# SCATTERING THEORY ON SL(3)/ SO(3): CONNECTIONS WITH QUANTUM 3-BODY SCATTERING

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Abstract. In this paper we continue our program of extending the methods of geometric scattering theory to encompass the analysis of the Laplacian on symmetric spaces of rank greater than one and their geometric perturbations. Our goal here is to explain how analysis of the Laplacian on the globally symmetric space $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})$ is very closed related to quantum three-body scattering. In particular, we adapt geometric constructions from recent advances in that field by one of us (A.V.), as well as from our previous work [14] concerning resolvents for product spaces, to give a precise description of the resolvent and the spherical functions on this space. Amongst the many technical advantages, these methods give results which are uniform up to the walls of the Weyl chambers.

## 1. Introduction

It has long been observed that there are formal similarities between the spectral theory for Laplacians on (locally and globally) symmetric spaces of rank greater than one and Hamiltonians associated to quantum $N$-body interactions. Our contention is that these similarities have deep-seated explanations, rooted in the geometry of certain natural compactifications of the spaces involved and the asymptotic structure of these operators, and that the methods of geometric scattering theory constitute a natural set of techniques with which to study both problems. In the present paper we use these methods to provide an alternate perspective on mostly well-known results concerning the Laplacian on the globally symmetric space

$$M = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R}).$$

Besides giving a new set of methods to study scattering theory on this space which are not constrained by the algebraic rigidity and structure, this more general approach has benefits even in this classical framework. Specifically, starting from the perturbation expansion methods of Harish-Chandra, as explained in [10], and continuing through recent developments by Anker and Ji [1], [2], [3], it has always been problematic to obtain uniformity of various analytic objects near the walls of the Weyl chambers. We obtain this uniformity as a simple by-product of our method.

Let us now briefly set this work in perspective. The recent advances in quantum $N$-body scattering from the point of view of geometric scattering, to which we alluded above, are detailed in [24], [25] and [26], and we shall not say much more about this work here. Next, there are very many applications of geometric scattering theory to scattering on asymptotically Euclidean spaces and locally and globally symmetric spaces of rank one, [20], [9], [12], [5], to name just a very few (and concentrating on those most relevant to the present discussion). More

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recently there has been progress on geometric scattering on higher rank spaces. For example, Vaillant [23] has extended Müller's well-known $L^2$ index theorem for spaces with $\mathbb{Q}$-rank one ends to a general geometric setting. Most germane to the present work is [14], which contains the beginnings of a serious approach to dealing with the main technical problem of 'corners at infinity' which arise in higher rank geometry. That paper focuses on the special cases of products of hyperbolic, or more generally, asymptotically hyperbolic spaces, and produces a thorough analysis of the resolvent of the Laplacian on such spaces, including such features as its meromorphic continuation and the fine structure of its asymptotics at infinity. This analysis includes the construction of a geometric compactification of the double-space of the product space, on which the resolvent naturally lives as a particularly simple distribution, and which we call the resolvent double space. The methods of that paper rely heavily on the product structure, and an interesting representation formula for the resolvent in terms of the resolvents on the factors which is afforded by this structure. While not perhaps apparent there, the final results are in fact independent of this product structure and obtain in much more general situations.

Before embarking on a general development of the analysis of the resolvent for spaces with 'asymptotically rank two (or higher) geometry', we have thought it worthwhile to explain in detail how these methods apply to this specific example, $M = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})$, since it is a natural model space for the more general situation and is of substantial interest in and of itself. The methods here should apply more generally without any new ideas, just a bit more sweat and tears! Our aim is several-fold. At the very least we wish to emphasize the resolvent double-space, which is a compactification of $M \times M$ as a manifold with corners, and its utility for obtaining and most naturally phrasing results about the asymptotics of the resolvent; we also wish to show how the seemingly special product analysis of [14] emerges as the 'model analysis' in this non-product setting.

Let us now describe our results in more detail. Fix an invariant metric $g$ on $M$, and let $\Delta_g$ and $R(\lambda) = (\Delta_g - \lambda)^{-1}$ denote its Laplacian and resolvent. We wish to examine the structure of the Schwartz kernel of $R(\lambda)$, and in particular to determine its asymptotics as the spatial variables tend to infinity in $M$. We do this here for $\lambda$ in the resolvent set; with additional work (which is not done here) this can also be carried out for $\lambda$ approaching $\mathrm{spec}\,(\Delta_g)$. As in the traditional approach, the invariance properties of $\Delta$ allow us to reduce the analysis to that on the flats, and this turns out to be very close to three-body scattering. The central new feature of the analysis is to replace the perturbation series expansion of Harish-Chandra by $L^2$-based scattering theory methods in the spirit of [24]. As noted earlier, the results are easily seen to be uniform across the walls of the Weyl chambers. In future work we shall study the resolvent for spaces with 'asymptotically rank-two geometry', which is only slightly more difficult; unlike there, however, the present analysis is an explicit mixture of algebra (the reduction) and geometric scattering theory (the three-body problem).

In order to describe the structure of $R(\lambda)$, we first define a compactification $\overline{M}$ of $M$ itself. Recall that $M$ is identified with the set of 3-by-3 positive definite matrices of determinant 1; it is five-dimensional, and its compactification $\overline{M}$ is a $\mathcal{C}^\infty$ manifold with corners of codimension two. $\overline{M}$ has two boundary hypersurfaces, $H_\sharp$ and $H^\sharp$, in the interior of each of which the ratio of the smaller two, respectively the larger two, eigenvalues of the representing matrix is bounded. Correspondingly, either of these boundary faces is characterized by the fact that the ratio of the appropriate two eigenvalues extends to vanish on that face, hence gives a local boundary defining function. The subspace of diagonal matrices is identified with the flat $\exp(\mathfrak{a})$, and the Weyl group $W = S_3$ acts on it by permutations; its closure in $\overline{M}$ is a hexagon, the faces of which are permuted by the action of $W$. The fixed point sets of elements of the Weyl group partition $\mathfrak{a}$ into the Weyl chambers; the fixed point sets themselves constitute the Weyl chamber walls, and the closure of the chambers in $\overline{M}$ are the sides of the hexagon. Adjacent sides of the hexagon lie in different boundary hypersurfaces of $\overline{M}$. The boundary hypersurfaces of $\overline{M}$ are equipped with a fibration with fibers $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2,\mathbb{R}) = \mathbb{H}^2$. For example, two interior points of $H^\sharp$ are in the same fiber if the sum of the eigenspaces of the two larger eigenvalues (whose ratio is, by assumption, bounded in this region) is the same. This gives $\overline{M}$ a boundary fibration structure, similar to (but more complicated than) ones considered in [20, 17, 13].

To give the reader a feeling for what it means for a function to be smooth relative to this choice of smooth structure on the compactification, consider the two-dimensional flat $\exp(\mathfrak{a})$ of diagonal matrices $A$ as above. Let $\lambda_1, \lambda_2, \lambda_3$ be the diagonal entries (hence the eigenvalues) of $A$. The walls of the Weyl chambers are described by the equations $\lambda_i = \lambda_j$, $i \neq j$. We choose the positive Weyl chamber so that $\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_3} \in (0,1)$ on it. We may then use $\mu = \frac{\lambda_1}{\lambda_2}$ and $\nu = \frac{\lambda_2}{\lambda_3}$ as valid coordinates on the closure of this chamber in $\overline{M}$ away from the closure of the walls, and in particular near the corner. Thus, an SO(3)-invariant function $f$ is smooth away from the walls if it is a $\mathcal{C}^\infty$ function of $\mu$ and $\nu$ in the positive chamber. In particular, such a function has a Taylor series expansion around the corner $\mu = \nu = 0$:

$$f(\mu,\nu) \sim \sum_{j,k=0}^{\infty} a_{jk}\mu^j\nu^k$$

where the coefficients $a_{jk} = \partial_\mu^j\,\partial_\nu^k f(0,0)/j!k!$ are given by the usual formula from Taylor's theorem. $\overline{M}$ is a real analytic manifold with corners and this series converges when $f$ is real analytic, but in general, the assertion that $f$ is smooth on $\overline{M}$ means that $f$ has a complete asymptotic expansion. We refer to Section 2 for a detailed discussion of $\overline{M}$.

We must blow up $\overline{M}$ further in order to describe the resolvent efficiently. The motivation for this is that since the flat is a Euclidean space, its most natural compactification is the usual radial one, also known as the geodesic compactification. Unfortunately, this is not directly compatible with the structure of $\overline{M}$. Denote by $\rho_\sharp$ and $\rho^\sharp$ the boundary defining functions for $H_\sharp$ and $H^\sharp$ (given in terms of ratios of eigenvalues, as above). Then we replace these with the 'slow variables' $-1/\log\rho_\sharp$ and $-1/\log\rho^\sharp$, respectively. If we use *these* as new boundary defining functions on $\overline{M}$, then we obtain a new smooth structure, containing many more 'smooth' functions. We denote the resulting space by $\overline{M}_{\log}$, and call it the logarithmic blow-up of $\overline{M}$. Every smooth function on $\overline{M}$ is smooth on $\overline{M}_{\log}$, or in other words, the map $\iota : \overline{M}_{\log} \to \overline{M}$ is $\mathcal{C}^\infty$. To see this, note that the functions $\bar{\mu} = -1/\log(\lambda_1/\lambda_2)$ and $\bar{\nu} = -1/\log(\lambda_2/\lambda_3)$ give coordinates on the closure of the positive chamber in $\overline{M}_{\log}$, and $\mu = \exp(-1/\bar{\mu})$ and $\nu = \exp(-1/\bar{\nu})$ are $\mathcal{C}^\infty$ as functions of $\bar{\mu}, \bar{\nu} \geq 0$. On the other hand, $\iota^{-1}$ is a homeomorphism, but *not* $\mathcal{C}^\infty$, since for example $-1/\log\mu$ is not a smooth function of $\mu$ at $\mu = 0$! As another example of the effect of this change of $\mathcal{C}^\infty$ structure, note that if $f, g \in \mathcal{C}^\infty(\overline{M})$ and $f = g$ on $\partial\overline{M}$ then $f - g$ vanishes to all orders at $\partial(\overline{M}_{\log})$, i.e. its complete Taylor series in $(\bar{\mu}, \bar{\nu})$ vanishes at the corner of $\overline{M}_{\log}$, but this certainly need not be the case for the expansion in terms of $(\mu, \nu)$.

After the logarithmic blow-up, we perform the normal (spherical) blow-up of the corner $H_\sharp \cap H^\sharp$ in the space $\overline{M}_{\log}$. This sequence of operations results in the final 'single space'

$$\widetilde{M} = [\overline{M}_{\log}; H_\sharp \cap H^\sharp] \tag{1.1}$$

To check that this fulfills the goal of being compatible with the radial compactification of the flat, note that if $r$ is a Euclidean radial variable on $\mathfrak{a}$ (outside a compact set), then its inverse $r^{-1} \equiv x \in \mathcal{C}^\infty(\widetilde{M})$ is a total boundary defining function of $\widetilde{M}$, i.e. vanishes simply on all faces. Note that, with the previous notation, $r$ is a constant multiple of $\left((\log \lambda_1)^2 + (\log \lambda_2)^2 + (\log \lambda_3)^2\right)^{1/2}$, which may be easily expressed in terms of $\bar{\mu}$ and $\bar{\nu}$, if we also take advantage of the determinant condition $\lambda_1\lambda_2\lambda_3 = 1$. We also let $x_\sharp$, resp. $x^\sharp$, be defining functions of the lifts of $H_\sharp$ and $H^\sharp$ in $\widetilde{M}$ (so these are comparable to $-1/\log \rho_\sharp$, $-1/\log \rho^\sharp$, respectively, in the interiors of these faces). Finally, we denote by mf the new 'front face' created in this blow-up.

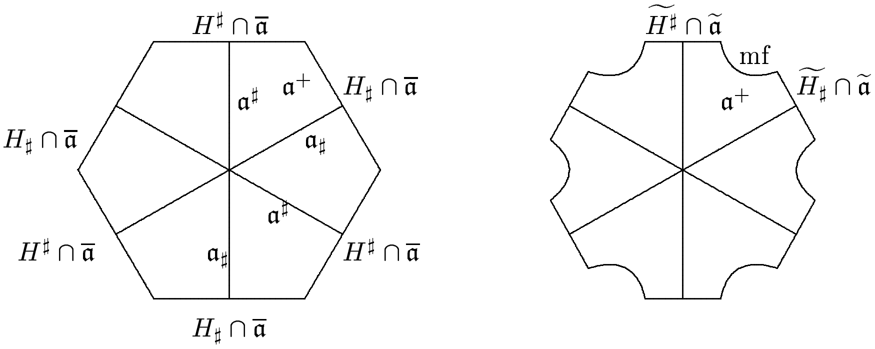


FIGURE 1. The closure of $\mathfrak{a}$ in the compactifications $\bar{M}$ and $\widetilde{M}$ of $M$.

A standard preliminary result in scattering theory concerns the far-field behaviour of $R(\lambda)f$ where $f \in \mathcal{C}_c^\infty(M)$, initially when $\lambda$ lies in the resolvent set, and later when it approaches the spectrum, cf. [18]. The geometry of $\tilde{M}$ has been set up precisely so that the analogous result here has a fairly simple form:

**Theorem.** *(See Corollary 5.5.) Suppose $f \in \mathcal{C}_c^\infty(M)$, and $\lambda \notin \operatorname{spec}(\Delta)$. Then, with $\lambda_0 = 1/3$,*

$$R(\lambda)f = \rho_\sharp\rho^\sharp x^{1/2} x_\sharp x^\sharp \exp\left(-i\sqrt{\lambda - \lambda_0}/x\right) g,$$

*where $g \in \mathcal{C}^\infty(\widetilde{M})$. The square root in the exponential is the one having negative imaginary part in the resolvent set $\lambda \in \mathbb{C} \setminus [\lambda_0, \infty)$. If $f \in \mathcal{C}_c^{-\infty}(M)$, then a similar statement holds (away from* sing supp *$f$).*

*Remark* 1.1. As already indicated, there is an extension of this result which describes the behavior as $\lambda \to \operatorname{spec}(\Delta)$. Thus, taking the limit as $\lambda$ approaches the spectrum from below,

$$R(\lambda - i0)f = \rho_\sharp \rho^\sharp x^{1/2} e^{-i\sqrt{\lambda-\lambda_0}/x} g_-, \qquad g_- \in \mathcal{C}^\infty(\widetilde{M}),$$

and there is an analogous formula for $R(\lambda + i0)f$, $\lambda > \lambda_0$, involving some function $g_+ \in \mathcal{C}^\infty(\widetilde{M})$. As before, the fact that $g_\pm \in \mathcal{C}^\infty(\widetilde{M})$ means that $R(\lambda - i0)f$ has a complete expansion (in terms of the appropriate defining functions). This extension allows one to analyze the range of the map $\mathcal{C}_c^\infty(M) \ni f \mapsto g_\pm|_{\partial\widetilde{M}}$. For example, one can show that the range of this map is dense in $\mathcal{C}^\infty(\mathrm{mf})$, but for simplicity we shall not discuss this here since, just as in the three-body setting, it requires a more elaborate phase space analysis.

*Remark* 1.2. Notice that the functions $r^{-k} = x^k$, $k \in \mathbb{N}$ are smooth on $\widetilde{M}$, but not on $\overline{M}$. On the other hand, later we shall briefly discuss Harish-Chandra's spherical functions, and the coefficients of the six oscillatory terms appearing there are actually in $\mathcal{C}^\infty(\overline{M})$. In fact, the expansions for these functions originally constructed by Harish-Chandra converge near the corners of $\overline{\mathfrak{a}}$.

One of the corollaries of this theorem is the identification of the Martin compactification of $M$ with $\widetilde{M}$. Recall that the Martin boundary of $M$, relative to some eigenvalue $\lambda \in \mathbb{R}$, $\lambda < \lambda_0$, is the set of equivalence classes of sequences of eigenfunctions of the form $U_j := R(\lambda; w, w_j)/R(\lambda; w_0, w_j)$. Here $R(\lambda; w, w')$ is the Schwartz kernel of $R(\lambda)$ at $(w, w') \in M \times M$ and $w_j$ is some sequence of points tending to infinity. The functions $U_j$ satisfy $U_j(w_0) = 1$ and $(\Delta - \lambda)U_j = 0$ on $M \setminus \{w_j\}$, so by standard elliptic theory at least some subsequence tends to limit which is a nontrivial eigenfunction; two sequences are said to be equivalent if the limiting eigenfunctions are the same. A sufficiently fine description of the asymptotics of the resolvent will determine exactly when and how the sequences $U_j$ can converge. To say that the Martin compactification is identified with $\widetilde{M}$ means specifically that equivalence classes of sequences are in one-to-one correspondence with points $q \in \partial\widetilde{M}$. In other words, for any such $q$, if $w_j$ is any sequence converging to $q$, then $U_j$ necessarily converges to an eigenfunction $U_q$ which depends only on $q$, and moreover, $U_q \neq U_{q'}$ if $q \neq q'$. These eigenfunctions might be called the plane wave solutions for $\Delta - \lambda$ on $M$. In any case, this will prove the

**Theorem.** *For any $\lambda < \lambda_0$, this procedure described above gives a natural isomorphism of the Martin compactification $\overline{M}_{\mathrm{Mar}}(\lambda)$ with $\widetilde{M}$.*

A key step in this identification is to understand the leading coefficients in the expansions of $R(\lambda)f$, i.e. the values of the corresponding function $g$, at the various boundary faces of $\widetilde{M}$ for any $f \in \mathcal{C}_c^{-\infty}(M)$, since in particular $R(\lambda; \cdot, w_j) = R(\lambda)\delta_{w_j}$. While we do not obtain an explicit formula for these leading coefficients, we can at least describe the range of the map sending $f$ to these boundary data. This is somewhat simpler than the corresponding statements in the on-spectrum case, to which we alluded above; thus in Section 6 we show that this map has dense range when $f$ varies over $\mathcal{C}_c^\infty(M)$, and letting $f$ vary over a slightly larger space, then this map is surjective onto an appropriate space of $\mathcal{C}^\infty$ functions on $\partial\widetilde{M}$. This is also of independent interest.

The identification of the Martin boundary for $M = \mathrm{SL}(3)/\mathrm{SO}(3)$ was initially due to Guivarc'h, Ji and Taylor [7], although the smooth structure of $\widetilde{M}$ plays no role there. Their arguments rely on certain estimates due to Anker and Ji [1, 2] which control the behaviour of the resolvent kernel at the Weyl chamber walls. In fact, the estimates of [2], see also [7, Section 8.10], when $\lambda$ is real and in the resolvent set, amount to upper and lower bounds for $R(\lambda)f$ by expressions of the same form as in our theorem. In later work, Anker and Ji use algebraic methods to give a uniform description of the leading term of the asymptotics. On the other hand, our analytic approach automatically gives uniform asymptotics, and this leads directly to this theorem about the Martin compactification, just as in our previous work [14].

There is yet another approach, due to Trombi and Varadarajan [22], which is intermediate between our approach and that of Harish-Chandra. They construct spherical functions as sums of polyhomogeneous conormal functions on $\overline{M}$ by constructing their Taylor series at all boundary hypersurfaces of $\overline{M}$. By comparison, Harish-Chandra's method amounts to constructing the spherical functions in Taylor series at the corner of $\overline{M}$. Owing to the algebraic nature of the space, these Taylor series actually converge in the appropriate regions, but of course this does not hold in more geometric settings.

As another application of our resolvent estimates, we also take up the construction of the spherical functions. This construction is essentially just that of Trombi and Varadarajan, but instead of appealing to convergence of the Taylor series, we use the resolvent to remove the error term, and this results in an additional term with the same asymptotics as the Green function. However, since the Taylor series actually converges, the error term vanishes, and hence the Green function asymptotics do not appear in the asymptotics of the spherical function; this is the extent to which algebra enters into our analysis.

To set this discussion into the language of Euclidean scattering, and in particular to compare with the language of three-body scattering, the spherical functions on $M$ are analogues of (reflected) 'plane waves' on the flats, corresponding to colliding particles, although here the eigenvalues collide; on the other hand, the Green function for $\Delta$ on $M$ is the analogue of a 'spherical wave' in Euclidean scattering. The conflict of terminology is somewhat unfortunate.

**Overview of the parametrix construction**

We now sketch in outline some details of our methods and constructions. The goal is to construct the Schwartz kernel of the resolvent as a distribution with quite explicit singular structure on some compactification of $\overline{M}^2$. This is accomplished by constructing a sequence of successively finer approximations to $(\Delta - \lambda)^{-1}$, where 'fineness' is measured by the extent to which these operators map into spaces with better regularity and decay at infinity. These parametrices lie in certain 'calculi' of pseudodifferential operators which are defined by fixing the possible singular structures of the Schwartz kernels of their elements both at the diagonal, but more interestingly, near the boundary of $\overline{M}^2$.

The first step is the construction of a parametrix in the 'small calculus', which we also call the edge-to-edge calculus, of pseudodifferential operators on $M$. In fact, this is defined on any manifold with corners up to codimension two which has fibrations on its boundary faces analogous to those of $\mathrm{SL}(3)/\mathrm{SO}(3)$. A more general development of this calculus will appear elsewhere. The parametrix $G(\lambda)$ for $\Delta-\lambda$ in this calculus has the property that the error $E(\lambda) = G(\lambda)(\Delta-\lambda)-\mathrm{Id}$ is smoothing but does not increase the decay rate of functions, hence is not compact on $L^2(M, dV_g)$. The constructions within this small calculus are merely a systematic way of organizing the local elliptic parametrix construction uniformly to infinity, and this parametrix gives scale-invariant estimates uniform to $\partial M$. To amplify on this last statement, we may use this calculus to define the Sobolev spaces $H^m_{\mathrm{ee}}(\overline{M}) = \{u \in L^2(M, dV_g) : \ \Delta^{m/2}u \in L^2(M, dV_g)\}$, $m \geq 0$. These spaces reflect the basic scaling structure of $\overline{M}$ near its boundaries.

At this point we use the group structure to simplify matters by effectively reducing to the flat. For $p \in M$, let $K_p$ denote the subgroup of SL(3) fixing this point; we may as well assume that $p$ is the identity matrix, which identifies the subspace of $K_p$-invariant functions with the space of Weyl group invariant functions on the flat $\exp(\mathfrak{a})$ (or equivalently, of functions on diagonal matrices invariant under permutation of the diagonal entries). Since the Green function (for $\Delta - \lambda$) with pole at $p$ is $K_p$-invariant, we may as well consider only parametrices which respect this structure. This reduction is certainly helpful, but not essential; it is the key point where our restriction to the actual symmetric space makes a difference in terms of simplifying the presentation.

Denote by $H^m_{\mathrm{ee}}(\overline{M})^{K_p}$ the invariant elements of the Sobolev space $H^m_{\mathrm{ee}}(\overline{M})$. The initial parametrix $G(\lambda)$ constructed in the first step may not preserve $K_p$ invariance, but this is easily remedied by averaging it over $K_p$; this produces an operator $G_p(\lambda)$ which satisfies

$$\tilde{E}(\lambda) = G_p(\lambda)(\Delta - \lambda) - \mathrm{Id} : L^2(M, dV_g)^{K_p} \longrightarrow H^m_{\mathrm{ee}}(\overline{M})^{K_p} \qquad \text{for all } m.$$

A second step is needed to get a parametrix with a decaying (as well as smoothing) error term. Thus we wish to construct another parametrix $\tilde{R}(\lambda)$ for $\Delta - \lambda$, $\lambda \notin \mathrm{spec}(\Delta)$, which acts on these $K_p$-invariant function spaces and satisfies

$$\tilde{R}(\lambda)(\Delta - \lambda) - \mathrm{Id} : H^m_{\mathrm{ee}}(\overline{M})^{K_p} \longrightarrow x^s H^m_{\mathrm{ee}}(\overline{M})^{K_p} \qquad \text{for all } s.$$

Granting this for a moment, we combine these two operators to get

$$\left(G_p(\lambda) + \tilde{E}(\lambda)\tilde{R}(\lambda)\right)(\Delta - \lambda) - \mathrm{Id} : L^2(M, dV_g)^{K_p} \to x^s H^m_{\mathrm{ee}}(\overline{M})^{K_p}$$

for any $s > 0$. This error term is now compact on $L^2(M, dV_g)^{K_p}$, and so using the simplest spectral properties of $\Delta$, we may remove it and obtain an inverse to $\Delta - \lambda$ acting on $K_p$-invariant functions. Since, as remarked before, $(\Delta - \lambda)^{-1}$ is necessarily $K_p$-invariant, we have captured the full resolvent.

The main subtleties in this paper center on the construction of the parametrix $\tilde{R}(\lambda)$, which we now outline. *This step, as implemented here, crucially uses the fact that we can reduce to spaces of $K_p$-invariant functions.*

Recall the single space $\widetilde{M}$. Denote by $\widetilde{H_\sharp}$ and $\widetilde{H^\sharp}$ the lifts of the boundary faces $H_\sharp$ and $H^\sharp$ from $\overline{M}$ to $\widetilde{M}$, and let $\mathfrak{a}_\sharp$ and $\mathfrak{a}^\sharp$ be the Weyl chamber walls intersecting these faces, respectively. Choose a $\mathcal{C}^\infty(\widetilde{M})^{K_p}$ partition of unity, $\chi_\sharp + \chi^\sharp + \chi_0 = 1$ on $\widetilde{M}$ such that $\operatorname{supp} \chi_\sharp$ is disjoint from $\widetilde{H^\sharp} \cap \mathfrak{a}^\sharp$ and $\operatorname{supp} \chi^\sharp$ is disjoint from $\widetilde{H_\sharp} \cap \mathfrak{a}_\sharp$, and with $\chi_0 \in \mathcal{C}^\infty_c(M)$. (These can be constructed on the closure of $\widetilde{\mathfrak{a}}$ and extended to $K_p$-invariant functions on $\widetilde{M}$.) Let $\psi_\sharp$, $\psi^\sharp$ be $K_p$-invariant cutoffs which are

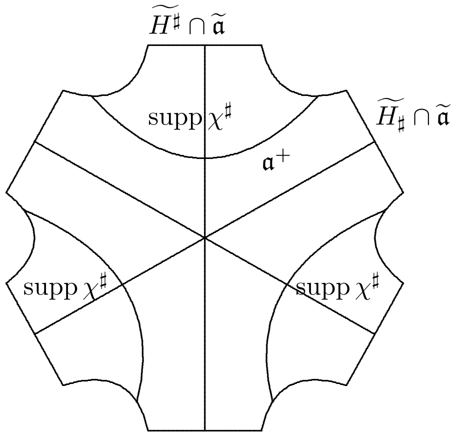


FIGURE 2. The intersection of $\operatorname{supp}\chi^\sharp$ with the flat $\widetilde{\mathfrak{a}}$.

identically 1 on $\operatorname{supp}\chi_\sharp$ and $\operatorname{supp}\chi^\sharp$, respectively, and which vanish on $\widetilde{H^\sharp} \cap \mathfrak{a}^\sharp$ and $\widetilde{H_\sharp} \cap \mathfrak{a}_\sharp$, respectively.

Along $\widetilde{H^\sharp}$, $\Delta$ is well approximated by the product operator $L^\sharp = \frac{3}{4}(sD_s)^2 + i\frac{3}{2}(sD_s) + \Delta_{\mathbb{H}^2}$, where $s = \rho^\sharp$ and $\Delta_{\mathbb{H}^2}$ is the Laplacian on the fiber $\mathbb{H}^2$ of $\widetilde{H^\sharp}$. More precisely,

$$\Delta - L^\sharp : H^m_{\mathrm{ee}}(\overline{M})^{K_p} \longrightarrow \rho^\sharp H^{m-1}_{\mathrm{ee}}(\overline{M}). \tag{1.2}$$

There is an analogous product operator $L_\sharp$ which approximates $\Delta$ near $\widetilde{H_\sharp}$. One small complication is that because of its structure at $\mathfrak{a}_\sharp$, $L^\sharp$ does not preserve $K_p$-invariance, but this is not serious since $\psi^\sharp L^\sharp \chi^\sharp$ does preserve this invariance. The fact that we can approximate $\Delta$ by product operators is the big gain, and is one of the remarkable things accomplished by passing to $\widetilde{M}$, for in fact the representation formula for the resolvent on a product space from [14] gives a precise description of $(L^\sharp - \lambda)^{-1}$ and $(L_\sharp - \lambda)^{-1}$. We use these as local models for the structure of the improved parametrix, and put

$$\tilde{R}(\lambda) = \psi^\sharp (L^\sharp - \lambda)^{-1} \chi^\sharp + \psi_\sharp (L_\sharp - \lambda)^{-1} \chi_\sharp.$$

It is not difficult to see that this has all the desired properties.

One simplification in this $K_p$-invariant setting is that one may bypass the first 'small calculus' step of the parametrix construction. The reason is that, acting on $K_p$-invariant functions supported near $\widetilde{H^\sharp}$, $\Delta - L^\sharp$ not only improves decay, but is a first order differential operator. (The latter fails on non $K_p$-invariant functions.) This is indicated already in the Sobolev mapping properties (1.2): there is a loss of only one derivative, even though $\Delta$ is second order. Because of this, one can obtain a parametrix with compact remainder by coupling a standard interior parametrix in a compact subset of $M$ with the parametrices for $L^\sharp$ and $L_\sharp$ as above. This observation is actually quite important in in our subsequent work [16] because, under complex scaling, the full Laplacian ceases to be elliptic, while its radial part retains this property. We are presenting the more general parametrix construction here simply to indicate how our methods can be adapted in a more general setting (of 'edge-to-edge structures'). Nonetheless, the reader may well wish to keep in mind this simplification, cf. Remark 5.3.

As noted above, one may also extend this construction to let $\lambda$ approach the spectrum, so as to obtain the structure of the limiting values of the resolvent $R(\lambda \pm i0) = (\Delta - (\lambda \pm i0))^{-1}$. This does require rather more work, albeit using well-understood techniques, and so is omitted in this paper for reasons of brevity. It is worth making some brief comments on a few consequences of this extension. The main difficulty is that we must keep track of the 'propagation of singularities' along $\partial\widetilde{\exp(\mathfrak{a})}$, which is very much as in in three-body scattering [24]. The notion of 'singularity' now refers to a microlocal description of the lack of rapid decay at infinity. An explicit iteration allows us to construct successively finer parametrices, leaving error terms which map $L^2(M, dV_g)^{K_p} \to x^k H^m_{\mathrm{ee}}(\bar{M})^{K_p}$ for higher and higher values of $k$ and $m$. The terms in this iterative series can be used to show that the singularities of generalized eigenfunctions reflect from the walls at infinity. Keeping track of these reflections more carefully shows that, just as in three-body scattering, only three reflections really occur.

