts to replacing the z-axis by its spherical normal bundle, $\mathbb{R}_z \times \mathbb{S}^1_\theta$. Thus, one replaces $\mathbb{R}^3$ by $\mathbb{R}_z \times [0,\infty)_r \times \mathbb{S}^1_\theta$, with the two identified away from the z axis, resp. the 'front face' $r = 0$ (i.e. $\mathbb{R}_z \times \{0\} \times \mathbb{S}^1$) by the diffeomorphism $\Phi(z, r, \theta) = (r\cos\theta, r\sin\theta, z)$, i.e. by introducing cylindrical coordinates. The boundary is then $r = 0$ (so $x = r$), the fiber Z is $\mathbb{S}^1$, and the Euclidean metric becomes $dz^2 + dr^2 + r^2\,d\theta^2$.

A more interesting case is if M is a manifold with corners, and $\tilde{M}$ 'total boundary blow up' (blow up all corners in the manner sketcked above for a single submanifold, starting with the corner of lowest dimension). In this case the fibers Z have a boundary, so this does not quite fit previous framework, e.g. one has $\theta \in [0, \beta]$ rather than $\theta \in \mathbb{S}^1$ when one blows up the corner of a wedge domain in $\mathbb{R}^2$. However, as long as one stays away from bicharacteristics hitting the face F° in question tangentially to the other faces, the methods used in the analysis of edge metrics still work.

For edge manifolds the compressed characteristic sets, generalized broken bicharacteristics, geometric broken bicharacteristics (where now the induced curve $\tilde{c}$ is an unbroken geodesic of length π on Z) and the *non-geometric* diffracted front are defined analogously to manifolds with corners (its regular part is now all of it), and the analogue of Theorem 22 holds:

Theorem 24 (Melrose, Vasy and Wunsch, Corollary 1.4 of [15])**.** *Suppose (M, g) is an edge manifold, $X = M \times \mathbb{R}_t$. Suppose that $o \in X^\circ$ is sufficiently close to ∂X, and let u be the forward fundamental solution of the wave equation with pole at o.*

Then microlocally *near the non-geometric part $\dot{\mathcal{F}}_{O,NG,\Lambda}$ of the diffractive front emanating from the flow-out Λ of N^*o, u is in $H^{s'+(k-1)/2}_{\mathrm{loc}}(X^\circ)$ for all $s' < -n/2 + 1$, $n = \dim M$, i.e. $\mathrm{WF}^{s'+(k-1)/2}(u) \cap \mathcal{F}_{O,NG,\Lambda} = \emptyset$.*

In the setting of edge metrics, resp. manifolds with corners with smooth metrics, let γ_0 be a bicharacteristic segment on $[0, s_0)$, $s_0 > 0$, $\gamma_0(0) \in {}^bT^*_{\partial X}X$, resp. $\gamma_0(0) \in {}^bT^*_{F^\circ}X$. Let Γ denote the set of all generalized broken bicharacteristics extending γ_0 (extending backwards is the interesting part here). The theorem on the propagation of singularities states that if

$$\Gamma_{-\epsilon} = \bigcup\{\gamma((-\epsilon, 0)) :\ \gamma \in \Gamma\} \tag{43}$$

is disjoint from $\mathrm{WF}_b(u)$, then so is the image of γ_0; similarly for $\mathrm{WF}^m_b(u)$.

In the edge manifold setting, let $\dot{\mathcal{F}}_I$ be the b-flow-in of ∂X, defined analogously to Definition 16. Thus, $\dot{\mathcal{F}}_I \setminus {}^bT^*_{\partial X}X$ is a smooth conic *coisotropic* submanifold of $T^*X^\circ \setminus o$, analogously to the manifolds with corners case. Note that as the fibers in the boundary have no boundaries themselves, there is no analogue of the normal incidence considerations required in the manifolds with corners setting.

We now recall the definition of coisotropic distributions. Let S be a conic coisotropic submanifold of T^*X°. Let $\mathcal{M}$ be the set of first order ps.d.o's with symbol vanishing along S, and let $\mathcal{M}^j$ be the set of finite sums of products of at most j factors, each of which is in $\mathcal{M}$.

Definition 25. We say that a distribution u is H^ℓ-coisotropic associated to S if for all N, and all $A_j \in \mathcal{M}$, $j = 1, \dots, N$, $A_1 \dots A_N u \in H^\ell$.

This definition is applicable to all coisotropic submanifolds S (with $\mathcal{M}$ defined as the set of first order ps.d.o's with symbols vanishing on S), in particular when $S = \Lambda$ is a conic Lagrangian; then one calls H^ℓ-coisotropic distributions associated to Λ *Lagrangian.* An example of a Lagrangian distribution is the fundamental solution of the wave equation for small non-zero times and pole o in X°; this is associated to the flowout of N^*o in Σ under H_p.

