SCATTERING THEORY ON $\mathrm{SL}(3)/\mathrm{SO}(3)$: CONNECTIONS WITH QUANTUM 3-BODY SCATTERING

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Abstract. In this paper we continue our program of extending the methods of geometric scattering theory to encompass the analysis of the Laplacian on symmetric spaces of rank greater than one and their geometric perturbations. Our goal here is to explain how analysis of the Laplacian on the globally symmetric space $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})$ is very closed related to quantum three-body scattering. In particular, we adapt geometric constructions from recent advances in that field by one of us (A.V.), as well as from our previous work [13] concerning resolvents for product spaces, to give a precise description of the resolvent and the spherical functions on this space. Amongst the many technical advantages, these methods give results which are uniform up to the walls of the Weyl chambers.

1. Introduction

It has long been observed that there are formal similarities between the spectral theory for Laplacians on (locally and globally) symmetric spaces of rank greater than one and Hamiltonians associated to quantum N-body interactions. Our contention is that these similarities have deep-seated explanations, rooted in the geometry of certain natural compactifications of the spaces involved and the asymptotic structure of these operators, and that the methods of geometric scattering theory constitute a natural set of techniques with which to study both problems. In the present paper we use these methods to provide an alternate perspective on mostly well-known results concerning the Laplacian on the globally symmetric space

$$M = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R}).$$

Besides giving a new set of methods to study scattering theory on this space which are not constrained by the algebraic rigidity and structure, this more general approach has benefits even in this classical framework. Specifically, starting from the perturbation expansion methods of Harish-Chandra, as explained in [9], and continuing through recent developments by Anker and Ji [1], [2], [3], it has always been problematic to obtain uniformity of various analytic objects near the walls of the Weyl chambers. We obtain this uniformity as a simple by-product of our method.

Let us now briefly set this work in perspective. The recent advances in quantum N-body scattering from the point of view of geometric scattering, to which we alluded above, are detailed in [20], [21] and [22], and we shall not say much more about this work here. Next, there are very many applications of geometric scattering theory to scattering on asymptotically Euclidean spaces and locally and globally symmetric spaces of rank one, [17], [8], [11], [5], to name just a very few (and concentrating on those most relevant to the present discussion). More

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recently there has been progress on geometric scattering on higher rank spaces. For example, Vaillant [19] has extended Müller's well-known L^2 index theorem for spaces with $\mathbb{Q}$-rank one ends to a general geometric setting. Most germane to the present work is [13], which contains the beginnings of a serious approach to dealing with the main technical problem of 'corners at infinity' which arise in higher rank geometry. That paper focuses on the special cases of products of hyperbolic, or more generally, asymptotically hyperbolic spaces, and produces a thorough analysis of the resolvent of the Laplacian on such spaces, including such features as its meromorphic continuation and the fine structure of its asymptotics at infinity. This analysis includes the construction of a geometric compactification of the double-space of the product space, on which the resolvent naturally lives as a particularly simple distribution, and which we call the resolvent double space. The methods of that paper rely heavily on the product structure, and an interesting representation formula for the resolvent in terms of the resolvents on the factors which is afforded by this structure. While not perhaps apparent there, the final results are in fact independent of this product structure and obtain in much more general situations.

Before embarking on a general development of the analysis of the resolvent for spaces with 'asymptotically rank two (or higher) geometry', we have thought it worthwhile to explain in detail how these methods apply to this specific example, $M = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})$, since it is a natural model space for the more general situation and is of substantial interest in and of itself. The methods here should apply more generally without any new ideas, just a bit more sweat and tears! Our aim is several-fold. At the very least we wish to emphasize the resolvent double-space, which is a compactification of $M \times M$ as a manifold with corners, and its utility for obtaining and most naturally phrasing results about the asymptotics of the resolvent; we also wish to show how the seemingly special product analysis of [13] emerges as the 'model analysis' in this non-product setting.

Let us now describe our results in more detail. Fix an invariant metric g on M, and let Δ_g and $R(\lambda) = (\Delta_g - \lambda)^{-1}$ denote its Laplacian and resolvent. We wish to examine the structure of the Schwartz kernel of $R(\lambda)$, and in particular to determine its asymptotics as the spatial variables tend to infinity in M. We do this here for λ in the resolvent set, and with additional work (which is not done here) this can also be carried out for λ approaching $\mathrm{spec}\,(\Delta_g)$. As in the traditional approach, the invariance properties of Δ allow us to reduce the analysis to that on the flats, and this turns out to be very close to three-body scattering. The central new feature of the analysis is to replace the perturbation series expansion of Harish-Chandra by L^2-based scattering theory methods in the spirit of [20]. As noted earlier, the results are easily seen to be uniform across the walls of the Weyl chambers. In future work we shall study the resolvent for spaces with 'asymptotically rank-two geometry', which is only slightly more difficult; unlike there, however, the present analysis is an explicit mixture of algebra (the reduction) and geometric scattering theory (the three-body problem).

In order to describe the structure of $R(\lambda)$, we first define a compactification $\overline{M}$ of M itself. Recall that M is identified with the set of 3-by-3 positive definite matrices of determinant 1; it is five-dimensional, and its compactification $\overline{M}$ is a $\mathcal{C}^\infty$ manifold with corners of codimension two. $\overline{M}$ has two boundary hypersurfaces, $H_\sharp$ and $H^\sharp$, in the interior of each of which the ratio of the smaller two, respectively the larger two, eigenvalues of the representing matrix is bounded. Correspondingly, either of these boundary faces is characterized by the fact that the ratio of the appropriate two eigenvalues extends to vanish on that face, hence gives a local boundary defining function. The subspace of diagonal matrices is identified with the flat $\exp(\mathfrak{a})$, and the Weyl group $W = S_3$ acts on it by permutations; its closure in $\overline{M}$ is a hexagon, the faces of which are permuted by the action of W. The fixed point sets of elements of the Weyl group partition $\mathfrak{a}$ into the Weyl chambers; the fixed point sets themselves constitute the Weyl chamber walls, and the closure of the chambers in $\overline{M}$ are the sides of the hexagon. Adjacent sides of the hexagon lie in different boundary hypersurfaces of $\overline{M}$. The boundary hypersurfaces of $\overline{M}$ are equipped with a fibration with fibers $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}) = \mathbb{H}^2$. For example, two interior points of $H^\sharp$ are in the same fiber if the sum of the eigenspaces of the two larger eigenvalues (whose ratio is, by assumption, bounded in this region) is the same. This gives $\overline{M}$ a boundary fibration structure, similar to (but more complicated than) ones considered in [17, 14, 12].

To give the reader a feeling for what it means for a function to be smooth relative to this choice of smooth structure on the compactification, consider the two-dimensional flat $\exp(\mathfrak{a})$ of diagonal matrices A as above. Let $\lambda_1, \lambda_2, \lambda_3$ be the diagonal entries (hence the eigenvalues) of A. The walls of the Weyl chambers are described by the equations $\lambda_i = \lambda_j$, $i \neq j$. We choose the positive Weyl chamber so that $\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_3} \in (0,1)$ on it. We may then use $\mu = \frac{\lambda_1}{\lambda_2}$ and $\nu = \frac{\lambda_2}{\lambda_3}$ as valid coordinates on the closure of this chamber in $\overline{M}$ away from the closure of the walls, and in particular near the corner. Thus, an SO(3)-invariant function f is smooth away from the walls if it is a $\mathcal{C}^\infty$ function of μ and ν in the positive chamber. In particular, such a function has a Taylor series expansion around the corner $\mu = \nu = 0$:

$$f(\mu,\nu) \sim \sum_{j,k=0}^{\infty} a_{jk}\mu^j\nu^k$$

where the coefficients $a_{jk} = \partial_\mu^j \partial_\nu^k f(0,0)/j!k!$ are given by the usual formula from Taylor's theorem. $\overline{M}$ is a real analytic manifold with corners and this series converges when f is real analytic, but in general, the assertion that f is smooth on $\overline{M}$ means that f has a complete asymptotic expansion. We refer to Section 2 for a detailed discussion of $\overline{M}$.

We must blow up $\overline{M}$ further in order to describe the resolvent efficiently. The motivation for this is that since the flat is a Euclidean space, its most natural compactification is the usual radial one, also known as the geodesic compactification. Unfortunately, this is not directly compatible with the structure of structure of $\overline{M}$. Denote by $\rho_\sharp$ and $\rho^\sharp$ the boundary defining functions for $H_\sharp$ and $H^\sharp$ (given in terms of ratios of eigenvalues, as above). Then we replace these with the 'slow variables' $-1/\log\rho_\sharp$ and $-1/\log\rho^\sharp$, respectively. If we use *these* as new boundary defining functions on $\overline{M}$, then we obtain a new smooth structure, containing many more 'smooth' functions. We denote the resulting space by $\overline{M}_{\log}$, and call it the logarithmic blow-up of $\overline{M}$. Every smooth function on $\overline{M}$ is smooth on $\overline{M}_{\log}$, or in other words, the map $\iota : \overline{M}_{\log} \to \overline{M}$ is $\mathcal{C}^\infty$. To see this, note that the functions $\bar{\mu} = -1/\log(\lambda_1/\lambda_2)$ and $\bar{\nu} = -1/\log(\lambda_2/\lambda_3)$ give coordinates on the closure of the positive chamber in $\overline{M}_{\log}$, and $\mu = \exp(-1/\bar{\mu})$ and $\nu = \exp(-1/\bar{\nu})$ are $\mathcal{C}^\infty$ as functions of $\bar{\mu}, \bar{\nu} \geq 0$. On the other hand, ι^{-1} is a homeomorphism, but *not* $\mathcal{C}^\infty$, since for example $-1/\log\mu$ is not a smooth function of μ at $\mu = 0$! As another example of the effect of this change of $\mathcal{C}^\infty$ structure, note that if $f, g \in \mathcal{C}^\infty(\overline{M})$ and $f = g$ on $\partial\overline{M}$ then $f - g$ vanishes to all orders at $\partial(\overline{M}_{\log})$, i.e. its complete Taylor series in $(\bar\mu, \bar\nu)$ vanishes at the corner of $\overline{M}_{\log}$, but this certainly need not be the case for the expansion in terms of (μ, ν).

After the logarithmic blow-up, we perform the normal (spherical) blow-up of the corner $H_\sharp \cap H^\sharp$ in the space $\overline{M}_{\log}$. This sequence of operations results in the final 'single space'

$$\widetilde{M} = [\overline{M}_{\log}; H_\sharp \cap H^\sharp] \tag{1.1}$$

To check that this fulfills the goal of being compatible with the radial compactification of the flat, note that if r is a Euclidean radial variable on $\mathfrak{a}$ (outside a compact set), then its inverse $r^{-1} \equiv x \in \mathcal{C}^\infty(\widetilde{M})$ is a total boundary defining function of $\widetilde{M}$, i.e. vanishes simply on all faces. Note that, with the previous notation, r is a constant multiple of $\big((\log\lambda_1)^2 + (\log\lambda_2)^2 + (\log\lambda_3)^2\big)^{1/2}$, which may be easily expressed in terms of $\bar\mu$ and $\bar\nu$, if we also take advantage of the determinant condition $\lambda_1\lambda_2\lambda_3 = 1$. We also let $x_\sharp$, resp. $x^\sharp$, be defining functions of the lifts of $H_\sharp$ and $H^\sharp$ in $\widetilde{M}$ (so these are comparable to $-1/\log\rho_\sharp$, $-1/\log\rho^\sharp$, respectively, in the interiors of these faces). Finally, we denote by mf the new 'front face' created in this blow-up.

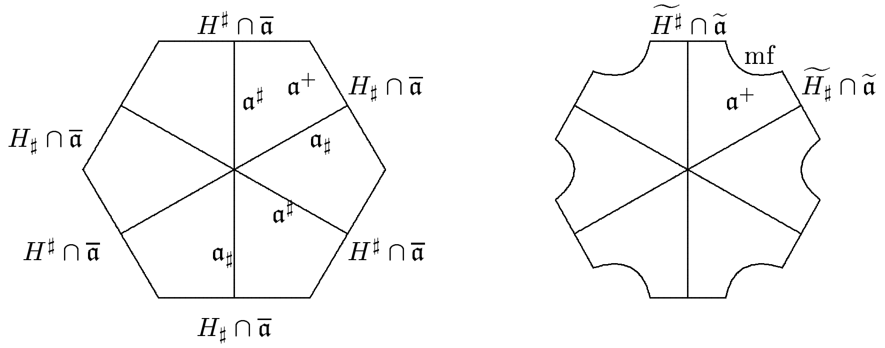

FIGURE 1. The closure of $\mathfrak{a}$ in the compactifications $\bar{M}$ and $\widetilde{M}$ of M.

A standard preliminary result in scattering theory concerns the far-field behaviour of $R(\lambda)f$ where $f \in \mathcal{C}_c^\infty(M)$, initially when λ lies in the resolvent set, and later when it approaches the spectrum, cf. [15]. The geometry of $\tilde{M}$ has been set up precisely so that the analogous result here has a fairly simple form:

Theorem. *(See Corollary 5.3.) Suppose $f \in \mathcal{C}_c^\infty(M)$, and $\lambda \notin \operatorname{spec}(\Delta)$. Then, with $\lambda_0 = 1/3$,*

$$R(\lambda)f = \rho_\sharp\rho^\sharp x^{1/2} x_\sharp x^\sharp \exp\left(-i\sqrt{\lambda - \lambda_0}/x\right) g,$$

where $g \in \mathcal{C}^\infty(\widetilde{M})$. The square root in the exponential is the one having negative imaginary part in the resolvent set $\lambda \in \mathbb{C} \setminus [\lambda_0, \infty)$. If $f \in \mathcal{C}_c^{-\infty}(M)$, then a similar statement holds (away from sing supp f*).*

Remark 1.1. As already indicated, there is an extension of this result which describes the behavior as $\lambda \to \operatorname{spec}(\Delta)$. Thus, taking the limit as λ approaches the spectrum from below,

$$R(\lambda - i0)f = \rho_\sharp \rho^\sharp x^{1/2} e^{-i\sqrt{\lambda-\lambda_0}/x} g_-, \qquad g_- \in \mathcal{C}^\infty(\widetilde{M}),$$

and there is an analogous formula for $R(\lambda + i0)f$, $\lambda > \lambda_0$, involving some function $g_+ \in \mathcal{C}^\infty(\widetilde{M})$. As before, the fact that $g_\pm \in \mathcal{C}^\infty(\widetilde{M})$ means that $R(\lambda - i0)f$ has a complete expansion (in terms of the appropriate defining functions). This extension allows one to analyze the range of the map $\mathcal{C}_c^\infty(M) \ni f \mapsto g_\pm|_{\partial\widetilde{M}}$. For example, one can show that the range of this map is dense in $\mathcal{C}^\infty(\mathrm{mf})$, but for simplicity we shall not discuss this here since, just as in the three-body setting, it requires a more elaborate phase space analysis.

Remark 1.2. Notice that the functions $r^{-k} = x^k$, $k \in \mathbb{N}$ are smooth on $\widetilde{M}$, but not on $\overline{M}$. On the other hand, later we shall briefly discuss Harish-Chandra's spherical functions, and the coefficients of the six oscillatory terms appearing there are actually in $\mathcal{C}^\infty(\overline{M})$. In fact, the expansions for these functions originally constructed by Harish-Chandra converge near the corners of $\overline{\mathfrak{a}}$.

One of the corollaries of this theorem is the identification of the Martin compactification of M with $\widetilde{M}$. Recall that the Martin boundary of M, relative to some eigenvalue $\lambda \in \mathbb{R}$, $\lambda < \lambda_0$, is the set of equivalence classes of sequences of eigenfunctions of the form $U_j := R(\lambda; w, w_j)/R(\lambda; w_0, w_j)$. Here $R(\lambda; w, w')$ is the Schwartz kernel of $R(\lambda)$ at $(w, w') \in M \times M$ and w_j is some sequence of points tending to infinity. The functions U_j satisfy $U_j(w_0) = 1$ and $(\Delta - \lambda)U_j = 0$ on $M \setminus \{w_j\}$, so by standard elliptic theory at least some subsequence tends to limit which is a nontrivial eigenfunction; two sequences are said to be equivalent if the limiting eigenfunctions are the same. A sufficiently fine description of the asymptotics of the resolvent will determine exactly when and how the sequences U_j can converge. To say that the Martin compactification is identified with $\widetilde{M}$ means specifically that equivalence classes of sequences are in one-to-one correspondence with points $q \in \partial\widetilde{M}$. In other words, for any such q, if w_j is any sequence converging to q, then U_j necessarily converges to an eigenfunction U_q which depends only on q, and moreover, $U_q \neq U_{q'}$ if $q \neq q'$. These eigenfunctions might be called the plane wave solutions for $\Delta - \lambda$ on M. In any case, this will prove the

Theorem. *For any $\lambda < \lambda_0$, this procedure described above gives a natural isomorphism of the Martin compactification $\overline{M}_{\mathrm{Mar}}(\lambda)$ with $\widetilde{M}$.*

A key step in this identification is to understand the leading coefficients in the expansions of $R(\lambda)f$, i.e. the values of the corresponding function g, at the various boundary faces of $\widetilde{M}$ for any $f \in \mathcal{C}_c^{-\infty}(M)$, since in particular $R(\lambda; \cdot, w_j) = R(\lambda)\delta_{w_j}$. While we do not obtain an explicit formula for these leading coefficients, we can at least describe the range of the map sending f to these boundary data. This is somewhat simpler than the corresponding statements in the on-spectrum case, to which we alluded above; thus in Section 6 we show that this map has dense range when f varies over $\mathcal{C}_c^\infty(M)$, and letting f vary over a slightly larger space, then this map is surjective onto an appropriate space of $\mathcal{C}^\infty$ functions on $\partial\widetilde{M}$. This is also of independent interest.

The identification of the Martin boundary for $M = \mathrm{SL}(3)/\mathrm{SO}(3)$ was initially due to Guivarc'h, Ji and Taylor [7], although the smooth structure of $\widetilde{M}$ plays no role there. Their arguments rely on certain estimates due to Anker and Ji [1, 2] which control the behaviour of the resolvent kernel at the Weyl chamber walls. In fact, the estimates of [2], see also [7, Section 8.10], when λ is real and in the resolvent set, amount to upper and lower bounds for $R(\lambda)f$ by expressions of the same form as in our theorem. In later work, Anker and Ji use algebraic methods to give a uniform description of the leading term of the asymptotics. On the other hand, our analytic approach automatically gives uniform asymptotics, and this leads directly to this theorem about the Martin compactification, just as in our previous work [13].