The spherical functions centered at $p$ are also $K_p$ invariant, and are parametrized by incoming directions $\xi$, $|\xi|^2 = \lambda - \lambda_0$. These are constructed as perturbations of the plane waves

$$u_\xi(z) = \rho_\sharp(z)\rho^\sharp(z)e^{-i\xi\cdot z}$$

on $\mathfrak{a}$ (which we identify with $\exp(\mathfrak{a})$), where $z$ a Euclidian variable. In fact, on $\widetilde{M}$, $(\Delta - \lambda)u_\xi$ decreases rapidly away from $\widetilde{H^\sharp}$ and $\widetilde{H_\sharp}$. More importantly, if $\xi \notin \mathfrak{a}_\sharp \cap \mathfrak{a}^\sharp$, then $(\Delta - \lambda)u_\xi$ is nowhere incoming, in the sense of the scattering wave front set, so that $R(\lambda + i0)$ can be applied to it. The detailed structure of $R(\lambda + i0)$ discussed above leads to reflected plane waves. There are six such terms, corresponding to the six elements of the Weyl group. These correspond precisely to the six terms in Harish-Chandra's construction of spherical functions. The coefficients of the leading terms, which are Harish-Chandra's $\mathfrak{c}$-function, correspond to the scattering matrices of the 'two-body problems', in this case the scattering matrices on $\mathbb{H}^2$. As before, this analysis would allow us to let $\xi$ approach the walls. Certain aspects of this still appear in the construction of off-spectrum spherical functions in §6 below.

The remainder of this paper is organized as follows. In §2 we review the geometry of $M = \mathrm{SL}(3)/\mathrm{SO}(3)$ and its compactification $\overline{M}$. The small calculus of pseudodifferential operators on $M$ is defined in §3 through the properties of the Schwartz kernels of its elements on the resolvent compactification of $M \times M$. This leads to the first parametrix for $\Delta - \lambda$, which captures the diagonal singularity of $R(\lambda)$ uniformly to infinity. In §4, we discuss a model problem on $\mathbb{R} \times \mathbb{H}^2$, which is used for the construction of the finer parametrix in §5. In §6 we consider spherical functions, and discuss the extent to which algebra plays a role in their asymptotics. The Appendix contains a summary of results from [14] concerning resolvents for product problems.

As noted earlier, since this paper was initially written, we have completed two other papers [16] and [15], which study the analytic continuation properties of the resolvent on symmetric spaces (first on SL(3)/ SO(3), then in the general noncompact setting). Those papers contain a simplification of the parametrix construction which relies strongly on the symmetric space structure. This is viable because such analytic continuation results require much less information than details about the precise asymptotics of the Schwartz kernel of the resolvent. Although the original version of the present paper was our first studying the Laplacian on irreducible higher rank symmetric spaces, its publication has been delayed and will appear later than the others. It still remains important, however, in our general program, and we have incorporated some of the simplifications from the later works here.

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## 2. Geometric preliminaries

Our goal in this section is to analyze the structure of $M$ in various neighbourhoods of infinity. In the first two subsections we discuss the differential topology and metric structure in these neighbourhoods, and this leads in §2.3 to the definition of the preliminary compactification $\overline{M}$. The salient properties of this compactification are then abstracted in the definition of an ee-structure.

We refer to [4], [10] and [11] for nice general discussions of non-compact symmetric spaces, each with a slightly different emphasis; however, all the essential ingredients required here are discussed in this section.

2.1. **Geometry.** Any element $A \in \mathrm{SL}(3)$ admits a unique polar decomposition $A = BR$, where $B = (AA^t)^{1/2}$ is positive definite and symmetric, with $\det B = 1$, and $R \in \mathrm{SO}(3)$. This leads to the standard identification of the symmetric space $\mathrm{SL}(3)/\mathrm{SO}(3)$ with the space $M$ of positive definite 3-by-3 matrices with determinant 1, via

$$\mathrm{SL}(3)/SO(3) \ni [A] = A \cdot \mathrm{SO}(3) \stackrel{\Phi}{\longmapsto} B = (AA^t)^{1/2} \in M.$$

The action of $\mathrm{SL}(3)$ on $M$ is described by

$$\mathrm{SL}(3) \times M \ni (A, B) \longmapsto \phi_A(B) = (AB^2A^t)^{1/2}.$$

In a moment we shall use that $M \subset \mathrm{SL}(3)$ and $\phi_B(I) = B$.

Since $M$ is a submanifold of the space of symmetric matrices, elements of $T_BM$ can be regarded as symmetric matrices too. In particular, $T_{\mathrm{Id}}M$ consists of the symmetric matrices of trace 0, and we shall use the Killing form $g(W, W) = 6\,\mathrm{Tr}(WW^t) = 6\sum_{ij} w_{ij}^2$ as the metric on this vector space. The differential $(\phi_B^{-1})_*$ identies $T_BM$ with $T_{\mathrm{Id}}M$, and

$$g(W_1, W_2) = 6\,\mathrm{Tr}((\phi_B^{-1})_* W_1\, (\phi_B^{-1})_* W_2), \qquad W_1, W_2 \in T_BM, \tag{2.1}$$

gives an invariant Riemannian metric $g$ on all of $M$. Later, we shall compute this using the explicit matrix formula

$$(\phi_B^{-1})_*\big|_B (W) = \frac{1}{2}\left(B^{-1}W + WB^{-1}\right). \tag{2.2}$$

The first key point is that, away from some lower dimensional strata, the space $M$ is diffeomorphic to a product of an open set $\mathfrak{a}^*$ in a Euclidean space $\mathfrak{a}$ and a compact factor; this gives a globally well-defined sense of the 'radial' and 'angular' (or rotational) parts of $M$ (once the base-point $o$ has been fixed). To explain this, take any $B \in M$, $B \neq \mathrm{Id}$ and diagonalize it by writing $B = O\Lambda O^t$, where $\Lambda$ is diagonal with positive entries and $O \in \mathrm{SO}(3)$. We define $\mathfrak{a}$ as the 2-dimensional vector space of diagonal matrices with trace zero, so $\exp(\mathfrak{a}) = M \cap \mathrm{Diag}_3$ is the space of positive diagonal matrices with determinant 1, and $\mathfrak{a}^*$ as the open subset where the diagonal entries are all different. Neither $O$ nor $\Lambda$ are uniquely determined in this decomposition of $B$ since the neither the ordering of the entries of $\Lambda$ nor the sign of the entries of $O$ are fixed. These indeterminacies may be understood as follows. Let $\mathcal{P}$ denote the subgroup of all signed permutation matrices in SO(3). Thus for any $P \in \mathcal{P}$, $P\Lambda P^t$ is again diagonal with positive entries, and in fact $O\Lambda O^t = (OP^t)(P\Lambda P^t)(OP^t)^t$. This is the full extent of the ambiguity, and it is not hard to show that the matrices $\Lambda$ and $O$ appearing in the decomposition of $B$ are determined up to the action $P \cdot (\Lambda, O) = (P\Lambda P^t, OP^t)$, $P \in \mathcal{P}$.

Now, $\exp(\mathfrak{a}^*)$ is the subset of $\exp(\mathfrak{a})$ consisting of matrices with distinct diagonal entries, and we define $\exp(\mathfrak{a}^*)_+$ to be the smaller subset where the entries (in order descending along the diagonal) satisfy $\lambda_1 < \lambda_2 < \lambda_3$. This is stabilized by the subgroup $\mathcal{P}' \subset \mathcal{P}$ of signed diagonal permutation matrices with determinant 1, i.e. which have two $-1$'s and one $+1$ on the diagonal. It is now clear that

$$M^* = \exp(\mathfrak{a}^*)_+ \times \mathrm{SO}(3)/\mathcal{P}'$$

is a dense open set in $M$. The complement $\mathcal{C} = M \setminus M^*$ consists of matrices where at least two eigenvalues are the same. Again in analogy with three-body scattering, we think of the walls $\exp(\mathfrak{a}) \setminus \exp(\mathfrak{a}^*)$, which we identify with the subset $\mathfrak{w} \subset \mathfrak{a}$, as 'collision planes'. In fact, $\mathfrak{w}$ is the union of three lines, each making an angle of $2\pi/3$ with the next, which divide $\mathfrak{a}$ into six chambers.

The group $\mathcal{P}$ consists of orthogonal matrices which permute (and possibly reflect) the factors of the decomposition $\mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$; the subgroup $\mathcal{P}'$ consists of those elements which fix the factors. The quotient $\mathcal{P}/\mathcal{P}'$ is known as the Weyl group $W$ for $M$, and is identified with the full symmetric group $S_3$. (Note that $|\mathcal{P}| = 24$, and $|\mathcal{P}'| = 4$, and in fact $\mathcal{P}' = \mathbb{Z}_2 \times \mathbb{Z}_2$.) The Weyl group permutes the components of $\mathfrak{a} \setminus \mathfrak{w}$. The compact cross-section $\mathrm{SO}(3)/\mathcal{P}'$ is a compact locally symmetric space.

Let us now examine the structure of $M$ near the Weyl chamber walls. Suppose that the matrix $B$ lies in the neighbourhood $\mathcal{U}$ where the diagonal entries (again listed in order descending along the diagonal) of the corresponding matrix $\Lambda$ satisfy

$$c < \lambda_1/\lambda_2 < c^{-1}, \quad \lambda_3 > 1/c,$$

for some fixed $c \in (0,1)$. Recalling that $\lambda_3 = 1/\lambda_1\lambda_2$, we have

$$\lambda_1 = (\lambda_1/\lambda_2)^{1/2}\lambda_3^{-1/2} < 1, \quad \lambda_2 = (\lambda_2/\lambda_1)^{1/2}\lambda_3^{-1/2} < 1, \qquad \lambda_3 > 1$$

for $B \in \mathcal{U}$. This corresponds to the decomposition $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$, where $\mathbb{R}^2 = E_{12}$ and $\mathbb{R} = E_3$ are the sum of the first two eigenspaces and the third eigenspaces for $\Lambda$, respectively. We only keep track of the sum of the first two eigenspaces in this neighbourhood because they are indistinguishable when $\lambda_1 = \lambda_2$, whereas in this same neighbourhood the eigenspace for $\lambda_3$ is always well-defined. Completely equivalent to this is a different factorization of $B$ as $OCO^t$, where $C$ is block-diagonal preserving the splitting $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$. There is a larger ambiguity in this factorization since $C$ can be conjugated by an element of $\mathrm{O}(2)$, where the embedding $\mathrm{O}(2) \subset \mathrm{SO}(3)$ is as the subgroup of 2-by-2 orthogonal matrices in the top left corner, with the bottom right entry equal to $\pm 1$ appropriately. Denote by $C'$ the upper left block of $C$; the lower right entry is $\lambda_3$ and $\det C = \det C' \lambda_3 = 1$, so $C'' = \lambda_3^{1/2} C'$ is positive definite and symmetric with determinant 1, hence represents an element of $\mathrm{SL}(2)/\mathrm{SO}(2) \equiv \mathbb{H}^2$. Thus there is a neighbourhood $\mathcal{V}$ of [Id] in $\mathrm{SL}(2)/\mathrm{SO}(2)$ so that $\mathcal{U}$ is identified with $(\mathcal{V} \times \mathrm{SO}(3))/\mathrm{O}(2) \times (1/c, \infty)$. The action of $\mathrm{O}(2)$ on $\mathcal{V}$ has [Id] as a fixed point, but its action on $\mathrm{SO}(3)$, hence on the product, is free. The neighbourhood $\mathcal{V}$ can be chosen larger when $\lambda_3$ is larger, and the in the limit as $C \to \infty$, the 'cross-section' $\lambda_3 = C$ tends to $(\mathbb{H}^2 \times \mathrm{SO}(3))/\mathrm{O}(2)$. This is the total space of a fibre bundle over $\mathrm{SO}(3)/\mathrm{O}(2) = \mathbb{R}P^2$ with fibre $\mathbb{H}^2$. The Weyl chamber wall in this neighbourhood is the set of points fixed by the $\mathrm{O}(2)$ action, i.e. is the product of $(1/c, \infty)$ with the canonical section of this bundle consisting of the origins $o$ in each fibre $\mathbb{H}^2$.

There is an analogous product representation for the set of matrices $B$ for which $\lambda_1 < c < \lambda_2/\lambda_3 < 1/c$ for some fixed $c \in (0, 1)$.

Altogether, we have identified three neighbourhoods of infinity in $M$: the first, which we denote $\mathcal{U}_E$ is identified with the product of the two-dimensional Euclidean sector $\mathfrak{a}^*_+$ and the compact manifold $\mathrm{SO}(3)/\mathcal{P}'$. The other two, denoted $\mathcal{U}_\sharp$ and $\mathcal{U}^\sharp$, respectively, are identified with the product of a half-line $(1/c, \infty)$ and a neighbourhood $\mathcal{V} \subset (\mathbb{H}^2 \times \mathrm{SO}(3))/\mathrm{O}(2)$ which is invariant with respect to rotations of the $\mathbb{H}^2$ fibres. (The dependence on $c$ is omitted from the notation.)

2.2. **Coordinates and metric.** We now discuss some useful coordinate systems on $M$, particularly in the various neighbourhoods of infinity. It is sufficient to work in a neighbourhood of a fixed diagonal matrix $B_0 \in M$, and we shall define coordinates using the mapping $(B, O) \mapsto OBO^t$, where $B$ is a symmetric matrix near to $B_0$, and $O \in \mathrm{SO}(3)$ is close to the identity.

First suppose $B_0 \in \exp(\mathfrak{a}^*)$; we then restrict $B$ to lie in some neighbourhood $B_0$ in $\exp(\mathfrak{a}^*)$ where its diagonal entries remain distinct, and write these as $\lambda_1 < \lambda_2 < \lambda_3$. Since we assume $O \approx \mathrm{Id}$, we may neglect the $\mathcal{P}'$ quotient, and hence identify $O$ with the above-diagonal entries $c_{12}, c_{13}, c_{23}$ of its logarithm, i.e. the corresponding skew-symmetric matrix in $\mathfrak{so}(3) = T_{\mathrm{Id}}\,\mathrm{SO}(3)$ which exponentiates to $O$. A valid coordinate system is obtained by choosing any coordinates on $\exp(\mathfrak{a}^*)$, for example any two of the $\lambda_i$ (remember that $\lambda_1\lambda_2\lambda_3 = 1$), augmented by the $c_{ij}$.

Using (2.2) and (2.1), a straightforward calculation now gives that

$$\begin{aligned} g|_{B_0} = 6\left(\frac{d\lambda_1^2}{\lambda_1^2} + \frac{d\lambda_2^2}{\lambda_2^2} + \frac{d\lambda_3^2}{\lambda_3^2}\right) \\ +3\left(\frac{\lambda_1}{\lambda_2} - \frac{\lambda_2}{\lambda_1}\right)^2 dc_{12}^2 + 3\left(\frac{\lambda_1}{\lambda_3} - \frac{\lambda_3}{\lambda_1}\right)^2 dc_{13}^2 + 3\left(\frac{\lambda_2}{\lambda_3} - \frac{\lambda_3}{\lambda_2}\right)^2 dc_{23}^2 \end{aligned}$$

As expected, this expression is singular if any two of the $\lambda_j$ coincide, but as we verify below, this is only a polar coordinate singularity (which is obvious since the metric is smooth on $M$).

The first part of this formula involving the $\lambda_j$ must be reduced further, depending on the specific choice of coordinates on $\exp(\mathfrak{a}^*)$. For example, suppose we restrict to the subregion of $\mathcal{U}_E$ determined by the stronger inequalities

$$\mu \equiv \lambda_1/\lambda_2 < c, \qquad \nu \equiv \lambda_2/\lambda_3 < c,$$

for some $c < 1$. We have $\lambda_1 = \mu^{2/3}\nu^{1/3}$, $\lambda_2 = \mu^{-1/3}\nu^{1/3}$ and $\lambda_3 = \mu^{-1/3}\nu^{-2/3}$, and so

$$\begin{aligned} g = 4((d\mu/\mu)^2 + (d\mu/\mu)(d\nu/\nu) + (d\nu/\nu)^2) \\ +3(\mu - \mu^{-1})^2 dc_{12}^2 + 3(\mu\nu - \mu^{-1}\nu^{-1})^2 dc_{13}^2 + 3(\nu - \nu^{-1})^2 dc_{23}^2. \end{aligned} \tag{2.3}$$

Significantly, this expression is valid uniformly as $\mu, \nu \searrow 0$.

For reasons that will become clear later, it will also be advantageous to use the coordinates $\mu = \lambda_1/\lambda_2$ and $s = \lambda_3^{-3/2}$. Then

$$\lambda_1 = \mu^{1/2}\lambda_3^{-1/2} = \mu^{1/2}s^{1/3},\ \lambda_2 = \mu^{-1/2}\lambda_3^{-1/2} = \mu^{1/2}s^{1/3},$$

so that

$$\lambda_2/\lambda_3 = \mu^{-1/2}s,\ \lambda_1/\lambda_3 = \mu^{1/2}s.$$

In terms of these, the metric takes the form

$$\begin{aligned} g =& 3(d\mu/\mu)^2 + 4(ds/s)^2 + 3(\mu - \mu^{-1})^2\, dc_{12}^2 \\ &+3s^{-2}(\mu^{1/2}s^2 - \mu^{-1/2})^2\, dc_{13}^2 + 3s^{-2}(\mu^{-1/2}s^2 - \mu^{1/2})^2\, dc_{23}^2. \end{aligned} \tag{2.4}$$

This expression is valid in $\mathcal{U}_\sharp$, in particular uniformly as $s \searrow 0$, but a priori only away from the set $\{\mu = 1, s^2, s^{-2}\}$.

To resolve these apparent singularities in the coefficients in (2.4), suppose that the initial diagonal matrix $B_0$ has $\lambda_1 = \lambda_2 < 1$. The stabilizer of $B_0$ in SO(3) is O(2), and the orbit of $\mathfrak{a}$ under this subgroup consists of upper 2-by-2 block matrices, so it is natural to restrict $B$ to lie in the space of symmetric matrices in this block form, where the upper left block is written as $s^{1/3}B_1$, the lower right corner equals $s^{-2/3}$, and with 0 in the remaining entries. $B_1$ is symmetric with determinant 1, hence lies in $\mathbb{H}^2$, and we may use any coordinate system $(z_1, z_2)$ we please on this piece. We also restrict the orthogonal matrix $O$ to be the exponential of a skew-symmetric matrix with $c_{12} = 0$. Altogether, we use $(s, z_1, z_2, c_{13}, c_{23})$ as a coordinate system near $B_0$. It is clear from (2.4) that in the corresponding coordinate expression for $g$, the coefficients of all terms involving $ds$ and $dz_i$ are smooth, and that there are no cross-terms involving the $dc_{j3}$, so it remains to check that the coefficients of $dc_{i3}\, dc_{j3}$ are smooth as functions of $s$ and $z$. In fact, it even suffices to check their smooth dependence on $(s, z)$ at $c_{13} = c_{23} = 0$. This requires a calculation.

Set $E_{j3} = (e_{j3} - e_{3j})$ and define $O_j(\epsilon) = \exp(\epsilon E_{j3})$, so that $O_j'(0) = E_{j3}$. The corresponding tangent vector to $M$ at $B$ is

$$W_j = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} O_j(\epsilon)BO_j(\epsilon)^t = O_j'(0)B + BO_j'(0)^t = (E_{j3}B - BE_{j3}),$$

which pushes forward to

$$V_j = \frac{1}{2}\left(B^{-1}W_j + W_jB^{-1}\right) = \frac{1}{2}\left(B^{-1}E_{j3}B - BE_{j3}B^{-1}\right) \in T_{\mathrm{Id}}M.$$

Hence, denoting the entries of $B_1$ and $B_1^{-1}$ by $b_{ij}$ and $b^{ij}$, respectively, $i,j=1,2$, we get

$$g(W_i,W_j)=6\,\mathrm{Tr}(V_iV_j)$$
$$=3\left((sb_{1i}-s^{-1}b^{1i})(sb_{1j}-s^{-1}b^{1j})+(sb_{2i}-s^{-1}b^{2i})(sb_{2j}-s^{-1}b^{2j})\right).$$

Since (by definition), the $b_{ij}$ and $b^{ij}$ depend smoothly on $(s,z)$, so does this entire expression, and in fact $s^2g(W_i,W_j)$ induces a nondegenerate metric in the $c_{13},c_{23}$ directions at $s=0$.

We now write out the Laplacian in each of these coordinate systems. In the former region, we have

$$\begin{aligned}\Delta_g=\frac{1}{3}\big((\mu D_\mu)^2+(\nu D_\nu)^2-(\mu D_\mu)(\nu D_\nu)+i(\mu D_\mu)+i(\nu D_\nu)\\+(\mu D_{c_{12}})^2+(\nu D_{c_{23}})^2+(\mu\nu D_{c_{13}})^2\big)+E,\end{aligned}\tag{2.5}$$

where $E$ is the collection of all terms which are higher order when $\mu$ and $\nu$ are small. In other words, it is a sum of smooth multiples of a product of up to two of the vector fields $\mu D_\mu$, $\nu D_\nu$, $\mu D_{c_{12}}$, $\nu D_{c_{23}}$ and $\mu\nu D_{c_{13}}$, where the smooth multiple has at least one extra factor of $\mu$ or $\nu$.

There is a 'radial part' of this operator, which in these coordinates simply corresponds to $\Delta$ acting on functions which are independent of the $c_{ij}$; it is

$$\Delta_{\mathrm{rad}}=\frac{1}{3}\left((\mu D_\mu)^2+(\nu D_\nu)^2-(\mu D_\mu)(\nu D_\nu)+i(\mu D_\mu)+i(\nu D_\nu)+E'\right),\tag{2.6}$$

where $E'$ is an error term as above.

We shall only write out the radial part, rather than the full Laplacian in the other coordinate system, in $\mathcal{U}_\sharp$ near the Weyl chamber wall; it is

$$\begin{aligned}\Delta_{\mathrm{rad}}=&\frac{1}{3}\left((\mu D_\mu)^2-\left(\frac{\mu+\mu^{-1}}{\mu-\mu^{-1}}-\frac{s^2(\mu-\mu^{-1})}{s^4-s^2(\mu+\mu^{-1})+1}\right)i\mu D_\mu\right)\\&+\frac{1}{4}\left((sD_s)^2-\frac{2(s^4-1)}{s^4-s^2(\mu+\mu^{-1})+1}isD_s\right).\end{aligned}\tag{2.7}$$

2.3. **The compactification $\overline{M}$.** We are now in a position to describe the preliminary compactification $\overline{M}$ of $M$. It is obtained by adjoining to $M$ certain boundary hypersurfaces at infinity. These arise quite naturally from either the geometric description of $M$ in §2.1 or the coordinate systems in §2.2.

Consider first the neighbourhood $\mathcal{U}_E$, where $\lambda_1<\lambda_2<\lambda_3$, and suppose that $0<\mu,\nu<c<1$. The angular part, $\mathrm{SO}(3)/\mathcal{P}'$ is just carried along as a factor in this region, and $(\mu,\nu,c_{12},c_{13},c_{23})$ is a coordinate chart. We compactify by adjoining the hypersurfaces $\mu=0$ and $\nu=0$. These intersect at the corner $\mu=\nu=0$ at infinity, which is a copy of $\mathrm{SO}(3)/\mathcal{P}'$.

On the other hand, the region $\mathcal{U}_\sharp$ where $c<\mu<c^{-1}$, $\lambda_3>c^{-1}$ is identified with $(\mathcal{V}\times\mathrm{SO}(3))/\,\mathrm{O}(2)\times[1/c,\infty)$, where $\mathcal{V}$ is a ball around $o$ in $\mathbb{H}^2$. We compactify by adding the face $s'\equiv\lambda_3^{-1}=0$, i.e. we (partially) compactify this neighbourhood as $(\mathcal{V}\times\mathrm{SO}(3))/\,\mathrm{O}(2)\times[0,c)_{s'}$. As $c$ decreases, this forms a nested family, and its union as $c\searrow 0$ is an open boundary hypersurface which we denote $H_\sharp$. The analogous construction in $\mathcal{U}^\sharp$ yields a boundary hypersurface $H^\sharp$. These two faces intersect the corner neighbourhood from the first step in the regions where $\nu=0$ and $\mu=0$, respectively, and taken all together these constitute the boundary at infinity of $\overline{M}$. Since no corners are added in the second step, the final compactification of $\mathfrak{a}$

obtained by these two steps may be regarded as the hexagon in the left picture in Figure 1 (or rather, the quotient of this hexagon by the Weyl group $W$, though it is more convenient to picture this hexagon instead), and $\overline{M}$ is also a manifold with corners of codimension two.

We must show that these various regions are smoothly compatible in the region of overlap, so that $\overline{M}$ becomes a compact $\mathcal{C}^\infty$ manifold with corners up to codimension 2. This entails showing that the transition map, say when $c < \mu < 1$, is smooth. This is certainly clear in the interior, for this transition map is given by diagonalizing the 2-by-2 block $C''$ (cf. the discussion at the end of §2.1) and changing coordinates on the flat, and this is smooth away from the boundary at infinity. However, the boundary defining functions for this face ($\nu$ in the first chart and $s'$ in the second) are not smoothly related. In fact, since $\nu\mu^{1/2} = \lambda_3^{-3/2}$ and we are supposing that $\mu > c$ and $\nu \searrow 0$, we see that it is necessary to use $s = \lambda_3^{-3/2} = (s')^{-3/2}$ as the smooth boundary defining function for this face.

In summary, the manifold $\overline{M}$ is a compact manifold with a corner of codimension 2, diffeomorphic to $\mathrm{SO}(3)/\mathcal{P}'$, and two boundary hypersurfaces, $H_\sharp$ and $H^\sharp$, which are the closures of the parts of the boundary where the ratio of the two smaller, respectively the two larger, eigenvalues is bounded (or more directly, as the parts of the boundary where the ratio of the larger two, respectively the smaller two, eigenvalues vanishes). In the region $\mu, \nu \le c < 1$, where $\mu = \lambda_1/\lambda_2$, $\nu = \lambda_2/\lambda_3$, we have

$$H_\sharp = \{\nu = 0\}, \qquad H^\sharp = \{\mu = 0\}.$$

The interior of each of these faces is the total space of a fibre bundle, with base space $\mathrm{SO}(3)/\mathrm{O}(2)$ and fibre $\mathbb{H}^2$. The closure of each face is again a fibration, with the same base space and fibre obtained by compactifying $\mathbb{H}^2$ as a closed ball $\overline{\mathbb{H}^2} = \overline{B^2}$. We denote these two fibrations by $\phi_\sharp$ and $\phi^\sharp$, respectively.

We note in passing that there are other more directly group-theoretic procedures to obtain this same compactification. Let us denote by $Q$ the standard minimal parabolic subgroup of SL(3), consisting of upper triangular matrices of determinant 1. Then $\mathrm{SL}(3)/Q$ is identified with $\mathrm{SO}(3)/\mathcal{P}'$, which is the same as the corner of $\overline{M}$. Next, let $Q_{21}$ and $Q_{12}$ be the two maximal parabolic subgroups consisting of matrices which preserve the flags $\mathbb{R}^2 \subset \mathbb{R}^2 \oplus \mathbb{R}$ and $\mathbb{R}^2 \subset \mathbb{R} \oplus \mathbb{R}^2$, respectively. Then $\mathrm{SL}(3)/Q_{21}$ is the same as $\mathrm{SO}(3)/\mathrm{O}(2)$, where O(2) is embedded as the upper left hand block (with the lower right entry set to $\pm 1$ appropriately). This is the base space of the fibration $\phi_\sharp$ of $H_\sharp$; the $\mathbb{H}^2$ fibres are known as boundary components (even though each one constitutes only a small piece of each boundary hypersurface).

It remains to understand the relationship between these fibrations at the corner. We begin by considering the restrictions of these two fibrations to the corner, i.e. as maps

$$\phi_\sharp, \phi^\sharp : \mathrm{SO}(3)/\mathcal{P}' \to \mathrm{SO}(3)/\mathrm{O}(2).$$

The targets of these two maps are different copies of the same manifold, since the quotient of SO(3) in each case is by a different embedding of O(2). In any case, these maps are (quotients under the finite group action of) two independent Hopf fibrations. The fibres of either map are SO(3)-invariant and at each point, the tangent spaces of the fibres from the two families are independent. Each fibre is a circle (identified as the boundary of the corresponding fibre $\overline{\mathbb{H}^2}$). The sum of the tangent spaces of the two fibres at each point of $\mathrm{SO}(3)/\mathcal{P}'$ defines an everywhere nonintegrable plane-field distribution. However, if we fix a point $p' \in \mathrm{SO}(3)/\mathcal{P}'$, the fibers through $p'$ can be locally linearized, i.e. there is a neighborhood of $p'$ such that in suitable local coordinates the fiber of $\phi_\sharp$, resp. $\phi^\sharp$, through $p'$ is given by the vanishing of some of the coordinate functions. Indeed, for $p' = [\mathrm{Id}]$, this is immediate as the fibers are (images of) block diagonal matrices given by $c_{13} = c_{23} = 0$, resp. $c_{12} = c_{13} = 0$, and the case of general $p'$ follows by translation to Id. Moreover, as this translation can be done smoothly, we can make these local coordinates depend smoothly on $p'$.

This entire structure carries over to a full neighbourhood of the corner in $\overline{M}$. In other words, these fibrations extend to the neighbourhood of the corner $[0, c)_\mu \times [0, c)_\nu \times \mathrm{SO}(3)/\mathcal{P}'$ so that the fibres of the extension of $\phi_\sharp$ are identified with the product of $[0, c)_\mu$ and its fibres on the corner, and so that its base is extended to $\mathrm{SO}(3)/\mathrm{O}(2) \times [0, c)_\nu$, and similarly for $\phi^\sharp$.

We conclude this discussion by remarking that the metric $g$ naturally induces a metric on the fibers of each of the faces. For example, from (2.4), we see that $g$ induces a metric on $H_\sharp$, where $s = 0$, which restricts to each fibre on this face as (3 times) the standard hyperbolic metric.

2.4. **The boundary fibration structure.** The differential topological structure which we have defined on $\overline{M}$ is an example of a boundary fibration structure in the sense of [19]. This particular boundary fibration structure, which we christen the edge-to-edge (or ee) structure, is described in general as follows:

**Definition 2.1.** Suppose that $\overline{X}$ is a compact manifold with corners of codimension 2. Then we say that $\overline{X}$ is equipped with an edge-to-edge structure if

- (i) Each boundary hypersurface $H$ is the total space of a fibration $\phi_H : H \to B_H$, with fibre $F_H$, a manifold with boundary, transversal to $\partial H$, so that the restriction of $\phi_H$ to $\partial H$ is again a fibration with the same base and with fibre $\partial F_H$.
- (ii) The fibrations are independent at the corners, i.e. at any corner $H_1 \cap H_2$, the tangent spaces of the fibres $\partial F_j$ of $\phi_j|_{H_1 \cap H_2} : \partial H_j \to B_j$ are independent.