Non-focusing is the dual condition to coisotropy. For edge manifolds, it takes the following form:

Definition 26. Suppose (M, g) is an edge manifold, $\mathcal{M}$ the module corresponding to $\dot{\mathcal{F}}_I \cap T^*X^\circ$. We say that a distribution u satisfies the non-focusing condition of order ℓ for $q \in \mathcal{H}$ if for some $\epsilon > 0$, microlocally near $\dot{\mathcal{F}}_{I,q}(\epsilon)$, and for some N,

$$u = \sum A_j v_j, \;\; A_j \in \mathcal{M}^N, \;\; v_j \in H^\ell. \tag{44}$$

For manifolds with corners, the non-focusing statement does not make sense for non-normally incident bicharacteristics, so we need to add an assumption.

Definition 27. Suppose (M, g) is a manifold with corners with a smooth metric, F a codimension $k \geq 2$ corner, $\mathcal{M}$ the module corresponding to $\mathcal{F}_{I,\mathrm{reg}}$. We say that a distibution u saisfies the non-focusing condition of order ℓ for $q \in \mathcal{H} \cap {}^bT^*_{F^\circ}X$ if for some $\epsilon > 0$,

(1) if $\gamma : (-\epsilon, 0] \to \dot{\Sigma}$ satisfies $\gamma(0) = q$ and γ is not normally approaching F, then $\mathrm{WF}^\ell_b(u) \cap \gamma|_{(-\epsilon,0)} = \emptyset$, and
(2) microlocally near $\mathcal{F}_{I,q,\mathrm{reg}}(\epsilon)$, and for some N,

$$u = \sum A_j v_j, \;\; A_j \in \mathcal{M}^N, \;\; v_j \in H^\ell. \tag{45}$$

Thus, $u \in H^{\ell-N}$ only, but along $\mathcal{F}_{I,\mathrm{reg}}$ it is 'better' in the sense of (45): 'better' refers to the A_j being products of operators with vanishing principal symbols at $\mathcal{F}_{I,\mathrm{reg}}$, hence are lower order than their order indicates 'at' (rather than 'near') $\mathcal{F}_{I,\mathrm{reg}}$. If $u \in H^s$ but u is H^ℓ-non-focusing for $\ell > s$, we call $\ell - s$ the *non-focusing improvement.*

A Lagrangian distribution satisfies a non-focusing condition if the Lagrangian Λ intersects the coisotropic manifold $\dot{\mathcal{F}}_I$ transversally inside $\Sigma(P)$, see [15, Section 14]. In fact, inside Λ, the codimension of this intersection is the codimension k of the corner, minus 1, which implies that u satisfies the non-focusing condition with an *improvement* of $(k-1)/2 - \delta$ for all $\delta > 0$. Very roughly speaking, one can think of a Lagrangian distribution u associated to Λ is smooth along Λ, so one can divide u by some first order factors vanishing at $\dot{\mathcal{F}}_I \cap \Lambda$ (symbols of ps.d.o.'s) and still improve Sobolev regularity – for the precise argument see [15, Proposition 14.2].

Theorem 28. *[Melrose-Vasy-Wunsch,* [14]*, analogue of* [15, Theorem 1.3] *for manifolds with edge metrics] Suppose that (M, g) is a manifold with a smooth metric, $X = M \times \mathbb{R}$, and u is an admissible solution of $Pu = 0$, $P = D_t^2 - \Delta$. Let F be a corner of X, let $\gamma_0 : [0, s_0) \to \dot{\Sigma}$ be a normally incident bicharacteristic segment (with s_0 small), and suppose that u satisfies the non-focusing assumption of order ℓ for $\gamma_0(0)$.*

Then for $R < \ell$, $\gamma_0|_{(0,s_0)} \cap \mathrm{WF}^R(u) = \emptyset$ provided that, for some $\epsilon > 0$, all geometric *generalized broken bicharacteristics $\gamma \in \Gamma$ extending γ_0 satisfy $\gamma((-\epsilon, 0)) \cap \mathrm{WF}^R(u) = \emptyset$.*

That is, singularities of order $R < \ell$ can only propagate into γ_0 from geometric generalized broken bicharacteristics extending it; note the contrast with the propagation of singularities result: in (43) *all* extension of γ are needed. Theorems 22 and 24 are immediate consequences of this theorem, together with the non-focusing property of the flowout of N^*o, as well as the fact that over $T^*_{U\setminus\mathrm{ff}}X$, the flow-in of $\gamma_0(0)$ is regular, $\dot{\mathcal{F}}_{I,\gamma_0(0)}(T) \cap {}^bT^*_{U\setminus\mathrm{ff}}X \subset \mathcal{F}_{I,\gamma_0(0),\mathrm{reg}}$, so generalized broken bicharacteristics other than γ_0 cannot go through q and N^*o.

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