There is yet another approach, due to Trombi and Varadarajan [18], which is intermediate between our approach and that of Harish-Chandra. They construct spherical functions as sums of polyhomogeneous conormal functions on $\overline{M}$ by constructing their Taylor series at all boundary hypersurfaces of $\overline{M}$. By comparison, Harish-Chandra's method amounts to constructing the spherical functions in Taylor series at the corner of $\overline{M}$. Owing to the algebraic nature of the space, these Taylor series actually converge in the appropriate regions, but of course this does not hold in more geometric settings.

As another application of our resolvent estimates, we also take up the construction of the spherical functions. This construction is essentially just that of Trombi and Varadarajan, but instead of appealing to convergence of the Taylor series, we use the resolvent to remove the error term, and this results in an additional term with the same asymptotics as the Green function. However, since the Taylor series actually converges, the error term vanishes, and hence the Green function asymptotics do not appear in the asymptotics of the spherical function; this is the extent to which algebra enters into our analysis.

To set this discussion into the language of Euclidean scattering, and in particular to compare with the language of three-body scattering, the spherical functions on M are analogues of (reflected) 'plane waves' on the flats, corresponding to colliding particles, although here the eigenvalues collide; on the other hand, the Green function for Δ on M is the analogue of a 'spherical wave' in Euclidean scattering. The conflict of terminology is somewhat unfortunate.

Overview of the parametrix construction

We now sketch in outline some details of our methods and constructions. The goal is to construct the Schwartz kernel of the resolvent as a distribution with quite explicit singular structure on some compactification of $\overline{M}^2$. This is accomplished by constructing a sequence of successively finer approximations to $(\Delta-\lambda)^{-1}$, where 'fineness' is measured by the extent to which these operators map into spaces with better regularity and decay at infinity. These parametrices lie in certain 'calculi' of pseudodifferential operators which are defined by fixing the possible singular structures of the Schwartz kernels of their elements both at the diagonal, but more interestingly, near the boundary of $\overline{M}^2$.

The first step is the construction of a parametrix in the 'small calculus', which we also call the edge-to-edge calculus, of pseudodifferential operators on M. In fact, this is defined on any manifold with corners up to codimension two which has fibrations on its boundary faces analogous to those of $\mathrm{SL}(3)/\mathrm{SO}(3)$. A more general development of this calculus will appear elsewhere. The parametrix $G(\lambda)$ for $\Delta-\lambda$ in this calculus has the property that the error $E(\lambda)=G(\lambda)(\Delta-\lambda)-\mathrm{Id}$ is smoothing but does not increase the decay rate of functions, hence is not compact on $L^2(M,dV_g)$. The constructions within this small calculus are merely a systematic way of organizing the local elliptic parametrix construction uniformly to infinity, and this parametrix gives scale-invariant estimates uniform to ∂M. To amplify on this last statement, we may use this calculus to define the Sobolev spaces $H^m_{\mathrm{ee}}(\overline{M})=\{u\in L^2(M,dV_g):\ \Delta^{m/2}u\in L^2(M,dV_g)\}$, $m\geq 0$. These spaces reflect the basic scaling structure of $\overline{M}$ near its boundaries.

At this point we use the group structure to simplify matters by effectively reducing to the flat. For $p\in M$, let K_p denote the subgroup of SL(3) fixing this point; we may as well assume that p is the identity matrix, which identifies the subspace of K_p-invariant functions with the space of Weyl group invariant functions on the flat $\exp(\mathfrak{a})$ (or equivalently, of functions on diagonal matrices invariant under permutation of the diagonal entries). Since the Green function (for $\Delta-\lambda$) with pole at p is K_p-invariant, we may as well consider only parametrices which respect this structure. This reduction is certainly helpful, but not essential; it is the key point where our restriction to the actual symmetric space makes a difference in terms of simplifying the presentation.

Denote by $H^m_{\mathrm{ee}}(\overline{M})^{K_p}$ the invariant elements of the Sobolev space $H^m_{\mathrm{ee}}(\overline{M})$. The initial parametrix $G(\lambda)$ constructed in the first step may not preserve K_p invariance, but this is easily remedied by averaging it over K_p; this produces an operator $G_p(\lambda)$ which satisfies

$$\tilde{E}(\lambda)=G_p(\lambda)(\Delta-\lambda)-\mathrm{Id}:L^2(M,dV_g)^{K_p}\longrightarrow H^m_{\mathrm{ee}}(\overline{M})^{K_p}\qquad\text{for all } m.$$

A second step is needed to get a parametrix with a decaying (as well as smoothing) error term. Thus we wish to construct another parametrix $\tilde{R}(\lambda)$ for $\Delta-\lambda$, $\lambda\notin\mathrm{spec}(\Delta)$, which acts on these K_p-invariant function spaces and satisfies

$$\tilde{R}(\lambda)(\Delta-\lambda)-\mathrm{Id}:H^m_{\mathrm{ee}}(\overline{M})^{K_p}\longrightarrow x^sH^m_{\mathrm{ee}}(\overline{M})^{K_p}\qquad\text{for all } s.$$

Granting this for a moment, we combine these two operators to get

$$\left(G_p(\lambda)+\tilde{E}(\lambda)\tilde{R}(\lambda)\right)(\Delta-\lambda)-\mathrm{Id}:L^2(M,dV_g)^{K_p}\to x^sH^m_{\mathrm{ee}}(\overline{M})^{K_p}$$

for any $s>0$. This error term is now compact on $L^2(M,dV_g)^{K_p}$, and so using the simplest spectral properties of Δ, we may remove it and obtain an inverse to $\Delta-\lambda$ acting on K_p-invariant functions. Since, as remarked before, $(\Delta-\lambda)^{-1}$ is necessarily K_p-invariant, we have captured the full resolvent.

The main subtleties in this paper center on the construction of $\tilde{R}(\lambda)$, to which we now turn. We return to the single space $\widetilde{M}$. Denote by $\widetilde{H_\sharp}$ and $\widetilde{H^\sharp}$ the lifts of the boundary faces $H_\sharp$ and $H^\sharp$ from $\overline{M}$ to $\widetilde{M}$, and let $\mathfrak{a}_\sharp$ and $\mathfrak{a}^\sharp$ be the Weyl chamber walls intersecting these faces, respectively. Choose a $\mathcal{C}^\infty(\widetilde{M})^{K_p}$ partition of unity, $\phi_\sharp+\phi^\sharp+\phi_0=1$ on $\widetilde{M}$ such that $\operatorname{supp}\phi_\sharp$ is disjoint from $\widetilde{H^\sharp}\cap\mathfrak{a}^\sharp$ and $\operatorname{supp}\phi^\sharp$ is disjoint from $\widetilde{H_\sharp}\cap\mathfrak{a}_\sharp$, and with $\phi_0\in\mathcal{C}^\infty_c(M)$. (These can be constructed on the closure of $\widetilde{\mathfrak{a}}$ and extended to K_p-invariant functions on $\widetilde{M}$.) Let $\psi_\sharp$, $\psi^\sharp$ be K_p-invariant cutoffs which are identically 1 on $\operatorname{supp}\phi_\sharp$ and $\operatorname{supp}\phi^\sharp$, respectively, and which vanish on $\widetilde{H^\sharp}\cap\mathfrak{a}^\sharp$ and $\widetilde{H_\sharp}\cap\mathfrak{a}_\sharp$, respectively.

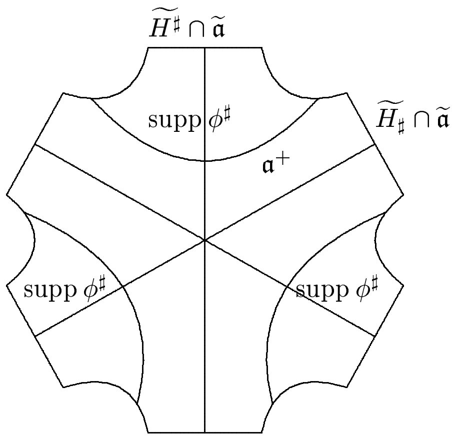

FIGURE 2. The intersection of supp $\phi^\sharp$ with the flat $\widetilde{\mathfrak{a}}$.

Along $\widetilde{H^\sharp}$, Δ is well approximated by the product operator $L^\sharp = \frac{3}{4}(sD_s)^2 + i\frac{3}{2}(sD_s) + \Delta_{\mathbb{H}^2}$, where $s = \rho^\sharp$ and $\Delta_{\mathbb{H}^2}$ is the Laplacian on the fiber $\mathbb{H}^2$ of $\widetilde{H^\sharp}$. More precisely,

$$\Delta - L^\sharp : H^m_{\mathrm{ee}}(\overline{M})^{K_p} \longrightarrow \rho^\sharp H^{m-1}_{\mathrm{ee}}(\overline{M}).$$

There is an analogous product operator $L_\sharp$ which approximates Δ near $\widetilde{H_\sharp}$. One small complication is that because of its structure at $\mathfrak{a}_\sharp$, $L^\sharp$ does not preserve K_p-invariance, but this is not serious since $\psi^\sharp L^\sharp \phi^\sharp$ does preserve this invariance. The fact that we can approximate Δ by product operators is the big gain, and is one of the remarkable things accomplished by passing to $\widetilde{M}$, for in fact the representation formula for the resolvent on a product space from [13] gives a precise description of $(L^\sharp - \lambda)^{-1}$ and $(L_\sharp - \lambda)^{-1}$. We use these as local models for the structure of the improved parametrix, and put

$$\tilde{R}(\lambda) = \psi^\sharp (L^\sharp - \lambda)^{-1} \phi^\sharp + \psi_\sharp (L_\sharp - \lambda)^{-1} \phi_\sharp.$$

It is not difficult to see that this has all the desired properties.

As noted earlier in this introduction, this construction can be extended to let λ approach the spectrum, so as to obtain the structure of the limiting values of the resolvent $R(\lambda \pm i0) = (\Delta - (\lambda \pm i0))^{-1}$. This is not done here for reasons of brevity only. We comment briefly, in this paragraph and the next, on some consequences of this extension. The issue is that we must keep track of the 'propagation of singularities' along $\partial\widetilde{\exp(\mathfrak{a})}$, as in in three-body scattering [20]. In this context, the term 'singularity' refers to a microlocal description of the lack of rapid decay at infinity. An explicit iteration allows us to construct successively finer parametrices, leaving error terms which map $L^2(M, dV_g)^{K_p} \to x^k H^m_{\mathrm{ee}}(\bar{M})^{K_p}$ for higher and higher values of k and m. The terms in this iterative series correspond to singularities reflecting from the walls at infinity. In fact, just as in three-body scattering, there are only finitely many (in fact three) such reflections.

The spherical functions centered at p are also K_p invariant, and are parametrized by incoming directions ξ, $|\xi|^2 = \lambda - \lambda_0$. These are constructed as perturbations of the plane waves

$$u_\xi(z) = \rho_\sharp(z)\rho^\sharp(z)e^{-i\xi\cdot z}$$

on $\mathfrak{a}$ (which we identify with $\exp(\mathfrak{a})$), where z a Euclidian variable. In fact, on $\widetilde{M}$, $(\Delta-\lambda)u_\xi$ decreases rapidly away from $\widetilde{H^\sharp}$ and $\widetilde{H_\sharp}$. More importantly, if $\xi \notin \mathfrak{a}_\sharp \cap \mathfrak{a}^\sharp$, then $(\Delta - \lambda)u_\xi$ is nowhere incoming, in the sense of the scattering wave front set, so that $R(\lambda+i0)$ can be applied to it. The detailed structure of $R(\lambda+i0)$ discussed above leads to reflected plane waves. There are six such terms, corresponding to the six elements of the Weyl group. These correspond precisely to the six terms in Harish-Chandra's construction of spherical functions. The coefficients of the leading terms, which are Harish-Chandra's $\mathfrak{c}$-function, correspond to the scattering matrices of the 'two-body problems', in this case the scattering matrices on $\mathbb{H}^2$. As before, this analysis would allow us to let ξ approach the walls. Certain aspects of this still appear in the construction of off-spectrum spherical functions in §6 below.

The remainder of this paper is organized as follows. In §2 we review the geometry of $M = \mathrm{SL}(3)/\mathrm{SO}(3)$ and its compactification $\overline{M}$. The small calculus of pseudodifferential operators on M is defined in §3 through the properties of the Schwartz kernels of its elements on the resolvent compactification of $M \times M$. This leads to the first parametrix for $\Delta - \lambda$, which captures the diagonal singularity of $R(\lambda)$ uniformly to infinity. In §4, we discuss a model problem on $\mathbb{R} \times \mathbb{H}^2$, which is used for the construction of the finer parametrix in §5. In §6 we consider spherical functions, and discuss the extent to which algebra plays a role in their asymptotics. The Appendix contains a summary of results from [13] concerning resolvents for product problems.

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2. Geometric preliminaries

We refer to [4], [9] and [10] for a nice general discussion of non-compact symmetric spaces, though all the essential ingredients are discussed below.

2.1. **Geometry.** Any element $A \in \mathrm{SL}(3)$ admits a unique polar decomposition $A = VR$, where $V = (AA^t)^{1/2}$ is positive definite and symmetric, with $\det V = 1$, and $R \in \mathrm{SO}(3)$. This yields the standard identification of the symmetric space $\mathrm{SL}(3)/\mathrm{SO}(3)$ with the set M of positive definite 3-by-3 matrices of determinant 1, where

$$\mathrm{SL}(3)/SO(3) \ni A \cdot \mathrm{SO}(3) \longmapsto (AA^t)^{1/2} \in M.$$

We write $[A]$ for the image of A in M. The action of $\mathrm{SL}(3)$ on M is described by

$$\mathrm{SL}(3) \times M \ni (B, V) \longmapsto \phi_B(V) = (BV^2B^t)^{1/2}.$$

Thus $(\phi_B)_*$ identifies $T_{\mathrm{Id}}M$ with $T_{[B]}M$. $T_{\mathrm{Id}}M$ consists of symmetric matrices of trace 0, and we shall use the Killing form $g(A, A) = 6\,\mathrm{Tr}(AA^t) = 6\sum_{ij} a_{ij}^2$ as the metric on this subspace. This induces a unique invariant Riemannian metric g on all of M in the usual way.

We need to analyze the structure of M at infinity, and so we proceed as follows. This space is almost a product, except along some lower dimensional strata. To understand this, diagonalize any $B \neq \mathrm{Id}$, i.e. write $B = O\Lambda O^t$, where Λ is diagonal with positive entries and $O \in \mathrm{SO}(3)$. We write $\mathfrak{a}$ for the subspace of diagonal matrices of trace zero, so that $\exp(\mathfrak{a}) = M \cap \mathrm{Diag}_3$ is the space of diagonal matrices with positive entries. Neither O nor Λ are uniquely determined here since the ordering of the entries of Λ is not fixed. Let $\mathcal{P}$ denote the subgroup of all signed permutation matrices in $\mathrm{SO}(3)$, i.e. orthogonal matrices of determinant 1 which preserve the subspace decomposition $\mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ rather than just permuting basis elements. If $P \in \mathcal{P}$ then $P\Lambda P^t$ is again diagonal with positive entries. Conversely, Λ and O determined up to the action of $P \in \mathcal{P}$. Here $\mathcal{P}$ acts on Λ by conjugation and on $\mathrm{SO}(3)$ by right multiplication by the inverse: $O\Lambda O^t = (OP^t)(P\Lambda P^t)(OP^t)^t$. Let $(\exp(\mathfrak{a}))^*$ be the subset of $\exp(\mathfrak{a})$ where the diagonal entries are all distinct. The action on $\mathrm{SO}(3)$, hence on the product $(\exp(\mathfrak{a}))^* \times \mathrm{SO}(3)$ is free, so $((\exp(\mathfrak{a}))^* \times \mathrm{SO}(3))/\mathcal{P}$ may be identified with $M \setminus \mathcal{C}$, where $\mathcal{C}$ is the subset of elements $[A] \in M$ where at least two of the eigenvalues coincide.

Let $\mathcal{P}'$ be the subgroup of $\mathcal{P}$ that preserves each summand $\mathbb{R}$ in $\mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$. If $P \in \mathcal{P}'$ then $P\Lambda P^t = \Lambda$, i.e. the true indeterminacy in the choice of Λ is measured by the Weyl group $W = \mathcal{P}/\mathcal{P}'$, which in this case is just the full symmetric group S_3. (Note that $|\mathcal{P}| = 24$, and $|\mathcal{P}'| = 4$). Again in analogy with three-body scattering, we think of the walls $\exp(\mathfrak{a}) \setminus (\exp(\mathfrak{a}))^*$ as 'collision planes', where two eigenvalues are equal, and which we identify with the subset $\mathfrak{w} \subset \mathfrak{a}$. There are three collision planes, and these divide $\mathfrak{a}$ into six chambers.