We write this structure as $(\overline{X}, \phi)$, where $\phi$ is the collection of all of the mappings $\phi_H$.

The independence assumption in (ii) gives a weak local product form:

**Lemma 2.2.** *For each corner $H_1 \cap H_2$, the fibrations are jointly linearizable in the following weak sense: there is a family of diffeomorphisms $\Phi(p')$, depending smoothly on $p' \in H_1 \cap H_2$, such that each $\Phi(p')$ maps a neighborhood of $p'$ in $H_1 \cap H_2$ to a neighborhood of the origin in $\mathbb{R}^{N_1+N_2+N_3}$ in such a way that the fibers of $\phi_1|_{H_1 \cap H_2}$ and $\phi_2|_{H_1 \cap H_2}$ passing through $p'$ are mapped to relatively open neighbourhoods in $\mathbb{R}^{N_1} \times \{0\} \times \{0\}$ and $\{0\} \times \mathbb{R}^{N_2} \times \{0\}$, respectively.*

We emphasize that $\Phi(p')$ does not simultaneously put into product form all the fibers of $\phi_1$ and $\phi_2$ near $p'$, which is generally impossible, but only those fibers passing through $p'$.

*Proof.* Fix $p'_0$, and take any local bases of sections $Y_{i,1}$, $i = 1, \dots, N_1$, $Y_{j,2}$, $j = 1, \dots, N_2$ for the fibers of $\phi_1|_{H_1 \cap H_2}$ and $\phi_2|_{H_1 \cap H_2}$. Choose additional vector fields $Z_\ell$, $\ell = 1, \dots N_3$ so that in a neighborhood $\mathcal{U}$ of $p'_0$, $\{Y_{i,1}, Y_{j,2}, Z_\ell\}$ give a local basis of sections of $T(H_1 \cap H_2)$. For any $p' \in \mathcal{U}$, define $\Phi(p')$ as the inverse of the diffeomorphism provided by the exponential map

$$\begin{aligned} &\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3} \supset \mathcal{V} \ni (t_1, t_2, t_3) \mapsto \\ &\exp\left(\sum_{j=1}^{2}\sum_{i=1}^{N_j} t_{i,j} Y_{i,j} + \sum_{\ell=1}^{N_3} t_{\ell,3} Z_\ell\right)(p') \in H_1 \cap H_2. \end{aligned} \tag{2.8}$$

For each $p'$ this is a diffeomorphism (since the differential at $t = 0$ is an isomorphism) with all the required properties, and clearly depends smoothly on $p'$. □

Fixing $p'$, we write the coordinates induced by $\Phi(p')$ as $y_{1i}$, $y_{2j}$ and $z_\ell$, $i = 1, \dots N_1$, $j = 1, \dots N_2$ and $\ell = 1, \dots N_3$. The fibers of $\phi_1$ and $\phi_2$ through $p'$ are given by $\{y_{2j} = z_\ell = 0\}$ and $\{y_{1i} = z_\ell = 0\}$. When $M = \mathrm{SL}(3)/\mathrm{SO}(3)$ and $\phi_1 = \phi_\sharp$, $\phi_2 = \phi^\sharp$, we have $N_1 = N_2 = N_3 = 1$, so we may omit the indices $i, j, \ell$, and then write $y_1 = \tilde{c}_{12}$, $y_2 = \tilde{c}_{23}$ and $z = \tilde{c}_{13}$. (The tildes here are meant to be a reminder that the coordinate system depends on $p'$.)

We associate to any ee structure on $\overline{X}$ the Lie algebra of all smooth vector fields on $\overline{X}$ which are unconstrained in the interior and which are required to lie tangent not only to the boundaries $H$ (hence also to the corners $H_i \cap H_j$), but also to the fibres $F_H$ of the fibration $\phi_H$ along boundary hypersurface $H$.

**Definition 2.3.** If $(\overline{X}, \phi)$ is an edge-to-edge structure, then the associated Lie algebra of $\mathcal{C}^\infty$ vector fields which are tangent to the fibers of $\phi_H$ for all $H$ is denoted by $\mathcal{V}_{\mathrm{ee}}(\overline{X})$. The class of differential operators $\mathrm{Diff}^*_{\mathrm{ee}}(\overline{X})$ obtained by taking all possible finite sums of products of elements of $\mathcal{V}_{\mathrm{ee}}(\overline{X})$ is the enveloping algebra.

For the symmetric space $M$, these ee vector fields can be expressed in terms of the coordinates $\mu, \nu$, and the left invariant vector fields $X_{ij}$ on $\mathrm{SO}(3)/\mathcal{P}'$ induced from the Lie algebra coordinates $c_{ij}$ centered at the point $p'$. Indeed,

$$\mu\partial_\mu, \nu\partial_\nu,\ \mu X_{12},\ \nu X_{23}, \mu\nu X_{13}. \tag{2.9}$$

comprise a spanning set of sections for $\mathcal{V}_{\mathrm{ee}}(\overline{M})$. To be even more explicit, using the coordinates $\mu, \nu, \tilde{c}_{ij}$ around any $p'$, there exist smooth functions $b_{ij}$, $a_{ijk}$, $e_{ij}$ such that

$$X_{12} = b_{12}\partial_{\tilde{c}_{12}} + \sum_{i,j=1,2} a_{ij3}\tilde{c}_{i3}\partial_{\tilde{c}_{j3}}, \tag{2.10}$$

$$X_{23} = b_{23}\partial_{\tilde{c}_{23}} + \sum_{i,j=2,3} a_{ij1}\tilde{c}_{1i}\partial_{\tilde{c}_{1j}}, \tag{2.11}$$

$$X_{13} = b_{13}\partial_{\tilde{c}_{13}} + e_{12}\partial_{\tilde{c}_{12}} + e_{23}\partial_{\tilde{c}_{23}}, \tag{2.12}$$

with $b_{12} = 1$ when $\tilde{c}_{13} = \tilde{c}_{23} = 0$, $b_{23} = 1$ when $\tilde{c}_{12} = \tilde{c}_{13} = 0$ (so $X_{12} = \partial_{\tilde{c}_{12}}$ and $X_{23} = \partial_{\tilde{c}_{23}}$ on the fibers of $\phi_\sharp$ and $\phi^\sharp$ through $p'$), and $b_{13}(p') \neq 0$.

We wish to write the analogous vector fields for a general ee structure. To do this, observe first that there is a diffeomorphism from a neighborhood $\mathcal{V}$ of the corner $H_1 \cap H_2$ in $\overline{X}$ to $[0, c)_\mu \times [0, c)_\nu \times (H_1 \cap H_2)$, mapping $H_1$ to $\nu = 0$ and $H_2$ to $\mu = 0$, which is fiber preserving in the sense that each $\phi_j|_{\mathcal{V}\cap H_j}$ factors through the induced projection to $H_1 \cap H_2$. (Thus, the factors $[0, c)_\mu$, resp. $[0, c)_\nu$ extend the fibers of $\phi_1|_{H_1\cap H_2}$, resp. $\phi_2|_{H_1\cap H_2}$, to define $\phi_1 : H_1 \cap \mathcal{V} \to B_1$, resp. $\phi_2 : H_2 \cap \mathcal{V} \to B_2$.) This diffeomorphism may be obtained by exponentiating appropriate vector fields as above.

Now, near any $p' \in H_1 \cap H_2$, choose local bases of sections $Y_{i,j}$, $i = 1, \ldots, N_j$, $j = 1, 2$ for the fibres of $\phi_j|_{H_1 \cap H_2}$ and extend these to $\mathcal{V}$ using this diffeomorphism. The extensions will still be denoted $Y_{ij}$; but note that at $H_j$, each $Y_{i,j}$ is still tangent to the fibres of $\phi_j|_{\mathcal{V}}$. Complement these with vector fields $Z_\ell$, $\ell = 1, \ldots N_3$, transverse to both fibers at $p'$. It is then clear that $\mathcal{V}_{\mathrm{ee}}(\overline{X})$ is generated over $\mathcal{C}^\infty(\overline{X})$ by

(2.13)
$$\mu\partial_\mu,\ \nu\partial_\nu,\ \mu Y_{i,1},\ \nu Y_{j,2},\ \mu\nu Z_\ell,$$
$$i = 1, \ldots, N_1,\ j = 1, \ldots, N_2,\ \ell = 1, \ldots, N_3 = \dim M - N_1 - N_2 - 2.$$

Furthermore, by Lemma 2.2, the vector fields $Y_{i,j}$ and $Z_\ell$ can be chosen to be of a special form around each $p'$, analogously to (2.10), (2.11), (2.12). In particular, using coordinates $\mu, \nu$, $y_{i,j}$ and $z_\ell$ on $\mathcal{V}$, $Y_{i,j} = \partial_{y_{i,j}}$ on the fiber of $\phi_j$ through $p'$.

The space $\mathcal{V}_{\mathrm{ee}}(\bar{X})$ is the full set of $\mathcal{C}^\infty$ sections of a vector bundle ${}^{\mathrm{ee}}T\overline{X}$ over $\overline{X}$, called the ee-tangent bundle. Its dual, the ee-cotangent bundle, is denoted ${}^{\mathrm{ee}}T^*\overline{X}$. The following result is is then almost tautological from the preceding definitions:

**Proposition 2.4.** *Let $\overline{M}$ be the compactification of $M = \mathrm{SL}(3)/\,\mathrm{SO}(3)$ described above, and let $g$ be the invariant metric. Then $g \in \mathcal{C}^\infty(\overline{M}; S^2({}^{ee}T^*\overline{M}))$, and furthermore, $\Delta_g \in \mathrm{Diff}^2_{ee}(\overline{M})$.*

## 3. The edge-to-edge small calculus

In this section we discuss a general construction of parametrices in the setting of ee structures with a view toward future applications. The reader should take note, however, that as already observed in the introduction, for the immediate purposes of this paper certain simplifying features obviate various parts of this more general parametrix construction, cf. also Section 5, especially Remark 5.3. Nonetheless, if nothing else, the discussion here should indicate the flexibility and scope of our overall methods.

We follow a general strategy which has proved successful in many analogous situations, whereby a boundary fibration structure leads to an adapted calculus of pseudodifferential operators, which are used in turn to investigate the analytic properties of the elliptic operators associated to that boundary fibration structure. We give only a 'minimal' development of such a theory here, and in particular discuss only those parts of the theory of ee-pseudodifferential operators needed to understand the resolvent of $\Delta_g$. This requires only slightly more than solving the 'model problems' corresponding to any more general elliptic ee operators, and hence simplifies the presentation. The more general theory will be taken up elsewhere.

In this section we construct the 'small calculus' of ee-pseudodifferential operators, which are characterized in terms of the conormal behaviour of their Schwartz kernels on a resolution of the double space $\overline{X} \times \overline{X}$ (which is very closely related to the construction of the calculus $\Psi_{p0}(\overline{X})$ in [14]). We also discuss the mapping properties of these operators.

3.1. **The ee double space.** The preliminary step in this construction is to resolve via blow-up the double-space $\overline{X} \times \overline{X}$ to obtain the 'edge-to-edge double space' $\overline{X}^2_{\mathrm{ee}}$. This is done first and most carefully for the symmetric space $X = M$, but after that we briefly sketch the construction for a general ee structure. The main criterion for this space, which is what we verify in this subsection, is that the lift of the

Laplacian, first to either the left or right factor of $\overline{X}^2$ and then to this blow-up, is transversely elliptic with respect to the lifted diagonal, uniformly to all boundary faces. Since $\Delta$ is an elliptic combination of vector fields in $\mathcal{V}_{\rm ee}(\overline{X})$, this will be true provided the lifts of the ee structure vector fields span the normal bundle of the lifted diagonal, uniformly on the closure of this submanifold. (In contrast, these vector fields vanish at the boundary of the diagonal on $\overline{X}^2$.) What this guarantees is that one can define a class of pseudodifferential operators, the elements of which have Schwartz kernels supported (or concentrated) near the diagonal, and this class is sufficiently large as to contain parametrices for any symbol-elliptic operator, such as $\Delta_g - \lambda$. Such a parametrix captures the smoothing properties of the resolvents of elliptic operators, for instance, but does not contain sufficient information to describe many things we wish to understand, including for example the off-diagonal asymptotics of the Green function.

If one were to proceed to construct a 'refined' parametrix (reflecting the asymptotics) *purely* using the double-space, which we do *not* do in this paper, we would need a second criterion, namely that the Schwartz kernel of the resolvent of the Laplacian is in fact well behaved (in terms of its conormal, or better, polyhomogeneous behaviour) on this space. Here we avoid this by working directly with operators acting on $K_p$-invariant functions for $X = M$, in which case one only needs to describe the double space for a product model $M_1 \times M_2$, see Section 4, which has been accomplished in [14].

Now set $X = M$. The space $\overline{M}^2$ is a manifold with corners up to codimension 4, and the diagonal intersects its boundary in the corners of codimension 2 and 4 only. The intersection near the corners of codimension 2, and the way to resolve the space here, is exactly the same as for the edge calculus on a manifold with boundary, cf. [17]. Namely, we blow up the fiber diagonal. The situation near the maximal codimension corner is a bit more complicated. To be concrete, we describe the situation in terms of the coordinates $(\mu, \nu, c_{ij})$ on each copy of $\overline{M}$. In fact, let $(\mu, \nu, c_{ij})$ and $(\mu', \nu', c'_{ij})$ denote lifts of identical sets of coordinates from the left and right factors of $\overline{M}^2$, respectively. For each point $p' = (0, 0, c'_{ij})$ of the corner of $\overline{M}$ we choose adapted coordinates $\tilde{c}_{ij}$ on $H^\sharp \cap H_\sharp$ as in the last section, depending smoothly on $p'$, so that the fibers of $\phi^\sharp$, resp. $\phi_\sharp$ through $p'$ are given by $\tilde{c}_{12} = \tilde{c}_{13} = 0$ and $\tilde{c}_{23} = \tilde{c}_{13} = 0$, respectively. Then $(\mu, \nu, \mu', \nu', \tilde{c}_{ij}, c'_{ij})$ is a full set of coordinates on $\overline{M}^2$ near its maximal codimension corner. (Note that the double space $(H^\sharp \cap H_\sharp)^2$ is the natural place to use these adapted coordinates.)

In the edge calculus on a manifold with boundary, the edge double space is obtained by blowing up the fibre diagonal of the boundary. We should do the same thing away from the corner. Thus, for $\nu, \nu' \geq c > 0$ but $\mu, \mu' \to 0$, we blow up the fiber diagonal

$$\begin{aligned} \mathcal{F}^\sharp &= \{(q, q') \in H^\sharp \times H^\sharp : \phi^\sharp(q) = \phi^\sharp(q')\} \\ &= \{\mu = \mu' = 0, \tilde{c}_{12} = 0, \tilde{c}_{13} = 0\}. \end{aligned}$$

This has the effect of desingularizing the lifts (from the left factor) of the vector fields $\mu\partial_\mu$, $\mu X_{12}$ and $\mu\nu X_{13}$ near $\mu = 0$. Note that the remaining vector fields $\nu\partial_\nu$ and $\nu X_{23}$ in the spanning set for $\mathcal{V}_{\rm ee}$ are tangent to this fiber diagonal, but disjoint from its $b$ normal bundle, hence do not become more singular in this blowup. Similarly, in the region where $\mu, \mu' \geq c > 0$ but $\nu, \nu' \to 0$, we blow up the other fiber diagonal

$$\begin{aligned}\mathcal{F}_\sharp &= \{(q,q') \in H_\sharp \times H_\sharp : \phi_\sharp(q) = \phi_\sharp(q')\}\\ &= \{\nu = \nu' = 0, \tilde{c}_{23} = 0, \tilde{c}_{13} = 0\}\end{aligned}$$

to desingularize the lifts of $\nu\partial_\nu$, $\nu X_{23}$ and $\mu\nu X_{13}$.

Unfortunately, these two submanifolds do not intersect transversely, and hence it matters in which orders the blow-ups are performed. In other words, the two spaces obtained when one first blows up $\mathcal{F}^\sharp$ and then the lift of $\mathcal{F}_\sharp$ to the resulting space, or when these operations are done in the reverse order, are not naturally diffeomorphic, in the sense that the 'identity' map in the interior does not extend smoothly to the boundaries. In general, one should deal with this problem by first blowing up the intersection $\mathcal{F}_\sharp \cap \mathcal{F}^\sharp$ (with respect to an appropriate parabolic scaling); this has the effect of separating $\mathcal{F}_\sharp$ from $\mathcal{F}^\sharp$, and the two blow-ups are now independent of one another. On this big double space, lifts of the ee-vector fields certainly span the normal bundle of the diagonal, and all the standard pseudodifferential constructions proceed without difficulty.

However, this space is not 'minimally resolved', i.e. there are other spaces with the aforementioned spanning property, which are locally blow-downs of this fully blown-up space. This presents some complications for our present purposes, hence we proceed in what seems to be a less natural manner, simply choosing an order in which to blow up the two faces; to be concrete, blow up $\mathcal{F}^\sharp$ first and after that, $\mathcal{F}_\sharp$. Denote the resulting space by $\overline{M}^2_{\mathrm{ee}}$; the space obtained by reversing the order of the blow-ups is denoted $(\overline{M}^2_{\mathrm{ee}})'$. These are not naturally diffeomorphic, but for now the difference is immaterial since the identity map in the interior *does* extend smoothly to the whole interior of the front faces, i.e. the set of boundary hypersurfaces which intersect the lifted diagonal, as well as to the interior of the corner where they intersect. Even more strongly, the spaces of smooth functions on $\overline{M}^2_{\mathrm{ee}}$ and $(\overline{M}^2_{\mathrm{ee}})'$ which vanish to infinite order on all other boundary hypersurfaces except these front faces are naturally isomorphic. The same is true for spaces of distributions conormal to the diagonal with the same infinite order vanishing away from the front faces. Since only spaces of this type are used in the construction of the small calculus, this is a reasonable compromise.

We say a few words about these various equivalences in this last paragraph. It obviously suffices to prove the first statement, about the smooth extendability of the identity map. In either space $(\overline{M}^2_{\mathrm{ee}})$ or $(\overline{M}^2_{\mathrm{ee}})'$, there are two front faces, one arising from the blowup of $\mathcal{F}^\sharp$ and the other arising from the blowup of $\mathcal{F}_\sharp$. In either case, the interiors of these faces are the bundles of inward pointing unit normal vectors over $\mathcal{F}^\sharp$ and $\mathcal{F}_\sharp$, respectively, so the fibres are open quarter-spheres. This structure is natural, and can be used to identify these faces away from the corner where they intersect. The interior of that corner again corresponds to a bundle of inward pointing unit normal vectors, now over a base space which is the minimal fibre diagonal $\{c_{12} = c'_{12}, c_{13} = c'_{13}, c_{23} = c'_{23}\}$ (or, in 'adapted coordinates' $\{\tilde{c}_{12} = 0,\ \tilde{c}_{13} = 0,\ \tilde{c}_{23} = 0\}$) at $\{\mu = \mu' = \nu = \nu' = 0\}$, with fibres diffeomorphic to an open orthant of a sphere of one lower dimension than before and which is naturally identified with one of the open boundary faces of the fibres over the front faces. Again, these identifications are natural, so this proves the claim. We note in passing that the 'big' double space mentioned earlier is obtained by blowing up this corner.

Let us recast this more generally. Let $(\overline{X}, \phi)$ be an ee structure. If $H$ is a boundary hypersurface, define its fiber diagonal

$$\mathrm{diag}_{H,\phi_H} = \mathrm{diag}_H = \{(p, p') \in H \times H : \ \phi_H(p) = \phi_H(p')\}.$$

Fix an ordering $<$ on the set of boundary hypersurfaces, $\{H_i\}_i$. This induces an order on $\mathrm{diag}_{H_i,\phi_{H_i}}$. In terms of this ordering, we define

$$\overline{X}^2_{\mathrm{ee}} = \left[\overline{X}^2; \ \bigcup_i \mathrm{diag}_{H_i}\right],$$

where the ordering determines the sequence of blowups. For simplicity of notation, we omit the choice of ordering from the notation.

*Coordinates on the blow-up.* Each of these blow-ups can be realized by the introduction of appropriate polar coordinates. However, such coordinates are usually quite messy, and in practice projective coordinates are far more convenient. In general, in any local coordinate system $(x_1, \ldots, x_k, y_1, \ldots, y_\ell)$ near a corner of codimension $k$ (so each $x_j \geq 0$), if a product submanifold is specified by $x_1 = \ldots = x_r = 0$, $y_1 = \ldots = y_s = 0$ for some $r \leq k$, $s \leq \ell$, then projective coordinates for its blow-up are obtained by choosing any one of these boundary defining functions $x_j$ and defining $\xi_i = x_i/x_j$, $i \leq r$, $i \neq j$, and $u_i = y_i/x_j$, $i \leq s$. The full projective coordinate system then is

$$(x_j, \xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots \xi_r, x_{r+1}, \ldots, x_k, u_1, \ldots, u_s, y_{s+1}, \ldots, y_\ell).$$

These are undefined on the face in the original space where $x_j = 0$, but are valid in (a neighborhood of) the closure, in this blown up space, of each region where $\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_r < C$ and $x_j > 0$. In particular, the resulting coordinate systems, as $j$ varies, cover the entire blown up space. In this region of definition, the equation $x_j = 0$ defines the new face obtained in the blow-up.

To return to our specific setting, introduce a set of projective coordinates which is nonsingular away from the lift of the face where $\mu' = 0$ (which is the copy of $H^\sharp$ on the second factor of $\overline{M}$). Now define coordinates near the lift of $\mathcal{F}^\sharp$:

$$\begin{gathered} \mu', \ \mu/\mu', \ \nu, \ \nu', c'_{12}, c'_{13}, c'_{23}, \tilde{c}_{23} \\ \tilde{c}_{12}/\mu', \tilde{c}_{13}/\mu'; \end{gathered}$$

in particular, $\mu' = 0$ defines the new boundary hypersurface in this blowup. The lift of the diagonal is given by

$$\{\mu/\mu' = 1, \ \nu = \nu', \ \tilde{c}_{12}/\mu' = 0, \tilde{c}_{13}/\mu' = 0, \ \tilde{c}_{23} = 0\}.$$

This is completely contained within the region of validity of these coordinates.

In these coordinates,

$$\mathcal{F}_\sharp = \{\nu = \nu' = 0, \tilde{c}_{23} = 0, \tilde{c}_{13}/\mu' = 0\}.$$

This is a p-submanifold (i.e. product submanifold) of $[\overline{M}_{\mathrm{ee}}; \mathcal{F}^\sharp]$, so it makes sense to blow it up, and doing so produces the space $[\overline{M}_{\mathrm{ee}}; \mathcal{F}^\sharp; \mathcal{F}_\sharp]$. Using $\nu'$ as the scaling coordinate gives coordinates

$$\begin{gathered} \mu', \ s_2 = \mu/\mu', \ \nu', \ s_1 = \nu/\nu', c'_{12}, c'_{13}, c'_{23} \\ C_{12} = \tilde{c}_{12}/\mu', C_{13} = \tilde{c}_{13}/(\mu'\nu'), \ C_{23} = \tilde{c}_{23}/\nu'. \end{gathered} \tag{3.1}$$

The lifts of $\mathcal{F}^\sharp$ and $\mathcal{F}_\sharp$ are defined by $\mu' = 0$ and $\nu' = 0$, respectively. On the other hand, $\mu/\mu' = 0$ (i.e. $s_2 = 0$) and $\nu/\nu' = 0$ (i.e. $s_1 = 0$) define other boundary hypersurfaces (these are the lifts of $\mu = 0$ and $\nu = 0$, respectively); since they do not intersect the lifted diagonal, we refer to them as side faces. As before, the lifted diagonal lies within the region of validity of this second projective coordinate system and is given by

$$\mathrm{diag}_{\mathrm{ee}} = \{s_1 = 1,\ s_2 = 1,\ C_{12} = 0,\ C_{13} = 0,\ C_{23} = 0\}.$$

Interchanging the order of blow-ups simply amounts to reversing the role of $\mu$ and $\nu$, $\mu'$ and $\nu'$, $\tilde{c}_{12}$ and $\tilde{c}_{23}$, $c'_{12}$ and $c'_{23}$, respectively. From these calculations, it is clear that in a neighborhood of the diagonal the two spaces are naturally diffeomorphic; in other words, (3.1) defines valid coordinates in both cases. In fact, more is true: the spaces of functions vanishing to infinite order at all boundary hypersurfaces except the two front faces are also isomorphic.

*Vector fields.* An immediate benefit of the use of these projective coordinates is that it is very simple to compute the lifts of the basic vector field (2.9) to $\overline{M}^2_{\mathrm{ee}}$. Lifting each of the vector fields (2.9), first to the left factor of $\overline{M}$ in $\overline{M}^2$ and then to the blowup, gives

$$\begin{aligned}
\mu\partial_\mu &= s_2\partial_{s_2},\\
\nu\partial_\nu &= s_1\partial_{s_1},\\
\mu X_{12} &= b_3 s_2\partial_{C_{12}} + a_{113}C_{13}s_2\mu'\partial_{C_{13}} + a_{123}C_{13}s_2(\mu')^2\partial_{C_{23}}\\
&\qquad + a_{213}C_{23}s_2\partial_{C_{13}} + a_{223}C_{23}s_2\mu'\partial_{C_{23}},\\
\nu X_{23} &= b_1 s_1\partial_{C_{23}} + a_{221}C_{12}s_1\nu'\partial_{C_{12}} + a_{231}C_{12}s_1\partial_{C_{13}}\\
&\qquad + a_{321}C_{13}s_1(\nu')^2\partial_{C_{12}} + a_{331}C_{13}s_1\nu'\partial_{C_{13}},\\
\mu\nu X_{13} &= b_2 s_1 s_2\partial_{C_{13}} + a_{12}s_1 s_2\nu'\partial_{C_{12}} + a_{23}s_1 s_2\mu'\partial_{C_{23}}.
\end{aligned}$$

We claim that these vector fields span the normal bundle to $\mathrm{diag}_{\mathrm{ee}}$, which is $\{s_1 = s_2 = 1,\ C_{ij} = 0\}$, uniformly to the boundary and corners. This is clear in the interior, as well as at the codimension one boundaries where it reduces to the corresponding property for edge vector fields on a manifold with boundary. On the other hand, at the corner, these vector fields become

$$\begin{aligned}
\mu\partial_\mu &= s_2\partial_{s_2},\\
\nu\partial_\nu &= s_1\partial_{s_1},\\
\mu X_{12} &= b_3 s_2\partial_{C_{12}} + a_{213}C_{23}s_2\partial_{C_{13}},\\
\nu X_{23} &= b_1 s_1\partial_{C_{23}} + a_{231}C_{12}s_1\partial_{C_{13}},\\
\mu\nu X_{13} &= b_2 s_1 s_2\partial_{C_{13}},
\end{aligned}$$

and since the $b_j$ are nonvanishing, we see that they do indeed span the normal bundle of the lifted diagonal.

3.2. **ee-pseudodifferential operators.** Let $(\overline{X}, \phi)$ be an ee structure, and let $\overline{X}^2_{\mathrm{ee}}$ be the associated ee double space. We now define the space of ee-pseudodifferential operators $\Psi^*_{\mathrm{ee}}(X)$ to consist of those pseudodifferential operators $A$ on $X$ whose Schwartz kernel $\kappa_A$ has the following properties. $\kappa_A$ is a distribution on $X^2$, and we require that it lift to $\overline{X}^2_{\mathrm{ee}}$ to be polyhomogeneous conormal with respect to the lifted diagonal $\mathrm{diag}_{\mathrm{ee}}$, with singularities smoothly extendible across all boundary faces of $\overline{X}^2_{\mathrm{ee}}$ which meet $\mathrm{diag}_{\mathrm{ee}}$ (i.e. a polyhomogeneous conormal distribution in the usual sense on a manifold with corners, conormal to a submanifold that intersects transversally all boundary faces which it intersects), and which vanishes to all orders at all boundary faces which do not meet $\mathrm{diag}_{\mathrm{ee}}$. For later reference, we write $\mathcal{H}'$ and $\mathcal{H}''$ for the union of boundary faces which do or do not, respectively, meet $\mathrm{diag}_{\mathrm{ee}}$.

The preceding computations concerning the $\mathcal{V}_{\mathrm{ee}}$ vector fields immediately give the

**Proposition 3.1.** *If $L$ is any ee differential operator, then the Schwartz kernel of $L$ lifts to a (differentiated) delta section in $\overline{X}^2_{ee}$ supported along* $\mathrm{diag}_{ee}$.

As in ordinary pseudodifferential theory, there is a symbol map

$$\sigma_m : \Psi^m_{\mathrm{ee}}(\overline{X}) \longmapsto S^m_{\mathrm{hom}}({}^{\mathrm{ee}}T^*\overline{X});$$

indeed, this is simply Hörmander's principal symbol map for conormal distributions. There is the usual short exact sequence of symbols (again, this *is* simply the short exact sequence for conormal distributions):

$$0 \to \Psi^{m-1}_{\mathrm{ee}}(\overline{X}) \hookrightarrow \Psi^m_{\mathrm{ee}}(\overline{X}) \to S^m_{\mathrm{hom}}({}^{\mathrm{ee}}T^*\overline{X}) \to 0;$$

the two maps in the center of this sequence are the inclusion and symbol map, respectively.

An operator $L \in \Psi^m_{\mathrm{ee}}(\overline{X})$ is called ee-elliptic if $\sigma_m(L)(z,\zeta)$ is invertible for $\zeta \neq 0$. Using this, the standard elliptic parametrix construction can be mimicked to give

**Proposition 3.2.** *If $L \in \Psi^m_{ee}(\overline{X})$ is elliptic then there exists a parametrix $G \in \Psi^{-m}_{ee}(\overline{X})$ such that $LG - \mathrm{Id}, GL - \mathrm{Id} \in \Psi^{-\infty}_{ee}(\overline{X})$.*

Let us specialize again to the symmetric space $M$. For any $\lambda \in \mathbb{C}$, $\Delta - \lambda$ is elliptic in this sense, and so this proposition yields a parametrix $G(\lambda) \in \Psi^m_{\mathrm{ee}}(\overline{X})$ which depends holomorphically on $\lambda$, so that both $(\Delta-\lambda)G(\lambda) - \mathrm{Id}$ and $G(\lambda)(\Delta-\lambda) - \mathrm{Id}$ are of order $-\infty$ in this small calculus. It is convenient to modify this parametrix slightly. In fact, if $p \in M$ and $K_p$ is the stabilizer subgroup of SL(3) fixing this point, then $\Delta - \lambda$ is $K_p$-invariant. It would be nice to have a $K_p$-invariant parametrix, and this is easy to arrange: simply define $G_p(\lambda) = \int_{K_p} \phi_O^* G(\lambda)(\phi_O^{-1})^*\, dg_{K_p}(O)$, where $dg_{K_p}$ denotes the normalized invariant measure, and $\phi_O$ right multiplication by the element $O \in K_p$.