We next examine the structure of M near the Weyl chamber walls. Write the eigenvalues of $B \in M$, i.e. the diagonal entries of Λ in the decomposition for B above, as λ_1, λ_2 and λ_3. Suppose that B lies in a small neighbourhood $\mathcal{U}$ where

$$c < \lambda_1/\lambda_2 < c^{-1}, \quad \lambda_3 > 1/c,$$

for some fixed $c \in (0,1)$. Recall also that $\lambda_3 = 1/\lambda_1\lambda_2$. These inequalities imply that $\lambda_1 = (\lambda_1/\lambda_2)^{1/2}\lambda_3^{-1/2} < 1$ and $\lambda_2 = (\lambda_2/\lambda_1)^{1/2}\lambda_3^{-1/2} < 1$, and $\lambda_3 > 1$ in $\mathcal{U}$. Hence there is a well-defined decomposition $\mathbb{R}^3 = E_{12} \oplus E_3$ for any $B \in \mathcal{U}$, where E_{12} is the sum of the first two eigenspaces and E_3 is the eigenspace corresponding to λ_3, regardless of whether or not λ_1 and λ_2 coincide. We could write equivalently $B = OCO^t$, where C is block-diagonal, preserving the splitting of $\mathbb{R}^3$. The ambiguity in this factorization is that C can be conjugated by an element of $\mathrm{O}(2)$ (acting in the upper left block), and $\mathrm{O}(2)$ can be included in the top left corner of $\mathrm{SO}(3)$ (the bottom right entry being set equal to ± 1 appropriately). Let C' denote the upper-left block of C; the bottom right entry of C is just λ_3, and so $\lambda_3 \det C' = 1$. In other words, $C' = \lambda_3^{-1/2} C''$ where C'' is positive definite and symmetric with determinant 1, hence represents an element of $\mathrm{SL}(2)/\mathrm{SO}(2) \equiv \mathbb{H}^2$. Hence for an appropriate neighbourhood $\mathcal{V}$ of $[\mathrm{Id}]$ in $\mathrm{SL}(2)/\mathrm{SO}(2)$, the neighbourhood $\mathcal{U}$ is identified with $(\mathcal{V} \times \mathrm{SO}(3))/\mathrm{O}(2) \times (1/c, \infty)$. Note that although the action of $\mathrm{O}(2)$ on $\mathcal{V}$ has a fixed point (namely $[\mathrm{Id}]$), its action on $\mathrm{SO}(3)$, and hence on the product, is free. The neighbourhood $\mathcal{V}$ can be chosen larger when λ_3 is larger, and the limiting 'cross-section' $\lambda_3 = C$ has the form $(\mathbb{H}^2 \times \mathrm{SO}(3))/\mathrm{O}(2)$. This space is a fibre bundle over $\mathrm{SO}(3)/\mathrm{O}(2)$ $(= \mathbb{R}P^2)$ with fibre $\mathbb{H}^2$. Notice that the Weyl chamber wall corresponds to the origin (i.e. the point fixed by the SO(2) action) in $\mathbb{H}^2$.

2.2. **Coordinates and metric.** We now discuss some useful coordinate systems on $M \setminus \mathcal{C}$ using local coordinates on $\exp(\mathfrak{a})$ and SO(3). It is sufficient to work near a diagonal element $B \in M$, and we write the diagonal elements of elements in $\exp(\mathfrak{a})$ (near B) as $\lambda_1, \lambda_2, \lambda_3$ (in this order). Of course, these functions are not independent since their product equals 1. More general points in M close to B have SO(3) factor in their polar decomposition close to Id, and so we augment these diagonal entries with the matrix entries c_{12}, c_{13} and c_{23} which parametrize the space of skew-symmetric matrices $\mathfrak{so}(3) = T_{\mathrm{Id}}\,\mathrm{SO}(3)$. Then a straightforward calculation shows that at B,

$$\begin{gathered} g = 6\left(\frac{d\lambda_1^2}{\lambda_1^2} + \frac{d\lambda_2^2}{\lambda_2^2} + \frac{d\lambda_3^2}{\lambda_3^2}\right) \\ +3\left(\frac{\lambda_1}{\lambda_2} - \frac{\lambda_2}{\lambda_1}\right)^2 dc_{12}^2 + 3\left(\frac{\lambda_1}{\lambda_3} - \frac{\lambda_3}{\lambda_1}\right)^2 dc_{13}^2 + 3\left(\frac{\lambda_2}{\lambda_3} - \frac{\lambda_3}{\lambda_2}\right)^2 dc_{23}^2 \end{gathered}$$

Of course, a true coordinate system is obtained by fixing two independent functions of the λ_j as coordinates on $\exp(\mathfrak{a})$. For reasons that will become clear later, an advantageous choice is $\mu = \lambda_1/\lambda_2$ and $s = \lambda_3^{-3/2}$. We have

$$\lambda_1 = \mu^{1/2}\lambda_3^{-1/2} = \mu^{1/2}s^{1/3},\ \lambda_2 = \mu^{-1/2}\lambda_3^{-1/2} = \mu^{1/2}s^{1/3},$$

and hence

$$\lambda_2/\lambda_3 = \mu^{-1/2}s,\ \lambda_1/\lambda_3 = \mu^{1/2}s.$$

This gives

$$\begin{aligned} g =& 3(d\mu/\mu)^2 + 4(ds/s)^2 + 3(\mu - \mu^{-1})^2\, dc_{12}^2 \\ &+3s^{-2}(\mu^{1/2}s^2 - \mu^{-1/2})^2\, dc_{13}^2 + 3s^{-2}(\mu^{-1/2}s^2 - \mu^{1/2})^2\, dc_{23}^2. \end{aligned} \tag{2.1}$$

Notice that while this expression for the metric has coordinate singularities at the the Weyl chamber wall $\mu = 1$, g is necessarily smooth across $\mathfrak{w}$. In fact, this is just a polar coordinate singularity. To see this, write $\mu = e^r$, so $\mu = 1$ corresponds to $r = 0$ and

$$\begin{aligned} g =& 3\left(dr^2 + \sinh^2 r\, d\bar{c}_{12}^2\right) \\ &+\frac{4}{s^2}\left(ds^2 + \frac{3}{4}(e^{r/2}s^2 - e^{-r/2})^2\, dc_{13}^2 + \frac{3}{4}(e^{-r/2}s^2 - e^{r/2})^2\, dc_{23}^2\right), \end{aligned} \tag{2.2}$$

where $\bar{c}_{12} = 2c_{12}$. It is clear that this is smooth across $r = 0$ if $\theta = \bar{c}_{12} \in [0, 2\pi]$, and in fact the first part of this metric is just a multiple of the metric on hyperbolic space.

We also wish to examine the metric g in the region where

$$\mu \equiv \lambda_1/\lambda_2 < c, \qquad \nu \equiv \lambda_2/\lambda_3 < c,$$

for $c < 1$. We have $\lambda_1 = \mu^{2/3}\nu^{1/3}$, $\lambda_2 = \mu^{-1/2}\nu^{1/3}$ and $\lambda_3 = \mu^{-1/3}\nu^{-2/3}$, and so

$$\begin{aligned} g =& 4((d\mu/\mu)^2 + (d\mu/\mu)(d\nu/\nu) + (d\nu/\nu)^2) \\ &+3(\mu - \mu^{-1})^2 dc_{12}^2 + 3(\mu\nu - \mu^{-1}\nu^{-1})^2 dc_{13}^2 + 3(\nu - \nu^{-1})^2 dc_{23}^2. \end{aligned} \tag{2.3}$$

The Laplacian in these coordinates is of the form

$$\begin{aligned}\Delta_g = &\frac{1}{3}\left((\mu D_\mu)^2 + (\nu D_\nu)^2 - (\mu D_\mu)(\nu D_\nu) + i(\mu D_\mu) + i(\nu D_\nu)\right.\\ &\left.+(\mu D_{c_{12}})^2 + (\nu D_{c_{23}})^2 + (\mu\nu D_{c_{13}})^2\right) + E,\end{aligned} \tag{2.4}$$

where E is the collection of all terms which are higher order when μ and ν are small, i.e. it is a sum of up to two products of the vector fields μD_μ, νD_ν, $\mu D_{c_{12}}$, $\nu D_{c_{23}}$ and $\mu\nu D_{c_{13}}$, with at least one extra factor of μ or ν. We note that the 'radial part' of this operator, which in these coordinates simply corresponds to the operator acting on functions which are independent of the c_{ij}, is

$$\Delta_{\mathrm{rad}} = \frac{1}{3}\left((\mu D_\mu)^2 + (\nu D_\nu)^2 - (\mu D_\mu)(\nu D_\nu) + i(\mu D_\mu) + i(\nu D_\nu) + E'\right), \tag{2.5}$$

where E' is an error term as above. In addition, using the coordinates μ and s which are valid in the interior of $H_\sharp$,

$$\begin{aligned}\Delta_{\mathrm{rad}} =&\frac{1}{3}\left((\mu D_\mu)^2 - \left(\frac{\mu+\mu^{-1}}{\mu-\mu^{-1}} - \frac{s^2(\mu-\mu^{-1})}{s^4 - s^2(\mu+\mu^{-1})+1}\right) i\mu D_\mu\right)\\ &+\frac{1}{4}\left((sD_s)^2 - \frac{2(s^4-1)}{s^4-s^2(\mu+\mu^{-1})+1} isD_s\right).\end{aligned} \tag{2.6}$$

2.3. **The compactification $\overline{M}$.** We now proceed to describe the first compactification $\overline{M}$ of M. Consider first the interior of a Weyl chamber away from the walls. Fix the chamber where $\lambda_1 < \lambda_2 < \lambda_3$, and suppose that $0 < \mu, \nu < c < 1$. As we have seen, $\mathrm{SO}(3)$ is just carried along as a factor in this region, and $(\mu, \nu, c_{12}, c_{13}, c_{23})$ is a coordinate chart. We compactify by adding the faces $\mu = 0$ and $\nu = 0$. These intersect at the corner $\mu = \nu = 0$ at infinity.

Now consider the region $c < \mu < c^{-1}$, $\lambda_3 > c^{-1}$. We showed in §2.1 that this can be identified with $(\mathcal{V} \times \mathrm{SO}(3))/\mathrm{O}(2) \times [1/c, \infty)$, where $\mathcal{V}$ is a ball in $\mathbb{H}^2$. We compactify this by adding the face $s' = 0$ where $s' \equiv \lambda_3^{-1}$, or in other words, we (partially) compactify this neighbourhood as $(\mathcal{V} \times \mathrm{SO}(3))/\mathrm{O}(2) \times [0, c)_{s'}$.

No new corners are added in this second step, and so the induced compactification of $\mathfrak{a}$ is as a hexagon, cf. the left picture in Figure 1.

It remains to make sure that these two partial compactifications are compatible in the region of overlap, so that they fit together to give $\overline{M}$ the structure of a compact $\mathcal{C}^\infty$ manifold with corners of codimension 2. In other words, we must show that the transition map, say when $c < \mu < 1$, is smooth. But this is clear, for the transition map is given by diagonalizing the 2-by-2 block C'' (cf. the discussion at the end of §2.1) and changing coordinates on the flat, which is certainly smooth away from the boundary at infinity. Notice, however, that the boundary defining functions for this face (ν in the first chart and s' in the second) are not smoothly related. In fact, since $\nu\mu^{1/2} = \lambda_3^{-3/2}$ and we are supposing that $\mu > c$ and $\nu \searrow 0$, we must replace s' by $s = \lambda_3^{-3/2}$ as the $\mathcal{C}^\infty$ defining function near this face.

We summarize what we have done. The manifold $\overline{M}$ is a compact manifold with a corner of codimension 2. It has two boundary hypersurfaces, $H_\sharp$ and $H^\sharp$, which are the closures of the parts of the boundary where the ratio of the smaller two, respectively the larger two, eigenvalues is bounded, or more directly, as the parts of the boundary where the ratio of the larger two, respectively the smaller two, eigenvalues vanishes. In the region $\mu, \nu \le c < 1$, where $\mu = \lambda_1/\lambda_2$, $\nu = \lambda_2/\lambda_3$, then

$$H_\sharp = \{\nu = 0\}, \qquad H^\sharp = \{\mu = 0\}.$$

The projection $((\mathbb{H}^2 \times \mathrm{SO}(3))/\,\mathrm{O}(2)) \mapsto \mathrm{SO}(3)/\,\mathrm{O}(2)$ at $\lambda_3^{-1} = 0$ is a fibration of the interior of $H_\sharp$ with base $\mathrm{SO}(3)/\,\mathrm{O}(2)$ and fibre $\mathbb{H}^2$ and this extends to a fibration $\phi_\sharp$ of $H_\sharp$ with fibres $\overline{\mathbb{H}^2} = \overline{B^2}$; there is an analogous fibration $\phi^\sharp : H^\sharp \to \mathrm{SO}(3)/\,\mathrm{O}(2)$.

We remark that the metric g induces a metric on the fibers of each of the faces. For example, from (2.2), we find that g induces a metric on $H_\sharp$, where $s = 0$, which restricts to give (3 times) the standard hyperbolic metric on the $\mathbb{H}^2$ fibres.

2.4. **The boundary fibration structure.** This differential topological structure on $\overline{M}$ is an example of a boundary fibration structure in the sense of [16]. This particular boundary fibration structure, which we christen the edge-to-edge (or ee) structure, is described in general as follows:

Definition 2.1. Suppose that $\overline{X}$ is a compact manifold with corners of codimension 2. Then we say that $\overline{X}$ is equipped with an edge-to-edge structure if

(i) Each boundary hypersurface H is the total space of a fibration $\phi_H : H \to B_H$ which is transversal to ∂H and has fiber F_H, a manifold with boundary;
(ii) Near any corner $H_1 \cap H_2$, there is a fibre-preserving diffeomorphism of $\overline{X}$ to $F_{H_1} \times F_{H_2} \times \mathcal{U}$, where $\mathcal{U}$ is an open neighbourhood in some Euclidean space.

We write this structure as $(\overline{X}, \phi)$, where ϕ is the collection of all of the mappings ϕ_H.

Associated to an ee structure on $\overline{X}$ is the Lie algebra of all smooth vector fields on $\overline{X}$ which are arbitrary in the interior and are tangent not only to the boundaries H (and hence to the corners $H_i \cap H_j$), but also the the fibres of the fibration F_H along each codimension one boundary face H. These can be easily expressed in our particular case of interest, using the coordinates μ, ν, c_{ij}. The vector fields which meet all of these restrictions are in the span over $\mathcal{C}^\infty(\overline{M})$ of the basic vector fields

$$\mu\partial_\mu, \nu\partial_\nu,\ \mu\partial_{c_{12}},\ \nu\partial_{c_{23}}, \mu\nu\partial_{c_{13}}. \tag{2.7}$$

Definition 2.2. If $(\overline{X}, \phi)$ is an edge-to-edge structure, then the associated Lie algebra of $\mathcal{C}^\infty$ vector fields which are tangent to the fibers of ϕ_H for all H is denoted by $\mathcal{V}_{\mathrm{ee}}(\overline{X})$. The enveloping algebra of this Lie algebra consists of all finite sums of products of elements of $\mathcal{V}_{\mathrm{ee}}(\overline{X})$, and hence consists of a particular class of differential operators, which we denote $\mathrm{Diff}^*_{\mathrm{ee}}(\overline{X})$.

The space $\mathcal{V}_{\mathrm{ee}}(\bar{X})$ is the full set of $\mathcal{C}^\infty$ sections of a vector bundle ${}^{\mathrm{ee}}T\overline{X}$ over $\overline{X}$, called the ee-tangent bundle. Its dual, the ee-cotangent bundle, is denoted ${}^{\mathrm{ee}}T^*\overline{X}$.

Proposition 2.3. *Let $\overline{M}$ be the compactification of $M = \mathrm{SL}(3)/\,\mathrm{SO}(3)$ described above, and let g be the invariant metric. Then $g \in \mathcal{C}^\infty(\overline{M}; S^2({}^{ee}T^*\overline{M}))$, and furthermore, $\Delta_g \in \mathrm{Diff}^2_{ee}(\overline{M})$.*

3. The edge-to-edge small calculus

The general strategy we shall follow, which has proved successful in many analogous situations, is that a boundary fibration structure leads to the introduction of an adapted calculus of pseudodifferential operators, and these operators in turn are used to investigate the analytic properties of the elliptic operators associated to that boundary fibration structure. As explained in the introduction, our aim here is more modest in that we shall give a minimal development, including only those parts of the theory of ee-pseudodifferential operators which are needed to understand the resolvent of Δ_g on the space M. This simplifies the presentation considerably since it amounts to only a little bit more than solving the 'model problems' which arise for more general operators (or for the Laplacian on spaces endowed with an ee-metric, rather than the specific invariant one). We shall take up the more general theory elsewhere.

In this section we construct the 'small calculus' of ee-pseudodifferential operators. These are characterized in terms of the behaviour of their Schwartz kernels as conormal distributions on a resolution of the double space $\overline{X} \times \overline{X}$. This is very closely related to the construction of the calculus $\Psi_{p0}(\overline{X})$ in [13].

3.1. **The ee double space.** We first describe the resolution process for $\overline{M}$, where we have already chosen good coordinates; after that we briefly describe the process in the general situation. Thus the immediate goal is to find a suitable blow-up $\overline{M}^2_{\mathrm{ee}}$ of $\overline{M}^2$, which we call the edge-to-edge double space, where suitable means that the lift of the Laplacian, first to either factor of $\overline{M}^2$ and then to this blow-up, is transversely elliptic with respect to the lifted diagonal, uniformly to all boundary faces. Since Δ is an elliptic combination of vector fields in $\mathcal{V}_{\mathrm{ee}}(\overline{M})$, this last property, in turn, will be guaranteed provided the lifts of these vector fields span the normal bundle of the lifted diagonal, uniformly on the closure of this submanifold. (This is in contrast to what happens when these vector fields are lifted only to $\overline{M}^2$, for on this space they all vanish at the boundary of the diagonal.)

It will be simplest to describe the construction of $\overline{M}^2_{\mathrm{ee}}$ using local coordinates. We use two sets of coordinates, $(\mu, \nu, c_{ij}, \mu', \nu', c'_{ij})$ on $\overline{M}^2$ and consider especially the region near the product of the corner with itself. We recall the simpler situation in the edge calculus [14], which is the special case of the ee-calculus when there is only one boundary hypersurface, which is the total space of a fibration, and no corners. The appropriate submanifold to blow up there is the fibre diagonal of the boundary, and accordingly, away from the corner, e.g. where $\nu, \nu' \geq c > 0$ but $\mu, \mu' \to 0$, we should blow up the set

$$\mathcal{F}^\sharp = \{\mu = \mu' = 0, c_{12} = c'_{12}, c_{13} = c'_{13}\}.$$

The effect of this blowup would be to desingularize the lifts of the vector fields $\mu\partial_\mu$, $\mu\partial_{c_{12}}$ and $\mu\nu\partial_{c_{13}}$; the other vector fields $\nu\partial_\nu$ and $\nu\partial_{c_{23}}$ are tangent to the submanifold being blown up and are not affected. Similarly, in the region where $\mu, \mu' \geq c > 0$ but $\nu, \nu' \to 0$, we should blow up

$$\mathcal{F}_\sharp = \{\nu = \nu' = 0, c_{23} = c'_{23}, c_{13} = c'_{13}\}$$

to desingularize the lifts of $\nu\partial_\nu$, $\nu\partial_{c_{23}}$ and $\mu\nu\partial_{c_{13}}$. Unfortunately, these two submanifolds do not intersect transversely, and to remedy this we must first blow up their intersection

$$\mathcal{F}^\sharp \cap \mathcal{F}_\sharp = \{\mu = \mu' = \nu = \nu' = 0, c_{ij} = c'_{ij} \ \ \forall i, j\}.$$

The picture is slightly more complicated because to resolve the quadratic singularity of the vector field $\mu\nu\partial_{c_{13}}$ it is necessary to perform a parabolic blow-up of this final submanifold; this blow-up corresponds to the parabolic dilation

$$\begin{aligned}&(\mu,\mu',\nu,\nu',c_{12}-c'_{12},c'_{12},c_{23}-c'_{23},c'_{23},c_{13}-c'_{13},c'_{13})\\&\longmapsto(\lambda\mu,\lambda\mu',\lambda\nu,\lambda\nu',\lambda(c_{12}-c'_{12}),c'_{12},\lambda(c_{23}-c'_{23}),c'_{23},\lambda^2(c_{13}-c'_{13}),c'_{13}),\end{aligned}$$

for $\lambda>0$.

Now let us recast this more generally. Let $(\overline{X},\phi)$ be an ee structure. If H is a boundary hypersurface, consider its fiber diagonal

$$\mathrm{diag}_{H,\phi_H}=\mathrm{diag}_H=\{(p,p')\in H\times H:\ \phi_H(p)=\phi_H(p')\}.$$

For each (nontrivial) corner $K_{12}=H_1\cap H_2$, the submanifolds diag_{H_1} and diag_{H_2} intersect normally in a subset of $K_{12}\times K_{12}$. Thus the first step is to blow up each of these submanifolds $\mathrm{diag}_{H_1}\cap\mathrm{diag}_{H_2}$ parabolically, with the direction of quadratic scaling given by the subbundle $S_{12}=(T\,\mathrm{diag}_{H_1}+T\,\mathrm{diag}_{H_2})\cap T_{\mathrm{diag}}$, where $\mathrm{diag}\subset\overline{X}^2$ is the diagonal. This makes the lifts of diag_{H_j} disjoint, and so we may now blow these up in either order. Symbolically, if we enumerate the boundary faces of $\overline{X}$ as H_i, then the ee double space of $\overline{X}$ is by definition the space

$$\overline{X}^2_{\mathrm{ee}}=\left[\overline{X}^2;\ \bigcup_{i,j}K_{ij},\ \bigcup_{i,j}S_{ij};\ \bigcup_i\mathrm{diag}_{H_i}\right].$$

Coordinates on the blow-up. Each of these blow-ups can be realized by the introduction of appropriate polar coordinates. However, such coordinates are usually quite messy, and in practice it is far more convenient to work with projective coordinates as follows, even though these have the defect that they are singular at one of the boundary faces. Thus for example, let us introduce a set of projective coordinates which is nonsingular away from the lift of the face where $\mu'=0$ (i.e. the copy of $H^\sharp$ on the second copy of $\overline{M}$). Then we may use

$$\begin{gathered}\mu',\ \mu/\mu',\ \nu/\mu',\ \nu'/\mu',c'_{12},c'_{13},c'_{23}\\(c_{12}-c'_{12})/\mu',(c_{23}-c'_{23})/\mu',(c_{13}-c'_{13})/(\mu')^2\end{gathered}$$

as coordinates near the lift of $\mathcal{F}^\sharp\cap\mathcal{F}_\sharp$. Notice that in these coordinates $\mu'=0$ defines the 'front face', i.e. the new boundary hypersurface created in this blowup. We continue by blowing up the lift of $\mathcal{F}_\sharp$; in these coordinates,

$$\mathcal{F}_\sharp=\{\nu/\mu'=0,\ \nu'/\mu'=0,\ (c_{13}-c'_{13})/(\mu')^2=0,\ (c_{23}-c'_{23})/\mu'=0\},$$

and we shall use ν'/μ' as the scaling coordinate. This gives coordinates

$$\begin{gathered}\mu',\ s_1=(\nu/\mu')/(\nu'/\mu')=\nu/\nu',\ s_2=\mu/\mu',\ \nu'/\mu',c'_{12},\ c'_{13},\ c'_{23}\\C_{12}=(c_{12}-c'_{12})/\mu',\ C_{23}=(c_{23}-c'_{23})/\nu',C_{13}=(c_{13}-c'_{13})/(\mu'\nu').\end{gathered}$$

Here there are two boundary defining functions, μ' and ν'/μ'; the former continues to define the front face from the first blow-up, while the latter defines the new side-front face introduced in this second blow-up. All other coordinates are either angular or lateral along each of these faces. To obtain coordinates near the lift of $\mathcal{F}^\sharp$ it is most convenient to interchange the roles of μ' and ν' in the preceding discussion.

Vector fields. The virtue of using these projective coordinates is immediately evident when we compute the lifts of the basic vector field (2.7) to $\overline{M}^2_{\mathrm{ee}}$. Thus, lifting each of the vector fields (2.7) first to the left factor of $\overline{M}$ in $\overline{M}^2$ and then to the blowup gives

$$\mu\partial_\mu \to s_2\partial_{s_2},\ \nu\partial_\nu \to s_1\partial_{s_1},$$
$$\mu\partial_{c_{12}} \to s_2\partial_{C_{12}},\ \nu\partial_{c_{23}} \to s_1\partial_{C_{23}},\ \mu\nu\partial_{c_{13}} \to s_1 s_2\partial_{C_{13}}.$$

The lifted diagonal $\mathrm{diag}_{\mathrm{ee}}$ in $\overline{M}^2\mathrm{ee}$ is described by the equations $\{s_1 = s_2 = 1,\ C_{ij} = 0\}$, and from this it is clear that these vector fields span the normal bundle to $\mathrm{diag}_{\mathrm{ee}}$, uniformly to the boundary and corners.

3.2. **ee-pseudodifferential operators.** Let $(\overline{X}, \phi)$ be an ee structure, and let $\overline{X}^2_{\mathrm{ee}}$ be the associated ee double space. We now define the space of ee-pseudodifferential operators $\Psi^*_{\mathrm{ee}}(X)$ to consist of those pseudodifferential operators A on X whose Schwartz kernel κ_A has the following properties. κ_A is a distribution on X^2, and we require that it lift to $\overline{X}^2_{\mathrm{ee}}$ to be polyhomogeneous with respect to the lifted diagonal $\mathrm{diag}_{\mathrm{ee}}$, with singularities smoothly extendible across all boundary faces of $\overline{X}^2_{\mathrm{ee}}$ which meet $\mathrm{diag}_{\mathrm{ee}}$, and which vanishes to all orders at all boundary faces which do not meet $\mathrm{diag}_{\mathrm{ee}}$. For later reference, we write $\mathcal{H}'$ and $\mathcal{H}''$ for the union of boundary faces which do or do not, respectively, meet $\mathrm{diag}_{\mathrm{ee}}$.

The preceding computations concerning the $\mathcal{V}_{\mathrm{ee}}$ vector fields immediately give the

Proposition 3.1. *If L is any ee differential operator, then the Schwartz kernel of L lifts to a (differentiated) delta section in $\overline{X}^2_{ee}$ supported along* diag_{ee}.

As in ordinary pseudodifferential theory, there is a symbol map

$$\sigma_m : \Psi^m_{\mathrm{ee}}(\overline{X}) \longmapsto S^m_{\mathrm{hom}}({}^{\mathrm{ee}}T^*\overline{X});$$

an operator $L \in \Psi^m_{\mathrm{ee}}(\overline{X})$ is called ee-elliptic if $\sigma_m(L)(z,\zeta)$ is invertible for $\zeta \neq 0$. Using this, the standard elliptic parametrix construction can be mimicked to give

Proposition 3.2. *If $L \in \Psi^m_{ee}(\overline{X})$ is elliptic then there exists a parametrix $G \in \Psi^{-m}_{ee}(\overline{X})$ such that $LG - \mathrm{Id}, GL - \mathrm{Id} \in \Psi^{-\infty}_{ee}(\overline{X})$.*

Let us specialize again to the symmetric space M. For any $\lambda \in \mathbb{C}$, $\Delta - \lambda$ is elliptic in this sense, and so this proposition yields a parametrix $G(\lambda) \in \Psi^m_{\mathrm{ee}}(\overline{X})$ which depends holomorphically on λ, so that both $(\Delta-\lambda)G(\lambda)$ and $G(\lambda)(\Delta-\lambda)$ are of order $-\infty$ in this small calculus. It is convenient to modify this parametrix slightly. In fact, if $p \in M$ and K_p is the stabilizer subgroup of $\mathrm{SL}(3)$ fixing this point, then $\Delta-\lambda$ is K_p-invariant. It would be nice to have a K_p-invariant parametrix, and this is easy to arrange: simply define $G_p(\lambda) = \int_{K_p} \phi_O^* G(\lambda)(\phi_O^{-1})^*\, dg_{K_p}(O)$, where dg_{K_p} denotes the normalized invariant measure, and ϕ_O right multiplication by the element $O \in K_p$.

Proposition 3.3. *For any $\lambda \in \mathbb{C}$ and $p \in M$, there exists an operator $G_p(\lambda) \in \Psi^{-2}_{ee}(\overline{X})$ which is K_p-invariant and which satisfies*

$$(\Delta-\lambda)G_p(\lambda) - \mathrm{Id},\quad G_p(\lambda)(\Delta-\lambda) - \mathrm{Id} \in \Psi^{-\infty}_{ee}(\overline{X})$$

Furthermore, $G(\lambda)$ depends holomorphically on λ.

As already explained in the introduction, this parametrix is not the final one for the simple reason that the error terms it leaves are not compact on L^2; removing these is the principal motivation for defining the more elaborate large calculus.

We conclude this section with a description of the regularity properties of this parametrix. We fix an ee-metric g, i.e. one for which the generating vector fields of the structure are of bounded length. Boundedness of operators of order 0 on $L^2(X, dV_g)$ may be deduced using the usual combination of a symbol calculus argument to reduce to showing boundedness of operators of order $-\infty$ and then proving this case directly by Schur's inequality; cf. [14] for an example of this argument. Next, for any $m \in \mathbb{R}$, define the Sobolev space

$$H^m_{\mathrm{ee}}(\overline{X}) = \{u : Au \in L^2(X; dV_g)\ \forall A \in \Psi^m_{\mathrm{ee}}(\overline{X})\}. \tag{3.1}$$

If $m \in \mathbb{N}$, an equivalent formulation is

$$H^m_{\mathrm{ee}}(\overline{X}) = \{u : V_1 \dots V_j u \in L^2(X, dV_g), \text{ whenever } V_i \in \mathcal{V}_{\mathrm{ee}} \text{ and } j \leq m\}.$$

From the existence of the parametrix in the last proposition, we have

Proposition 3.4. *For any $m \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, the parametrix $G_p(\lambda)$ satisfies*

$$G_p(\lambda) : H^m_{ee}(\overline{M}) \to H^{m+2}_{ee}(\overline{M}),$$
$$G_p(\lambda) : H^m_{ee}(\overline{M})^{K_p} \to H^{m+2}_{ee}(\overline{M})^{K_p}.$$

In addition, if $E_p(\lambda) = G_p(\lambda)(\Delta - \lambda) - \mathrm{Id}$ and $F_p(\lambda) = (\Delta - \lambda)G_p(\lambda) - \mathrm{Id}$ are the error terms, then

$$E_p(\lambda),\ F_p(\lambda) : H^m_{ee}(\overline{M})^{K_p} \to H^\infty_{ee}(\overline{M})^{K_p}$$

for all m.

To understand what these statements mean, suppose that $(\Delta - \lambda)u = f \in \mathcal{C}^\infty_0(M)$. If neither u nor f are K_p-invariant, then from $u = G_p(\lambda)f + E_p(\lambda)u$ and assuming that $u \in L^2(M, dV_g)$, we get

$$(\mu\partial_\mu)^j(\nu\partial_\nu)^k(\mu\partial_{c_{12}})^r(\nu\partial_{c_{23}})^s(\mu\nu\partial_{c_{13}})^t u \in L^2(M, dV_g) \text{ for all } j, k, r, s, t \geq 0.$$

Thus u has full tangential regularity only in directions tangent to the fibres of the boundary fibration. On the other hand, suppose that both u and f are K_p-invariant (and f is still $\mathcal{C}^\infty_0$). Then the vector fields $\partial_{c_{ij}}$ annihilate u and we see that u restricts to a conormal function on the compactified flat $\overline{\exp(\mathfrak{a})}$. In other words,

$$(\mu\partial_\mu)^j(\nu\partial_\nu)^k u \in L^2(M, dV_g) \text{ for all } j, k \geq 0.$$

4. Product models

The key remaining issue is to find a correction for the parametrix $G_p(\lambda)$ which solves away as much of the error terms $E_p(\lambda)$ and $F_p(\lambda)$ (as in Proposition 3.4) as possible. In fact, it will suffice to find a new parametrix for which the corresponding error terms are not only regularizing, but which map $H^m_{\mathrm{ee}}(\overline{M})^{K_p}$ into spaces of functions which decay at some definite rate at infinity in the flat. As already indicated in the introduction, this is done by solving away the expansions of the Schwartz kernels of these operators at the boundary faces $H_\sharp$ and $H^\sharp$ of $\overline{\exp(\mathfrak{a})}$. However, the main difficulty is caused by the fact that the Laplacian does not have a product decomposition at the corner $H_\sharp \cap H^\sharp$. This problem appears challenging, but as explained in detail in §5, the lift of Δ_g to the logarithmic blow-up of $\overline{M}$ is (quite remarkably) well-approximated by product type operators at each of the new corners of this blown up space. This means that the main technical difficulties involve the analysis of the product operators arising in this process, and this is the subject of the current section.

We shall be using the results and methods of [13], concerning the detailed analysis of the resolvent of the Laplacian on a product space $X = M_1 \times M_2$, which we review below. Although that paper focused particularly on the case where both factors (M_j, g_j) are conformally compact (asymptotically hyperbolic), our application here requires that we let the factors be $(\mathbb{R}^+_s, ds^2/s^2)$ and $(\mathbb{H}^2, h)$, respectively (after all, the first is just the one-dimensional hyperbolic space). We shall both extend and refine the results in this setting.

4.1. **Geometry and compactification of the product.** One of the main conclusions of [13] is that for purposes of analyzing its resolvent, or more precisely its Green function with given pole, the best compactification of $X = M_1 \times M_2$ is given by

$$\widetilde{X} = [(\overline{M}_1)_{\log} \times (\overline{M}_2)_{\log}; \partial\overline{M}_1 \times \partial\overline{M}_2]. \tag{4.1}$$

Recall that this means that if ρ_j, $j = 1, 2$, are smooth boundary defining functions for M_j, then we replace these by $-1/\log\rho_j$ and then perform the standard blow-up of the corner. The function

$$x = \sqrt{(\log\rho_1)^{-2} + (\log\rho_2)^{-2}}$$

is a total boundary defining function for $\widetilde{X}$, i.e. is smooth and vanishes simply at all the boundary faces with respect to this new smooth structure.

Now let

$$(M_1, g_1) = \left(\mathbb{R}^+_s, 4\frac{ds^2}{s^2}\right), \qquad \text{and} \qquad (M_2, g_2) = (\mathbb{H}^2, 3h),$$

where h is the standard (curvature -1) metric on hyperbolic space. If δ_j is the Riemannian distance function on each of these spaces, then the distance between pairs of points $(s, z), (s', z') \in X$ is given by

$$\delta((s, z), (s', z')) = \sqrt{\delta_1(s, s')^2 + \delta_2(z, z')^2}.$$

In particular, fixing the point $o = (1, q) \in X$, then

$$\delta_1(s, 1) = 2\,|\log s|, \qquad \delta_2(z, q) = \sqrt{3}\,|\log\mu|,$$

where μ is a suitable defining function on $\mathbb{H}^2$ (we include the factors 2 and $\sqrt{3}$ to keep track of the scaling factors on the metrics). Neglecting for the moment the fact that these functions are only smooth away from $s = 1$ and $z = q$, we set $\rho_1 = e^{-\delta_1(s,1)}$, $\rho_2 = e^{-\delta_2(z,q)}$. Hence

$$x = \frac{1}{\delta(z, o)} = [4(\log s)^2 + 3(\log\mu)^2]^{-1/2} \tag{4.2}$$

is a total boundary defining function for $\widetilde{X}$ and

$$\begin{aligned} x_1 &= \frac{x}{(4(\log s)^2 + 1)^{-1/2}} \sim 2x\,|\log s| = \frac{\delta_1(1, s)}{\delta(o, z)}, \\ x_2 &= \frac{x}{(3(\log\mu)^2 + 1)^{-1/2}} \sim \sqrt{3}x\,|\log\mu| = \frac{\delta_2(q, z_2)}{\delta(o, z)} \end{aligned} \tag{4.3}$$

are defining functions for the two side faces. Note that in terms of the "eigenvalue coordinates" $(\lambda_1, \lambda_2, \lambda_3)$ on M,

$$x^{-2} = 6((\log\lambda_1)^2 + (\log\lambda_2)^2 + (\log\lambda_3)^2).$$

Returning to address the fact that the δ_j and hence δ are not smooth everywhere, we replace them by smoothed versions, $\tilde\delta_j, \tilde\delta \in \mathcal{C}^\infty$, which are chosen so that $\tilde\delta \geq 1$, $\delta \leq \tilde\delta \leq \delta + 2$, and $\tilde\delta = \delta$ if $\delta \geq 3$, and similarly for $\tilde\delta_j$. Although these no longer satisfy the triangle inequality, their failure to do so is bounded:

$$\tilde\delta(z,z') + \tilde\delta(z',z'') - \tilde\delta(z,z'') \geq \delta(z,z') + \delta(z',z'') - (\delta(z',z'') + 2) \geq -2,$$

which is all we require in later estimates, and which we continue to call the triangle inequality. Combining it with the fact that $\tilde\delta \geq 1$, we have

$$\tilde\delta(z,z'') \leq \tilde\delta(z,z') + \tilde\delta(z',z'') + 2 \leq 2\tilde\delta(z,z')\tilde\delta(z',z'') + 2 \leq 4\tilde\delta(z,z')\tilde\delta(z',z''),$$

and so

$$\frac{\tilde\delta(z,z'')}{\tilde\delta(z,z')\tilde\delta(z',z'')} \leq 4. \tag{4.4}$$

We now replace our previous defining functions by

$$x_1 = \frac{\tilde\delta_1(1,s)}{\tilde\delta(o,z)},\ x_2 = \frac{\tilde\delta_2(q,z_2)}{\tilde\delta(o,z)},\ x = \tilde\delta(o,z)^{-1},$$

which are smooth and globally defined.