**Proposition 3.3.** *For any $\lambda \in \mathbb{C}$ and $p \in M$, there exists an operator $G_p(\lambda) \in \Psi^{-2}_{ee}(\overline{X})$ which is $K_p$-invariant and which satisfies*

$$(\Delta - \lambda)G_p(\lambda) - \mathrm{Id}, \quad G_p(\lambda)(\Delta - \lambda) - \mathrm{Id} \in \Psi^{-\infty}_{ee}(\overline{X})$$

*Furthermore, $G(\lambda)$ depends holomorphically on $\lambda$.*

As already explained in the introduction, this parametrix is not the final one for the simple reason that the error terms it leaves are not compact on $L^2$; removing these is the principal motivation for defining the more elaborate large calculus.

We conclude this section with a description of the regularity properties of this parametrix. We fix an ee-metric $g$, i.e. one for which the generating vector fields of the structure are of bounded length. Boundedness of operators of order 0 on $L^2(X, dV_g)$ may be deduced using the usual combination of a symbol calculus argument to reduce to showing boundedness of operators of order $-\infty$ and then proving this case directly by Schur's inequality; cf. [17] for an example of this argument. Next, for any $m \in \mathbb{R}$, define the Sobolev space

$$H^m_{\mathrm{ee}}(\overline{X}) = \{u : Au \in L^2(X; dV_g)\ \forall A \in \Psi^m_{\mathrm{ee}}(\overline{X})\}. \tag{3.2}$$

If $m \in \mathbb{N}$, an equivalent formulation is

$$H^m_{\mathrm{ee}}(\overline{X}) = \{u : V_1 \dots V_j u \in L^2(X, dV_g), \quad \text{whenever } V_i \in \mathcal{V}_{\mathrm{ee}}(\overline{X}) \text{ and } j \le m\}.$$

From the existence of the parametrix in the last proposition, we have

**Proposition 3.4.** *For any $m \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, the parametrix $G_p(\lambda)$ satisfies*

$$G_p(\lambda) : H^m_{ee}(\overline{M}) \to H^{m+2}_{ee}(\overline{M}),$$
$$G_p(\lambda) : H^m_{ee}(\overline{M})^{K_p} \to H^{m+2}_{ee}(\overline{M})^{K_p}.$$

*In addition, if $E_p(\lambda) = G_p(\lambda)(\Delta - \lambda) - \mathrm{Id}$ and $F_p(\lambda) = (\Delta - \lambda)G_p(\lambda) - \mathrm{Id}$ are the error terms, then*

$$E_p(\lambda),\ F_p(\lambda) : H^m_{ee}(\overline{M})^{K_p} \to H^\infty_{ee}(\overline{M})^{K_p}$$

*for all $m$.*

To understand what these statements mean, suppose that $(\Delta - \lambda)u = f \in \mathcal{C}^\infty_0(M)$. If neither $u$ nor $f$ are $K_p$-invariant, then from $u = G_p(\lambda)f + E_p(\lambda)u$ and assuming that $u \in L^2(M, dV_g)$, we get

$$(\mu\partial_\mu)^j(\nu\partial_\nu)^k(\mu X_{12})^r(\nu X_{23})^s(\mu\nu X_{13})^t u \in L^2(M, dV_g) \ \text{ for all } j, k, r, s, t \ge 0.$$

Thus $u$ has full tangential regularity only in directions tangent to the fibres of the boundary fibration. On the other hand, suppose that both $u$ and $f$ are $K_p$-invariant (and $f$ is still $\mathcal{C}^\infty_0$). Then the vector fields $X_{ij}$ annihilate $u$ and hence $u$ restricts to a conormal function on the compactified flat $\overline{\exp(\mathfrak{a})}$. In other words,

$$(\mu\partial_\mu)^j(\nu\partial_\nu)^k u \in L^2(M, dV_g) \ \text{ for all } j, k \ge 0.$$

We also need the mapping properties of the parametrix, or more general elements of $\Psi^m_{\mathrm{ee}}(\overline{X})$, on weighted spaces. Our weights correspond to the geometry of $\tilde{X}$. For this purpose, it is convenient to perform two further blow-ups: first, a logarithmic blow-up of the two front faces of $\overline{X}^2_{\mathrm{ee}}$, and following this a spherical blow-up of their intersection. We denote the resulting space by $\widetilde{X}^2_{\mathrm{ee}}$. A neighborhood of the lifted diagonal, $\widetilde{\mathrm{diag}}_{\mathrm{ee}}$, is diffeomorphic to a neighborhood of the zero section of a vector bundle over $\widetilde{X}$, namely of the pullback of ${}^{\mathrm{ee}}T\overline{X}$ by the blowdown map $\widetilde{X} \to \overline{X}$. We let $\Psi^m_{\mathrm{ee}}(\widetilde{X})$ be the space of pseudodifferential operators with Schwartz kernel vanishing to infinite order at all boundary faces except those intersecting the lifted diagonal. The principal symbol map is

$$\sigma_m : \Psi^m_{\mathrm{ee}}(\widetilde{X}) \longmapsto S^m_{\mathrm{hom}}({}^{\mathrm{ee}}T^*\widetilde{X}),$$

where ${}^{\mathrm{ee}}T^*\widetilde{X}$ is the pull-back of ${}^{\mathrm{ee}}T^*\overline{X}$ by the blow-down map, and again, all of the standard properties of pseudodifferential algebras hold, in particular the $L^2$-boundedness of 0th order operators. Since the blow-down map $\widetilde{X} \to \overline{X}$ is smooth, we have $\Psi^m_{\mathrm{ee}}(\overline{X}) \subset \Psi^m_{\mathrm{ee}}(\tilde{X})$. Furthermore, an elliptic element of $\Psi^m_{\mathrm{ee}}(\overline{X})$ remains elliptic in $\Psi^m_{\mathrm{ee}}(\tilde{X})$. A consequence of this is that the Sobolev spaces associated to either of these algebras of pseudodifferential operators are the same. Indeed, $u \in H^m_{\mathrm{ee}}(\overline{X})$, respectively $H^m_{\mathrm{ee}}(\widetilde{X})$, provided $Au \in L^2(X; dV_g)$ for a single elliptic element in the corresponding algebra, and the claim follows since we can choose $A$ in the smaller algebra $\Psi^m_{\mathrm{ee}}(\overline{X})$.

The following is a key result for us.

**Lemma 3.5.** *Suppose $A \in \Psi^m_{ee}(\widetilde{X})$. Let $x$ be a total boundary defining function of $\tilde{X}$. Then for all $\alpha \in \mathbb{R}$,*

$$e^{\alpha/x} A e^{-\alpha/x} \in \Psi^m_{ee}(\widetilde{X}).$$

*Proof.* The Schwartz kernel $\kappa$ of $e^{\alpha/x} A e^{-\alpha/x}$ is $e^{\alpha(1/x - 1/x')}\kappa_A$, where $\kappa_A$ is the Schwartz kernel of $A$, and $x$, resp. $x'$, are (with a slight abuse of notation) the lift of $x$ from either factor of $\overline{X}^2$ to $\widetilde{X}^2_{\mathrm{ee}}$. It is straightforward to check that this is a smooth function near the diagonal, with appropriate bounds (polynomial in the defining functions – recall that the side faces have not been blown up logarithmically) at all other faces of $\widetilde{X}^2_{\mathrm{ee}}$. Since $\kappa_A$ vanishes to infinite order at these other faces, the result follows. □

**Corollary 3.6.** *Suppose $A \in \Psi^m_{ee}(\widetilde{X})$. Let $x$ be a total boundary defining function of $\tilde{X}$. Then for all $\alpha \in \mathbb{R}$, $k \in \mathbb{R}$,*

$$A : e^{\alpha/x} H^k_{ee}(\overline{X}) \to e^{\alpha/x} H^{k-m}_{ee}(\overline{X}).$$

*Proof.* The statement is equivalent to $e^{-\alpha/x} A e^{\alpha/x}$ being bounded from $H^{k-m}_{\mathrm{ee}}(\overline{X})$ to $H^k_{\mathrm{ee}}(\overline{X})$, which holds as $e^{-\alpha/x} A e^{\alpha/x} \in \Psi^m_{\mathrm{ee}}(\widetilde{X})$. □

There is a slightly more elaborate version of this result, showing that our ps.d.o's preserve expansions:

**Proposition 3.7.** *Suppose $A \in \Psi^m_{ee}(\widetilde{X})$. Let $x$ be a total boundary defining function of $\tilde{X}$. Then for all $\alpha \in \mathbb{R}$,*

$$A : e^{\alpha/x}\mathcal{C}^\infty(\widetilde{X}) \to e^{\alpha/x}\mathcal{C}^\infty(\widetilde{X});$$

*$\mathcal{C}^\infty(\widetilde{X})$ can be replaced by any space of polyhomogeneous distributions here.*

*Proof.* As above, this amounts to checking that $A \in \Psi^m_{\mathrm{ee}}(\widetilde{X})$ preserves $\mathcal{C}^\infty(\widetilde{X})$, which is straightforward. □

## 4. Product models

The key remaining issue is to find a correction for the parametrix $G_p(\lambda)$ which solves away as much of the error terms $E_p(\lambda)$ and $F_p(\lambda)$ (as in Proposition 3.4) as possible. In fact, it will suffice to find a new parametrix for which the corresponding error terms are not only regularizing, but which map $H^m_{\mathrm{ee}}(\overline{M})^{K_p}$ into spaces of functions which decay at some definite rate at infinity in the flat. As already indicated in the introduction, this is done by solving away the expansions of the Schwartz kernels of these operators at the boundary faces $H_\sharp$ and $H^\sharp$ of $\overline{\exp(\mathfrak{a})}$. However, the main difficulty is caused by the fact that the Laplacian does not have a product decomposition at the corner $H_\sharp \cap H^\sharp$. This problem appears challenging, but as explained in detail in §5, the lift of $\Delta_g$ to the logarithmic blow-up of $\overline{M}$ is (quite remarkably) well-approximated by product type operators at each of the new corners of this blown up space. This means that the main technical difficulties involve the analysis of the product operators arising in this process, and this is the subject of the current section.

We shall be using the results and methods of [14], concerning the detailed analysis of the resolvent of the Laplacian on a product space $X = M_1 \times M_2$, which we review below. Although that paper focused particularly on the case where both factors $(M_j, g_j)$ are conformally compact (asymptotically hyperbolic), our application here requires that we let the factors be $(\mathbb{R}^+_s, ds^2/s^2)$ and $(\mathbb{H}^2, h)$, respectively. We shall both extend and refine the results in this setting. We recall though that the basic idea of the analysis of [14] is to use an explicit contour integral for the resolvent of the Laplacian of a product space, and then use the well-known results on conformally compact spaces (including the actual hyperbolic plane, where all these are explicit) and a complex stationary phase argument to deduce the asymptotics.

4.1. **Geometry and compactification of the product.** One of the main conclusions of [14] is that for purposes of analyzing its resolvent, or more precisely its Green function with given pole, the best compactification of a product of conformally compact spaces $X = M_1 \times M_2$ is the space

$$\widetilde{X} = [(\overline{M}_1)_{\log} \times (\overline{M}_2)_{\log}; \partial\overline{M}_1 \times \partial\overline{M}_2]. \tag{4.1}$$

Recall that this means that if $\rho_j$, $j = 1, 2$, are smooth boundary defining functions for $M_j$, then we replace these by $-1/\log\rho_j$ and then perform the standard blow-up of the corner. The function

$$x = \sqrt{(\log\rho_1)^{-2} + (\log\rho_2)^{-2}}$$

is a total boundary defining function for $\widetilde{X}$, i.e. is smooth and vanishes simply at all the boundary faces with respect to this new smooth structure.

Now let

$$(M_1, g_1) = \left(\mathbb{R}^+_s, 4\frac{ds^2}{s^2}\right), \qquad \text{and} \qquad (M_2, g_2) = (\mathbb{H}^2, 3h),$$

where $h$ is the standard (curvature $-1$) metric on hyperbolic space. If $\delta_j$ is the Riemannian distance function on each of these spaces, then the distance between pairs of points $(s, z), (s', z') \in X$ is given by

$$\delta((s, z), (s', z')) = \sqrt{\delta_1(s, s')^2 + \delta_2(z, z')^2}.$$

In particular, fixing the point $o = (1, q) \in X$, then

$$\delta_1(s, 1) = 2\,|\log s|, \qquad \delta_2(z, q) = \sqrt{3}\,|\log\mu|,$$

where $\mu$ is a suitable defining function on $\mathbb{H}^2$ (we include the factors 2 and $\sqrt{3}$ to keep track of the scaling factors on the metrics). Neglecting for the moment the fact that these functions are only smooth away from $s = 1$ and $z = q$, we set $\rho_1 = e^{-\delta_1(s,1)}$, $\rho_2 = e^{-\delta_2(z,q)}$. Hence

$$x = \frac{1}{\delta(z, o)} = [4(\log s)^2 + 3(\log\mu)^2]^{-1/2} \tag{4.2}$$

is a total boundary defining function for $\widetilde{X}$ and

$$\begin{aligned} x_1 &= \frac{x}{(4(\log s)^2 + 1)^{-1/2}} \sim 2x\,|\log s| = \frac{\delta_1(1, s)}{\delta(o, z)}, \\ x_2 &= \frac{x}{(3(\log\mu)^2 + 1)^{-1/2}} \sim \sqrt{3}x\,|\log\mu| = \frac{\delta_2(q, z_2)}{\delta(o, z)} \end{aligned} \tag{4.3}$$

are defining functions for the two side faces, namely the lifts of $\overline{M}_1 \times \partial M_2$, resp. $\partial M_1 \times \overline{M}_2$. Note that in terms of the "eigenvalue coordinates" $(\lambda_1, \lambda_2, \lambda_3)$ on $M$,

$$x^{-2} = 6((\log\lambda_1)^2 + (\log\lambda_2)^2 + (\log\lambda_3)^2).$$

Returning to address the fact that the $\delta_j$ and hence $\delta$ are not smooth everywhere, we replace them by smoothed versions, $\tilde{\delta}_j, \tilde{\delta} \in \mathcal{C}^\infty$, which are chosen so that $\tilde{\delta} \geq 1$, $\delta \leq \tilde{\delta} \leq \delta + 2$, and $\tilde{\delta} = \delta$ if $\delta \geq 3$, and similarly for $\tilde{\delta}_j$. Although these no longer satisfy the triangle inequality, their failure to do so is bounded:

$$\tilde{\delta}(z, z') + \tilde{\delta}(z', z'') - \tilde{\delta}(z, z'') \geq \delta(z, z') + \delta(z', z'') - (\delta(z', z'') + 2) \geq -2,$$

which is all we require in later estimates, and which we continue to call the triangle inequality. Combining it with the fact that $\tilde{\delta} \geq 1$, we have

$$\tilde{\delta}(z, z'') \leq \tilde{\delta}(z, z') + \tilde{\delta}(z', z'') + 2 \leq 2\tilde{\delta}(z, z')\tilde{\delta}(z', z'') + 2 \leq 4\tilde{\delta}(z, z')\tilde{\delta}(z', z''),$$

and so

$$\frac{\tilde{\delta}(z, z'')}{\tilde{\delta}(z, z')\tilde{\delta}(z', z'')} \leq 4. \tag{4.4}$$

We can also arrange that $\tilde{\delta}(o, .)$ and $\tilde{\delta}_2(q, .)$ are SO(2)-invariant. We now replace our previous defining functions by

$$x_1 = \frac{\tilde{\delta}_1(1, s)}{\tilde{\delta}(o, z)},\ x_2 = \frac{\tilde{\delta}_2(q, z_2)}{\tilde{\delta}(o, z)},\ x = \tilde{\delta}(o, z)^{-1},$$

which are smooth, globally defined and SO(2)-invariant.

*Remark* 4.1. We shall often identify $K_p$-invariant functions on $M = \mathrm{SL}(3)/\mathrm{SO}(3)$ supported in either of the regions where the cutoff functions $\chi_\sharp$ or $\chi^\sharp$ are nonzero with SO(2)-invariant functions on the compactified product space $\tilde{X}$ supported in an analogous neighborhood of the lift of $\partial\overline{M}_1 \times \overline{M}_2$. Note that the 'transplantation' of a function supported in one of these sets vanishes near the lift of $\overline{M}_1 \times \partial\overline{M}_2$.

4.2. **Resolvent asymptotics.** The analysis of $\Delta_g$ on $K_p$-invariant functions ultimately reduces near the face $H_\sharp$ to that for the operator

$$L_\sharp = \frac{1}{4}(sD_s)^2 + i\frac{1}{2}(sD_s) + \frac{1}{3}\Delta_h,$$

which is self-adjoint on

$$\mathcal{H} = L^2(\mathbb{R}^+_s \times \mathbb{H}^2, s^{-3} ds\, dV_h). \tag{4.5}$$

It is computationally simpler to use its conjugate

$$L_0 = s^{-1} L_\sharp s = \frac{1}{4}(sD_s)^2 + \frac{1}{3}\Delta_h + \frac{1}{4}, \tag{4.6}$$

which is self-adjoint on

$$L^2_{p0}(\mathbb{R}^+_s \times \mathbb{H}^2) = L^2(\mathbb{R}^+_s \times \mathbb{H}^2, dV_{g'}) = L^2(\mathbb{R}^+_s \times \mathbb{H}^2, s^{-1} ds\, dV_h).$$

Note that $L_0 = \Delta_{g'} + \frac{1}{4}$, where

$$g' = 4\frac{ds^2}{s^2} + 3g_{\mathbb{H}^2}, \tag{4.7}$$

which explains our choice of factors $(M_j, g_j)$ above, and that

$$\mathcal{H} = sL^2_{p0}(\mathbb{R}^+_s \times \mathbb{H}^2). \tag{4.8}$$

We now quote results from [14] concerning the structure of the resolvent

$$R_0(\lambda) = (L_0 - \lambda)^{-1} = (\Delta_{g'} - (\lambda - \frac{1}{4}))^{-1}.$$

Note that

$$\inf \operatorname{spec}(\Delta_{g'}) = 0 + \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12} \Longrightarrow \inf \operatorname{spec}(L_0) \equiv \lambda_0 = \frac{1}{12} + \frac{1}{4} = \frac{1}{3},$$

and correspondingly, the formulæ below frequently involve the quantity $\sqrt{\lambda - \lambda_0}$. We always use the branch of the square root which has *negative* imaginary part on $\mathbb{C} \setminus [0, \infty)$. These formulæ also involve the Poisson operator $P_j(\lambda)$, or rather its adjoint $P^t_j(\lambda)$, and the spherical function $S_j(\lambda)$, on each of the factors. Of course, for the one-dimensional factor $M_1 = \mathbb{R}^+$, these objects are particularly simple and quite explicit.

We begin with the asymptotics of $R_0(\lambda)f$, where $f \in \dot{\mathcal{C}}^\infty(\overline{X})$:

**Proposition 4.2** ([14] Proposition 7.7)**.** *Let $f \in \dot{\mathcal{C}}^\infty(\overline{X})$ and $\lambda \in \mathbb{C} \setminus [\lambda_0, \infty)$. Then on $\widetilde{X}$,*

$$R_0(\lambda) f = \mu^{1/2} x_2 \, x^{1/2} \exp(-i\sqrt{\lambda - \lambda_0}/x)\, h, \quad h \in \mathcal{C}^\infty(\widetilde{X}). \tag{4.9}$$

*Moreover, setting*

$$\sigma = x_2/x_1 = \tilde{\delta}_2(q, z_2)/\tilde{\delta}_1(1, s),$$

$$\lambda^0_1(\sigma) = \frac{\lambda - \lambda_0}{1 + \sigma^2}, \tag{4.10}$$

*the restriction of $h$ to the boundary is given by*

$$\begin{aligned} & a(\lambda,\sigma)\left(P^t_1(\lambda^0_1(\sigma)) \otimes P^t_2(\lambda - \frac{1}{4} - \lambda^0_1(\sigma))\right) f \ \textit{on the front face}, \\ & a'(\lambda)\left(S_1(0) \otimes P^t_2(\lambda - \frac{1}{4})\right) f \ \textit{on the lift of } \overline{M}_1 \times \partial \overline{M}_2 \ (\textit{ where } \sigma \to +\infty), \\ & a''(\lambda)\left(P^t_1(\lambda - \lambda_0) \otimes S_2(\frac{1}{12})\right) f \ \textit{on the lift of } \partial \overline{M}_1 \times \overline{M}_2 \ (\textit{ where } \sigma = 0), \end{aligned} \tag{4.11}$$

*where $a, a'$ and $a''$ are all nonvanishing.*

*Remark* 4.3. These formulæ are different from those in [14] in one minor point: the resolvent $((sD_s)^2 - \lambda)^{-1}$ has a singularity of the form $\lambda^{-1/2}$ at the threshold branch point 0, instead of $\lambda^{1/2}$, which occurs when $M_1$ is higher dimensional. However, as follows directly from the proofs presented in [14, Section 7], this makes a difference only along $M_1 \times \partial \overline{M}_2$, where $s$ is finite. (The precise explanation is that only for points on this face is it necessary to choose the contour of integration for the stationary phase calculation which yields the asymptotics to run through the threshold branch point of $((sD_s)^2 - \lambda)^{-1}$.) This simply causes the order of the leading term of the asymptotic expansions at $M_1 \times \partial \overline{M}_2$ to change by 1. In other words, this removes a factor of $x_1$, but not $x_2$, in (4.9), as compared to [14, Proposition 7.7]. Indeed, this can be seen from the proof of [14, Lemma 7.3]: on the last line of p. 1047, one has a factor of $(\mu_1 - k_1^2/4)^{-1/2}$ instead of $\sqrt{\mu_1 - k_1^2/4}$, which in the first displayed formula on p. 1048 eliminates the factor of $\tau^2$, and then the following change of variable explains the missing factor of $x_1$ with our notation (which is, unfortunately, denoted by $-1/\log x_2$ there).

In fact, the asymptotics at the lift of $\overline{M}_1 \times \partial \overline{M}_2$ is even simpler than what we have stated since the $\mathbb{R}^+$-invariance of $(sD_s)^2$ can be used to show that $R(\lambda)f$ is still polyhomogeneous even when this face is blown down. However, this does not affect the way we apply our results later, so we do not take advantage of this.

We explicitly state a corollary of this proposition, namely that if $f$ is invariant under the SO(2)-action on $M_2$ (with SO(2) identified as the subgroup of SL(2) fixing our origin $q \in \mathbb{H}^2$) then the leading term on $\partial\overline{M}_1 \times \overline{M}_2$ is, suitably interpreted, a family of spherical functions on $M_2$, centered at $q$, parametrized by $\partial\overline{M}_1$. More specifically, in these asymptotics, a factor of $\mu^{1/2}\tilde{\delta}_2(q, z_2)$ was removed from $h$; if this factor had been left in, the leading asymptotic coefficient would be an actual spherical function.

In the present setting, $\partial\overline{M}_1$ is a two-point set, and we will usually focus on the component corresponding to $s = 0$.

**Definition 4.4.** Let $\mathcal{C}^\infty_S(\widetilde{X})$ ($\mathcal{C}^0_S(\widetilde{X})$) denote the space of smooth (continuous) functions on $\widetilde{X}$ whose restriction to each fiber of $\partial\overline{M}_1 \times M_2 \to \partial\overline{M}_1$ is of the form $\mu^{-1/2}\tilde{\delta}_2(q, z_2)^{-1}$ times a generalized eigenfunction of $\Delta_{M_2}$ associated to the bottom of the spectrum of the Laplacian. In other words, up to the factor $\mu^{-1/2}\tilde{\delta}_2(q, z_2)^{-1}$, this restriction should be a generalized eigenfunction of $\Delta_{\mathbb{H}^2}$ (the Laplacian with respect to the standard hyperbolic metric with curvature $-1$) with eigenvalue $1/4$.

The subspaces $\mathcal{C}^\infty_S(\widetilde{X})^{\mathrm{SO}(2)}$ and $\mathcal{C}^0_S(\widetilde{X})^{\mathrm{SO}(2)}$ contain all smooth, respectively continuous, SO(2)-invariant functions on $\widetilde{X}$ whose restriction to each fiber of $\partial\overline{M}_1 \times M_2 \to \partial\overline{M}_1$ is, up to the factor $\mu^{-1/2}\tilde{\delta}_2(q, z_2)^{-1}$, a spherical (SO(2)-invariant) function on $M_2$ associated to the bottom of the spectrum of the Laplacian of $\Delta_{\mathbb{H}^2}$ with eigenvalue $1/4$.

**Corollary 4.5.** *Suppose that $f \in \dot{\mathcal{C}}^\infty(\overline{X})$, $\lambda \in \mathbb{C} \setminus [\lambda_0, \infty)$. Then the function $h$ which appears as the leading asymptotic coefficient in* (4.9) *is an element of $\mathcal{C}^\infty_S(\widetilde{X})$. If $f \in \dot{\mathcal{C}}^\infty(\overline{X})^{\mathrm{SO}(2)}$, then $h \in \mathcal{C}^\infty_S(\widetilde{X})^{\mathrm{SO}(2)}$.*

*Proof.* The restriction of $\mu^{1/2}\tilde{\delta}_2(z_2, q)h$ to $\partial\overline{M}_1 \times M_2$ is equal to $(P^t_1(\lambda - \lambda_0) \otimes S_2(\frac{1}{12}))f$, so the corollary follows since the range of $S_2(\frac{1}{12})$ lies in the space of generalized eigenfunctions of $\Delta_{\mathbb{H}^2}$ with eigenvalue $1/4$. If $f$ is SO(2)-invariant, then so is $h$, and hence its restriction too. ☐

From here, [14] goes on to deduce the full structure of the Schwartz kernel of the resolvent in the product setting. This kernel is a sum of two terms, $R'_0(\lambda) + R''_0(\lambda)$; the first term is in the small product-0 calculus (which is a simple case of the ee calculus) and contains the full diagonal singularity, while the second is smooth in the interior but has a more complicated singularity structure at the boundary which is resolved by passing to a further blow-up, the resolvent double space $\widetilde{X}^2_{\mathrm{res}}$, as we now explain. (A similar decomposition is true for the Schwartz kernel of the resolvent on the full space $M$; this will emerge as part of the construction in the next section.)

Operators in the small product-0 calculus $\Psi_{p0}(\overline{X})$ (for $X = M_1 \times M_2$), are characterized by the fact that their Schwartz kernels lift to the product-0 double space

$$\overline{X}^2_{p0} \equiv (\overline{M}_1)^2_0 \times (\overline{M}_2)^2_0$$

to be conormal to the lifted diagonal, smoothly extendible across the front faces $\mathrm{ff}(\overline{M}_1)^2_0 \times (\overline{M}_2)^2_0$ and $(\overline{M}_1)^2_0 \times \mathrm{ff}(\overline{M}_2)^2_0$ and vanishing to infinite order at all other boundary faces. Here $M_1 = \mathbb{R}^+$ and $M_2 = \mathbb{H}^2$, so that $\overline{M}_1 = I$ is the radial compactification of the half-line as an interval and $\overline{\mathbb{H}^2}$ is the ball $B^2$.

The resolvent double space $\widetilde{X}^2_{\mathrm{res}}$ is obtained from $\overline{X}^2_{p0}$ by judiciously blowing up a certain minimal collection of corners, so that $R_0''(\lambda)$ lifts to be polyhomogeneous (when $\lambda$ is in the resolvent set).

A more detailed discussion of the structure of $\widetilde{X}^2_{\mathrm{res}}$ has been relegated to the end of this paper in an appendix because, since we have reduced to studying the restriction of the resolvent on $M$ to the flat, it suffices to consider only the action of $R_0(\lambda)$ on SO(2)-invariant functions (with respect to a fixed point $q$) on the $\mathbb{H}^2$ factor. We regard such functions as depending on the variables $s \in \mathbb{R}^+$ and $\mu \in (0,1)$, where $\mu$ is the boundary defining function on $\overline{\mathbb{H}^2}$ used earlier. The apparent 'boundary' $\mu = 1$ is artificial (it corresponds to the point $q$ in polar coordinates), and we systematically ignore it (for example by only considering functions which are supported away from this set). In fact, set

$$\mathfrak{b}^+ = \mathbb{R}^+_s \times (0,1)_\mu \subset \mathfrak{b} = \mathbb{R}^+_s \times \mathbb{R}^+_\mu.$$

If $\phi \in \mathcal{C}^\infty(\mathbb{R}^+)$, $\phi = 0$ near $\mu = 0$ and $\phi = 1$ for $\mu \geq 1/2$, then the Schwartz kernel of

$$(1-\phi(\mu))R_0(\lambda)(1-\phi(\mu))$$

lifts to the resolvent double-space $\widetilde{\mathfrak{b}}^2_{\mathrm{res}}$ and is supported on the closure of $\mathfrak{b}^+ \times \mathfrak{b}^+$. Note that the Schwartz kernels of $\phi(\mu)R_0(\lambda)(1-\phi(\mu))$ and $(1-\phi)(\mu)R_0(\lambda)\phi(\mu)$ also lift trivially since they are supported away from the corner $\mu = \mu' = 0$.