4.2. **Resolvent asymptotics.** The analysis of Δ_g on K_p-invariant functions ultimately reduces near the face $H_\sharp$ to that for the operator

$$L_\sharp = \frac{1}{4}(sD_s)^2 + i\frac{1}{2}(sD_s) + \frac{1}{3}\Delta_h,$$

which is self-adjoint on

$$L^2(\mathbb{R}^+_s \times \mathbb{H}^2, s^{-3}ds\, dV_h).$$

It is computationally simpler to use its conjugate

$$L_0 = s^{-1}L_\sharp s = \frac{1}{4}(sD_s)^2 + \frac{1}{3}\Delta_h + \frac{1}{4}, \tag{4.5}$$

which is self-adjoint on

$$L^2(\mathbb{R}^+_s \times \mathbb{H}^2, dV_{g'}) = L^2(\mathbb{R}^+_s \times \mathbb{H}^2, s^{-1}ds\, dV_h).$$

Note that $L_0 = \Delta_{g'} + \frac{1}{4}$, where

$$g' = 4\frac{ds^2}{s^2} + 3g_{\mathbb{H}^2}. \tag{4.6}$$

(This explains our choice of factors (M_j, g_j) above.)

We now quote results from [13] concerning the structure of the resolvent

$$R_0(\lambda) = (L_0 - \lambda)^{-1} = (\Delta_{g'} - (\lambda - \frac{1}{4}))^{-1}.$$

The formulæ below frequently involve the quantity $\sqrt{\lambda - \lambda_0}$, which is natural since

$$\inf\operatorname{spec}(\Delta_{g'}) = 0 + \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12} \Longrightarrow \inf\operatorname{spec}(L_0) \equiv \lambda_0 = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}.$$

We always use the branch of the square root which has *negative* imaginary part on $\mathbb{C} \setminus [0,\infty)$. These formulæ also involve the Poisson operator $P_j(\lambda)$, or rather its adjoint $P_j^t(\lambda)$, and the spherical function $S_j(\lambda)$, on each of the factors. Of course, for the one-dimensional factor $M_1 = \mathbb{R}^+$, these objects are particularly simple and quite explicit.

For simplicity, we begin with the asymptotics of $R_0(\lambda)f$, where $f \in \dot{\mathcal{C}}^\infty(\overline{X})$:

Proposition 4.1 ([13], Proposition 7.7)**.** *Let $f \in \dot{\mathcal{C}}^\infty(\overline{X})$ and $\lambda \in \mathbb{C} \setminus [\lambda_0, \infty)$. Then on $\widetilde{X}$,*

$$R_0(\mu)f = \mu^{1/2} x_2\, x^{1/2} \exp(-i\sqrt{\lambda - \lambda_0}/x)\, h, \quad h \in \mathcal{C}^\infty(\widetilde{X}). \tag{4.7}$$

Moreover, setting

$$\lambda_1^0(s) = \frac{\lambda - \lambda_0}{1+s^2}, \tag{4.8}$$

then the restriction of h to the boundary is given by

$$\begin{aligned} &a(\lambda, s) \left(P_1^t(\lambda_1^0(s)) \otimes P_2^t(\lambda - \frac{1}{4} - \lambda_1^0(s)) \right) f \text{ on the front face,} \\ &a'(\lambda) \left(S_1(0) \otimes P_2^t(\lambda - \frac{1}{4}) \right) f \text{ on the lift of } \overline{M}_1 \times \partial \overline{M}_2, \\ &a''(\lambda) \left(P_1^t(\lambda - \lambda_0) \otimes S_2(\frac{1}{12}) \right) f \text{ on the lift of } \partial\overline{M}_1 \times \overline{M}_2, \end{aligned} \tag{4.9}$$

where a, a' and a'' are all nonvanishing.

Remark 4.2. These formulæ are different from the ones in [13] in one point: the resolvent $((sD_s)^2 - \lambda)^{-1}$ has a singularity of the form $\lambda^{-1/2}$ at the threshold branch point 0, instead of one of the form $\lambda^{1/2}$ which occurs when this factor is higher dimensional. This simply causes the order of the leading term of the asymptotic expansions at the side faces where s is finite to change by 1, and this produces an extra factor x_2, but not x_1, in (4.7). In fact, the asymptotics at the lift of $\overline{M}_1 \times \partial\overline{M}_2$ is even simpler than what we have stated since the $\mathbb{R}^+$-invariance of $(sD_s)^2$ can be used to show that $R(\lambda)f$ is still polyhomogeneous even when this face is blown down. However, this does not affect the way we apply our results later, so we do not take advantage of this.

From here, [13] goes on to deduce the full structure of the Schwartz kernel of the resolvent in the product setting. Just as for the (more complicated) situation we are in the process of establishing, that the resolvent for Δ on M decomposes as a sum of two terms, $G_p(\lambda) + \tilde{R}(\lambda)$, this kernel is a sum of two terms, $R_0'(\lambda) + R_0''(\lambda)$. The first term is in the small product-0 calculus (a simple case of the ee calculus) and contains the full diagonal singularity, while the second is smooth on the interior but has a somewhat more complicated structure at the boundary which is resolved by passing to a further blow-up, the resolvent double space $\widetilde{X}^2_{\mathrm{res}}$.

Operators in the small product-0 calculus $\Psi_{p0}(\overline{X})$ (for $X = M_1 \times M_2$), are characterized by the fact that their Schwartz kernels lift to the product-0 double space

$$\overline{X}^2_{p0} \equiv (\overline{M}_1)^2_0 \times (\overline{M}_2)^2_0$$

to be conormal to the lifted diagonal, smoothly extendible across the front faces $\mathrm{ff}(\overline{M}_1)^2_0 \times (\overline{M}_2)^2_0$ and $(\overline{M}_1)^2_0 \times \mathrm{ff}(\overline{M}_2)^2_0$ and vanishing to infinite order at all other boundary faces. Here $M_1 = \mathbb{R}^+$ and $M_2 = \mathbb{H}^2$, so that $\overline{M}_1 = I$ is the radial compactification of the half-line as an interval and $\overline{\mathbb{H}^2}$ is the ball B^2.

The resolvent double space $\widetilde{X}^2_{\mathrm{res}}$ is obtained from $\overline{X}^2_{p0}$ by judiciously blowing up a certain (minimal) collection of corners, so that (at least when λ is in the resolvent set) $R_0''(\lambda)$ lifts to be polyhomogeneous.

Recall that we have reduced to studying the restriction of the resolvent on M to the flat; because of this, we only need to understand certain parts of the structure of the resolvent $R_0(\lambda)$ for L_0 here. We have relegated the more detailed discussion of the structure of $\widetilde{X}^2_{\mathrm{res}}$ to the end of this paper in an appendix. More specifically, it clearly suffices to focus on the action of $R_0(\lambda)$ on SO(2)-invariant functions (with respect to a fixed point q) on the $\mathbb{H}^2$ factor. We regard these functions as depending on the variables $s \in \mathbb{R}^+$ and $\mu \in (0,1)$, where μ is the boundary defining function on $\overline{\mathbb{H}^2}$ used earlier. The apparent 'boundary' $\mu = 1$ here is artificial, and corresponds to the point q in polar coordinates, and we systematically ignore it (for example by only considering functions which are supported away from this set). In fact, set

$$\mathfrak{b}^+ = \mathbb{R}^+_s \times (0,1)_\mu \subset \mathfrak{b} = \mathbb{R}^+_s \times \mathbb{R}^+_\mu .$$

If $\phi \in \mathcal{C}^\infty(\mathbb{R}^+)$, $\phi = 0$ near $\mu = 0$ and $\phi = 1$ for $\mu \geq 1/2$, then the Schwartz kernel of

$$(1-\phi(\mu))R_0(\lambda)(1-\phi(\mu))$$

lifts to the resolvent double-space $\widetilde{\mathfrak{b}}^2_{\mathrm{res}}$ and is supported on the closure of $\mathfrak{b}^+ \times \mathfrak{b}^+$. Note that the Schwartz kernels of $\phi(\mu)R_0(\lambda)(1-\phi(\mu))$ and $(1-\phi)(\mu)R_0(\lambda)\phi(\mu)$ also lift trivially since they are supported away from the corner $\mu = \mu' = 0$.

The advantage gained by this reduction is that the geometry of both the product-0 double space $\overline{(\mathfrak{b})}^2_{p0}$ and the resolvent double space $\overline{\mathfrak{b}}^2_{\mathrm{res}}$ are simpler than when the second factor has dimension bigger than one. Thus the 0-double space of $I = \overline{\mathbb{R}^+}$ is obtained from I^2 by blowing up the boundary of the diagonal, $\partial\,\mathrm{diag}$, in I^2, $I^2_0 = [I^2; \partial\,\mathrm{diag}]$. This has the effect of separating the left and right boundary faces. (Strictly speaking, this is not quite true because they still intersect at the off-diagonal corners of this square, but since all kernels are supported away from these points, we studiously ignore this small untruth.) When $\dim M > 1$, the left and right faces of the 0-double space M^2_0 are no longer separated, and necessitates some extra blow-ups in the resolvent double space. In any case, if lf_j, rf_j, $j = 1,2$, are the left and right faces on the two factors I^2_0 in $\overline{\mathfrak{b}}^2_{p0}$, then let $\mathcal{S}$ be the collection of codimension two corners $\mathrm{lf}_1 \times \mathrm{lf}_2$, $\mathrm{lf}_1 \times \mathrm{rf}_2$, $\mathrm{rf}_1 \times \mathrm{lf}_2$, $\mathrm{rf}_1 \times \mathrm{rf}_2$; by construction (and again neglecting the off-diagonal corners), these do not intersect. Now replace the defining functions ρ_{lf_j} and ρ_{rf_j} at these faces in each factor with $\mathcal{R}_{\mathrm{lf}_j} = -1/\log \rho_{\mathrm{lf}_j}$, $\mathcal{R}_{\mathrm{rf}_j} = -1/\log \rho_{\mathrm{rf}_j}$. Notice that we are not changing the defining function at the front faces ff_j. The product-0 space with these defining functions is now denoted $\overline{\mathfrak{b}}^2_{p0,\log}$. We finally define

$$\widetilde{\mathfrak{b}}^2_{\mathrm{res}} = [\overline{\mathfrak{b}}^2_{p0}; \mathcal{S}].$$

The following theorem follows directly from [13], cf. the Appendix below.

Theorem 4.3. *If $\lambda \in \mathbb{C} \setminus \mathrm{spec}(L_0)$, then*

$$(L_0 - \lambda)^{-1} = R_0(\lambda) = R_0'(\lambda) + R_0''(\lambda), \tag{4.10}$$

where $R_0'(\lambda) \in \Psi^{-2}_{p0}(\overline{X})$ and

$$R_0''(\lambda) = (\rho_{\mathrm{lf}_2}\rho_{\mathrm{rf}_2})^{1/2} \exp(-i\sqrt{\lambda - \lambda_0}\; \tilde{\delta}(z,z'))\, F(\lambda), \qquad \lambda_0 = \frac{1}{3}. \tag{4.11}$$

The kernel $F(\lambda)$ on the right here is of the form

$$F(\lambda) = \tilde{\delta}(z,z')^{-3/2}\,\tilde{\delta}_2(z_2,z_2')F'(\lambda)\frac{ds'}{s'}\,dV_h'$$

where the primes on the density factor mean that it is pulled back from the second (z') factor. The function $F'(\lambda)$ is bounded and smooth on $X \times X$ and lies in $\mathcal{C}^0(\widetilde{X} \times X)$. In fact, if $\phi(\mu) \in \mathcal{C}_0^\infty(\mathbb{R}^+)$, $\phi = 1$ for μ near 1, then for any $\epsilon > 0$,

$$e^{-\epsilon\tilde{\delta}(z',o)}\phi(\mu)\,F'(\lambda) \in \mathcal{C}^\infty(\widetilde{X}; L^\infty(X)). \tag{4.12}$$

(Here z and z' lie in $\widetilde{X}$ and X, respectively.) Of course, F' is considerably more regular than this, but its L^2 mapping properties are determined by what is stated here.

The restriction of this operator to SO(2)*-invariant functions in the second factor can be obtained by averaging $R_0''(\lambda)$ with respect to Haar measure on* SO(2)*, and can then be regarded as living on $\mathfrak{b}^2$ as above. Writing $F_0'(\lambda)$ as the average of $F'(\lambda)$, then for all $\epsilon > 0$,*

$$e^{-\epsilon\tilde{\delta}(z',o)}(1-\phi(\mu))F_0'(\lambda) \in \mathcal{C}^\infty\left(\widetilde{\mathfrak{b}^+}; L_b^2(\mathfrak{b}^+; \frac{ds'}{s'}\frac{d\mu'}{\mu'}\right). \tag{4.13}$$

A similar formula obtains when the first and second factors are interchanged.

4.3. **Boundedness of the resolvent on weighted spaces.** We now prove some refined mapping properties of the resolvent $R_0(\lambda)$ which are required later. These results reflect the fact that when λ is away from the spectrum, the Schwartz kernel of $R_0(\lambda)$ has 'off-diagonal' exponential decay of order $-\kappa$, where

$$\kappa = -\operatorname{Im}\sqrt{\lambda - \lambda_0} > 0, \qquad \lambda_0 = \frac{1}{3}.$$

For simplicity we phrase our theorems in terms of the decay or asymptotics of $R_0(\lambda)f$ for various classes of functions f, rather than in terms of the structure of the kernel itself.

There are three results in this direction. The first states that if $|\alpha| < \kappa$ and f decays (or grows) like $e^{\alpha/x}$ then so does $R_0(f)$. Next, if f decays like $e^{\alpha/x}$ with $\alpha < -\kappa$, then $R_0(\lambda)f$ decomposes as a sum of two terms, one decaying at the same rate as f and another which has an expansion, but decays only like $e^{-\kappa/x}$. Finally, we show that if f decays exponential (at a rate $\alpha \in (-\kappa,\kappa)$) in some sector, then $R_0(f)$ decays even faster in disjoint sectors, with rate depending on the angle between the two.

Proposition 4.4. *Suppose $|\alpha| < -\kappa$. Then*

$$R_0(\lambda) : e^{\alpha/x}H_{ee}^m(\overline{X}) \longrightarrow e^{\alpha/x}H_{ee}^{m+2}(\overline{X})$$

is bounded, where we are using the L^2 measure with respect to the metric g' and the ee Sobolev spaces defined in (3.1) (applied to the simple ee structure on $X = \mathbb{R}^+ \times \mathbb{H}^2$). Moreover, if $\alpha = -\kappa$, then

$$R_0(\lambda) : x^p e^{\alpha/x}H_{ee}^m(\overline{X}) \longrightarrow x^{-p}e^{\alpha/x}H_{ee}^{m+2}(\overline{X})$$

for any $p > 2$. (The restriction $p > 2$ is not optimal, but suffices for our later use.)

Proof. We shall only prove boundedness between weighted L^2 spaces; the boundedness between Sobolev spaces, and the gain of 2 in ee regularity, is a simple consequence of the fact that L_0 can be applied on either the left or right, cf. Lemma A.2, and arbitrary powers of it can be commuted through. We also consider only the case where $-\kappa < \alpha \leq 0$, since one need only reverse the roles of z and z' to handle the case $0 < \alpha < \kappa$.

The conclusion of the theorem is equivalent to the boundedness of the mapping

$$e^{-\alpha/x} R_0(\lambda) e^{\alpha/x} \in \mathcal{B}(L^2_{p0}(\overline{X})).$$

It can be verified by direct calculation that the lift of the function

$$A_\alpha \equiv e^{\alpha/x' - \alpha/x}$$

to $\overline{X}^2_{p0}$ is smooth up to the front faces. We can assume that the small-calculus part $R_0'(\lambda)$ of the resolvent has support not intersecting any of the boundaries except the front faces, and so A_α is smooth and bounded on its support. Thus we can focus on the term $R_0''(\lambda)$, or rather $A_\alpha R_0''(\lambda)$.

It is convenient to replace the measure $dV_{g'} = s^{-1}\mu^{-2}\, ds\, d\mu\, dy$, where y is the angular (tangential) variable in $\mathbb{H}^2$, by $s^{-1}\mu^{-1}\, ds\, d\mu\, dy$, so we set

$$L^2_b(\overline{X}) = L^2(X; \frac{ds\, d\mu\, dy}{s\mu}) = \mu^{-1/2} L^2(\overline{X}; dV_{g'}).$$

An operator A is bounded on $L^2(\overline{X}; dV_{g'})$ if and only if $\mu^{-1/2} A \mu^{1/2}$ is bounded on $L^2_b(\overline{X})$. Thus we must prove that

$$e^{\alpha/x' - \alpha/x} (\mu'/\mu)^{1/2} R(\lambda) \in \mathcal{B}(L^2_b(\overline{X})). \tag{4.14}$$

From Theorem 4.3, with $x(z) = \tilde{\delta}(z,o)^{-1}$, then this conjugated kernel has the form

$$e^{-\alpha(\tilde{\delta}(o,z) - \tilde{\delta}(o,z')) - i\sqrt{\lambda-\lambda_0}\tilde{\delta}(z,z')} e^{-(\tilde{\delta}_2(z_2,z_2') + \tilde{\delta}_2(q,z_2') - \tilde{\delta}_2(q,z_2))/2} F(\lambda)$$
$$= e^{-\alpha(\tilde{\delta}(o,z) - \tilde{\delta}(o,z') - \tilde{\delta}(z,z'))} e^{-(\tilde{\delta}_2(z_2,z_2') + \tilde{\delta}_2(q,z_2') - \tilde{\delta}_2(q,z_2))/2} e^{(-i\sqrt{\lambda-\lambda_0} + \alpha)\tilde{\delta}(z,z')} F(\lambda),$$

where F is as in (4.11). Using the triangle inequality to bound the exponents in the first two terms on the right, we can rewrite this as

$$e^{-\gamma\tilde{\delta}(z,z')} G, \qquad G \in L^\infty(X \times X),\ \gamma = \kappa + \alpha > 0.$$

This is an element of $L^\infty(X_z; L^1_b(X_{z'})) \cap L^\infty(X_{z'}; L^1_b(X_z))$, and hence the conclusion follows from Schur's lemma.

If $\alpha = -\kappa$ we can argue similarly, except now the kernel is rewritten as

$$\tilde{\delta}(z,o)^{-p} \tilde{\delta}(z',o)^{-p} G, \qquad G \in L^\infty(X^2).$$

Since $p > 2$, this is integrable as before. $\square$

The next result concerns the behaviour of $R_0(\lambda) f$ when $f \in e^{\alpha/x} L^2$ for some $\alpha < -\kappa$. As expected, this function is the sum of two terms, the first decaying at the same rate as f and the second having the decay of a homogeneous solution to $L_0 u = 0$. We note, however, that there are some subtleties in describing the precise regularity of this second term; these arise already for the resolvent on $\mathbb{H}^2$ [14] (hence a fortiori on $\mathbb{R}^+ \times \mathbb{H}^2$). More specifically, suppose that $f \in \mu^\gamma H^\infty_0(\overline{\mathbb{H}^2})$, where $\gamma > |\operatorname{Im} \sqrt{\lambda - 1/4}|$, or in other words, $(\mu\partial_\mu)^j (\mu\partial_y)^\ell f \in \mu^\gamma L^2(dV_h)$ for all $j, \ell \geq 0$. Setting $u = (\Delta_{\mathbb{H}^2} - \lambda)^{-1} f$, then the basic (small calculus) regularity result states that u also lies in $\mu^\gamma H^\infty_0(\overline{\mathbb{H}^2})$. However, u has no greater tangential regularity than f itself, i.e. we do not expect that $\partial_y^\ell u \in \mu^\gamma L^2$ for $\ell > 0$ unless the same is true for f too. Returning to $X = \mathbb{R}^+ \times \overline{\mathbb{H}^2}$, we are fortunately spared these considerations because our main interest is when f is $\mathrm{SO}(2)$ invariant.

In the following, let $H^m_{p0}(\overline{X})$ denote the space which we formerly called H^m_{ee}. We shall also use the logarithmically blown up single space

$$\widetilde{X} = [I_{\log} \times (\overline{\mathbb{H}^2})_{\log}; \partial I \times \partial \overline{\mathbb{H}^2}].$$

Proposition 4.5. *If $f \in e^{\alpha/x} H^m_{p0}(\overline{X})$ for some $\alpha < -\kappa$, then*

$$R(\lambda) f = \mu^{1/2} x^{1/2} x_2 e^{-i\sqrt{\lambda - \lambda_0}/x} h + e^{\alpha/x} H^{m+2}_{p0}(\overline{X});$$

In general, h is (at least) continuous on $\widetilde{X}$, but if f is $\mathrm{SO}(2)$-invariant, then h is smooth on $\widetilde{X}$. In particular, if $m = \infty$, and f is $\mathrm{SO}(2)$-invariant then

$$R(\lambda) f = \mu^{1/2} x^{1/2} x_2 e^{-i\sqrt{\lambda - \lambda_0}/x} h', \quad h' \in \mathcal{C}^\infty(\widetilde{X}).$$

Proof. We must show that

$$e^{i\sqrt{\lambda-\lambda_0}/x} \mu^{-1/2} x^{-1/2} x_2^{-1} R_0(\lambda) \mu^{1/2} e^{\alpha/x} \in \mathcal{B}(L^2_b(\overline{X}), C^0(\widetilde{X})), \tag{4.15}$$

and as in the preceeding proposition, we may immediately replace R_0 by R_0'', since the small calculus contribution R_0' causes no difficulties. The kernel of this operator then takes the form

$$K'' = e^{-i\sqrt{\lambda-\lambda_0}(\tilde\delta(z,z') - \tilde\delta(z,o)) - \kappa\tilde\delta(z',o)}$$
$$\times (e^{(\alpha+\kappa)\tilde\delta(z',o)} \tilde\delta(z',o)^{5/2}) \cdot e^{-(\tilde\delta_2(z_2,z_2') + \tilde\delta_2(q,z_2') - \tilde\delta_2(q,z_2))/2} \cdot \tilde F(\lambda),$$

where

$$\tilde F(\lambda) = x^{-1/2} x_2^{-1} \tilde\delta(z',o)^{-5/2} F(\lambda) = \tilde\delta(z,o)^{3/2} \tilde\delta_2(z_2,q)^{-1} \tilde\delta(z',o)^{-5/2} F(\lambda).$$

Now, dropping the singular measure, $\tilde F(\lambda)$ is bounded since on the one hand,

$$F(\lambda) = \tilde\delta(z,z')^{-3/2} \tilde\delta_2(z_2,z_2') F'(\lambda), \ F'(\lambda) \in L^\infty(X \times X),$$

but we also know that

$$\tilde\delta(z,o)^{3/2} \tilde\delta_2(z_2,q)^{-1} \tilde\delta(z',o)^{-5/2} \tilde\delta(z,z')^{-3/2} \tilde\delta_2(z_2,z_2')$$
$$= \left(\tilde\delta(z,o)\tilde\delta(z',o)^{-1}\tilde\delta(z,z')^{-1}\right)^{3/2} \left(\tilde\delta_2(z_2,q)^{-1}\tilde\delta_2(z_2,z_2')\tilde\delta(z',o)^{-1}\right)$$

is bounded using (4.4). In fact, using the triangle inequality we even get

$$e^{(-\alpha-\kappa)\tilde\delta(z',o)} \tilde\delta(z',o)^{-5/2} K'' \in L^\infty(\tilde X \times \bar X). \tag{4.16}$$

Each factor here is continuous on $\widetilde{X} \times X$ (note that we are making no claims about continuity at infinity in the second factor). Indeed, for the main term $\tilde F$ this follows from the asymptotics of $F(\lambda)$, while for the other factors this is true when z' lies in a compact set in X since the triangle inequality gives that both $\tilde\delta_2(z_2,z_2') - \tilde\delta_2(q,z_2)$ and $\tilde\delta(z,z') - \tilde\delta(z,o)$ are bounded continuous functions then. So we actually have

$$e^{-(\alpha+\kappa)\tilde\delta(z',o)} K'' \in C^0(\tilde X \times X). \tag{4.17}$$

Since in fact $e^{(\alpha+\kappa)\tilde\delta(z',o)} \tilde\delta(z',o)^{-5/2}$ is continuous on $\tilde X \times \overline{X}$ and vanishes at $\tilde X \times \partial\overline{X}$, we deduce from (4.16) and (4.17) that

$$e^{(\alpha+\kappa)\tilde\delta(z',o)/2} K'' \in C^0(\tilde X \times \overline{X}). \tag{4.18}$$

Finally, $e^{(\alpha+\kappa)\tilde{\delta}(z',o)/2}\tilde{\delta}(z',o)^{-5/2} \in L^2_b(\overline{X})$, and this gives

$$K'' \in C^0(\widetilde{X}; L^2_b(\overline{X})), \qquad \text{and so} \qquad K'' : L^2_b(\overline{X}) \to C^0(\widetilde{X})$$

is continuous. This proves the first part of the proposition.

To continue, note that we would obtain full asymptotic expansions in general if it were true that $K'' \in \mathcal{C}^\infty(\widetilde{X}; L^2_b(\overline{X}))$. However, this is false on account of the behaviour of $F(\lambda)$ at the front face $I_0^2 \times \mathrm{ff}(\overline{\mathbb{H}^2})$. However, choosing a cutoff function ϕ as in Theorem 4.3, then both $\phi(\mu)K''$ and $K''\phi(\mu')$ do satisfy this smoothness criterion, and map into functions with full expansions. It remains only to consider $(1-\phi(\mu))K''(1-\phi(\mu'))$.

Let us denote the corresponding kernel by $\tilde{K}''$, and work on the base space $\mathfrak{b}^+$. The desired result now follows easily from Theorem 4.3, since that result implies that if $P \in \mathrm{Diff}^m(\widetilde{\mathfrak{b}^+})$ and $\epsilon > 0$, then $e^{-(\alpha+\kappa)\tilde{\delta}(z',o)}e^{-\epsilon\tilde{\delta}(z',o)}P\tilde{K}''$ is continuous and bounded on $\widetilde{\mathfrak{b}^+} \times \mathfrak{b}^+$.

□

The final result concerns off-diagonal decay of $R_0(\lambda)$. To state this, choose SO(2)-invariant cutoff functions ϕ and ψ on the logarithmically blown up single space $\widetilde{X}$ with $\operatorname{supp}\phi \cap \operatorname{supp}\psi = \emptyset$. We can regard both as defined on $\mathfrak{b}_+$, or rather, its logarithmically blown up single space $\widetilde{\mathfrak{b}_+}$. As explained in the introduction, the front face of $\widetilde{\mathfrak{b}_+}$ can be identified with (a large sector in) the Euclidean radial compactification $\overline{\mathfrak{b}_r}$ of $\mathfrak{b}_+$. This allows us to define the angle θ between $\operatorname{supp}\phi$ and $\operatorname{supp}\psi$ on the sphere at infinity. If $\operatorname{supp}\phi$ and $\operatorname{supp}\psi$ are conic outside a compact subset of $\mathfrak{a}$, then this is simply the angle between these cones. We have already shown that for $\lambda \notin \mathrm{spec}(L_0)$, $\psi R(\lambda)\phi$ is bounded on the spaces $e^{\alpha/x}L^2_{p0}(\overline{X})$, $|\alpha| < \kappa$, but we now show that this cut off kernel actually improves decay.

Before proceeding, we note that this phenomenon is a very familiar one. Consider the Laplacian Δ on $\mathbb{R}^2_w$. The plane wave solutions of $(\Delta - \lambda)u = 0$ are those of the form $u(w) = e^{i\sqrt{\lambda}\omega_0\cdot w}$, $|\omega_0| = 1$. (As usual, we use the branch of the square root function with negative imaginary part on $\mathbb{C}\backslash\mathbb{R}^+$.) Now write w in polar coordinates as $r\omega'$. Fix ω and let $\cos\theta = \omega\cdot\omega'$, so θ is the angle between the source of the plane wave and w. Then $|u| = e^{-\operatorname{Im}\sqrt{\lambda}\cos\theta r}$, or in other words, the exponential rate of attenuation of u is proportional to $\cos\theta$. This effect is directly attributable to the structure of the resolvent $R_0(\lambda) = (\Delta-\lambda)^{-1}$ itself. Indeed, if $\chi \in \mathcal{C}^\infty(\mathbb{S}^1)$, $\chi \equiv 1$ near ω_0, then

$$u = \chi(\omega)e^{i\sqrt{\lambda}\omega_0\cdot w} - R_0(\lambda)(\Delta-\lambda)(\chi(\omega)e^{i\sqrt{\lambda}\omega_0\cdot w}).$$

We refer to [6] for an interesting discussion about a localization of this phenomenon (still in Euclidean space), concerning 'decay profiles' of solutions of $(\Delta-\lambda)u = 0$ which are only defined in cones.

When $\lambda \in \mathbb{R}^+$, one no longer obtains such decay, of course. In its place is a propagation phenomenon at infinity, seen already in [17], which plays a very important role in many-body scattering [22], [21]. The behaviour we are studying here, which might be called "dissipative propagation", should be understood as a sort of analytic continuation of these on-spectrum propagation results.

Proposition 4.6 (Dissipative propagation)**.** *Choosing cut-off functions ϕ, ψ as above, let $\theta \in (0,\pi/2)$ be less than the angle between their supports. For $\lambda \notin$*

$\operatorname{spec}(L_0)$ *and* $|\alpha| \le \kappa = -\operatorname{Im}\sqrt{\lambda-\lambda_0}$, *write* $\alpha = \kappa\cos\theta_0$ *for* $\theta_0 \in [0,\pi]$. *Choose* $\beta > \kappa\cos(\theta+\theta_0)$ *if* $\theta+\theta_0 \le \pi$, *otherwise choose* $\beta > -\kappa$. *Then for any* m,

$$\psi R_0(\lambda)\phi : e^{\alpha/x}L^2_{p0}(\overline{X}) \to e^{\beta/x}H^m_{p0}(\overline{X})$$

Proof. Since we can assume that the supports of ϕ and ψ do not intersect, we can immediately discard the on-diagonal term $R_0'(\lambda)$, and furthermore it also suffices to just prove boundedness into $e^{\beta/x}L^2_{p0}$.

Let $\beta_0 = \kappa\cos(\theta+\theta_0)$ if $\theta+\theta_0 \le \pi$, and otherwise let $\beta_0 = -\kappa$. The key point is the uniform boundedness of

$$e^{\alpha\tilde\delta(o,z')}e^{-\kappa\tilde\delta(z,z')}e^{-\beta_0\tilde\delta(o,z)} \tag{4.19}$$

when $z' \in \operatorname{supp}\phi$, $z \in \operatorname{supp}\psi$. It suffices to prove the analogous boundedness when these modified distance functions are replaced by the actual (nonsmooth) distance functions. For $z = (s, z_2) \in \mathbb{R}^+ \times \mathbb{H}^2$, define

$$w = (w_1, w_2) \in \mathbb{R}^2,\ w_1 = \log s = \delta_1(1,s),\ w_2 = \delta_2(q, z_2),$$

and define w' analogously, corresponding to $z' = (s', z_2')$. Then $\delta(o,z) = |w|$, $\delta(o,z') = |w'|$. The support conditions on ϕ and ψ mean that we restrict w and w' to lie in cones Γ and Γ' in $\mathbb{R}^2$ making an angle θ with one another. The triangle inequality gives $|\delta_2(z_2,o) - \delta_2(z_2',o)| \le \delta_2(z_2,z_2')$, hence

$$\begin{aligned}\delta(z,z') &= \sqrt{\delta_1(s,s')^2 + \delta_2(z_2,z_2')^2} \ge \sqrt{\delta_1(s,s')^2 + (\delta_2(z_2,o) - \delta_2(z_2',o))^2}\\ &= \sqrt{(w_1-w_1')^2 + (w_2-w_2')^2} = |w-w'|.\end{aligned}$$

Hence the boundedness of (4.19) follows from the estimate

$$\alpha|w'| - \kappa|w-w'| - \beta_0|w| \le 0, \qquad \forall\ w \in \Gamma,\ w' \in \Gamma', \tag{4.20}$$

or equivalently, dividing through by $\kappa > 0$,

$$\begin{aligned}&\cos\theta_0|w'| \le |w-w'| + \cos(\theta+\theta_0)|w|, &&\text{if } \theta+\theta_0 \le \pi,\\ &\cos\theta_0|w'| \le |w-w'| - |w|, &&\text{if } \theta+\theta_0 > \pi.\end{aligned} \tag{4.21}$$

These certainly hold when one or the other vector vanishes. On the other hand, if neither is zero, then it suffices to consider the case where the angle between them is exactly θ, since this configuration minimizes the right hand side and keeps the left hand side fixed.

Suppose first that $\theta+\theta_0 > \pi$. Then $\pi \ge \theta_0 > \pi - \tilde\theta \ge 0$ implies $\cos\theta_0 < \cos(\pi-\tilde\theta) = -\cos\theta$, so we must only prove that $|w| - \cos\tilde\theta|w'| \le |w-w'|$. But this follows from

$$|w|^2 = (w-w')\cdot w + w'\cdot w \le |w-w'||w| + |w'||w|\cos\tilde\theta$$

once we divide through by $|w|$.

If, on the other hand, $\theta+\theta_0 \le \pi$, then we let v'' be the unique unit vector such that $v'\cdot v'' = \cos\theta_0$ and $v\cdot v'' = \cos(\theta+\theta_0)$. Since $w'\cdot v'' = (w'-w)\cdot v'' + w\cdot v''$ and $(w'-w)\cdot v'' \le |w-w'|$ as well, we deduce that $|w'|\cos\theta_0 \le |w-w'| + |w|\cos(\theta+\theta_0)$, and this completes the proof. $\square$

Proposition 4.7. *With the same notation as above, if* $f \in e^{\alpha/x}H^m_{p0}(\overline{X})$ *and* $\theta+\theta_0 > \pi$, *then*

$$\psi R_0(\lambda)\phi f = \mu^{1/2}x^{1/2}x_2 e^{-i\sqrt{\lambda-\lambda_0}/x}h + e^{\alpha/x}H^{m+2}_{p0}(\overline{X}),$$

where h is continuous on $\widetilde{X}$. If f is SO(2)*-invariant, then $h \in \mathcal{C}^\infty(\widetilde{X})$. In particular, if $m = \infty$, and f is* SO(2)*-invariant, then*

$$\psi R_0(\lambda)\phi f = \mu^{1/2} x^{1/2} x_2 e^{-i\sqrt{\lambda-\lambda_0}/x} h', \quad h' \in \mathcal{C}^\infty(\widetilde{X}).$$

Proof. We must now consider the kernel

$$\begin{aligned}\tilde{K}'' =& e^{(\alpha+\epsilon)\delta(o,z')} e^{-i(\sqrt{\lambda-\lambda_0}\delta(z,z')-\delta(o,z))}\phi(z')\psi(z)\\ & \qquad e^{-(\delta_2(z_2,z_2')+\delta_2(q,z_2')-\delta_2(q,z_2))/2}\tilde{\delta}(z',o)^{5/2}\tilde{F}(\lambda),\end{aligned}$$

where

$$\tilde{F}(\lambda) = x^{-1/2} x_2^{-1} \tilde{\delta}(z',o)^{-5/2} e^{-\epsilon\delta(o,z')} F(\lambda).$$

Using (4.19) with κ in place of β_0, we deduce that the first factor is bounded if $z \in \operatorname{supp}\psi$, $z' \in \operatorname{supp}\phi$, and $\epsilon > 0$ is sufficiently small. Hence the arguments of Proposition 4.5 show that $\tilde{K}'' \in C^0(\widetilde{X}; L^2_b(\overline{X}))$, giving the first part of the conclusion. The rest follows as in Proposition 4.5. □

Of course, $R_0(\lambda)$ preserves SO(2)-invariance, and if f is rotationally invariant, then $R_0(\lambda)f$ also has an expansion.