The advantage of this reduction is that the geometry of both the product-0 double space $\overline{\mathfrak{b}}^2_{p0}$ and the resolvent double space $\overline{\mathfrak{b}}^2_{\mathrm{res}}$ are simpler than when the second factor has dimension bigger than one. Thus the 0-double space of $I = \overline{\mathbb{R}^+}$ is obtained from $I^2$ by blowing up the boundary of the diagonal, $\partial\,\mathrm{diag}$, in $I^2$, $I^2_0 = [I^2; \partial\,\mathrm{diag}]$. This has the effect of separating the left and right boundary faces. (Actually, they still intersect at the off-diagonal corners of this square, but since all the kernels we consider are supported away from these points, we ignore this.) When $\dim M > 1$, the left and right faces of the 0-double space $M^2_0$ are no longer separated, and this necessitates some extra blow-ups in the resolvent double space. In any case, let $\mathrm{lf}_j$, $\mathrm{rf}_j$, $j = 1, 2$ be the left and right faces on the two factors $I^2_0$ in $\overline{\mathfrak{b}}^2_{p0}$ and $\mathcal{S}$ be the collection of codimension two corners $\mathrm{lf}_1 \times \mathrm{lf}_2$, $\mathrm{lf}_1 \times \mathrm{rf}_2$, $\mathrm{rf}_1 \times \mathrm{lf}_2$, $\mathrm{rf}_1 \times \mathrm{rf}_2$; by construction (and again neglecting the off-diagonal corners), these do not intersect. Now replace the defining functions $\rho_{\mathrm{lf}_j}$ and $\rho_{\mathrm{rf}_j}$ at these faces in each factor with $\mathcal{R}_{\mathrm{lf}_j} = -1/\log \rho_{\mathrm{lf}_j}$, $\mathcal{R}_{\mathrm{rf}_j} = -1/\log \rho_{\mathrm{rf}_j}$. Notice that we are not changing the defining function at the front faces $\mathrm{ff}_j$. The product-0 space with these defining functions is denoted $\overline{\mathfrak{b}}^2_{p0,\log}$. Finally, define

$$\widetilde{\mathfrak{b}}^2_{\mathrm{res}} = [\overline{\mathfrak{b}}^2_{p0}; \mathcal{S}].$$

The following theorem follows directly from [14], cf. the Appendix below.

**Theorem 4.6.** *If $\lambda \in \mathbb{C} \setminus \mathrm{spec}(L_0)$, then*

$$(L_0 - \lambda)^{-1} = R_0(\lambda) = R_0'(\lambda) + R_0''(\lambda), \tag{4.12}$$

*where $R_0'(\lambda) \in \Psi^{-2}_{p0}(\overline{X})$ and*

$$R_0''(\lambda) = (\rho_{\mathrm{lf}_2}\rho_{\mathrm{rf}_2})^{1/2} \exp(-i\sqrt{\lambda - \lambda_0}\ \tilde{\delta}(z,z'))\, F(\lambda), \qquad \lambda_0 = \frac{1}{3}. \tag{4.13}$$

*The kernel $F(\lambda)$ on the right has the form*

$$F(\lambda) = \tilde{\delta}(z, z')^{-3/2}\, \tilde{\delta}_2(z_2, z_2')F'(\lambda)\frac{ds'}{s'}\, dV_h'$$

*where the primes on the density factor mean that it is pulled back from the second ($z'$) factor. The function $F'(\lambda)$ lies in $\mathcal{C}^\infty(X \times X) \cap L^\infty$ and extends continuously to $\widetilde{X} \times X$. ($F'$ is considerably more regular, but this suffices for its $L^2$ mapping properties.) In fact, choosing any $\phi(\mu) \in \mathcal{C}_0^\infty(\mathbb{R}^+)$ with $\phi = 1$ near $\mu = 1$, then for any $k$ there is an $r > 0$ such that*

$$\tilde{\delta}(z', o)^{-r}\phi(\mu)\, F'(\lambda) \in \mathcal{C}^k(\widetilde{X}; L^\infty(X)). \tag{4.14}$$

*To determine the structure of the restriction of this operator to* SO(2)*-invariant functions in the second factor, average $R_0''(\lambda)$ with respect to Haar measure on* SO(2)*, so as to regard it as living on $\mathfrak{b}^2$ as above. Writing the average of $F'(\lambda)$ as $F_0'(\lambda)$, then for any $k$ there is $r > 0$ such that*

$$\tilde{\delta}(z', o)^{-r}(1 - \phi(\mu))F_0'(\lambda) \in \mathcal{C}^k\left(\widetilde{\mathfrak{b}^+}; L_b^2\left(\mathfrak{b}^+; \frac{ds'}{s'}\frac{d\mu'}{\mu'}\right)\right). \tag{4.15}$$

*A similar formula obtains when the first and second factors are interchanged.*

4.3. **Boundedness of the resolvent on weighted spaces.** We now prove some refined mapping properties of the resolvent $R_0(\lambda)$ which are required later. These results reflect the fact that when $\lambda$ is away from the spectrum, the Schwartz kernel of $R_0(\lambda)$ has 'off-diagonal' exponential decay of order $-\kappa$, where

$$\kappa = -\operatorname{Im}\sqrt{\lambda - \lambda_0} > 0, \qquad \lambda_0 = \frac{1}{3}.$$

For simplicity we phrase these theorems in terms of the decay or asymptotics of $R_0(\lambda)f$ for various classes of functions $f$, rather than in terms of the structure of the kernel itself.

There are three results in this direction. The first states that if $|\alpha| < \kappa$ and $f$ decays (or grows) like $e^{\alpha/x}$ then so does $R_0(\lambda)f$. Next, if $f$ decays like $e^{\alpha/x}$ with $\alpha < -\kappa$, then $R_0(\lambda)f$ decomposes as a sum of two terms, one decaying at the same rate as $f$ and another which has an expansion, but decays only like $e^{-\kappa/x}$. Finally, we show that if $f$ decays exponentially (at a rate $\alpha \in (-\kappa, \kappa)$) in some sector, then $R_0(\lambda)f$ decays even faster in disjoint sectors, with rate depending on the angle between the two.

**Proposition 4.7.** *Suppose $|\alpha| < \kappa$. Then*

$$R_0(\lambda) : e^{\alpha/x}H_{p0}^m(\overline{X}) \longrightarrow e^{\alpha/x}H_{p0}^{m+2}(\overline{X})$$

*is bounded, where we are using the $L^2$ measure with respect to the metric $g'$ and the $p0$ Sobolev spaces are the ee Sobolev spaces defined in* (3.2)*, applied to the simple ee structure on $X = \mathbb{R}^+ \times \mathbb{H}^2$. Moreover, if $\alpha = -\kappa$, then*

$$R_0(\lambda) : x^a e^{\alpha/x}H_{p0}^m(\overline{X}) \longrightarrow x^{-a}e^{\alpha/x}H_{p0}^{m+2}(\overline{X})$$

*for any $a > 2$. (The restriction $a > 2$ is not optimal, but suffices for our later use.)*

*Proof.* We shall only prove boundedness between weighted $L^2$ spaces; the boundedness between Sobolev spaces, and the gain of 2 in $p0$ regularity, is follows since $L_0$ can be applied on either the left or right, cf. Lemma A.2, and arbitrary powers of it can be commuted through. We also consider only the case where $-\kappa < \alpha \leq 0$; to handle the case $0 < \alpha < \kappa$, simply reverse the roles of $z$ and $z'$.

The conclusion of the theorem is equivalent to the boundedness of the mapping

$$e^{-\alpha/x} R_0(\lambda) e^{\alpha/x} \in \mathcal{B}(L^2_{p0}(\overline{X})).$$

It can be verified by direct calculation that the lift of the function

$$A_\alpha \equiv e^{\alpha/x' - \alpha/x}$$

to $\overline{X}^2_{p0}$ is smooth up to the front faces. We can assume that the small-calculus part $R_0'(\lambda)$ of the resolvent has support not intersecting any of the boundaries except the front faces, and so $A_\alpha$ is smooth and bounded on its support. Thus we can focus on the term $A_\alpha R_0''(\lambda)$.

It is convenient to replace the measure $dV_{g'} = s^{-1}\mu^{-2}\, ds\, d\mu\, dy$, where $y$ is the angular (tangential) variable in $\mathbb{H}^2$, by $s^{-1}\mu^{-1}\, ds\, d\mu\, dy$, so we set

$$L^2_b(\overline{X}) = L^2(X; \frac{ds\, d\mu\, dy}{s\mu}) = \mu^{-1/2} L^2(\overline{X}; dV_{g'}).$$

An operator $A$ is bounded on $L^2(\overline{X}; dV_{g'})$ if and only if $\mu^{-1/2} A \mu^{1/2}$ is bounded on $L^2_b(\overline{X})$. Thus we must prove that

$$e^{\alpha/x' - \alpha/x} (\mu'/\mu)^{1/2} R(\lambda) \in \mathcal{B}(L^2_b(\overline{X})). \tag{4.16}$$

From Theorem 4.6, with $x(z) = \tilde{\delta}(z, o)^{-1}$, this conjugated kernel has the form

$$e^{-\alpha(\tilde{\delta}(o,z) - \tilde{\delta}(o,z')) - i\sqrt{\lambda - \lambda_0}\tilde{\delta}(z,z')} e^{-(\tilde{\delta}_2(z_2,z_2') + \tilde{\delta}_2(q,z_2') - \tilde{\delta}_2(q,z_2))/2} F(\lambda)$$

$$= e^{-\alpha(\tilde{\delta}(o,z) - \tilde{\delta}(o,z') - \tilde{\delta}(z,z'))} e^{-(\tilde{\delta}_2(z_2,z_2') + \tilde{\delta}_2(q,z_2') - \tilde{\delta}_2(q,z_2))/2} e^{(-i\sqrt{\lambda-\lambda_0} + \alpha)\tilde{\delta}(z,z')} F(\lambda),$$

where $F$ is as in (4.13). Using the triangle inequality to bound the exponents in the first two terms on the right, we can rewrite this as

$$e^{-\gamma\tilde{\delta}(z,z')} G, \qquad G \in L^\infty(X \times X),\ \gamma = \kappa + \alpha > 0.$$

This is an element of $L^\infty(X_z; L^1_b(X_{z'})) \cap L^\infty(X_{z'}; L^1_b(X_z))$; a similar argument interchanging $z$ and $z'$ produces an analogous statement. We may now apply Schur's lemma to obtain the conclusion.

If $\alpha = -\kappa$ we can argue similarly, except now the kernel is rewritten as

$$\tilde{\delta}(z, o)^{-a} \tilde{\delta}(z', o)^{-a} G, \qquad G \in L^\infty(X^2).$$

Since $a > 2$, this is integrable as before. $\square$

The next result concerns the behaviour of $R_0(\lambda) f$ when $f \in e^{\alpha/x} L^2$ for some $\alpha < -\kappa$. This function is the sum of two terms, the first decaying at the same rate as $f$ and the second having the decay of a homogeneous solution to $L_0 u = 0$. There are some subtleties in describing the precise regularity of this second term; these arise already for the resolvent on $\mathbb{H}^2$ [17] (hence a fortiori on $\mathbb{R}^+ \times \mathbb{H}^2$). More specifically, suppose that $f \in \mu^\gamma H^\infty_0(\overline{\mathbb{H}^2})$, where $\gamma > |\operatorname{Im} \sqrt{\lambda - 1/4}|$, or in other words, $(\mu\partial_\mu)^j (\mu\partial_y)^\ell f \in \mu^\gamma L^2(dV_h)$ for all $j, \ell \geq 0$. Setting $u = (\Delta_{\mathbb{H}^2} - \lambda)^{-1} f$, then the basic (small calculus) regularity result states that $u$ also lies in $\mu^\gamma H^\infty_0(\overline{\mathbb{H}^2})$. However, $u$ has no greater tangential regularity than $f$ itself, i.e. we do not expect that $\partial_y^\ell u \in \mu^\gamma L^2$ for $\ell > 0$ unless the same is true for $f$ too. Returning to $X = \mathbb{R}^+ \times \overline{\mathbb{H}^2}$, we are fortunately spared these considerations because our main interest is when $f$ is SO(2) invariant.

In the following, we shall use the logarithmically blown up single space

$$\widetilde{X} = [I_{\log} \times (\overline{\mathbb{H}^2})_{\log}; \partial I \times \partial \overline{\mathbb{H}^2}].$$

It will be also convenient to have a notation for the class of functions super-logarithmically decaying relative to the critical decay $e^{-\kappa/x}$, so we let

$$e^{\alpha/x} H^{m,a}_{p0}(\overline{X}) = x^a e^{\alpha/x} H^m_{p0}(\overline{X}) = \{u \in \mathcal{C}^{-\infty}(\overline{X}) :\ x^{-a} e^{-\alpha/x} u \in H^m_{p0}(\overline{X})\},$$
$$e^{\alpha/x} H^m_{p0}(\overline{X}) = \cap_a e^{\alpha/x} H^{m,a}_{p0}(\overline{X});$$

the latter made into a Fréchet space by the standard construction using the norms on $e^{\alpha/x} H^{m,a}_{p0}(\overline{X})$ for integers $a$. We shall use these spaces for $\alpha = -\kappa$ below.

**Proposition 4.8.** *If $f \in e^{\alpha/x} H^m_{p0}(\overline{X})$ for some $\alpha < -\kappa$, then*

$$R(\lambda) f = \mu^{1/2} x^{1/2} x_2 e^{-i\sqrt{\lambda-\lambda_0}/x} h + e^{\alpha/x} H^{m+2}_{p0}(\overline{X}).$$

*In general, $h \in \mathcal{C}^0_S(\widetilde{X})$ is (at least) continuous on $\widetilde{X}$ (with restriction to the lift of $\partial\overline{M}_1 \times \overline{M}_2$ being given by generalized eigenfunctions of $\Delta_{M_2}$ associated to the bottom of the spectrum), but if $f$ is* SO(2)*-invariant, then $h$ is smooth on $\widetilde{X}$. In particular, if $m = \infty$, and $f$ is* SO(2)*-invariant then*

$$R_0(\lambda) f = \mu^{1/2} x^{1/2} x_2 e^{-i\sqrt{\lambda-\lambda_0}/x} h', \quad h' \in \mathcal{C}^\infty_S(\widetilde{X})^{\mathrm{SO}(2)}. \tag{4.17}$$

*In addition, if the assumptions are weakened to $f \in x^a e^{-\kappa/x} H^m_{p0}(\overline{X})$ for some $a > 7/2$, then*

$$R_0(\lambda) f = \mu^{1/2} x^{1/2} x_2 e^{-i\sqrt{\lambda-\lambda_0}/x} h + x^a e^{-\kappa/x} H^{m+2}_{p0}(\overline{X}),$$

*with $h \in \mathcal{C}^0_S(\widetilde{X})$. If $f \in x^a e^{-\kappa/x} H^m_{p0}(\overline{X})$ for every $a$ and $f$ is* SO(2)*-invariant, then $h \in \mathcal{C}^\infty_S(\widetilde{X})$, so for $m = \infty$, (4.17) holds.*

*Proof.* As in the preceeding proposition, we may immediately replace $R_0$ by $R_0''$, since the small calculus contribution $R_0'$ causes no difficulties; it gives the terms $e^{\alpha/x} H^{m+2}_{p0}(\overline{X})$, resp. $x^a e^{-\kappa/x} H^{m+2}_{p0}(\overline{X})$, above.

In the two cases $f \in e^{\alpha/x} H^m_{p0}(\overline{X})$, resp. $f \in x^a e^{-\kappa/x} H^m_{p0}(\overline{X})$, we must show that

$$e^{i\sqrt{\lambda-\lambda_0}/x} \mu^{-1/2} x^{-1/2} x_2^{-1} R_0''(\lambda) \mu^{1/2} e^{\alpha/x} \in \mathcal{B}(L^2_b(\overline{X}), C^0(\widetilde{X})), \tag{4.18}$$

or

$$e^{i\sqrt{\lambda-\lambda_0}/x} \mu^{-1/2} x^{-1/2} x_2^{-1} R_0''(\lambda) \mu^{1/2} x^a e^{-\kappa/x} \in \mathcal{B}(L^2_b(\overline{X}), C^0(\widetilde{X})), \tag{4.19}$$

respectively. Since (4.19) (with $a > 7/2$) implies (4.18) (with $\alpha < -\kappa$), we only need to prove the latter.

The Schwartz kernel of the operator in (4.19) takes the form

$$K'' = e^{-i\sqrt{\lambda-\lambda_0}(\tilde\delta(z,z')-\tilde\delta(z,o))-\kappa\tilde\delta(z',o)}$$
$$\times \tilde\delta(z',o)^{-a+5/2} \cdot e^{-(\tilde\delta_2(z_2,z_2')+\tilde\delta_2(q,z_2')-\tilde\delta_2(q,z_2))/2} \cdot \tilde F(\lambda),$$

where

$$\tilde F(\lambda) = x^{-1/2} x_2^{-1} \tilde\delta(z',o)^{-5/2} F(\lambda) = \tilde\delta(z,o)^{3/2} \tilde\delta_2(z_2,q)^{-1} \tilde\delta(z',o)^{-5/2} F(\lambda).$$

Dropping the singular measure (namely the b-measure), $\tilde F(\lambda)$ is bounded since on the one hand,

$$F(\lambda) = \tilde\delta(z,z')^{-3/2} \tilde\delta_2(z_2,z_2') F'(\lambda),\ F'(\lambda) \in L^\infty(X \times X),$$

but we also know that

$$\begin{aligned}&\tilde\delta(z,o)^{3/2}\tilde\delta_2(z_2,q)^{-1}\tilde\delta(z',o)^{-5/2}\tilde\delta(z,z')^{-3/2}\tilde\delta_2(z_2,z_2')\\&=\Big(\tilde\delta(z,o)\tilde\delta(z',o)^{-1}\tilde\delta(z,z')^{-1}\Big)^{3/2}\Big(\tilde\delta_2(z_2,q)^{-1}\tilde\delta_2(z_2,z_2')\tilde\delta(z',o)^{-1}\Big)\end{aligned}$$

is bounded using (4.4). In fact, using the triangle inequality we even get

$$\tilde\delta(z',o)^{-5/2+a}K''\in L^\infty(\tilde X\times\bar X). \tag{4.20}$$

Now let $r\in(0,a-7/2)$. Each factor here is continuous on $\widetilde X\times X$ (we make no claims about continuity at infinity in the second factor). Indeed, for the main term $\tilde F$ this follows from the asymptotics of $F(\lambda)$, while for the other factors this is true when $z'$ lies in a compact set in $X$ since the triangle inequality gives that both $\tilde\delta_2(z_2,z_2')-\tilde\delta_2(q,z_2)$ and $\tilde\delta(z,z')-\tilde\delta(z,o)$ are bounded continuous functions then. So we actually have

$$\tilde\delta(z',o)^{-5/2+a-r}K''\in C^0(\tilde X\times X). \tag{4.21}$$

Since in fact $\tilde\delta(z',o)^{-5/2+a-r}$ is continuous on $\tilde X\times\overline{X}$ and vanishes at $\tilde X\times\partial\overline{X}$, we deduce from (4.20) and (4.21) that

$$\tilde\delta(z',o)^{-5/2+a-r}K''\in C^0(\tilde X\times\overline{X}). \tag{4.22}$$

Finally, $\tilde\delta(z',o)^{5/2-a+r}\in L^2_b(\overline{X})$, and this gives

$$K''\in C^0(\widetilde X;L^2_b(\overline{X})),\qquad\text{and so}\qquad K'':L^2_b(\overline{X})\to C^0(\widetilde X)$$

is continuous. This proves the first part of the proposition, with $h\in\mathcal{C}^0(\widetilde X)$.

Next, $h|_{\partial\overline{M}_1\times M_2}$ is given by restricting $K''$ to $\partial\overline{M}_1\times M_2$, considering it as a function with values in $L^2_{\mathrm b}(\overline{X})$. By (4.14), if we choose $a$ sufficiently large, this lies in $\mathcal{C}^k(\partial\overline{M}_1\times M_2;L^2_b(\overline{X}))$, so we deduce that $h|_{\partial\overline{M}_1\times M_2}$ is $\mathcal{C}^k$ (we are not making a uniform statement up to the boundary of $M_2$ yet), and indeed the map from $f$ to this restriction of $h$ is continuous from $x^ae^{-\kappa/x}H^m_{p0}(\overline{X})$ to $\mathcal{C}^k(\overline{M}_1\times M_2)$. With $a=\infty$, we can thus take $k=\infty$. Since various factors are automatically bounded then, for $f$ compactly supported, with support in a fixed compact set $K\subset X$, it is easy to see from (A.5) that $\mu^{1/2}\tilde\delta_2(z_2,q)h|_{\partial\overline{M}_1\times M_2}$ is annihilated by $\Delta_{\mathbb{H}_2}-\frac14$. In fact, this follows even more easily from Proposition 4.2: if $f\in e^{-\kappa/x}H^{m,\infty}_{p0}(\overline{X})$ and $f_j\in\mathcal{C}^\infty_c(X)$ converges to $f$ in $e^{-\kappa/x}H^{m,\infty}_{p0}(\overline{X})$ then the corresponding restrictions $h_j$ converge to $h$ in $\mathcal{C}^\infty$, hence $(\Delta_{\mathbb{H}_2}-\frac14)(\mu^{1/2}\tilde\delta_2(z_2,q)h_j)$ converges to $(\Delta_{\mathbb{H}_2}-\frac14)(\mu^{1/2}\tilde\delta_2(z_2,q)h)$ in $\mathcal{C}^\infty$, so we deduce that the latter vanishes as claimed.

Now suppose that $f$ is SO(2)-invariant. In order to avoid the behaviour of $F(\lambda)$ at the front face $I_0^2\times\mathrm{ff}(\overline{\mathbb{H}^2})$, introduce a cutoff function $\phi$ as in Theorem 4.6. Then both $\phi(\mu)K''$ and $K''\phi(\mu')$ are in $\mathcal{C}^\infty(\widetilde X;L^2_b(\overline{X}))$ and map into the space of functions with full asymptotic expansions.

It remains only to consider $(1-\phi(\mu))K''(1-\phi(\mu'))$. Denote the corresponding kernel by $\tilde K''$, and work on the base space $\mathfrak{b}^+$. The desired result follows easily from Theorem 4.6, since that result implies that if $P\in\mathrm{Diff}^m(\widetilde{\mathfrak{b}^+})$ then for sufficiently large $a>0$, $\tilde\delta(z',o)^{-a}P\tilde K''$ is continuous and bounded on $\widetilde{\mathfrak{b}^+}\times\mathfrak{b}^+$. $\square$

The final result concerns off-diagonal decay of $R_0(\lambda)$. To state this, choose SO(2)-invariant cutoff functions $\chi$ and $\psi$ on the logarithmically blown up single space $\widetilde X$ with $\operatorname{supp}\chi\cap\operatorname{supp}\psi=\emptyset$. We can regard both as defined on $\mathfrak{b}_+$, or rather, its logarithmically blown up single space $\widetilde{\mathfrak{b}_+}$. As explained in the introduction, the front face of $\widetilde{\mathfrak{b}_+}$ can be identified with (a large sector in) the Euclidean radial compactification $\overline{\mathfrak{b}_r}$ of $\mathfrak{b}_+$. This allows us to define the angle $\theta$ between $\operatorname{supp}\chi$ and $\operatorname{supp}\psi$ on the sphere at infinity. If $\operatorname{supp}\chi$ and $\operatorname{supp}\psi$ are conic outside a compact subset of $\mathfrak{a}$, then this is simply the angle between these cones. We have already shown that for $\lambda \notin \operatorname{spec}(L_0)$, $\psi R(\lambda)\chi$ is bounded on the spaces $e^{\alpha/x}L^2_{p0}(\overline{X})$, $|\alpha| < \kappa$, but we now show that this cut off kernel actually improves decay.

Before proceeding, we note that this phenomenon is a very familiar one. Consider the Laplacian $\Delta$ on $\mathbb{R}^2_w$. The plane wave solutions of $(\Delta - \lambda)u = 0$ are those of the form $u(w) = e^{i\sqrt{\lambda}\omega_0\cdot w}$, $|\omega_0| = 1$. (As usual, the branch of the square root function has negative imaginary part on $\mathbb{C} \setminus \mathbb{R}^+$.) Now write $w$ in polar coordinates as $r\omega'$. Fix $\omega$ and let $\cos\theta = \omega\cdot\omega'$, so $\theta$ is the angle between the source of the plane wave and $w$. Then $|u| = e^{-\operatorname{Im}\sqrt{\lambda}\cos\theta r}$, or in other words, the exponential rate of attenuation of $u$ is proportional to $\cos\theta$. This effect is directly attributable to the structure of the resolvent $R_0(\lambda) = (\Delta - \lambda)^{-1}$ itself. Indeed, if $\chi \in \mathcal{C}^\infty(\mathbb{S}^1)$, $\chi \equiv 1$ near $\omega_0$, then

$$u = \chi(\omega)e^{i\sqrt{\lambda}\omega_0\cdot w} - R_0(\lambda)(\Delta - \lambda)(\chi(\omega)e^{i\sqrt{\lambda}\omega_0\cdot w}).$$

We refer to [6] for an interesting discussion about a localization of this phenomenon (still in Euclidean space), concerning 'decay profiles' of solutions of $(\Delta - \lambda)u = 0$ which are only defined in cones.

When $\lambda \in \mathbb{R}^+$, one no longer obtains such decay, of course. In its place is a propagation phenomenon at infinity, seen already in [20], which plays a very important role in many-body scattering [26], [25]. The behaviour we are studying here, which might be called "dissipative propagation", should be understood as a sort of analytic continuation of these on-spectrum propagation results.

**Proposition 4.9** (Dissipative propagation)**.** *Choosing cut-off functions $\chi$, $\psi$ as above, let $\theta \in (0, \pi/2)$ be less than the angle between their supports. For $\lambda \notin \operatorname{spec}(L_0)$ and $|\alpha| \le \kappa = -\operatorname{Im}\sqrt{\lambda - \lambda_0}$, write $\alpha = \kappa\cos\theta_0$ for $\theta_0 \in [0, \pi]$. Choose $\beta > \kappa\cos(\theta + \theta_0)$ if $\theta + \theta_0 \le \pi$, otherwise choose $\beta > -\kappa$. Then for any $m$,*

$$\psi R_0(\lambda)\chi : e^{\alpha/x}L^2_{p0}(\overline{X}) \to e^{\beta/x}H^m_{p0}(\overline{X})$$

*Proof.* Since we can assume that the supports of $\chi$ and $\psi$ do not intersect, we can immediately discard the on-diagonal term $R_0'(\lambda)$, and furthermore it also suffices to just prove boundedness into $e^{\beta/x}L^2_{p0}$.

Let $\beta_0 = \kappa\cos(\theta + \theta_0)$ if $\theta + \theta_0 \le \pi$, and otherwise let $\beta_0 = -\kappa$. The key point is the uniform boundedness of

$$e^{\alpha\tilde{\delta}(o,z')}e^{-\kappa\tilde{\delta}(z,z')}e^{-\beta_0\tilde{\delta}(o,z)} \tag{4.23}$$

when $z' \in \operatorname{supp}\chi$, $z \in \operatorname{supp}\psi$. It suffices to prove the analogous boundedness when these modified distance functions are replaced by the actual (nonsmooth) distance functions. For $z = (s, z_2) \in \mathbb{R}^+ \times \mathbb{H}^2$, define

$$w = (w_1, w_2) \in \mathbb{R}^2,\ w_1 = \log s = \delta_1(1, s),\ w_2 = \delta_2(q, z_2),$$

and define $w'$ analogously, corresponding to $z' = (s', z_2')$. Then $\delta(o, z) = |w|$, $\delta(o, z') = |w'|$. The support conditions on $\chi$ and $\psi$ mean that we restrict $w$ and $w'$ to lie in cones $\Gamma$ and $\Gamma'$ in $\mathbb{R}^2$ making an angle $\theta$ with one another. The triangle inequality gives $|\delta_2(z_2,o)-\delta_2(z_2',o)|\leq\delta_2(z_2,z_2')$, hence

$$\begin{aligned}\delta(z,z')&=\sqrt{\delta_1(s,s')^2+\delta_2(z_2,z_2')^2}\geq\sqrt{\delta_1(s,s')^2+(\delta_2(z_2,o)-\delta_2(z_2',o))^2}\\&=\sqrt{(w_1-w_1')^2+(w_2-w_2')^2}=|w-w'|.\end{aligned}$$

Hence the boundedness of (4.23) follows from the estimate

$$\alpha|w'|-\kappa|w-w'|-\beta_0|w|\leq 0,\qquad \forall\, w\in\Gamma,\ w'\in\Gamma', \tag{4.24}$$

or equivalently, dividing through by $\kappa>0$,

$$\begin{aligned}&\cos\theta_0|w'|\leq|w-w'|+\cos(\theta+\theta_0)|w|, &&\text{if } \theta+\theta_0\leq\pi,\\&\cos\theta_0|w'|\leq|w-w'|-|w|, &&\text{if } \theta+\theta_0>\pi.\end{aligned} \tag{4.25}$$

These certainly hold when one or the other vector vanishes. On the other hand, if neither is zero, then it suffices to consider the case where the angle between them is exactly $\theta$, since this configuration minimizes the right hand side and keeps the left hand side fixed.

Suppose first that $\theta+\theta_0>\pi$. Then $\pi\geq\theta_0>\pi-\theta\geq 0$ implies $\cos\theta_0<\cos(\pi-\theta)=-\cos\theta$, so we must only prove that $|w|-\cos\theta|w'|\leq|w-w'|$. But this follows from

$$|w|^2=(w-w')\cdot w+w'\cdot w\leq|w-w'||w|+|w'||w|\cos\theta$$

once we divide through by $|w|$.

If, on the other hand, $\theta+\theta_0\leq\pi$, then we let $v''$ be the unique unit vector such that $v'\cdot v''=\cos\theta_0$ and $v\cdot v''=\cos(\theta+\theta_0)$. Since $w'\cdot v''=(w'-w)\cdot v''+w\cdot v''$ and $(w'-w)\cdot v''\leq|w-w'|$ as well, we deduce that $|w'|\cos\theta_0\leq|w-w'|+|w|\cos(\theta+\theta_0)$, and this completes the proof. □

**Proposition 4.10.** *With the same notation as above, if $f\in e^{\alpha/x}H^m_{p0}(\overline{X})$ and $\theta+\theta_0>\pi$, then*

$$\psi R_0(\lambda)\chi f=\mu^{1/2}x^{1/2}x_2e^{-i\sqrt{\lambda-\lambda_0}/x}h+e^{\alpha/x}H^{m+2}_{p0}(\overline{X}),$$

*where $h$ is continuous on $\widetilde{X}$. If $f$ is $\mathrm{SO}(2)$-invariant, then $h\in\mathcal{C}^\infty_S(\widetilde{X})^{\mathrm{SO}(2)}$. In particular, if $m=\infty$, and $f$ is $\mathrm{SO}(2)$-invariant, then*

$$\psi R_0(\lambda)\chi f=\mu^{1/2}x^{1/2}x_2e^{-i\sqrt{\lambda-\lambda_0}/x}h',\quad h'\in\mathcal{C}^\infty_S(\widetilde{X})^{\mathrm{SO}(2)}.$$

*Proof.* We must now consider the kernel

$$\begin{aligned}\tilde K'' =&e^{(\alpha+\epsilon)\delta(o,z')}e^{-i(\sqrt{\lambda-\lambda_0}\delta(z,z')-\delta(o,z))}\chi(z')\psi(z)\\&\qquad e^{-(\delta_2(z_2,z_2')+\delta_2(q,z_2')-\delta_2(q,z_2))/2}\tilde\delta(z',o)^{5/2}\tilde F(\lambda),\end{aligned}$$

where

$$\tilde F(\lambda)=x^{-1/2}x_2^{-1}\tilde\delta(z',o)^{-5/2}e^{-\epsilon\delta(o,z')}F(\lambda).$$

Using (4.23) with $\kappa$ in place of $\beta_0$, we deduce that the first factor is bounded if $z\in\operatorname{supp}\psi$, $z'\in\operatorname{supp}\chi$, and $\epsilon>0$ is sufficiently small. Hence the arguments of Proposition 4.8 show that $\tilde K''\in C^0(\widetilde{X};L^2_b(\overline{X}))$, giving the first part of the conclusion. The rest follows as in Proposition 4.8. □

Of course, $R_0(\lambda)$ preserves SO(2)-invariance, and if $f$ is rotationally invariant, then $R_0(\lambda)f$ also has an expansion.