We conclude by translating this result back to the original operator $L_\sharp$ and function space $\mathcal{H}$ on which it is self-adjoint by reintroducing the weight s:

Corollary 4.8. *The operator $(L_\sharp - \lambda)^{-1}$ is bounded on $e^{\alpha/x}\mathcal{H}$ for $|\alpha| < \kappa$. Moreover, for ϕ, ψ, θ, α, β as above,*

$$\psi(L_\sharp - \lambda)^{-1}\phi : e^{\alpha/x}\mathcal{H} \to e^{\beta/x}\mathcal{H}.$$

If $\alpha < -\kappa$ and $f \in e^{\alpha/x} H^\infty_{p0}(\overline{X})^{\mathrm{SO}(2)}$, then $(L_\sharp - \lambda)^{-1} f$ has a full asymptotic expansion on $\widetilde{X}$ of the form

$$(L_\sharp - \lambda)^{-1} f = s\mu^{1/2} x^{1/2} x_2 e^{-i\sqrt{\lambda-\lambda_0}/x} h', \ h' \in \mathcal{C}^\infty(\widetilde{X}).$$

5. Radial solutions and the final parametrix

We now return to our main problem, and apply the results of the last section to finish the construction of a parametrix for $\Delta - \lambda$ with compact remainder. As explained earlier, this requires finding a correction term for the (K_p-invariant) small-calculus parametrix $G_p(\lambda)$. Recall that the error terms $E_p(\lambda)$ and $F_p(\lambda)$ from Proposition 3.4 left after this first stage are K_p-invariant, hence can be regarded as acting on functions on $\mathfrak{a}$. They are in fact residual elements of the ee calculus on $\overline{\mathfrak{a}}$. Thus, at the very least, we must solve equations of the form $(\Delta - \lambda)u = f$, where f is polyhomogeneous on $\overline{\mathfrak{a}}$. We now turn to this task.

The idea is that near $H_\sharp$, Δ is well approximated by the product operator $L_\sharp$, and similarly it is well approximated near $H^\sharp$ by $L^\sharp$. However, to make this precise we must pass to the logarithmically blown up space $\widetilde{\mathfrak{a}}$ and localize there. To motivate this, recall from §2.2 the coordinates μ, ν near the corner on $\overline{\mathfrak{a}}$ and the expression (2.5) for the restriction of Δ to K_p-invariant functions, i.e. to $\mathfrak{a}$. Since Δ_{rad} is not product type, even asymptotically, near the corner, we do not know a priori how to invert it. However, working near $\nu = 0$, first note by (2.6) that

$$\Delta_{\mathrm{rad}} = \frac{1}{3}\left((\mu D_\mu)^2 + \frac{\mu + \mu^{-1}}{\mu - \mu^{-1}} i\mu D_\mu\right) + \frac{1}{4}(sD_s)^2 + \frac{1}{2} isD_s + s^2 E,$$

where $E = a\mu D_\mu + bsD_s + c$, $a, b, c \in \mathcal{C}^\infty$. The first term in parentheses on the right is just the radial part of the Laplacian on $\mathbb{H}^2$ (with respect to the metric $3h$), and were we to have kept better track of the angular derivatives, we would see the complete Laplacian on $\mathbb{H}^2$ here. Hence at least in the interior of $H_\sharp$, $\Delta - L_\sharp$ is small. Now consider the situation near the corner more carefully. The coordinates μ, ν are valid here, and we define the change of variables $t = \mu$, $\bar{s} = \mu^{1/2}\nu$, so $s = \bar{s}$, where s is the notation used in the introduction. Then

$$\mu D_\mu = tD_t + \frac{1}{2}\bar{s}D_{\bar{s}}, \quad \nu D_\nu = \bar{s}D_{\bar{s}},$$

and inserting these into (2.6) gives

$$\Delta_{\mathrm{rad}} = \frac{1}{3}((tD_t)^2 + itD_t) + \frac{1}{4}(\bar{s}D_{\bar{s}})^2 + \frac{i}{2}\bar{s}D_{\bar{s}} + E,$$

where E is a different error term, consisting of sums of of smooth multiples of the vector fields tD_t and $\bar{s}D_{\bar{s}}$ with additional factors of t or $\bar{s}$ and a polyhomogeneous function in these new coordinates vanishing at least to order one at the corner. Once again we see that the first term in parentheses is the radial part of the Laplacian and the main terms of this expression (i.e. omitting the error term) are the same as for $L_\sharp$.

This change of coordinates near the corner appears rather complicated, but in fact it represents a smooth change of coordinates near the entire closure of $\widetilde{H_\sharp}$ in $\widetilde{\mathfrak{a}}$. To see this, let $\bar{\mu} = -1/\log\mu$, $\bar{\nu} = -1/\log\nu$, $\tau = -1/\log t$, $\sigma = -1/\log\bar{s}$, from which

$$\tau = \bar{\mu}, \qquad \sigma = \frac{\bar{\mu}\bar{\nu}}{\frac{1}{2}\bar{\mu} + \bar{\nu}}.$$

The functions

$$r = \frac{1}{2}\bar{\mu} + \bar{\nu}, \ \alpha = \frac{\frac{1}{2}\bar{\mu} - \bar{\nu}}{\frac{1}{2}\bar{\mu} + \bar{\nu}}, \qquad r' = \tau + \sigma, \ \alpha' = \frac{\tau - \sigma}{\tau + \sigma}$$

give two sets of coordinates on $\widetilde{\mathfrak{a}}$ near this face, and we have finally

$$r' = \frac{1}{2}r(1+\alpha)(3-\alpha), \qquad \alpha' = (1+\alpha)/(3-\alpha).$$

This proves the claim that this coordinate change is smooth on $\widetilde{\mathfrak{a}}$ near the corner of $\widetilde{H_\sharp}$ since (r,α) and (r',α') are both smooth coordinate systems there.

Altogether, noting in particular that $\bar{s} \leq \nu$ near the corner, we have just established that

$$\Delta - L_\sharp : H^m_{\mathrm{ee}}(\bar{M})^{K_p} \to \rho_\sharp H^{m-1}_{\mathrm{ee}}(\bar{M})$$

near the closure of $H_\sharp$, and the proper venue for this approximation is $\widetilde{\mathfrak{a}}$.

Now choose a smooth K_p-invariant partition of unity $\phi_\sharp + \phi^\sharp + \phi_0 = 1$ on $\widetilde{M}$ such that $\operatorname{supp}\phi_\sharp$ and $\operatorname{supp}\phi^\sharp$ are disjoint from $\widetilde{H^\sharp} \cup \mathfrak{a}^\sharp$ and $\widetilde{H_\sharp} \cup \mathfrak{a}_\sharp$, respectively, and $\operatorname{supp}\phi_0$ is compact in M. (We do not introduce new notation for the lifts of $\mathfrak{a}^\sharp$ and $\mathfrak{a}_\sharp$ to $\widetilde{\mathfrak{a}}$ because the walls of the Weyl chambers are disjoint from where the blowup takes place.) Also, choose additional cutoffs $\psi_\sharp$ and $\psi^\sharp$ which are identically 1 on $\operatorname{supp}\phi_\sharp$ and $\operatorname{supp}\phi^\sharp$, respectively, and with supports disjoint from $\widetilde{H^\sharp} \cup \mathfrak{a}^\sharp$ and $\widetilde{H_\sharp} \cup \mathfrak{a}_\sharp$.

We now define

$$\tilde{R}(\lambda) = G_p(\lambda) - \psi^\sharp(L^\sharp - \lambda)^{-1}\phi^\sharp F(\lambda) - \psi_\sharp(L_\sharp - \lambda)^{-1}\phi_\sharp F(\lambda),$$

Notation is abused slightly here because $L^\sharp$ is not defined near $\mathfrak{a}_\sharp$, but this is why the additional cutoff $\psi^\sharp$ is included, and similarly for $L_\sharp$.

We compute that on K_p-invariant functions

$$
\begin{aligned}
(\Delta-\lambda)\tilde{R}(\lambda) &= \mathrm{Id} + (1-\phi_\sharp-\phi^\sharp)\,F(\lambda)\\
&\quad -(\Delta-L_\sharp)\,\psi_\sharp\,(L_\sharp-\lambda)^{-1}\,\phi_\sharp\,F(\lambda) - [L_\sharp,\psi_\sharp]\,(L_\sharp-\lambda)^{-1}\,\phi_\sharp\,F(\lambda)\\
&\quad -(\Delta-L^\sharp)\,\psi^\sharp\,(L^\sharp-\lambda)^{-1}\,\phi^\sharp\,F(\lambda) - [L^\sharp,\psi^\sharp]\,(L^\sharp-\lambda)^{-1}\,\phi^\sharp\,F(\lambda).
\end{aligned}
\tag{5.1}
$$

Let $x_\sharp$ and $x^\sharp$ be the defining functions of the lifts of $H_\sharp$ and $H^\sharp$ to $\widetilde{M}$, respectively.

Proposition 5.1. *The following maps are bounded*

i) *For* $|\alpha| < \kappa = -\operatorname{Im}\sqrt{\lambda-\lambda_0}$,

$$\tilde{R}(\lambda) : e^{\alpha/x} H^m_{ee}(M)^{K_p} \longrightarrow e^{\alpha/x} H^{m+2}_{ee}(M)^{K_p}$$

ii) *For* $\alpha < -\kappa$,

$$\tilde{R}(\lambda) : e^{\alpha/x} H^m_{ee}(\overline{M})^{K_p} \longrightarrow$$
$$e^{-i\sqrt{\lambda-\lambda_0}/x}\, x^{1/2}\, x_\sharp x^\sharp\, \rho_\sharp\, \rho^\sharp\, \mathcal{C}^\infty(\widetilde{M}) \;+\; e^{\alpha/x} H^{m+2}_{ee}(\overline{M})^{K_p}.$$

iii) *In particular, for* $\alpha < -\kappa$,

$$\tilde{R}(\lambda) : e^{\alpha/x} H^\infty_{ee}(\overline{M})^{K_p} \longrightarrow e^{-i\sqrt{\lambda-\lambda_0}/x}\, x^{1/2}\, x_\sharp x^\sharp\, \rho_\sharp\, \rho^\sharp\, \mathcal{C}^\infty(\widetilde{M}).$$

Now choose $\theta \in (0,\pi/2)$ *less than the angles between* $\operatorname{supp}\phi_\sharp$ *and* $\operatorname{supp} d\psi_\sharp$ *and between* $\operatorname{supp}\phi^\sharp$ *and* $\operatorname{supp} d\psi^\sharp$ *on the sphere at infinity. Then there exists a* $\gamma > 0$ *with the following properties:*

iv) *If* $|\alpha| \le \kappa$, *so that* $\alpha = \kappa\cos\theta_0$ *for some* $\theta_0 \in [0,\pi]$, *and if in addition* $\beta > \kappa\cos(\theta+\theta_0)$ *when* $\theta+\theta_0 \le \pi$ *and* $\beta > -\kappa$ *otherwise, then the error term* $\tilde{F}(\lambda) = (\Delta-\lambda)\tilde{R}(\lambda) - \mathrm{Id}$ *satisfies*

$$\tilde{F}(\lambda) : e^{\alpha/x} H^m_{ee}(M)^{K_p} \longrightarrow e^{\max(\alpha-\gamma,\beta)/x} H^{m'}_{ee}(M)^{K_p}$$

for any m, m'.

v) *If* $\theta+\theta_0 > \pi$ *and* $\alpha < -\kappa+\gamma$, *then*

$$\tilde{F}(\lambda) : e^{\alpha/x} L^2(M)^{K_p} \longrightarrow e^{-i\sqrt{\lambda-\lambda_0}/x}\, x^{1/2}\, x_\sharp x^\sharp\, \rho_\sharp\, \rho^\sharp\, \mathcal{C}^\infty(\widetilde{M}).$$

Proof. The mapping properties for $\tilde{R}(\lambda)$ follow directly from Corollary 4.8. As for the mapping properties for $\tilde{F}(\lambda)$, note that the terms $[L_\sharp,\psi_\sharp](L_\sharp-\lambda)^{-1}\phi_\sharp F(\lambda)$, $[L^\sharp,\psi^\sharp](L^\sharp-\lambda)^{-1}\phi^\sharp F(\lambda)$ in (5.1) are bounded as stated since $[L_\sharp,\psi_\sharp]$ is supported on $d\psi_\sharp$, and so we may apply Corollary 4.8. On the other hand, to handle the remaining terms $(\Delta-L_\sharp)\psi_\sharp(L_\sharp-\lambda)^{-1}\phi_\sharp F(\lambda)$, $(\Delta-L^\sharp)\psi^\sharp(L^\sharp-\lambda)^{-1}\phi^\sharp F(\lambda)$, observe that $(L^\sharp-\lambda)^{-1}\phi^\sharp F(\lambda)$ is bounded on the appropriate spaces, and then $(\Delta-L_\sharp)\psi_\sharp$ gives the additional decay. This last statement follows from the fact that $\rho_\sharp\psi_\sharp < C\,e^{-\gamma/x}$ for some $\gamma > 0$. $\square$

The following theorem is now almost immediate.

Theorem 5.2. *The resolvent* $R(\lambda) = (\Delta-\lambda)^{-1}$ *has the following mapping properties on* K_p*-invariant functions.*

i) *For* $|\alpha| < \kappa$, $m \in \mathbb{R}$,

$$R(\lambda) \in \mathcal{B}(e^{\alpha/x} H^m_{ee}(\overline{M})^{K_p}, e^{\alpha/x} H^{m+2}_{ee}(\overline{M})^{K_p});$$

ii) *for* $\alpha < -\kappa$,

$$R(\lambda) : e^{\alpha/x} H^m_{ee}(\overline{M})^{K_p} \longrightarrow e^{-i\sqrt{\lambda-\lambda_0}/x}\, x^{1/2}\, x_\sharp x^\sharp\, \rho_\sharp\, \rho^\sharp\, \mathcal{C}^\infty(\widetilde{M}) + e^{\alpha/x} H^{m+2}_{ee}(\overline{M})^{K_p};$$

iii) *in particular, still for* $\alpha < -\kappa$,

$$R(\lambda) : e^{\alpha/x} H^{\infty}_{ee}(\overline{M})^{K_p} \longrightarrow e^{-i\sqrt{\lambda-\lambda_0}/x}\, x^{1/2}\, x_\sharp x^\sharp\, \rho_\sharp\, \rho^\sharp\, \mathcal{C}^\infty(\widetilde{M}). \tag{5.2}$$

Proof. We use the parametrix identity

$$R(\lambda) = \tilde{R}(\lambda) - \tilde{R}(\lambda)\tilde{F}(\lambda) + \tilde{E}(\lambda)R(\lambda)\tilde{F}(\lambda), \tag{5.3}$$

where $\tilde{F}(\lambda)$ is as in Proposition 5.1 and $\tilde{E}(\lambda) = \tilde{R}(\lambda)(\Delta - \lambda) - I$.

Suppose first that $0 \geq \alpha > -\kappa$. By the preceeding proposition, the first two terms are bounded $e^{\alpha/x} H^m_{\mathrm{ee}}(M)^{K_p} \to e^{\alpha/x} H^{m+2}_{\mathrm{ee}}(M)^{K_p}$.

Next, write $\beta = \kappa\cos(\theta + \frac{\pi}{2})$ (corresponding to $\cos\theta_0 = 0 = 0/\kappa$), and note that $\theta + \frac{\pi}{2} < \pi$. We have $R(\lambda), \tilde{F}(\lambda) \in \mathcal{B}(L^2(M)^{K_p})$; in addition, if β is given by the usual prescription but with α replaced by 0, and if we set $\beta' = \max(-\gamma, \beta) < 0$, then $\tilde{E}(\lambda) : L^2(M)^{K_p} \to e^{\beta'/x} H^{m'}_{\mathrm{ee}}(\overline{M})^{K_p}$. Hence for any m', we deduce that

$$R(\lambda) : e^{\alpha/x} H^m_{\mathrm{ee}}(\overline{M})^{K_p} \longrightarrow e^{\alpha'/x} H^{m+2}_{\mathrm{ee}}(\overline{M})^{K_p}, \qquad \text{where} \quad \alpha' = \max(\alpha, \beta').$$

We now bootstrap to improve the image space. Thus suppose that

$$\alpha_0 = \inf\{\alpha' \in [\alpha, 0] : \ R(\lambda) \in \mathcal{B}(e^{\alpha/x} H^m_{\mathrm{ee}}(\overline{M})^{K_p},\, e^{\alpha'/x} H^{m+2}_{\mathrm{ee}}(\overline{M})^{K_p})\}.$$

We know that this set is non-empty, and that $\alpha_0 < 0$. If $\alpha_0 > \alpha$, then fix $\alpha' > \alpha_0$. Then by definition of this inf we know that

$$R(\lambda) \in \mathcal{B}(e^{\alpha/x} L^2(M)^{K_p}, e^{\alpha'/x} L^2(M)^{K_p});$$

Furthermore, both $\tilde{R}(\lambda)$ and $\tilde{F}(\lambda)$ are bounded on $e^{\alpha/x} L^2(M)^{K_p}$, while

$$\tilde{E}(\lambda) \in \mathcal{B}(e^{\alpha'/x} L^2(M)^{K_p}, e^{\alpha''/x} L^2(M)^{K_p}), \qquad \text{where} \quad \alpha'' = \min(\alpha' - \gamma, \beta').$$

Here β' is calculated from α' as usual. Choosing α' sufficiently close to α_0, this gives

$$R(\lambda) \in \mathcal{B}(e^{\alpha/x} L^2(M)^{K_p}, e^{\alpha''/x} L^2(M)^{K_p}), \qquad \text{where} \quad \alpha'' < \alpha_0.$$

But this is a contradiction, and hence necessarily $\alpha_0 = \alpha$. The same argument shows that $R(\lambda) \in \mathcal{B}(e^{\alpha/x} L^2(M)^{K_p})$.