Finally, we reintroduce the weight $s$ and translate back to the original operator $L_\sharp$ acting on the function space $\mathcal{H}$ from (4.5) on which it is self-adjoint. Writing $\mathcal{H}^m$ for the corresponding Sobolev spaces, so

$$\mathcal{H}^m = sH^m_{p0}(\overline{X}),\ \mathcal{H}^{m,a} = sH^{m,a}_{p0}(\overline{X})$$

our results in this section yield the following corollary.

**Corollary 4.11.** *The operator $(L_\sharp - \lambda)^{-1}$ is bounded from $e^{\alpha/x}\mathcal{H}^m$ to $e^{\alpha/x}\mathcal{H}^{m+2}$ for $|\alpha| < \kappa$. Moreover, for $\chi$, $\psi$, $\theta$, $\alpha$, $\beta$ as above,*

$$\psi(L_\sharp - \lambda)^{-1}\chi : e^{\alpha/x}\mathcal{H}^m \to e^{\beta/x}\mathcal{H}^{m+2}.$$

*If $f \in x^a e^{-\kappa/x}(\mathcal{H}^\infty)^{\mathrm{SO}(2)} = e^{-\kappa/x}(\mathcal{H}^{\infty,p})^{\mathrm{SO}(2)}$ for all $p$, then $(L_\sharp - \lambda)^{-1}f$ has a full asymptotic expansion on $\widetilde{X}$ of the form*

$$(L_\sharp - \lambda)^{-1}f = s\mu^{1/2}x^{1/2}x_2 e^{-i\sqrt{\lambda-\lambda_0}/x}h',\ h' \in \mathcal{C}^\infty_S(\widetilde{X})^{\mathrm{SO}(2)}.$$

## 5. Radial solutions and the final parametrix

We now return to our main problem, and apply the results of the last section to finish the construction of a parametrix for $\Delta - \lambda$ with compact remainder. As explained earlier, this requires finding a correction term for the ($K_p$-invariant) small-calculus parametrix $G_p(\lambda)$. Recall that the error terms $E_p(\lambda)$ and $F_p(\lambda)$ from Proposition 3.4 left after this first stage are $K_p$-invariant, hence can be regarded as acting on functions on $\mathfrak{a}$. They are in fact residual elements of the ee calculus on $\overline{\mathfrak{a}}$. Thus, at the very least, we must solve equations of the form $(\Delta - \lambda)u = f$, where $f$ is polyhomogeneous on $\overline{\mathfrak{a}}$. We now turn to this task.

The idea is that near $H_\sharp$, $\Delta$ is well approximated by the product operator $L_\sharp$, and similarly it is well approximated near $H^\sharp$ by $L^\sharp$. However, to make this precise we must pass to the logarithmically blown up space $\widetilde{\mathfrak{a}}$ and localize there. To motivate this, recall from §2.2 the coordinates $\mu$, $\nu$ near the corner on $\overline{\mathfrak{a}}$ and the expression (2.6) for the restriction of $\Delta$ to $K_p$-invariant functions, i.e. to $\mathfrak{a}$. Since $\Delta_{\mathrm{rad}}$ is not product type, even asymptotically, near the corner, we do not know a priori how to invert it. However, working near $\nu = 0$, first note by (2.7) that

$$\Delta_{\mathrm{rad}} = \frac{1}{3}\left((\mu D_\mu)^2 + \frac{\mu + \mu^{-1}}{\mu - \mu^{-1}} i\mu D_\mu\right) + \frac{1}{4}(sD_s)^2 + \frac{1}{2}isD_s + s^2E,$$

where $E = a\mu D_\mu + bsD_s + c$, $a, b, c \in \mathcal{C}^\infty$. The first term in parentheses on the right is just the radial part of the Laplacian on $\mathbb{H}^2$ (with respect to the metric $3h$), and were we to have kept better track of the angular derivatives, we would see the complete Laplacian on $\mathbb{H}^2$ here. Hence at least in the interior of $H_\sharp$, $\Delta - L_\sharp$ is small. Now consider the situation near the corner more carefully. The coordinates $\mu$, $\nu$ are valid here, and we define the change of variables $t = \mu$, $\bar{s} = \mu^{1/2}\nu$, so $s = \bar{s}$, where $s$ is the notation used in the introduction. Then

$$\mu D_\mu = tD_t + \frac{1}{2}\bar{s}D_{\bar{s}}, \quad \nu D_\nu = \bar{s}D_{\bar{s}},$$

and inserting these into (2.7) gives

$$\Delta_{\mathrm{rad}} = \frac{1}{3}((tD_t)^2 + itD_t) + \frac{1}{4}(\bar{s}D_{\bar{s}})^2 + \frac{i}{2}\bar{s}D_{\bar{s}} + E,$$

where $E$ is a different error term, consisting of sums of of smooth multiples of the vector fields $tD_t$ and $\bar{s}D_{\bar{s}}$ with additional factors of $t$ or $\bar{s}$ and a polyhomogeneous function in these new coordinates vanishing at least to order one at the corner. Once again we see that the first term in parentheses is the radial part of the Laplacian and the main terms of this expression (i.e. omitting the error term) are the same as for $L_\sharp$.

This change of coordinates near the corner appears rather complicated, but in fact it represents a smooth change of coordinates near the entire closure of $\widetilde{H_\sharp}$ in $\widetilde{\mathfrak{a}}$. To see this, let $\bar{\mu} = -1/\log\mu$, $\bar{\nu} = -1/\log\nu$, $\tau = -1/\log t$, $\sigma = -1/\log\bar{s}$; then

$$\tau = \bar{\mu}, \qquad \sigma = \frac{\bar{\mu}\bar{\nu}}{\frac{1}{2}\bar{\mu} + \bar{\nu}}.$$

The functions

$$r = \frac{1}{2}\bar{\mu} + \bar{\nu},\ \alpha = \frac{\frac{1}{2}\bar{\mu} - \bar{\nu}}{\frac{1}{2}\bar{\mu} + \bar{\nu}}, \qquad r' = \tau + \sigma,\ \alpha' = \frac{\tau - \sigma}{\tau + \sigma}$$

give two sets of coordinates on $\widetilde{\mathfrak{a}}$ near this face, and we have finally

$$r' = \frac{1}{2}r(1+\alpha)(3-\alpha), \qquad \alpha' = (1+\alpha)/(3-\alpha).$$

This proves the claim that this coordinate change is smooth on $\widetilde{\mathfrak{a}}$ near the corner of $\widetilde{H_\sharp}$ since $(r,\alpha)$ and $(r',\alpha')$ are both smooth coordinate systems there.

Altogether, noting in particular that $\bar{s} \leq \nu$ near the corner, we have just established that

$$\Delta - L_\sharp : H^m_{\mathrm{ee}}(\bar{M})^{K_p} \to \rho_\sharp H^{m-1}_{\mathrm{ee}}(\bar{M})$$

near the closure of $H_\sharp$, and the proper venue for this approximation is $\widetilde{\mathfrak{a}}$.

Now choose a smooth $K_p$-invariant partition of unity $\chi_\sharp + \chi^\sharp + \chi_0 = 1$ on $\widetilde{M}$ such that $\operatorname{supp}\chi_\sharp$ and $\operatorname{supp}\chi^\sharp$ are disjoint from $\widetilde{H^\sharp} \cup \mathfrak{a}^\sharp$ and $\widetilde{H_\sharp} \cup \mathfrak{a}_\sharp$, respectively, and $\operatorname{supp}\chi_0$ is compact in $M$. (We do not introduce new notation for the lifts of $\mathfrak{a}^\sharp$ and $\mathfrak{a}_\sharp$ to $\widetilde{\mathfrak{a}}$ because the walls of the Weyl chambers are disjoint from where the blowup takes place.) Also, choose additional cutoffs $\psi_\sharp$ and $\psi^\sharp$ which are identically 1 on $\operatorname{supp}\chi_\sharp$ and $\operatorname{supp}\chi^\sharp$, respectively, and with supports disjoint from $\widetilde{H^\sharp} \cup \mathfrak{a}^\sharp$ and $\widetilde{H_\sharp} \cup \mathfrak{a}_\sharp$.

Now define

$$\tilde{R}(\lambda) = G_p(\lambda) - \psi^\sharp(L^\sharp - \lambda)^{-1}\chi^\sharp F(\lambda) - \psi_\sharp(L_\sharp - \lambda)^{-1}\chi_\sharp F(\lambda),$$

(This involves a slight abuse of notation because $L^\sharp$ is not defined near $\mathfrak{a}_\sharp$, and similarly for $L_\sharp$; this makes sense however because we have included the additional cutoffs $\psi^\sharp$ and $\psi_\sharp$.

On $K_p$-invariant functions

$$\begin{aligned}(\Delta-\lambda)\tilde{R}(\lambda) &= \mathrm{Id} + (1 - \chi_\sharp - \chi^\sharp)\, F(\lambda)\\ &\quad - (\Delta - L_\sharp)\,\psi_\sharp\,(L_\sharp - \lambda)^{-1}\,\chi_\sharp\,F(\lambda) - [L_\sharp, \psi_\sharp]\,(L_\sharp - \lambda)^{-1}\,\chi_\sharp\,F(\lambda)\\ &\quad - (\Delta - L^\sharp)\,\psi^\sharp\,(L^\sharp - \lambda)^{-1}\,\chi^\sharp\,F(\lambda) - [L^\sharp, \psi^\sharp]\,(L^\sharp - \lambda)^{-1}\,\chi^\sharp\,F(\lambda).\end{aligned} \tag{5.1}$$

Let $x_\sharp$ and $x^\sharp$ be the $K_p$-invariant defining functions of the lifts of $H_\sharp$ and $H^\sharp$ to $\widetilde{M}$, respectively.

In order to state the mapping properties of $\tilde{R}(\lambda)$ and $\tilde{F}(\lambda)$, we introduce spaces of functions whose restrictions to the lifts of $H_\sharp$ and $H^\sharp$ are spherical functions on each $\mathbb{H}^2$ fiber corresponding to the bottom of the spectrum of $\Delta_{\mathbb{H}^2}$.

**Definition 5.1.** Let $\mathcal{C}^\infty_S(\widetilde{M})$, resp. $\mathcal{C}^0_S(\widetilde{M})$, be the spaces of smooth, resp. continuous, functions on $\widetilde{M}$ whose restriction to each fiber of $\phi^\sharp$ and $\phi_\sharp$ is $\mu^{-1/2}\tilde{\delta}_2(.,q)^{-1}$ times a generalized eigenfunction of $\Delta_{\mathbb{H}^2}$ with eigenvalue 1/4. (Note that $\Delta_{\mathbb{H}^2}$ is the Laplacian of the standard hyperbolic metric $h$ of curvature $-1$.)

In particular, let $\mathcal{C}^\infty_S(\widetilde{M})^{K_p}$, resp. $\mathcal{C}^0_S(\widetilde{M})^{K_p}$, be the spaces of smooth, resp. continuous, functions on $\widetilde{M}$ whose restriction to each fiber of $\phi^\sharp$ and $\phi_\sharp$ is $\mu^{-1/2}\tilde{\delta}_2(.,q)^{-1}$ times a spherical function on $\mathbb{H}^2$ associated to the bottom of the spectrum of the Laplacian, i.e. is an SO(2)-invariant generalized eigenfunction of $\Delta_{\mathbb{H}^2}$ of eigenvalue 1/4.

We also need the spaces with polynomial and exponential weights:

$$e^{\alpha/x}H^{m,r}_{\mathrm{ee}}(\overline{M}) = x^r e^{\alpha/x}H^m_{\mathrm{ee}}(\overline{M}) = \{u \in \mathcal{C}^{-\infty}(\overline{M}) :\ x^{-r}e^{-\alpha/x}u \in H^m_{\mathrm{ee}}(\overline{M})\}.$$

**Proposition 5.2.** *The following maps are bounded*

i) *For* $|\alpha| < \kappa = -\operatorname{Im}\sqrt{\lambda-\lambda_0}$,

$$\tilde{R}(\lambda) : e^{\alpha/x}H^m_{ee}(M)^{K_p} \longrightarrow e^{\alpha/x}H^{m+2}_{ee}(M)^{K_p}$$

ii) *For* $\alpha < -\kappa$,

$$\tilde{R}(\lambda) : e^{\alpha/x}H^m_{ee}(\overline{M})^{K_p} \longrightarrow$$
$$e^{-i\sqrt{\lambda-\lambda_0}/x}\, x^{1/2}\, x_\sharp x^\sharp\, \rho_\sharp\, \rho^\sharp\, \mathcal{C}^\infty_S(\widetilde{M})^{K_p} \ +\ e^{\alpha/x}H^{m+2}_{ee}(\overline{M})^{K_p}.$$

iii) *In particular, for* $\alpha < -\kappa$,

$$\tilde{R}(\lambda) : e^{\alpha/x}H^\infty_{ee}(\overline{M})^{K_p} \longrightarrow e^{-i\sqrt{\lambda-\lambda_0}/x}\, x^{1/2}\, x_\sharp x^\sharp\, \rho_\sharp\, \rho^\sharp\, \mathcal{C}^\infty_S(\widetilde{M})^{K_p}.$$

*In (ii) and (iii), one can replace* $e^{\alpha/x}H^k_{ee}(\overline{M})^{K_p}$ *by* $e^{-\kappa/x}H^{k+2,\infty}_{ee}(\overline{M})^{K_p}$, *where* $k = m$ *or* $\infty$ *(and in the latter case* $k+2=\infty$ *as well).*

*Now choose* $\theta \in (0,\pi/2)$ *less than the angles between* $\operatorname{supp}\chi_\sharp$ *and* $\operatorname{supp} d\psi_\sharp$ *and between* $\operatorname{supp}\chi^\sharp$ *and* $\operatorname{supp} d\psi^\sharp$ *on the sphere at infinity. Then there exists a* $\gamma > 0$ *with the following properties:*

iv) *If* $|\alpha| \le \kappa$, *so that* $\alpha = \kappa\cos\theta_0$ *for some* $\theta_0 \in [0,\pi]$, *and if in addition* $\beta > \kappa\cos(\theta+\theta_0)$ *when* $\theta+\theta_0 \le \pi$ *and* $\beta > -\kappa$ *otherwise, then the error term* $\tilde{F}(\lambda) = (\Delta-\lambda)\tilde{R}(\lambda) - \mathrm{Id}$ *satisfies*

$$\tilde{F}(\lambda) : e^{\alpha/x}H^m_{ee}(M)^{K_p} \longrightarrow e^{\max(\alpha-\gamma,\beta)/x}H^{m'}_{ee}(M)^{K_p}$$

*for any* $m, m'$.

v) *If* $\theta+\theta_0 > \pi$ *and* $\alpha < -\kappa+\gamma$, *then*

$$\tilde{F}(\lambda) : e^{\alpha/x}L^2(M)^{K_p} \longrightarrow e^{-i\sqrt{\lambda-\lambda_0}/x}\, x^{1/2}\, x_\sharp x^\sharp\, \rho_\sharp\, \rho^\sharp\, \mathcal{C}^\infty_S(\widetilde{M})^{K_p}.$$

*Proof.* The mapping properties for $\tilde{R}(\lambda)$ follow directly from Corollary 4.11. As for the mapping properties for $\tilde{F}(\lambda)$, note that the terms $[L_\sharp,\psi_\sharp](L_\sharp-\lambda)^{-1}\chi_\sharp F(\lambda)$, $[L^\sharp,\psi^\sharp](L^\sharp-\lambda)^{-1}\chi^\sharp F(\lambda)$ in (5.1) are bounded as stated since $[L_\sharp,\psi_\sharp]$ is supported on $d\psi_\sharp$, and so we may apply Corollary 4.11. On the other hand, to handle the remaining terms $(\Delta-L_\sharp)\psi_\sharp(L_\sharp-\lambda)^{-1}\chi_\sharp F(\lambda)$, $(\Delta-L^\sharp)\psi^\sharp(L^\sharp-\lambda)^{-1}\chi^\sharp F(\lambda)$, observe that $(L^\sharp-\lambda)^{-1}\chi^\sharp F(\lambda)$ is bounded on the appropriate spaces, and then $(\Delta-L_\sharp)\psi_\sharp$

gives the additional decay. This last statement follows from the fact that $\rho_\sharp\psi_\sharp < C\,e^{-\gamma/x}$ for some $\gamma > 0$. □

*Remark* 5.3. As mentioned in the introduction, it is possible to eliminate the first step of the parametrix, or rather, to combine the first and second steps, as in [16]. Namely, we take as a parametrix

$$\psi^\sharp(L^\sharp-\lambda)^{-1}\chi^\sharp+\psi_\sharp(L_\sharp-\lambda)^{-1}\chi_\sharp+\psi_0\tilde G(\lambda)\chi_0;$$

hhere $\tilde G(\lambda)$ is a $K_p$-invariant pseudodifferential operator on $M$ which is a parametrix for $\Delta-\lambda$ near $\operatorname{supp}\chi_0$, and $\psi_0\in\mathcal{C}^\infty_c(M)^{K_p}$ is identically 1 near $\operatorname{supp}\chi_0$. (We could require $\tilde G(\lambda)$ to have a compactly supported Schwartz kernel in $M\times M$, in which case $\psi_0$ may even be dropped.) With this choice, iv) above is weakened, as we have $m'=m+1$, but this suffices for compactness of the error term, and it still allows the proof of Theorem 5.4 to go through.

There is a similar right parametrix, $\tilde R'(\lambda)$ given by

$$\tilde R'(\lambda)=G_p(\lambda)-E(\lambda)\chi^\sharp(L^\sharp-\lambda)^{-1}\psi^\sharp-E(\lambda)\chi_\sharp(L_\sharp-\lambda)^{-1}\psi_\sharp.$$

Indeed, then on $K_p$-invariant functions

$$\begin{aligned}\tilde R'(\lambda)(\Delta-\lambda)&=\mathrm{Id}+E(\lambda)(1-\chi_\sharp-\chi^\sharp)\\&-E(\lambda)\,\chi_\sharp\,(L_\sharp-\lambda)^{-1}\,\psi_\sharp\,(\Delta-L_\sharp)-E(\lambda)\,\chi_\sharp\,(L_\sharp-\lambda)^{-1}\,[\psi_\sharp,L_\sharp]\\&-E(\lambda)\,\chi^\sharp\,(L^\sharp-\lambda)^{-1}\,\psi^\sharp\,(\Delta-L^\sharp)-E(\lambda)\,\chi^\sharp\,(L^\sharp-\lambda)^{-1}\,[\psi^\sharp,L^\sharp].\end{aligned}\tag{5.2}$$

In particular, $\tilde R'(\lambda)$ and $\tilde E'(\lambda)=\tilde R'(\lambda)(\Delta-\lambda)-\mathrm{Id}$ have the same mapping properties as $\tilde R(\lambda)$ and $\tilde F(\lambda)$, respectively, stated in the preceeding proposition.

The following theorem is now almost immediate.

**Theorem 5.4.** *The resolvent $R(\lambda)=(\Delta-\lambda)^{-1}$ has the following mapping properties on $K_p$-invariant functions.*

i) *For $|\alpha|<\kappa$, $m\in\mathbb{R}$,*

$$R(\lambda)\in\mathcal{B}(e^{\alpha/x}H^m_{ee}(\overline{M})^{K_p},e^{\alpha/x}H^{m+2}_{ee}(\overline{M})^{K_p});$$

ii) *for $\alpha<-\kappa$,*

$$\begin{aligned}&R(\lambda):e^{\alpha/x}H^m_{ee}(\overline{M})^{K_p}\longrightarrow\\&\qquad e^{-i\sqrt{\lambda-\lambda_0}/x}\,x^{1/2}\,x_\sharp x^\sharp\,\rho_\sharp\,\rho^\sharp\,\mathcal{C}^\infty_S(\widetilde{M})^{K_p}+e^{\alpha/x}H^{m+2}_{ee}(\overline{M})^{K_p}\end{aligned}$$

*and*

$$\begin{aligned}&R(\lambda):e^{-\kappa/x}H^{m,\infty}_{ee}(\overline{M})^{K_p}\longrightarrow\\&\qquad e^{-i\sqrt{\lambda-\lambda_0}/x}\,x^{1/2}\,x_\sharp x^\sharp\,\rho_\sharp\,\rho^\sharp\,\mathcal{C}^\infty_S(\widetilde{M})^{K_p}+e^{-\kappa/x}H^{m+2,\infty}_{ee}(\overline{M})^{K_p};\end{aligned}$$

iii) *in particular,*

$$R(\lambda):e^{-\kappa/x}H^{\infty,\infty}_{ee}(\overline{M})^{K_p}\longrightarrow e^{-i\sqrt{\lambda-\lambda_0}/x}\,x^{1/2}\,x_\sharp x^\sharp\,\rho_\sharp\,\rho^\sharp\,\mathcal{C}^\infty_S(\widetilde{M})^{K_p},\tag{5.3}$$

*which in turn implies the analogous mapping property with domain space $e^{\alpha/x}H^\infty_{ee}(\overline{M})^{K_p}$, $\alpha<-\kappa$.*

*Proof.* We use the parametrix identity

$$R(\lambda) = \tilde{R}(\lambda) - \tilde{R}'(\lambda)\tilde{F}(\lambda) + \tilde{E}'(\lambda)R(\lambda)\tilde{F}(\lambda), \tag{5.4}$$

where $\tilde{F}(\lambda)$ is as in Proposition 5.2 and $\tilde{E}'(\lambda) = \tilde{R}'(\lambda)(\Delta - \lambda) - \mathrm{Id}$, as above.

Suppose first that $0 \geq \alpha > -\kappa$. By the preceeding proposition, the first two terms are bounded $e^{\alpha/x}H^m_{\mathrm{ee}}(M)^{K_p} \to e^{\alpha/x}H^{m+2}_{\mathrm{ee}}(M)^{K_p}$.

Next, write $\beta = \kappa\cos(\theta + \frac{\pi}{2})$ (corresponding to $\cos\theta_0 = 0 = 0/\kappa$), and note that $\theta + \frac{\pi}{2} < \pi$. We have $R(\lambda), \tilde{F}(\lambda) \in \mathcal{B}(L^2(M)^{K_p})$; in addition, if $\beta$ is given by the usual prescription but with $\alpha$ replaced by 0, and if we set $\beta' = \max(-\gamma, \beta) < 0$, then $\tilde{E}'(\lambda) : L^2(M)^{K_p} \to e^{\beta'/x}H^{m'}_{\mathrm{ee}}(\overline{M})^{K_p}$. Hence for any $m'$, we deduce that

$$R(\lambda) : e^{\alpha/x}H^m_{\mathrm{ee}}(\overline{M})^{K_p} \longrightarrow e^{\alpha'/x}H^{m+2}_{\mathrm{ee}}(\overline{M})^{K_p}, \qquad \text{where} \quad \alpha' = \max(\alpha, \beta').$$

We now bootstrap to improve the image space. Thus suppose that

$$\alpha_0 = \inf\{\alpha' \in [\alpha, 0] : \ R(\lambda) \in \mathcal{B}(e^{\alpha/x}H^m_{\mathrm{ee}}(\overline{M})^{K_p},\ e^{\alpha'/x}H^{m+2}_{\mathrm{ee}}(\overline{M})^{K_p})\}.$$

We know that this set is non-empty, and that $\alpha_0 < 0$. If $\alpha_0 > \alpha$, then fix $\alpha' > \alpha_0$. Then by definition of this inf we know that

$$R(\lambda) \in \mathcal{B}(e^{\alpha/x}L^2(M)^{K_p}, e^{\alpha'/x}L^2(M)^{K_p});$$

Furthermore, both $\tilde{R}(\lambda)$ and $\tilde{F}(\lambda)$ are bounded on $e^{\alpha/x}L^2(M)^{K_p}$, while

$$\tilde{E}'(\lambda) \in \mathcal{B}(e^{\alpha'/x}L^2(M)^{K_p}, e^{\alpha''/x}L^2(M)^{K_p}), \qquad \text{where} \quad \alpha'' = \max(\alpha' - \gamma, \beta').$$

Here $\beta'$ is calculated from $\alpha'$ as usual. Choosing $\alpha'$ sufficiently close to $\alpha_0$, this gives

$$R(\lambda) \in \mathcal{B}(e^{\alpha/x}L^2(M)^{K_p}, e^{\alpha''/x}L^2(M)^{K_p}), \qquad \text{where} \quad \alpha'' < \alpha_0.$$

But this is a contradiction, and hence necessarily $\alpha_0 = \alpha$. The same argument shows that $R(\lambda) \in \mathcal{B}(e^{\alpha/x}L^2(M)^{K_p})$.

Now suppose that $\alpha < -\kappa$. The first two terms of (5.4) map into

$$e^{-i\sqrt{\lambda-\lambda_0}/x}x^{1/2}\, x_\sharp x^\sharp\, \mathcal{C}^\infty(\widetilde{M}) + e^{\alpha/x}H^{m+2}_{\mathrm{ee}}(\overline{M})^{K_p}. \tag{5.5}$$

On the other hand,

$$\tilde{F}(\lambda) \in \mathcal{B}(e^{\alpha/x}L^2(M)^{K_p},\ e^{\alpha'/x}L^2(M)^{K_p})$$

for any $\alpha' > -\kappa$, and at the same time, $\tilde{R}(\lambda) \in \mathcal{B}(e^{\alpha'/x}L^2(M)^{K_p})$. But choosing $\alpha'$ sufficiently close to $-\kappa$ ensures that $\tilde{E}'(\lambda)$ maps this space to

$$e^{-i\sqrt{\lambda-\lambda_0}/x}x^{1/2}x_\sharp x^\sharp\mathcal{C}^\infty(\widetilde{M}).$$

The argument is similar if we replace $e^{\alpha/x}H^m_{\mathrm{ee}}(\overline{M})^{K_p}$ with $e^{-\kappa/x}H^{m,\infty}_{\mathrm{ee}}(\overline{M})^{K_p}$. This finishes the proof. ☐

**Corollary 5.5.** *If $\lambda \notin \mathrm{spec}(\Delta)$ and $f \in \mathcal{C}^\infty_c(M)$, then*

$$R(\lambda)f = \rho_\sharp\rho^\sharp x^{1/2}x_\sharp x^\sharp \exp\left(-i\sqrt{\lambda-\lambda_0}/x\right) g,$$

*where $g \in \mathcal{C}^\infty_S(\widetilde{M})$; the analogous statement holds if $f \in \mathcal{C}^{-\infty}_c(M)$, but then $g$ is only $\mathcal{C}^\infty$ away from* sing supp $f$*. In particular, this estimate holds when $f = \delta_p$, in which case $R(\lambda)f = R(\lambda; w, p)$ is the Green function with pole at $p$, in which case in addition away from $p$, $g$ is in $\mathcal{C}^\infty_S(\widetilde{M})^{K_p}$.*

*Proof.* If $f \in \mathcal{C}_c^\infty(M)^{K_p}$, the result follows from part (iii) of the theorem since $\mathcal{C}_c^\infty(M)^{K_p} \subset e^{\alpha/x} H_{\mathrm{ee}}^\infty(\overline{M})^{K_p}$ for all $\alpha$. Next, if $f \in \mathcal{C}_c^{-\infty}(M)^{K_p}$, then we first use a local parametrix, constructed by standard methods, for $\Delta$ near the (compact!) support of $f$ to obtain $u_0 \in \mathcal{C}_c^{-\infty}(M)^{K_p}$ with $f_0 = (\Delta - \lambda)u_0 - f \in \mathcal{C}_c^\infty(M)^{K_p}$; of course, $\operatorname{sing\,supp} f = \operatorname{sing\,supp} u_0$. Then $R(\lambda)f = u_0 - R(\lambda)f_0$, and so we have reduced to the previous case.

This last argument clearly also gives the asymptotic structure of the Schwartz kernel of $R(\lambda)$, i.e. (with an abuse of notation) the distribution $R(\lambda; p, p') = (R(\lambda)\delta_{p'})(p)$, and so this has exactly the same form. It is obvious that the asymptotics are uniform as the pole $p'$ varies over a compact set. From this we may now deduce the the general (non $K_p$-invariant) case from the expression $R(\lambda)f(p) = \int_M R(\lambda; p, p') f(p')\, dg(p')$, either when $f$ is smooth or a distribution, so long as it is compactly supported. □

## 6. Boundary asymptotics and the Martin compactification

As discussed in the introduction, the results in the preceding section may be used to identify $\widetilde{M}$ with the Martin compactification $\overline{M}_{\mathrm{Mar}}(\lambda)$ for any $\lambda < \lambda_0$ (which we henceforth write simply as $\overline{M}_{\mathrm{Mar}}$). The Martin boundary $\overline{M}_{\mathrm{Mar}} \setminus M$ is defined as the set of equivalence classes of sequences $U_j = R(\lambda; p, p_j)/R(\lambda; p_0, p_j)$, and we shall prove now that these equivalence classes are in bijective correspondence with points $q \in \partial\widetilde{M}$ via $U_j \to U_q$ if and only if $p_j \to q$.

We begin, however, with a somewhat more general discussion of the boundary data of functions $u = R(\lambda)f$ when $f$ is compactly supported. First note that, by the results of §5, any $f \in e^{-\kappa/x} H_{\mathrm{ee}}^{\infty,\infty}(M)$ determines a function

$$g = e^{i\sqrt{\lambda - \lambda_0}/x}\, x^{-1/2}\, (x_\sharp\, x^\sharp\, \rho_\sharp\, \rho^\sharp)^{-1} R(\lambda) f \in \mathcal{C}_S^\infty(\widetilde{M}), \tag{6.1}$$

where the restrictions of $g$ to the fibers of the lifts of $H_\sharp$ and $H^\sharp$ are (up to factors of $\mu^{-1/2}\tilde{\delta}_2(.,q)^{-1}$) generalized eigenfunctions on $\mathbb{H}^2$ with eigenvalue $\frac{1}{4}$ (and are SO(2)-invariant when $f$ is $K_p$-invariant). In particular, these restrictions are determined by their values at $\partial\overline{\mathbb{H}^2}$, and hence the restriction of $g$ to the face mf of $\widetilde{M}$ determines the restriction of $g$ to the entire boundary $\partial\widetilde{M}$.

**Lemma 6.1.** *Fix any $\lambda \in \mathbb{R}$ with $\lambda < \lambda_0$ and define the map*

$$A(\lambda) : e^{-\kappa/x} H_{ee}^{\infty,\infty}(\overline{M})^{K_p} \to \mathcal{C}^\infty(\mathrm{mf})^{K_p}, \qquad f \mapsto g|_{\mathrm{mf}},$$

*as in* (6.1)*. There exists an $f \in \mathcal{C}_c^\infty(M)$ such that $A(\lambda)f > 0$.*

*Proof.* First consider $\widetilde{X}$, and let $\sigma$ be the function defined in Proposition 4.2 which restricts to a projective angular coordinate on mf. We wish to study asymptotics of solutions near the lift of $\partial M_1 \times \overline{M}_2$. Fix any $0 < \sigma_1 < \sigma_0$; by [14], there exists a function $u_\sharp$ with $(L_\sharp - \lambda)u_\sharp \in e^{-\kappa/x}\mathcal{H}^{\infty,\infty}$ and such that the leading term $h_\sharp$ in the asymptotics of $u_\sharp$ satisfies the following properties: first, $h_\sharp \geq 0$; second, $\operatorname{supp} h_\sharp$ is contained in the region where $\sigma < \sigma_0$; finally, $h_\sharp > 0$ when $\sigma < \sigma_1$. It would be straightforward to find such a function $u_\sharp$ if we were not requiring the support condition on $h_\sharp$: namely, one could simply define $u_\sharp$ as the Green function for $L_\sharp - \lambda$ with pole at $o$, multiplied by a smooth cutoff function which is identically 1 near infinity and which vanishes near $o$. However, we need to perform further modifications to obtain a function for which the support condition is satisfied. Let $u_\sharp^0$ be this cutoff Green function. We assume that the cutoff function, and hence $u_\sharp^0$, are both SO(2)-invariant. To arrange that the support condition also holds, multiply $u_\sharp^0$ by a cutoff $\chi(\sigma)$, supported in $\sigma < \sigma_0$, which is identically 1 when $\sigma \leq \sigma_1$. Then $(L_\sharp - \lambda)(\chi(\sigma)u_\sharp^0) \notin e^{-\kappa/x}\mathcal{H}^{\infty,\infty}$, but rather, equals $e^{-i\sqrt{\lambda-\lambda_0}/x}\, x^{1/2+2}\, x_\sharp\, x^\sharp\, \rho_\sharp\, \rho^\sharp G$, where $G \in \mathcal{C}^\infty(\widetilde{X})$ vanishes to infinite order at all points of $\partial\widetilde{X}$ except along $\sigma_1 \leq \sigma \leq \sigma_0$, (i.e. on $\operatorname{supp} d\chi$). We can now successively subtract off the Borel sum of an infinite number of terms of the form $e^{-i\sqrt{\lambda-\lambda_0}/x}\, x^{1/2+k}\, x_\sharp\, x^\sharp\, \rho_\sharp\, \rho^\sharp g_k(\sigma)$, $k \in \mathbb{N}$, so that $(L_\sharp - \lambda)u_\sharp \in e^{-\kappa/x}\mathcal{H}^{\infty,\infty}$. We refer to [20, Proposition 12] for an essentially identical argument in the Euclidean setting (in fact, if $u_\sharp^0$ and the correction terms are all SO(2)-invariant, then we really are in the Euclidean setting).

If $\sigma_0$ is not too large, we can regard $u_\sharp$ as not only SO(2)-invariant, but actually $K_p$-invariant on $M$. Let us transfer the function $\sigma$ on $\widetilde{X}$ to a function $\sigma_\sharp$ on $\widetilde{M}$ under this identification. Then

$$f_\sharp := (\Delta - \lambda)u_\sharp = (L_\sharp - \lambda)u_\sharp + (\Delta - L_\sharp)u_\sharp \in e^{-\kappa/x}H_{\mathrm{ee}}^{\infty,\infty}(\widetilde{M}).$$

Since $R(\lambda)$ is the unique inverse of $\Delta - \lambda$ on $L^2(M, dg)$, necessarily $R(\lambda)f_\sharp = u_\sharp$. Defining $f^\sharp$ similarly and letting $f = f_\sharp + f^\sharp$, we deduce that $A(\lambda)f = A(\lambda)f_\sharp + A(\lambda)f^\sharp > 0$ provided that the two regions $\sigma_\sharp < \sigma_1$, resp. $\sigma^\sharp < \sigma_1$, cover mf, and this can certainly be arranged.

Since $\mathcal{C}_c^\infty(M)^{K_p}$ is dense in $e^{-\kappa/x}H_{\mathrm{ee}}^{\infty,\infty}(\overline{M})^{K_p}$, and $A(\lambda)$ is continuous from this space to $\mathcal{C}^\infty(\mathrm{mf})^{K_p}$, the existence of $f \in \mathcal{C}_c^\infty(M)$ with $A(\lambda)f > 0$ then follows. □

**Corollary 6.2.** *For $\lambda < \lambda_0$, the function $g = A(\lambda)\delta_p$ is strictly positive on $\widetilde{M}$.*

*Proof.* Since $f = \delta_p > 0$ in a distributional sense and $\lambda < \lambda_0$, we can apply the minimum principle to see that $R(\lambda)f$ is also strictly positive, and thus the corresponding function $g$ is nonnegative on $\widetilde{M}$ (and strictly positive in the interior). Next, by Lemma 6.1 we can choose $f_0 \in \mathcal{C}_c^\infty(M)^{K_p}$ with $A(\lambda)f_0 = g_0 > 0$. Let $O$ be a neighborhood of $\operatorname{supp} f_0 \cup \{p\}$ with compact closure. Since both $R(\lambda)f$ and $R(\lambda)f_0$ are strictly positive in $O$, we can choose $c > 0$ so that $cR(\lambda)f > R(\lambda)f_0$ on this neighborhood. On $M \setminus O$, the function $v = cR(\lambda)f - R(\lambda)f_0$ satisfies $\Delta v = 0$, $v \geq 0$ on $\partial O$, and $v \to 0$ at infinity. Hence $v \geq 0$ on $M \setminus O$, and so finally $g \geq c^{-1}g_0 > 0$ on $\partial\widetilde{M}$ as claimed. □

Using the symmetry of the Green function when $\lambda < \lambda_0$, we have now shown that for any such $\lambda$,

$$\begin{aligned} e^{i\sqrt{\lambda-\lambda_0}/x}x^{-1/2}(x_\sharp x^\sharp \rho_\sharp \rho^\sharp)^{-1} e^{i\sqrt{\lambda-\lambda_0}/x'}(x')^{-1/2}(x'_\sharp (x^\sharp)' \rho'_\sharp (\rho^\sharp)')^{-1} R(\lambda; w', w) \\ = g(\lambda; w', w) \in \mathcal{C}^\infty((M \times \widetilde{M}) \cup (\widetilde{M} \times M) \setminus \mathrm{diag}) \end{aligned}$$

and moreover, is strictly positive there. Hence if we restrict $w$ to lie in any compact set $K$ in $M$, and let $w_j \to q \in \partial\widetilde{M}$, then $U_j(w) = R(\lambda; w', w_j)/R(\lambda; w_0, w_j)$ converges to some smooth eigenfunction $U_q(w)$ which is nontrivial since $U_q(w_0) = 1$. As indicated in the introduction, if $U_j$ is any convergent sequence of quotients of Green functions which gives rise to a point $U$ in the Martin boundary, then the underlying sequence $w_j$ has a convergent subsequence $w_{j'} \to q \in \partial\widetilde{M}$; hence $U_{j'} \to U_q$, and so the limit $U$ of this initial sequence, which is unique, must equal this plan wave solution $U_q$. This proves the first half of the

**Theorem 6.3.** *If $U_j(w) = R(\lambda; w', w_j)/R(\lambda; w_0, w_j)$ converges to an eigenfunction $U(w)$ on $M$, then necessarily $U = U_q$ for some $q \in \partial\widetilde{M}$. If $q, q'$ are any two distinct points on $\partial\widetilde{M}$, then $U_q \neq U_{q'}$. Hence $\widetilde{M}$ is naturally identified with $\overline{M}_{\mathrm{Mar}}$.*

We must also prove injectivity, i.e. show that the plane wave solutions $U_q$ are distinct for different $q \in \widetilde{M}$. Let $\pi : \widetilde{M} \to \widehat{M}$ be the projection to the geodesic compactification of $M$. We show first that if $U_q = U_{q'}$, then $\pi(q) = \pi(q')$. This follows from estimates for the growth or decay of $U_q(w)$ as $w$ converges to different points $q' \in \partial\widetilde{M}$, specifically the asymptotics given in Corollary 5.5. Thus suppose that $\pi(q) \neq \pi(q')$ and let $\gamma_q$, $\gamma_{q'}$ be the (unit-speed) geodesics emanating from $w_0$ which converge to $\pi(q)$ and $\pi(q')$, respectively, and let $\theta$ be the angle between these geodesics at $w_0$. Setting $F = \rho_\sharp \rho^\sharp$, then we have $(FU_q)(\gamma_q(t)) \geq C_\epsilon e^{(\kappa-\epsilon)t}$, whereas $(FU_q)(\gamma_{q'}(t)) \leq C_\epsilon e^{(\kappa\cos\theta+\epsilon)t}$, for any $\epsilon > 0$ (cf. the law of cosines [4, Corollary 1.4.4(2)] applied to the geodesic triangle with vertices $w_0, w, w'$, and recall that $x^{-1}$ is the smoothed distance function from $p = w_0$). Hence, assuming $\pi(q) \neq \pi(q')$, then $FU_q \neq FU_{q'}$, and so $U_q \neq U_{q'}$, and this proves that $U_q$ determines $\pi(q)$. Since $\pi$ is injective in the interior of the face mf on $\widetilde{M}$, we have now shown that the map from this portion of $\partial\widetilde{M}$ to the Martin boundary is injective.

It remains to show that $U_q$ determines $q$ when $q \in \widetilde{H_\sharp}$ or $\widetilde{H^\sharp}$, and this requires a bit more work. Suppose $q \in \widetilde{H_\sharp}$, say, with associated eigenfunction $U_q$. We shall consider sequences of points $w'_j \in M$ converging to some other point $q' \in \widetilde{H_\sharp}$, $q' \neq q$, and show that the limit of $U_q(w'_j)$ distinguishes $q'$ from $q$. It suffices to consider only the case when $\pi(q) = \pi(q')$, because otherwise the discussion above applies. Thus let $q$ and $q'$ lie in the same fiber of $\pi|_{\widetilde{H_\sharp}}$, and recall that this fiber is a copy of $\mathbb{H}^2$. In terms of the matrix representation from §2.1, matrices associated to points in the same fibre of $\pi$ share the same eigenspaces $E_{12}$, corresponding to the two eigenvalues whose ratio is bounded (as well as $E_3 = E_{12}^\perp$ corresponding to the third eigenvalue). By conjugation we normalize so that $E_{12}$, $E_3$ are the standard summands $\mathbb{R}^2 \oplus \mathbb{R}$, and we also assume that $p = w_0$ corresponds to the identity matrix. We shall be considering the limits of ratios $R(\lambda; w', w)/R(\lambda; p, w)$ as $w \to q$, $w' \to q'$, and it is sufficient to restrict attention to points $w, w' \in \widetilde{M}$ associated to matrices in this particular block diagonal form. For such points, let $\pi_{12}$ be the corresponding projection onto points in $\mathbb{H}^2$. The asymptotic behaviour of these ratios depends on the rate of convergence to $q$, resp. $q'$, as well; we consider what happens if $w$ tends to infinity much faster than $w'$. Note that this is the relevant region for the pair of points $(w', w)$ in $M \times M$, since $U_q$ is precisely the limit of this ratio if we let $w \to \infty$ and $w'$ bounded. Thus, the behaviour of this ratio in this region allows us to study the asymptotics of $U_q(w')$ as $w' \to \infty$. (In other words, we are really interested in the behaviour of the resolvent kernel on a neighbourhood of the left faces in an appropriate compactification of $M \times M$, which we do not discuss here.)

So suppose that $w$ tends to infinity much faster than $w'$, i.e. $d(p, w') \to \infty$ and $d(p, w')/d(p, w) \to 0$. Then $d(p, w)/d(w, w') \to 1$, $d(w, w') \to \infty$ by the triangle inequality. The projections $z = \pi_{12}(w)$, $z' = \pi_{12}(w')$, converge to $z_0 = \pi_{12}(q)$ and $z'_0 = \pi_{12}(q')$, respectively, and so these sequences remain a bounded distance from one another in $\mathbb{H}^2$. Then letting $w, w' \to \infty$ and using the asymptotics of the Green function with pole at $w$, we have

$$R(\lambda; w', w)/R(\lambda; p, w) \sim e^{\kappa d(p,w')}(\rho_\sharp(w')\rho^\sharp(w'))^{-1}\frac{G(z_0', z_0)}{G(o, z_0)},$$

where $G(\cdot, z_0)$ is the spherical function on $\mathbb{H}^2$ centered at $z_0$ (which is an eigenfunction of $\Delta_{\mathbb{H}^2}$ with eigenvalue $\frac{1}{4}$) and $o = \pi_{12}(p)$. This means, more precisely, that given $\epsilon > 0$ there exists $\delta > 0$ such that if $d(p, w')/d(p, w) < \delta$, $d(p, w') > \delta^{-1}$, $\pi(w)$ and $\pi(w')$ are in a $\delta$-neighborhood of $\pi(q) = \pi(q')$ with respect to some metric on $\widehat{M}$ then

$$\left| e^{-\kappa d(p,w')}\rho_\sharp(w')\rho^\sharp(w')R(\lambda; w', w)/R(\lambda; p, w) - \frac{G(z_0', z_0)}{G(o, z_0)} \right| < \epsilon.$$

Note that for $w$, $w'$ satisfying the conditions above, and in particular $d(w, w') \to \infty$, the asymptotic expression for $R(\lambda; \cdot, w)$ involves a positive multiple of $G(\cdot, z_0)$, but this factor cancels when taking the quotient. By the very definition of $U_q(w')$, this quotient is also asymptotic to $U_q(w')$, and this shows that $U_q$ determines $\frac{G(z_0', z_0)}{G(o, z_0)}$. But as a function of $z_0'$, this is rotationally symmetric about $z_0$ and hence determines the point $z_0$. In other words, $U_q$ determines $q$ when $q$ lies in the interior of $H_\sharp$.

For the last part, note that the same asymptotics are valid when $w \to q \in \mathrm{mf} \cap H_\sharp$ and $d_{\mathbb{H}^2}(o, z)$ is much larger than $d_{\mathbb{H}^2}(z', z)$. In this case, the function $G$ must be replaced by the Poisson operator (or plane wave) centered at $z_0$, as in [14, Sections 7.1 and 10]. Hence here too $U_q$ determines $q$. This completes the proof.

## 7. Spherical functions

Generalized eigenfunctions of $\Delta$ with eigenvalue $\lambda$ are distributions $u$ on $M$, of exponential growth (or equivalently, of polynomial growth with respect to the smooth structure on $\overline{M}$), such that $(\Delta - \lambda)u = 0$. Generalized eigenfunctions which are $K_p$-invariant (for any fixed $p \in M$) are called spherical functions. These were introduced by Harish-Chandra, who constructed them as a convergent series in each of the six open Weyl chambers. The most delicate point is to match these up carefully along the walls in order to extend them as a global solution, which is accomplished rather miraculously by the introduction of the $\mathfrak{c}$-function. The notable defect of this approach is that these series only converge away from the Weyl chamber walls, so the behaviour of the spherical functions across the walls must be deduced indirectly. This has been accomplished by Trombi and Varadarajan [22], as well as Anker and Ji [3], Theorem 2.2.8, by purely algebraic methods. In this section we present two methods showing how the precise structure of the resolvent proved here can be used to determine the behaviour of at least certain of these spherical functions. Both methods proceed by constructing an initial approximation to the eigenfunction and then using the resolvent to solve away the error. The first method uses an easier ansatz, but it is difficult to detect too much about the structure of the correction term; the second method, on the other hand, requires a more elaborate preparation, but produces much finer information.

Suppose that $\xi \in \mathfrak{a}^*_{\mathbb{C}}$ is of the form $\xi = c\,\xi_0$, where $\xi_0 \in (\mathfrak{a}^*)^+$, $|\xi_0| = 1$, and $c \in \mathbb{C}$. Assume also that $\xi \cdot \xi = c^2 = \lambda - \lambda_0$ for some $\lambda \in \mathbb{C} \setminus [\lambda_0, \infty)$. We can certainly take $\operatorname{Im} c > 0$, in which case $\operatorname{Im} c = \kappa = -\operatorname{Im}\sqrt{\lambda - \lambda_0}$. In particular, $\operatorname{Re}(-i\,\xi \cdot z) = \operatorname{Im} c\,\xi_0 \cdot z \le \kappa/x$. This implies that

$$u_0(z) = \rho_\sharp(z)\rho^\sharp(z)e^{-i\xi\cdot z} \in e^{\kappa/x}H^{0,r}(M)^{K_p},\ r < -1,$$

where we regard $e^{-i\xi\cdot z}$ as a function on the closure of $\mathfrak{a}^+$, and extend it to the unique $K_p$-invariant function on $M$ with this restriction to $\exp(\mathfrak{a}^+)$. Thus, with $E'$ as in (2.6), $u_0$ solves $(\Delta_{\mathrm{rad}} - E' - \lambda)u_0 = 0$ away from the walls, but because of its behavior at the walls is not smooth on $\widetilde{M}$. To correct this, fix a cutoff function $\psi \in \mathcal{C}^\infty(\tilde{\mathfrak{a}})$ which is supported in $\mathfrak{a}^+$ (hence vanishes near the walls), and is identically 1 in an open cone around $\xi_0$. (It is here where we need that $\xi_0 \in (\mathfrak{a}^*)^+$.) As above, regard $\psi u_0$ as a $K_p$-invariant function on $M$; its restriction to the flat is then

$$z \mapsto \sum_{\sigma\in S_3} \rho_\sharp(\sigma z)\rho^\sharp(\sigma z)e^{-i\xi\cdot(\sigma z)}\psi(\sigma z),\ z \in \mathfrak{a}, \tag{7.1}$$

with each term supported in a different open Weyl chamber. Because $d\psi$ vanishes in a cone around $\xi_0$,

$$(\Delta - \lambda)(\psi u_0) \in e^{\alpha/x} H^\infty_{\mathrm{ee}}(\overline{M})^{K_p}$$

for some $\alpha < \kappa$. Because of this slightly reduced rate of growth, we can solve away this error term using $R(\lambda)$, to obtain the $K_p$-invariant eigenfunction

$$U = \psi u_0 - R(\lambda)(\Delta - \lambda)(\psi u_0);$$

this is precisely the spherical functions associated to the parameter $\xi$. This gives the preliminary result:

**Proposition 7.1.** *For $p \in M$, and $\xi = c\xi_0 \in \mathfrak{a}^*_{\mathbb{C}}$ as above, with $\xi_0 \in (\mathfrak{a}^*)^+$, $\operatorname{Im} c > 0$ and $\xi\cdot\xi = \lambda - \lambda_0$, there exists a $K_p$-invariant eigenfunction $U_\xi \in \mathcal{C}^\infty(M)^{K_p}$ such that for some $\alpha < \kappa$,*

$$|U(z) - u_0(z)| \le C\rho_\sharp(z)\rho^\sharp(z)e^{\alpha|z|},\ z \in \overline{\mathfrak{a}^+}. \tag{7.2}$$

*Moreover, $U_\xi$ is the unique $K_p$-invariant eigenfunction in $\mathcal{C}^{-\infty}(\overline{M})$ such that*

$$U(z) - u_0(z) \in \rho_\sharp(z)\rho^\sharp(z)e^{\alpha|z|}L^\infty(\mathfrak{a}^+).$$

Rather than pursuing this first method further, we describe the second method, which uses ideas from many body scattering more directly. In the symmetric space setting, this is essentially equivalent to that of Trombi and Varadarajan [22], but as will be clear, is more flexible since it does not require the precise symmetric structure.

The main idea in this second approach, in analogy with three-body scattering, cf. also [8] is to find the approximation $u$ to the spherical function $U$ as a sum of reflections

$$u(z) = \rho_\sharp(z)\rho^\sharp(z)\sum_{\sigma\in S_3} c_\sigma(z,\xi)e^{-i(\sigma\xi)\cdot z}, \tag{7.3}$$

where the coefficients $c_\sigma$ will be chosen carefully, using more global considerations than in the previous method, so that we have better control on the decay of the error term $(\Delta - \lambda)u = f$. The gain over (7.1) is that $\operatorname{supp} u$ now contains the Weyl chamber walls. To obtain this, however, we must consider interactions between the terms; indeed, depending on the value of $\operatorname{Im}\xi$, at least three of the six summands are not negligible in $\mathfrak{a}^+$ in the sense that they do not lie in $e^{(-\kappa+\epsilon)/x}L^2(\mathfrak{a}^+)$ for $\epsilon > 0$ sufficiently small. Having determined $u$, the correction term $v = R(\lambda)f$ is obtained as before, and has the same type of asymptotics as the Green function (5.3). The overall decomposition $U = u + v$ is then a sum of reflected plane waves and an outgoing spherical wave.

Before proceeding with a more careful description of this construction, we fix some notation. Let $\mathfrak{w}_\sharp$ and $\mathfrak{w}^\sharp$ denote the linear subspaces of $\mathfrak{a}$ corresponding to the two principal Weyl chamber walls (where $\mathfrak{w}_\sharp$ intersects $H_\sharp$ in the closure of $\exp(\mathfrak{a}^+)$, etc.; we usually drop exp below when it is clear what we mean, and identify $\mathfrak{a}$ with $\exp(\mathfrak{a})$), and let $\sigma_\sharp$ and $\sigma^\sharp$ be the the reflections across these subspaces Next, write the linear coordinate $z$ on $\mathfrak{a}$ as $(z^\sharp, z^\sharp_\perp)$, according to the orthogonal decomposition $\mathfrak{w} = \mathfrak{w}^\sharp \oplus \mathfrak{w}^\sharp_\perp$, and analogously $z = (z_\sharp, z_\sharp^\perp)$ for $\mathfrak{a} = \mathfrak{w}_\sharp \oplus \mathfrak{w}_\sharp^\perp$. The boundary hypersurface $\mathcal{F}_\sharp$, resp. $\mathcal{F}^\sharp$, of $\overline{\mathfrak{a}}$ is the one which intersects $\overline{\mathfrak{a}^+} \cap H_\sharp$, resp. $\overline{\mathfrak{a}^+} \cap H_\sharp$, and we write $\mathcal{F}_0 = \mathcal{F}^\sharp \cap \mathcal{F}_\sharp$. We identify the interiors of $\mathcal{F}^\sharp$, resp. $\mathcal{F}_\sharp$, with $\mathfrak{w}^\sharp_\perp$, resp. $\mathfrak{w}_\sharp^\perp$.

To make $u$ $W$-invariant, we consider its behaviour in $\mathfrak{a}^+$ and near the walls separately and then show that these definitions match up. Thus, in $\mathfrak{a}^+$ we use the expression (7.3) and demand that each $c_\sigma$ extends smoothly to a neighborhood of $\mathcal{F}_0$ in $\overline{\mathfrak{a}^+}$, while near the interior of $\mathcal{F}^\sharp$ we write

$$u = \rho^\sharp \rho_\sharp \sum_{[\sigma] \in \{1, \sigma^\sharp\} \backslash S_3} c^\sharp_\sigma(z)\, e^{-i(\sigma\xi)\cdot z^\sharp}, \tag{7.4}$$

where $z \in \mathfrak{a}^+$, $|z^\sharp_\perp| \leq C$, and demand that $c'_\sigma$ is smooth up to $\mathcal{F}^\sharp$ and invariant under the reflection $\sigma^\sharp$. This sum is over cosets, so there are only three terms, but note that $(\sigma\xi) \cdot z^\sharp$ is independent of the coset representative $\sigma$. There is an analogous expression near $\mathcal{F}_\sharp$. The compatibility between these various expressions (namely (7.3), and (7.4) for both $\mathcal{F}^\sharp$ and $\mathcal{F}_\sharp$) is simply that they must agree over their common domains of definition. Any function satisfying these smoothness and compatibility conditions can be extended first to a $W$-invariant function on $\mathfrak{a}$ and then to an SO(3)-invariant smooth function on $M$.

The coefficients $c_\sigma$ will be determined inductively on the boundary faces, according to the orders of growth or decay of $e^{-i(\sigma\xi)\cdot z}$ on $\mathfrak{a}^+$, which in turn corresponds to the angles between $\sigma \operatorname{Im} \xi$ and $\mathfrak{a}^+$. To begin, we suppose that $c_1|_{\mathcal{F}_0} = 1$, and shall now prove that this determines $c_1|_{\mathcal{F}^\sharp}$, $c_{\sigma^\sharp}|_{\mathcal{F}^\sharp}$ and $c^\sharp_1|_{\mathcal{F}_\sharp} = c^\sharp_{\sigma^\sharp}|_{\mathcal{F}_\sharp}$.

The key step is to find appropriate generalized eigenfunctions of $L_\sharp$ and $L^\sharp$, which is accomplished in the following lemma. We employ the notation of Section 4, so $\mathfrak{b}$ is the two-dimensional Euclidean space, with exponentiated Weyl chamber wall $\mathfrak{b}^\sharp = \mathbb{R}^+_s \times \{1\}$; the reflection across this wall is denoted by $\tau$, and we use the decomposition $(\zeta^\sharp, \zeta^\sharp_\perp)$ for $\zeta \in \mathfrak{b}^\sharp_\mathbb{C} + (\mathfrak{b}^\sharp_\perp)_\mathbb{C}$. (Here $\mathfrak{a}$ is identified with $\mathfrak{a}^*$ via the metric.) Since the notation in that section refers to a product $M_1 \times M_2$, $F^\sharp$ is the boundary hypersurface $\{s = 0\} \times \overline{M}_2$ and $F_0$ is the corner $\{s = 0\} \times \partial\overline{M}_2$. Finally, $\pi_j$ is the projection to the factor $M_j$ in $\overline{M}_1 \times \overline{M}_2$.

**Lemma 7.2.** *Fix* $\lambda \in \mathbb{C} \setminus [\lambda_0, \infty)$ *and choose* $\zeta \in \mathfrak{b}^*_\mathbb{C}$, *where* $\operatorname{Im}\zeta \in \mathfrak{b} \setminus \mathfrak{b}^\sharp$ *and* $\zeta \cdot \zeta = \lambda - \lambda_0$. *Suppose that* $\zeta^\sharp_\perp \cdot \zeta^\sharp_\perp + \lambda^\sharp_0$ *is not a resonance of* $(\mathbb{H}^2, 3h)$ *(where* $\lambda^\sharp_0$ *is the bottom of the spectrum of* $(\mathbb{H}^2, 3h)$*). Then, given any constant* $a^0_-$, *there are functions* $u^\sharp \in \mathcal{C}^\infty(\mathbb{H}^2)^{\mathrm{SO}(2)}$, $a_\pm \in \mathcal{C}^\infty(\overline{\mathbb{H}^2})^{\mathrm{SO}(2)}$, *so that* $a_-|_{\partial\mathbb{H}^2} = a^0_-$ *and*

$$u^\sharp = a_+ \mu^{1/2} e^{-i(\tau\zeta)\cdot z^\sharp_\perp} + a_- \mu^{1/2} e^{-i\zeta\cdot z^\sharp_\perp}.$$

*The function* $u = ((\pi_2)^* u^\sharp) s e^{-i\zeta\cdot z^\sharp}$ *then solves* $(L^\sharp - \lambda)u = 0$, *and is unique amongst generalized eigenfunctions of this form.*

*Moreover, if* $\lambda^\sharp = \zeta_\perp^\sharp \cdot \zeta_\perp^\sharp + \lambda_0^\sharp$ *and* $S^\sharp$ *is the scattering matrix on* $(\mathbb{H}^2, 3h)$*, then* $a_+|_{F_0} = S^\sharp(\lambda^\sharp) a_-|_{F_0}$.

Note that the restriction of $S^\sharp$ to SO(2)-invariant functions acts by multiplication by a number; this will be important below.

*Proof.* We first produce the SO(2)-invariant eigenfunction $u^\sharp$ on $(\mathbb{H}^2, 3h)$ with eigenvalue $\lambda^\sharp$ and given datum $a_-^0$. Regarding SO(2)-invariant functions on $\mathbb{H}^2$ as functions on $\mathbb{R}^+_{z_\perp^\sharp}$, there is a unique spherical function on $\mathbb{H}^2$ with eigenvalue $\lambda^\sharp$ which has the form

$$a_+\mu^{1/2}e^{i\zeta_\perp^\sharp \cdot z_\perp^\sharp} + a_-\mu^{1/2}e^{-i\zeta_\perp^\sharp \cdot z_\perp^\sharp}$$

when $z_\perp^\sharp$ is large. Here $a_\pm \in \mathcal{C}^\infty(\overline{\mathbb{H}^2})^{\mathrm{SO}(2)}$ and $a_-$ equals the given constant $a_-^0$ on $\partial\mathbb{H}^2$. The Taylor series of the functions $a_\pm$ are uniquely determined at $\partial\mathbb{H}^2$, but this decomposition of the spherical function is not unique in the interior.

The function $u$, defined as in the statement, has all desired properties. Its uniqueness in this class is also easy to see since it is the product of two functions pulled back from $M_1$ and $M_2$, respectively, and the latter is a spherical eigenfunction of $\Delta_{\mathbb{H}^2,3h}$ with eigenvalue $\lambda^\sharp$, which is unique up to scale. □

Now let us proceed to the determination of the $c_\sigma$. First set $c_1|_{\mathcal{F}_0} = 1$, and apply the Lemma with $\mathfrak{b} = \mathfrak{a}$, $\mathfrak{b}^\sharp = \mathfrak{w}^\sharp$. Here we identify $F^\sharp$ with $\mathcal{F}^\sharp$ (extending the natural identification of both their interiors with $\mathfrak{w}_\perp^\sharp$), and also use that $\rho^\sharp\rho_\sharp = \mu^{1/2}s$ near $\mathcal{F}^\sharp$ in $\overline{\mathfrak{a}^+}$. This gives

$$c_1|_{\mathcal{F}^\sharp} = a_-|_{F^\sharp}, \quad c_{\sigma^\sharp}|_{\mathcal{F}^\sharp} = a_+|_{F^\sharp}, \qquad \text{and} \quad c_1^\sharp|_{\mathcal{F}^\sharp} = u^\sharp.$$

In the same manner, we also define $c_1|_{\mathcal{F}_\sharp}$, $c_{\sigma_\sharp}|_{\mathcal{F}_\sharp}$ and $c_\sharp^1|_{\mathcal{F}_\sharp}$. In particular, note that $c_1$ is now determined on all of $\partial\overline{\mathfrak{a}^+} = \mathcal{F}_\sharp \cup \mathcal{F}^\sharp$. Note also that the Lemma may be applied only when $\xi_\perp^\sharp \cdot \xi_\perp^\sharp + \lambda_0^\sharp$ is not a resonance of the Laplacian on $(\mathbb{H}^2, 3h)$ (and analogously for $\xi_\sharp$); this is true, for example, when $|\operatorname{Im}\xi| < (1/2\sqrt{3})$ (the factor $\sqrt{3}$ corresponds to the factor 3 in the metric $3h$).

Now take any smooth extensions $C_\sigma \in \mathcal{C}^\infty(\overline{M})^{\mathrm{SO}(3)}$ for $c_\sigma$, and $C_\sigma^\sharp$ for $c_\sigma^\sharp$, and define

$$u^{(1)} = \rho_\sharp\rho^\sharp \sum_{\sigma=1,\sigma_\sharp,\sigma^\sharp} C_\sigma(z)e^{-i(\sigma\xi)\cdot z}$$

in $\mathfrak{a}^+$ away from the walls and $u^{(1)} = C_\sigma^\sharp(z)e^{-i\xi\cdot z^\sharp}$ near $\mathfrak{w}^\sharp$, and analogously near $\mathfrak{w}_\sharp$. This is a good first step toward constructing the approximate eigenfunction since in $\mathfrak{a}^+$ (away from the walls)

$$(\Delta - \lambda)u^{(1)} = \rho_\sharp\rho^\sharp \sum_{\sigma=1,\sigma_\sharp,\sigma^\sharp} A_\sigma(z)e^{-i(\sigma\xi)\cdot z},$$

with $A_\sigma$ extending smoothly to $\overline{\mathfrak{a}^+}$ and

$$A_1|_{\mathcal{F}_\sharp\cup\mathcal{F}^\sharp} = 0, \quad A_{\sigma_\sharp}\big|_{\mathcal{F}_\sharp} = 0, \qquad \text{and} \quad A_{\sigma^\sharp}|_{\mathcal{F}^\sharp} = 0.$$

Hence $A_1 = \mu\nu B_1$, $A_{\sigma_\sharp} = \mu B_{\sigma_\sharp}$ and $A_{\sigma^\sharp} = \nu B_{\sigma^\sharp}$ where $B_\sigma$ is smooth (and in particular, bounded) on $\overline{\mathfrak{a}^+}$.

The progress at this step is that $\mu\nu e^{-i\xi\cdot z}$, $\nu e^{-i\sigma^\sharp\cdot z}$, $\mu e^{-\sigma_\sharp\cdot z}$ are all smaller than $e^{-i\xi\cdot z}$ on $\mathfrak{a}^+$. This extra decay comes not only from the factors of $\mu$ and $\nu$, but also since $\operatorname{Im}\xi \in \mathfrak{a}^+$, so the reflections $\sigma_\sharp \operatorname{Im}\xi$ and $\sigma^\sharp \operatorname{Im}\xi$ lie outside $\mathfrak{a}^+$ and make an angle strictly larger than $\pi/3$ with the walls $\mathfrak{w}^\sharp$, resp. $\mathfrak{w}_\sharp$, where the factors $\nu$, resp. $\mu$, do not vanish. This is the dissipative propagation effect.

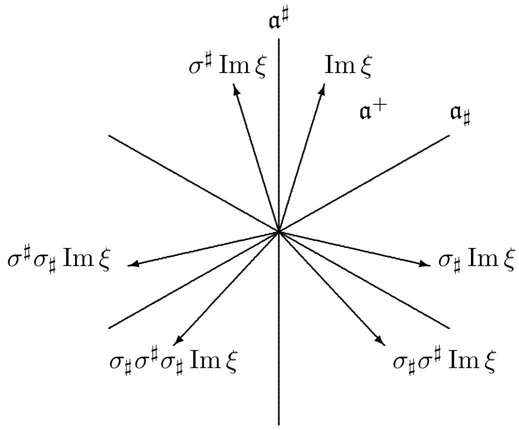


FIGURE 3. Reflections of $\operatorname{Im}\xi$ across the walls.

We now proceed to find the remaining coefficients in two further steps. First, from the value of $c_{\sigma_\sharp}$ at $\mathcal{F}^\sharp \cap \mathcal{F}_\sharp$ we use the same method to determine $c_{\sigma_\sharp}|_{\mathcal{F}^\sharp}$ and $c_{\sigma^\sharp\sigma_\sharp}|_{\mathcal{F}^\sharp}$, using $\sigma_\sharp\xi$ in place of $\xi$; $c_{\sigma^\sharp}|_{\mathcal{F}_\sharp}$, $c_{\sigma_\sharp\sigma^\sharp}|_{\mathcal{F}_\sharp}$ are determined the same way. Hence we have now found $c_1$, $c_{\sigma_\sharp}$ and $c_{\sigma^\sharp}$ on all of $\partial\overline{\mathfrak{a}^+} = \mathcal{F}_\sharp \cup \mathcal{F}^\sharp$.

Incorporating these terms into the next stage approximate eigenfunction $u^{(2)}$, then the error $(\Delta - \lambda)u^{(2)}$ is even smaller, corresponding to the dissipative propagation decay for angles larger than $2\pi/3$.

We go on to determine all the remaining terms $c_{\sigma_\sharp\sigma^\sharp}|_{\mathcal{F}^\sharp}$, $c_{\sigma^\sharp\sigma_\sharp\sigma^\sharp}|_{\mathcal{F}^\sharp}$, resp. $c_{\sigma^\sharp\sigma_\sharp}|_{\mathcal{F}_\sharp}$ and $c_{\sigma_\sharp\sigma^\sharp\sigma_\sharp}|_{\mathcal{F}_\sharp}$.

We have now defined the coefficients $c_\sigma$ at $\mathcal{F}^\sharp \cup \mathcal{F}_\sharp$. There is a compatibility issue at this last step, however, since $\sigma^\sharp\sigma_\sharp\sigma^\sharp = \sigma_\sharp\sigma^\sharp\sigma_\sharp$ in $S_3$, but there is no a priori reason why $c_{\sigma^\sharp\sigma_\sharp\sigma^\sharp}|_{\mathcal{F}^\sharp}$ and $c_{\sigma^\sharp\sigma_\sharp\sigma^\sharp}|_{\mathcal{F}_\sharp}$ should agree at $\mathcal{F}^\sharp \cap \mathcal{F}_\sharp$. To see that these do match up, note that these restrictions are both given as the product of three factors, each the S-matrix on $(\mathbb{H}^2, 3h)$ evaluated at a certain energy. Since these S-matrices are simply scalars, it suffices to check that the the energies are the same, albeit appearing in reverse order in the two terms.

The energy for the first reflection across $\mathfrak{a}^\sharp$ (leading to $c_1|_{\mathcal{F}^\sharp}$ and $c_{\sigma^\sharp}|_{\mathcal{F}^\sharp}$) is $\frac{1}{4}(\sigma^\sharp\xi - \xi)\cdot(\sigma^\sharp\xi - \xi) + \lambda_0^\sharp$; the energy corresponding to the final reflection going the other way, leading to the determination of $c_{\sigma^\sharp\sigma_\sharp}|_{\mathcal{F}_\sharp}$ and $c_{\sigma_\sharp\sigma^\sharp\sigma_\sharp}|_{\mathcal{F}_\sharp}$ from $c_{\sigma^\sharp\sigma_\sharp}|_{\mathcal{F}_0}$, is $\frac{1}{4}(\sigma_\sharp\sigma^\sharp\sigma_\sharp\xi - \sigma^\sharp\sigma_\sharp\xi)\cdot(\sigma_\sharp\sigma^\sharp\sigma_\sharp\xi - \sigma^\sharp\sigma_\sharp\xi) + \lambda_0^\sharp$. We claim that these two are equal; indeed, $\sigma_\sharp\sigma^\sharp\sigma_\sharp = \sigma^\sharp\sigma_\sharp\sigma^\sharp$, so $\sigma_\sharp\sigma^\sharp\sigma_\sharp\xi - \sigma^\sharp\sigma_\sharp\xi = \sigma^\sharp\sigma_\sharp(\sigma^\sharp\xi - \xi)$, and equality follows since reflections preserve the inner product. In a similar way one verifies that the energies corresponding to the first reflection across $\mathfrak{w}^\sharp$ and the final reflection going the other way are equal. Finally, the energies for the middle reflections are given by $\frac{1}{4}(\sigma_\sharp\sigma^\sharp\xi - \sigma^\sharp\xi)\cdot(\sigma_\sharp\sigma^\sharp\xi - \sigma^\sharp\xi) + \lambda_0^\sharp$ and by the analogous quantity with $\sigma^\sharp$ and $\sigma_\sharp$ interchanged. However, $\sigma_\sharp\sigma^\sharp\xi - \sigma^\sharp\xi = \sigma^\sharp(\sigma_\sharp\sigma^\sharp\sigma_\sharp\xi - \xi)$ and $\sigma_\sharp\sigma^\sharp\sigma_\sharp = \sigma^\sharp\sigma_\sharp\sigma^\sharp$, and the conclusion follows as before.

With these final choices of $c_\sigma$, take extensions $C_\sigma$ as above, and define

$$u = \rho_\sharp \rho^\sharp \sum_{\sigma \in S_3} C_\sigma(z) e^{-i(\sigma\xi)\cdot z}$$

in $\mathfrak{a}^+$ away from the walls, and $u = C_\sigma^\sharp(z) e^{-i\xi\cdot z^\sharp}$ near $\mathfrak{w}^\sharp$, with an analogous formula near $\mathfrak{w}_\sharp$. Thus away from the walls in $\mathfrak{a}^+$,

$$(\Delta - \lambda)u = \rho_\sharp \rho^\sharp \sum_{\sigma \in S_3} A_\sigma(z) e^{-i(\sigma\xi)\cdot z}$$

where $A_\sigma$ extends smoothly to $\overline{\mathfrak{a}^+}$ and vanishes at both $\mathcal{F}_\sharp$ and $\mathcal{F}^\sharp$, so in fact

$$(\Delta - \lambda)u = \rho_\sharp \rho^\sharp \sum_{\sigma \in S_3} \mu\nu B_\sigma(z) e^{-i(\sigma\xi)\cdot z}$$

with $B$ smooth (and hence bounded) on $\overline{\mathfrak{a}^+}$. (Note that $\mu\nu = \rho^\sharp\rho_\sharp$ in this region.) This gives $(\Delta-\lambda)u \in e^{\alpha/x} H^\infty_{\mathrm{ee}}(\overline{M})^{K_p}$ for $\alpha > |\operatorname{Im}\xi| - 1$, where we simply estimated all the exponentials by $e^{|\operatorname{Im}\xi||z|} = e^{|\operatorname{Im}\xi|/x}$.

Solving away the error term by the resolvent, i.e. by taking

$$U = u - R(\lambda)(\Delta - \lambda)u,$$

we deduce that $U = U_\xi$ has the form (7.3) (from an analytic point of view better expressed as (7.4) near the walls), plus a spherical wave of the form appearing on the right hand side of (5.3), provided $|\operatorname{Im}\xi| - 1 < -\kappa$. We have thus proved the following theorem:

**Theorem 7.3.** *If $p \in M$ and $\xi$ is chosen $\mathfrak{a}^*_{\mathbb{C}}$ with $\operatorname{Im}\xi \in (\mathfrak{a}^*)^+$, $|\operatorname{Im}\xi| < 1/(2\sqrt{3})$, $\xi\cdot\xi = \lambda - \lambda_0$, $\lambda \in \mathbb{C} \setminus [\lambda_0, \infty)$ and $|\operatorname{Im}\xi| + \kappa < 1$, $\kappa = -\operatorname{Im}\sqrt{\lambda - \lambda_0}$, then the spherical function $U = U_\xi$ can be written as $U = u + v$, $u$ a sum of plane waves associated to directions $\sigma\xi$ in the $W$-orbit of $\xi$ as in* (7.3)*, and $v$ a spherical wave of the form appearing on the right hand side of* (5.3)*.*

We have already seen one miracle in the course of proving this theorem: the last coefficient $c_\sigma$, determined from the two boundary hypersurfaces $\mathcal{F}^\sharp$ and $\mathcal{F}_\sharp$, gives consistent results at the corner. (Nothing too drastic would happen otherwise – see the remark below – but this is certainly rather nice!) This miracle could be seen and proved by our methods. The second miracle then is that in this Theorem we have $v = 0$. This follows from the results of Harish-Chandra which imply that $v = 0$ away from the walls, hence everywhere by the smoothness in (5.3), but it is by no means apparent from our approach. Indeed, with such flexible methods, one can at the very least expect a compactly supported error $(\Delta - \lambda)u$ – an even this gives rise to such a spherical wave $v$. It is quite possible, however, that there is a simple algebraic property that would prove $v = 0$ directly (by simple we mean one which does not need the full power of commuting invariant differential operators, etc); it would be very interesting to know if this is the case.

Some remarks are in order. First, Trombi and Varadarajan obtain estimates with respect to $\xi$, and also handle the cases $\operatorname{Im}\xi = 0$ and $\operatorname{Im}\xi$ large. The construction here can be modified in a straightforward way to work when $\operatorname{Im}\xi$ is large: one needs to determine more terms in the jet of $c_\sigma$ at the various faces. The only constraint is that one needs to avoid 'energies' $\lambda_0^\sharp$ in the discrete set of resonances on $\mathbb{H}^2$. Furthermore, with a finer analysis of the resolvent using the methods of [24], one would be able to extend the construction to also include the case when

$\lambda$ is real. Finally, the matching of the coefficients $c_\sigma$ in the argument above is a miracle of the precise symmetric space structure, but is only necessary if we want to prove that $u$ has the precise form (7.3). If these coefficients did not match up, one would expect that the precise singularity structure of $u$ as a Legendrian distribution would require a conic singularity in the appropriate associated Lagrangian in $T^*\mathfrak{a}$ describing this class of distributions, much as occurs in [21]. We hope to return to these issues elsewhere.

## Appendix A. Detailed description of the resolvent kernels

For convenience, we review the definition of the resolvent space from [14] which carries the Schwartz kernel of the resolvent for the operator $L_0$ considered in §4. Recall that $X = \mathbb{R}^+ \times \mathbb{H}^2$, and we sometimes write $M_1 = \mathbb{R}^+$, $M_2 = \mathbb{H}^2$, and also that $\overline{\mathbb{R}_+} = I$.

The product 0 double space $\overline{X}^2_{p0} = I^2_0 \times (\overline{\mathbb{H}^2})^2_0$ has six boundary hypersurfaces:

$$\mathrm{ff}_1 \times (\overline{M}_2)^2_0,\ \mathrm{lf}_1 \times (\overline{M}_2)^2_0,\ \mathrm{rf}_1 \times (\overline{M}_2)^2_0,\ (\overline{M}_1)^2_0 \times \mathrm{ff}_2,\ (\overline{M}_1)^2_0 \times \mathrm{lf}_2,\ (\overline{M}_1)^2_0 \times \mathrm{rf}_2;$$

here $\mathrm{ff}_j$ is the front face, and $\mathrm{lf}_j$ and $\mathrm{rf}_j$ the left and right faces of $(\overline{M}_j)^2_0$. We proceed from here by logarithmically blowing up each of the two side faces $\mathrm{lf}_j$ and $\mathrm{rf}_j$ of each factor, but *not* the front face. In other words, we just introduce the new logarithmic defining functions

$$\mathcal{R}_{\mathrm{lf}_j} = -1/\log \rho_{\mathrm{lf}_j}, \qquad \mathcal{R}_{\mathrm{rf}_j} = -1/\log \rho_{\mathrm{rf}_j}, \qquad j = 1, 2,$$

at these faces. The resulting space is denoted $(\overline{M}_j)^2_{0,\log}$. We next blow up

$$(\overline{M}_1)^2_{0,\log} \times (\mathrm{lf}_2 \cap \mathrm{rf}_2). \tag{A.1}$$

In general we would also need to blow up $(\mathrm{lf}_1 \cap \mathrm{rf}_1) \times (\overline{M}_2)^2_0$ now, but since $M_1$ is one dimensional, $\mathrm{lf}_1 \cap \mathrm{rf}_1 = \emptyset$.

The construction of the resolvent double space is completed by blowing up the collection of submanifolds covering

$$\begin{aligned}&(\mathrm{lf}_1 \times (\overline{M}_2)^2_{0,\log}) \cap ((\overline{M}_1)^2_{0,\log} \times \mathrm{lf}_2),\ (\mathrm{lf}_1 \times (\overline{M}_2)^2_{0,\log}) \cap ((\overline{M}_1)^2_{0,\log} \times \mathrm{rf}_2),\\ &(\mathrm{rf}_1 \times (\overline{M}_2)^2_{0,\log}) \cap ((\overline{M}_1)^2_{0,\log} \times \mathrm{lf}_2),\ (\mathrm{rf}_1 \times (\overline{M}_2)^2_{0,\log}) \cap ((\overline{M}_1)^2_{0,\log} \times \mathrm{rf}_2).\end{aligned} \tag{A.2}$$

These are mutually disjoint after the blow-up in the previous step, and so these final blowups may be done in any order. Note that (A.1)-(A.2) blow up *all five* codimension 2 corners coming from the intersections of the four 'side faces' of $\overline{X}^2_{p0}$: $\mathrm{lf}_1 \times (\overline{M}_2)^2_0$, $\mathrm{rf}_1 \times (\overline{M}_2)^2_0$, $(\overline{M}_1)^2_0 \times \mathrm{ff}_2$, $(\overline{M}_1)^2_0 \times \mathrm{lf}_2$, $(\overline{M}_1)^2_0 \times \mathrm{rf}_2$.

**Definition A.1.** The manifold with corners, $\overline{X}^2_{\mathrm{res}}$, obtained by the series of blow-ups of $\overline{X}^2_{p0}$ described above is called the resolvent compactification of $X^2$.

As proved in [14], the function

$$S = \frac{\delta_2(z_2, z_2')}{\delta_1(s, s')} \tag{A.3}$$

and its inverse $S^{-1}$ are smooth on those portions of $\overline{X}^2_{\mathrm{res}}$ where they are bounded. In addition,

$$\begin{aligned}\mathcal{R} = \delta(z, z')^{-1} &= \big(\delta_1(s, s')^2 + \delta_2(z_2, z_2')^2\big)^{-1/2}\\ &= \big((\mathcal{R}_{\mathrm{lf}_1}^{-1} + R_{\mathrm{rf}_1}^{-1})^2 + (\mathcal{R}_{\mathrm{lf}_2}^{-1} + \mathcal{R}_{\mathrm{rf}_2}^{-1})^2\big)^{-1/2}\end{aligned}$$

is also smooth on $\overline{X}^2_{\mathrm{res}}$.

**Lemma A.2.** *If $P \in \mathrm{Diff}^m_{p0}(\overline{X})$ then $P$ lifts to elements $P_L \in \mathrm{Diff}^m_b(\overline{X}^2_{\mathrm{res}})$ and $P_R \in \mathrm{Diff}^m_b(\overline{X}^2_{\mathrm{res}})$ from either the left or the right factors of the projection $\overline{X}\times\overline{X} \to \overline{X}$.*

Theorem 9.2 of [14] describes the structure of $(L_0-\lambda)^{-1}$:

**Theorem A.3.** *Suppose that $\lambda \in \mathbb{C} \setminus \mathrm{spec}(L_0)$. Then the Schwartz kernel of $R(\lambda) = (L_0-\lambda)^{-1}$ takes the following form:*

$$\begin{aligned} &R(\lambda) = R'(\lambda) + R''(\lambda),\ R'(\lambda) \in \Psi^{-2}_{p0}(\overline{X}),\\ &R''(\lambda) = (\rho_{\mathrm{lf},2}\rho_{\mathrm{rf},2})^{1/2}\exp(-i\sqrt{\lambda-\lambda_0}/\mathcal{R})F(\lambda),\ \lambda_0 = \frac{1}{3}, \end{aligned} \tag{A.4}$$

*$F(\lambda)$ is $\pi_R^*\Omega_{p0}\overline{X}$-valued polyhomogeneous on $\overline{X}^2_{\mathrm{res}}$, order $0$ on the lift of the two front faces $\mathrm{ff}_1 \times (\overline{M}_2)^2_{0,\log}$, $(\overline{M}_1)^2_{0,\log} \times \mathrm{ff}_2$, order $3/2$ on the lift of two of the 'side faces' $\mathrm{lf}_1 \times (\overline{M}_2)^2_0$, $\mathrm{rf}_1 \times (\overline{M}_2)^2_0$, as well as on the front face of the blow-up (A.1), and order $1/2$ on the other two 'side faces' $(\overline{M}_1)^2_0 \times \mathrm{lf}_2$, $(\overline{M}_1)^2_0 \times \mathrm{rf}_2$, as well as on the front faces of the four blow-ups (A.2). Moreover, the restriction of the leading term of $F(\lambda)$ to the boundary, i.e. its principal symbol, is*

(A.5)

$$a(\lambda,S)\left(P_1^t(\lambda_1^0(S)) \otimes P_2^t(\lambda - \frac{1}{4} - \lambda_1^0(S))\right) \text{ on the front face of the blow-up of } (\mathrm{lf}_1 \times (\overline{M}_2)^2_{0,\log}) \cap ((\overline{M}_1)^2_{0,\log} \times \mathrm{lf}_2),$$

$$a'(\lambda)\left(S_1(0) \otimes P_2^t(\lambda - \frac{1}{4})\right) \text{ on the lift of } (\overline{M}_1)^2_0 \times \mathrm{lf}_2,$$

$$a''(\lambda)\left(P_1^t(\lambda-\lambda_0) \otimes S_2(\frac{1}{12})\right) \text{ on the lift of } \mathrm{lf}_1 \times (\overline{M}_2)^2_0,$$

*with $\lambda_1^0(S)$ given by*

$$\lambda_1^0(S) = \frac{\lambda-\lambda_0}{1+S^2}, \tag{A.6}$$

*$a, a', a''$ never zero, $S$ as in (A.3).*

In fact, we are mostly interested in $R(\lambda)$ acting on SO(2)-invariant functions. Such functions can be regarded as living on

$$\exp(-\mathfrak{b}^+) = \mathbb{R}^+_s \times (0,1)_\mu,\ \mathfrak{b}^+ = \mathbb{R}_{w_1} \times (0,\infty)_{w_2},\ w_1 = -\log s,\ w_2 = -\log \mu;$$

the corresponding measure has a polar coordinate singularity at $\mu = 1$. As indicated by the notation, it is sometimes convenient to think of $\mathfrak{b}^+$ as a subset of

$$\mathfrak{b} = \mathbb{R}_{w_1} \times \mathbb{R}_{w_2},$$

though we always restrict our attention to $w_2 > 0$.

So let $\chi \in \mathcal{C}^\infty_c((0,\infty))$ be identically 1 near 1. Then the kernel of

$$(1-\chi(\mu))R(\lambda)(1-\chi(\mu))$$

can be described rather simply on a compactification of $\mathfrak{b}\times\mathfrak{b}$, supported in $\mathfrak{b}^+\times\mathfrak{b}^+$. More precisely, in complete analogy with the construction above, we can consider the resolved double space $\overline{\mathfrak{b}+}^2_{\text{res}}$ of

$$\overline{\mathfrak{b}+} = \overline{\mathbb{R}_s^+} \times [0,1)_\mu,$$

with the advantage that the left and right faces from the same factor never intersect. Then it is straightforward to describe the Schwartz kernel of $(1-\chi(\mu))R(\lambda)(1-\chi(\mu))$ as a distribution on $\overline{\mathfrak{b}+}^2_{\text{res}}$.

**Theorem A.4.** *Suppose that* $\lambda \in \mathbb{C}\backslash\operatorname{spec}(L_0)$ *and* $\chi(\mu) \in \mathcal{C}_c^\infty((0,\infty))$ *is identically* 1 *near* $\mu = 1$*. Let*

$$\overline{\mathfrak{b}+} = \overline{\mathbb{R}_s^+} \times [0,1)_\mu.$$

*Then the Schwartz kernel of* $(1-\chi(\mu))R(\lambda)(1-\chi(\mu))$ *acting on* $\mathrm{SO}(2)$*-invariant functions takes the following form:*

$$R'(\lambda) + R''(\lambda),\ R'(\lambda) \in \Psi_b^{-2}(\overline{\mathfrak{b}+}),$$

$$R''(\lambda) = \mu^{1/2}(\mu')^{-1/2}\tilde{\delta}(z,z')^{-3/2}\tilde{\delta}_2(z_2,z_2')\exp(-i\sqrt{\lambda-\lambda_0}/R)F_0(\lambda),\ \lambda_0 = \frac{1}{3},$$

$F_0(\lambda)$ *is* $\pi_R^*\Omega_b$*-valued smooth on* $\overline{\mathfrak{b}+}^2_{\text{res}}$*.*

*Proof.* This follows by integrating out the $\mathrm{SO}(2)$ variables in the preceeding theorem. The factor $(\mu')^{-1/2}$ appears as $(\mu')^{1/2}\pi_R^*\Omega_{p0} = (\mu')^{-1/2}\pi_R^*\Omega_b$. However, it is instructive to see this directly, by constructing a parametrix for the restriction of $L_0$ to $\mathrm{SO}(2)$-invariant functions, which is an element of $\mathrm{Diff}_b^2(\overline{\mathfrak{b}+})$. □

*Proof of Theorem 4.6.* In view of Lemma A.2, (4.12)-(4.14) follow immediately from Theorem A.3 and (4.15) follows from Theorem A.4. Indeed, all the derivatives in the statement of Theorem 4.6 (appearing as statements about $\mathcal{C}^k$ properties), when multiplied by a sufficiently large power of $x$, lie in $\mathrm{Diff}_{p0}(\overline{X})$. In fact, this is the very reason for treating (4.15) separately, for such a statement would not hold for derivatives along $\partial\mathbb{H}^2$ (one would need exponentially large weights), cf. the discussion before Proposition 4.8. □

## References


[1] J.-P. Anker. La forme exacte de l'estimation fondamentale de Harish-Chandra. *C. R. Acad. Sci. Paris*, Sér. I, 305:371–374, 1987.

[2] J.-P. Anker and L. Ji. Comportement exact du noyau de la chaleur et de la fonction de Green sur les espaces symétriques non-compacts. *C. R. Acad. Sci. Paris*, Sér. I, 326:153–156, 1998.

[3] J.-P. Anker and L. Ji. Heat kernel and Green function estimates on noncompact symmetric spaces. *Geom. Funct. Anal.*, 9(6):1035–1091, 1999.

[4] P. Eberlein. *Geometry of nonpositively curved manifolds*. University of Chicago Press, 1996.

[5] C. L. Epstein, R. B. Melrose, and G. A. Mendoza. Resolvent of the Laplacian on strictly pseudoconvex domains. *Acta Math.*, 167:1–106, 1991.

[6] R. Froese and I. Herbst. Patterns of exponential decay for solutions to second order elliptic equations in a sector of $\mathbf{r}^2$. *J. Analyse Math.*, 49:106–134, 1987.

[7] Y. Guivarc'h, L. Ji, and J. C. Taylor. *Compactifications of symmetric spaces*. Birkhäuser, Boston, 1998.

[8] A. Hassell. *Distorted plane waves for the* 3*-body Schrödinger operator. Trans. Amer. Math. Soc.*, 10(1):1-50 (2000).

[9] A. Hassell and A. Vasy. Intersecting Legendrians and blow-ups. *Math. Res. Letters*, 8:413–428, 2001.

[10] S. Helgason. *Groups and geometric analysis*. Academic Press, 1984.

[11] J. Jost. *Riemannian geometry and geometric analysis*. Springer, 1998.

[12] R. Mazzeo and R. B. Melrose. Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature. *J. Func. Anal.*, 75:260–310, 1987.

[13] R. Mazzeo and R. B. Melrose. Pseudodifferential operators on manifolds with fibred boundaries. *Asian Journal of Mathematics*, 2, 1998.
[14] R. Mazzeo and A. Vasy. Resolvents and Martin boundaries of product spaces. *Geom. Func. Anal.*, 12:1018–1079, 2002.
[15] R. Mazzeo and A. Vasy. Analytic continuation of the resolvent of the Laplacian on symmetric spaces of noncompact type. *J. Func. Anal.*, to appear.
[16] R. Mazzeo and A. Vasy. Analytic continuation of the resolvent of the Laplacian on SL(3)/SO(3). *Am. J. Math.*, 126:821-844, 2004.
[17] R. Mazzeo. Elliptic theory of differential edge operators. I. *Comm. Partial Differential Equations*, 16(10):1615–1664, 1991.
[18] R. B. Melrose. *Geometric scattering theory.* Cambridge University Press, 1995.
[19] R. B. Melrose. *Pseudodifferential Operators, Corners and Singular Limits* Proceedings of the International Congress of Mathematicians (Kyoto, 1990), Math. Soc. Japan 1991, 217-234.
[20] R. B. Melrose. *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces*. Marcel Dekker, 1994.
[21] R. B. Melrose and M. Zworski. *Scattering metrics and geodesic flow at infinity*, *Invent. Math.* 124(1-3):389-436 (1996).
[22] P. C. Trombi and V. S. Varadarajan. Spherical transforms of semisimple Lie groups. *Ann. of Math. (2)*, 94:246–303, 1971.
[23] B. Vaillant. Index and spectral theory for manifolds with generalized fibred cusps. *Preprint*, 2001. arXiv:math.DG/0102072.
[24] A. Vasy. Structure of the resolvent for three-body potentials. *Duke Math. J.*, 90:379–434, 1997.
[25] A. Vasy. Propagation of singularities in many-body scattering in the presence of bound states. *J. Func. Anal.*, 184:177-272 (2001).
[26] A. Vasy. Propagation of singularities in three-body scattering. *Astérisque*, 262, 2000.

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