Now suppose that $\alpha < -\kappa$. The first two terms of (5.3) map into

$$e^{-i\sqrt{\lambda-\lambda_0}/x} x^{1/2}\, x_\sharp x^\sharp\, \mathcal{C}^\infty(\widetilde{M}) + e^{\alpha/x} H^{m+2}_{\mathrm{ee}}(\overline{M})^{K_p}. \tag{5.4}$$

On the other hand,

$$\tilde{F}(\lambda) \in \mathcal{B}(e^{\alpha/x} L^2(M)^{K_p},\, e^{\alpha'/x} L^2(M)^{K_p})$$

for any $\alpha' > -\kappa$, and at the same time, $\tilde{R}(\lambda) \in \mathcal{B}(e^{\alpha'/x} L^2(M)^{K_p})$. But choosing α' sufficiently close to $-\kappa$ ensures that $\tilde{E}(\lambda)$ maps this space to

$$e^{-i\sqrt{\lambda-\lambda_0}/x} x^{1/2} x_\sharp x^\sharp \mathcal{C}^\infty(\widetilde{M}).$$

This finishes the proof. □

Corollary 5.3. *If* $\lambda \notin \mathrm{spec}(\Delta)$ *and* $f \in \mathcal{C}^\infty_c(M)$, *then*

$$R(\lambda)f = \rho_\sharp \rho^\sharp x^{1/2} x_\sharp x^\sharp \exp\left(-i\sqrt{\lambda - \lambda_0}/x\right) g,$$

where $g \in \mathcal{C}^\infty(\widetilde{M})$; *the analogous statement holds if* $f \in \mathcal{C}^{-\infty}_c(M)$, *but* g *is only* $\mathcal{C}^\infty$ *away from* sing supp f *then. In particular, this estimate holds when* $f = \delta_p$, *in which case* $R(\lambda)f = R(\lambda; w, p)$ *is the Green function with pole at* p.

Proof. If $f \in \mathcal{C}_c^\infty(M)^{K_p}$, the result follows from part (iii) of the theorem since $\mathcal{C}_c^\infty(M)^{K_p} \subset e^{\alpha/x} H_{\mathrm{ee}}^\infty(\overline{M})^{K_p}$ for all α. Next, if $f \in \mathcal{C}_c^{-\infty}(M)^{K_p}$, then we first use a local parametrix, constructed by standard methods, for Δ near the (compact!) support of f to obtain $u_0 \in \mathcal{C}_c^{-\infty}(M)^{K_p}$ with $f_0 = (\Delta - \lambda)u_0 - f \in \mathcal{C}_c^\infty(M)^{K_p}$; of course, sing supp f = sing supp u_0. Then $R(\lambda)f = u_0 - R(\lambda)f_0$, and so we have reduced to the previous case.

This last argument clearly also gives the asymptotic structure of the Schwartz kernel of $R(\lambda)$, i.e. (with an abuse of notation) the distribution $R(\lambda; p, p') = (R(\lambda)\delta_{p'})(p)$, and so this has exactly the same form. It is obvious that the asymptotics are uniform as the pole p' varies over a compact set. From this we may now deduce the the general (non K_p-invariant) case from the expression $R(\lambda)f(p) = \int_M R(\lambda; p, p') f(p')\, dg(p')$, either when f is smooth or a distribution, so long as it is compactly supported. □

6. Boundary asymptotics and the Martin compactification

As discussed in the introduction, the results in the preceding section may be used to identify $\widetilde{M}$ with the Martin compactification $\overline{M}_{\mathrm{Mar}}(\lambda)$ for any $\lambda < \lambda_0$ (which we henceforth write simply as $\overline{M}_{\mathrm{Mar}}$). The Martin boundary $\overline{M}_{\mathrm{Mar}} \setminus M$ is defined as the set of equivalence classes of sequences $U_j = R(\lambda; p, p_j)/R(\lambda; p_0, p_j)$, and we shall prove now that these equivalence classes are in bijective correspondence with points $q \in \partial\widetilde{M}$ via $U_j \to U_q$ if and only if $p_j \to q$.

We begin, however, with a somewhat more general discussion of the boundary data of functions $u = R(\lambda)f$ when f is compactly supported. First note that, by the results of §5, when $\alpha < -\kappa$, any $f \in e^{\alpha/x} H_{\mathrm{ee}}^\infty(M)$ determines a function

$$g = e^{i\sqrt{\lambda - \lambda_0}/x}\, x^{-1/2}\, (x_\sharp\, x^\sharp\, \rho_\sharp\, \rho^\sharp)^{-1} R(\lambda) f \in \mathcal{C}^\infty(\widetilde{M}). \tag{6.1}$$

It follows from the proofs there and also from the analysis of [13] that the restrictions of g to the lifts of $H_\sharp$ and $H^\sharp$ are eigenfunctions on $\mathbb{H}^2$ with eigenvalue $\frac{1}{4}$, the inf of spec $\Delta_{\mathbb{H}^2}$, (and are SO(2)-invariant when f is K_p-invariant), hence determined by their values at $\partial\overline{\mathbb{H}^2}$. This means that the restriction of g to the face mf of $\widetilde{M}$ determines the restriction of g to the entire boundary $\partial\widetilde{M}$.

Proposition 6.1. *There exists an $\epsilon > 0$ so that if $\alpha \in (-\kappa - \epsilon, -\kappa)$, then the map $A(\lambda) : e^{\alpha/x} H_{ee}^\infty(\overline{M}) \to \mathcal{C}^\infty(\mathrm{mf})$, $f \mapsto g|_{\mathrm{mf}}$, as determined by* (6.1)*, is surjective.*

Proof. Any $h \in \mathcal{C}^\infty(\mathrm{mf})$ can be decomposed as $h = \phi_\sharp h + \phi^\sharp h$, and so we may as well assume that h is supported away from $\widetilde{H_\sharp}$, say. Now identify h with a smooth function on $\widetilde{X}$. By [13], there exists a function $u_\sharp$ supported away from $H_\sharp$ with $(L_\sharp - \lambda)u_\sharp \in e^{\alpha/x} H_{\mathrm{ee}}^\infty(\overline{M})$ and such that h is the leading term in the asymptotics of $u_\sharp$. Then

$$f := (\Delta - \lambda)u_\sharp = (L_\sharp - \lambda)u_\sharp + (\Delta - L_\sharp)u_\sharp \in e^{(-\kappa-\gamma)/x} H_{\mathrm{ee}}^\infty(\widetilde{M})$$

for some sufficiently small $\gamma > 0$. Since $R(\lambda)$ is *the* inverse of $\Delta - \lambda$ on $L^2(M, dg)$, we deduce that $R(\lambda)f = u_\sharp$, so $A(\lambda)f = h$. □

Since $\mathcal{C}_c^\infty(M)$ is dense in $e^{\alpha/x} H_{\mathrm{ee}}^\infty(\overline{M})$, we immediately deduce the

Corollary 6.2. *The range of $A(\lambda) : \mathcal{C}_c^\infty(M)^{K_p} \ni f \mapsto g|_{\mathrm{mf}} \in \mathcal{C}^\infty(\mathrm{mf})^{K_p}$ is dense in $\mathcal{C}^\infty(\mathrm{mf})^{K_p}$.*

Corollary 6.3. *For $\lambda < \lambda_0$, the function $g = A(\lambda)\delta_p$ is strictly positive on $\widetilde{M}$.*

Proof. Since $f = \delta_p > 0$ in a distributional sense and $\lambda < \lambda_0$, we can apply the minimum principle to see that $R(\lambda)f$ is also strictly positive, and thus the corresponding function g is nonnegative on $\widetilde{M}$ (and strictly positive in the interior). Next, by Corollary 6.2 we can choose $f_0 \in \mathcal{C}_c^\infty(M)^{K_p}$ with $A(\lambda)f_0 = g_0 > 0$. Let O be a neighborhood of $\operatorname{supp} f_0 \cup \{p\}$ with compact closure. Since both $R(\lambda)f$ and $R(\lambda)f_0$ are strictly positive in O, we can choose $c > 0$ so that $cR(\lambda)f > R(\lambda)f_0$ on this neighborhood. On $M \setminus O$, the function $v = cR(\lambda)f - R(\lambda)f_0$ satisfies $\Delta v = 0$, $v \geq 0$ on ∂O, and $v \to 0$ at infinity. Hence $v \geq 0$ on $M \setminus O$, and so finally $g \geq c^{-1}g_0 > 0$ on $\partial\widetilde{M}$ as claimed. □

Using the symmetry of the Green function when $\lambda < \lambda_0$, we have now shown that for any such λ,

$$e^{i\sqrt{\lambda-\lambda_0}/x}x^{-1/2}(x_\sharp x^\sharp \rho_\sharp \rho^\sharp)^{-1}e^{i\sqrt{\lambda-\lambda_0}/x'}(x')^{-1/2}(x'_\sharp(x^\sharp)'\rho'_\sharp(\rho^\sharp)')^{-1}R(\lambda;w',w)$$
$$= g(\lambda;w',w) \in \mathcal{C}^\infty((M \times \widetilde{M}) \cup (\widetilde{M} \times M) \setminus \mathrm{diag})$$

and moreover, is strictly positive there. Hence if we restrict w to lie in any compact set K in M, and let $w_j \to q \in \partial\widetilde{M}$, then $U_j(w) = R(\lambda;w',w_j)/R(\lambda;w_0,w_j)$ converges to some smooth eigenfunction $U_q(w)$ which is nontrivial since $U_q(w_0) = 1$. As indicated in the introduction, if U_j is any convergent sequence of quotients of Green functions which gives rise to a point U in the Martin boundary, then the underlying sequence w_j has a convergent subsequence $w_{j'} \to q \in \partial\widetilde{M}$; hence $U_{j'} \to U_q$, and so the limit U of this initial sequence, which is unique, must equal this plan wave solution U_q. This proves the first half of the

Theorem 6.4. *If $U_j(w) = R(\lambda;w',w_j)/R(\lambda;w_0,w_j)$ converges to an eigenfunction $U(w)$ on M, then necessarily $U = U_q$ for some $q \in \partial\widetilde{M}$. If q, q' are any two distinct points on $\partial\widetilde{M}$, then $U_q \neq U_{q'}$. Hence $\widetilde{M}$ is naturally identified with $\overline{M}_{\mathrm{Mar}}$.*

We must also prove injectivity, i.e. show that the plane wave solutions U_q are distinct for different $q \in \widetilde{M}$. Let $\pi : \widetilde{M} \to \widehat{M}$ be the projection to the geodesic compactification of M. We show first that if $U_q = U_{q'}$, then $\pi(q) = \pi(q')$. This follows from estimates for the growth or decay of $U_q(w)$ as w converges to different points $q' \in \partial\widetilde{M}$, specifically the asymptotics given in Corollary 5.3. Thus suppose that $\pi(q) \neq \pi(q')$ and let γ_q, $\gamma_{q'}$ be the (unit-speed) geodesics emanating from w_0 which converge to $\pi(q)$ and $\pi(q')$, respectively, and let θ be the angle between these geodesics at w_0. Setting $F = \rho_\sharp\rho^\sharp$, then we have $(FU_q)(\gamma_q(t)) \geq C_\epsilon e^{(\kappa-\epsilon)t}$, whereas $(FU_q)(\gamma_{q'}(t)) \leq C_\epsilon e^{(\kappa\cos\theta+\epsilon)t}$, for any $\epsilon > 0$ (cf. the law of cosines [4, Corollary 1.4.4(2)] applied to the geodesic triangle with vertices w_0, w, w', and recall that x^{-1} is the smoothed distance function from $p = w_0$). Hence, assuming $\pi(q) \neq \pi(q')$, then $FU_q \neq FU_{q'}$, and so $U_q \neq U_{q'}$, and this proves that U_q determines $\pi(q)$. Since π is injective in the interior of the face mf on $\widetilde{M}$, we have now shown that the map from this portion of $\partial\widetilde{M}$ to the Martin boundary is injective.

It remains to show that U_q determines q when $q \in \widetilde{H}_\sharp$ or $\widetilde{H}^\sharp$, and this requires a bit more work. Suppose $q \in \widetilde{H}_\sharp$, say, with associated eigenfunction U_q. We shall consider sequences of points $w'_j \in M$ converging to some other point $q' \in \widetilde{H}_\sharp$, $q' \neq q$, and show that the limit of $U_q(w'_j)$ distinguishes q' from q. It suffices to consider only the case when $\pi(q) = \pi(q')$, because otherwise the discussion above applies. Thus let q and q' lie in the same fiber of $\pi|_{\widetilde{H_\sharp}}$, and recall that this fiber is a copy of $\mathbb{H}^2$. In terms of the matrix representation from §2.1, matrices associated to points in the same fibre of π share the same eigenspaces E_{12}, corresponding to the two eigenvalues whose ratio is bounded (as well as $E_3 = E_{12}^\perp$ corresponding to the third eigenvalue). By conjugation we normalize so that E_{12}, E_3 are the standard summands $\mathbb{R}^2 \oplus \mathbb{R}$, and we also assume that $p = w_0$ corresponds to the identity matrix. We shall be considering the limits of ratios $R(\lambda; w', w)/R(\lambda; p, w)$ as $w \to q$, $w' \to q'$, and it is sufficient to restrict attention to points $w, w' \in \widetilde{M}$ associated to matrices in this particular block diagonal form. For such points, let π_{12} be the corresponding projection onto points in $\mathbb{H}^2$. The asymptotic behaviour of these ratios depends on the rate of convergence to q, resp. q', as well; we consider what happens if w tends to infinity much faster than w'. Note that this is the relevant region for the pair of points (w', w) in $M \times M$, since U_q is precisely the limit of this ratio when $w \to \infty$ and w' remains bounded. The behaviour of this ratio in this region determines the asymptotics of $U_q(w')$ as $w' \to \infty$. (In other words, we are really interested in the behaviour of the resolvent kernel on a neighbourhood of the left faces of an appropriate compactification of $M \times M$.)

So suppose that w tends to infinity much faster than w', i.e. $d(p, w') \to \infty$ and $d(p, w')/d(p, w) \to 0$. Then $d(p, w)/d(w, w') \to 1$, $d(w, w') \to \infty$ by the triangle inequality. The projections $z = \pi_{12}(w)$, $z' = \pi_{12}(w')$, converge to $z_0 = \pi_{12}(q)$ and $z_0' = \pi_{12}(q')$, respectively, and so these sequences remain a bounded distance from one another in $\mathbb{H}^2$. Letting $w, w' \to \infty$ and using the asymptotics of the Green function with pole at w, we have

$$\frac{R(\lambda; w', w)}{R(\lambda; p, w)} \sim e^{-\kappa d(p, w')} (\rho_\sharp(w')\rho^\sharp(w'))^{-1} \frac{G(z_0', z_0)}{G(o, z_0)},$$

where $G(\cdot, z_0)$ is the spherical function on $\mathbb{H}^2$ centered at z_0 (which is an eigenfunction of $\Delta_{\mathbb{H}^2}$ with eigenvalue $\frac{1}{4}$) and $o = \pi_{12}(p)$. This means, more precisely, that given $\epsilon > 0$ there exists $\delta > 0$ such that if $d(p, w')/d(p, w) < \delta$, $d(p, w') > \delta^{-1}$, and if $\pi(w)$ and $\pi(w')$ are in a δ-neighborhood of $\pi(q) = \pi(q')$ with respect to some metric on $\widehat{M}$ then

$$\left| e^{\kappa d(p,w)} \rho_\sharp(w')\rho^\sharp(w') \frac{R(\lambda; w', w)}{R(\lambda; p, w)} - \frac{G(z_0', z_0)}{G(o, z_0)} \right| < \epsilon.$$

Note that if w and w' satisfy these conditions above, and in particular $d(w, w') \to \infty$, then the asymptotic expression for $R(\lambda; \cdot, w)$ involves a positive multiple of $G(\cdot, z_0)$, but this factor cancels when taking the quotient. By the very definition of $U_q(w')$, this quotient is also asymptotic to $U_q(w')$, and this shows that U_q determines $\frac{G(z_0', z_0)}{G(o, z_0)}$. But as a function of z_0', this is rotationally symmetric about z_0 and hence determines the point z_0. In other words, U_q determines q when q lies in the interior of $H_\sharp$.

For the last part, note that the same asymptotics are valid when $w \to q \in \mathrm{mf} \cap H_\sharp$ and $d_{\mathbb{H}^2}(o, z)$ is much larger than $d_{\mathbb{H}^2}(z', z)$. In this case, the function G must be replaced by the Poisson operator (or plane wave) centered at z_0, as in [13, Sections 7.1 and 10]. Hence here too U_q determines q. This completes the proof.

7. Spherical functions

The space of generalized eigenfunctions of Δ consists of the tempered distributions u on $\overline{M}$ such that $(\Delta-\lambda)u = 0$. If $p \in M$, then the K_p-invariant eigenfunctions are called spherical functions. These were introduced by Harish-Chandra, who analyzed them using a perturbation series. As noted in the introduction, the defect of this approach is that this series only converges away from the Weyl chamber walls. However, it is straightforward to use the information obtained about the resolvent to determine the behavior of these functions.

Suppose that $\xi \in \mathfrak{a}^*_{\mathbb{C}}$ with $\operatorname{Im}\xi \in (\mathfrak{a}^*)^+ \equiv \mathfrak{a}^+$ and $\xi\cdot\xi = \lambda - \lambda_0$ for some λ in the resolvent set $\mathbb{C}\setminus[\lambda_0,\infty)$. As always, set $\kappa = -\operatorname{Im}\sqrt{\lambda-\lambda_0} > 0$. Then

$$u_0(z) = \rho_\sharp(z)\rho^\sharp(z)e^{-i\xi\cdot z} \in e^{\kappa/x}x^r L^2(M)^{K_p},\ r < -1,$$

is an eigenfunction of Δ_{rad}. However, it is not smooth on $\widetilde{M}$ due to its behavior at the walls. Thus let ψ be a cutoff function which vanishes near the walls, but is identically 1 in an open cone around $\operatorname{Im}\xi$. (This is precisely where we are using that $\operatorname{Im}\xi \in (\mathfrak{a}^*)^+$.) Then

$$(\Delta-\lambda)(\psi u_0) \in e^{\alpha/x}H^\infty_{\mathrm{ee}}(\overline{M})^{K_p}$$

for some $\alpha < \kappa$, and we can thus use $R(\lambda)$ to solve away this error term. We obtain the K_p-invariant eigenfunction

$$U = \psi u_0 - R(\lambda)(\Delta-\lambda)(\psi u_0),$$

which is precisely the spherical functions associated to the parameter ξ. We have the preliminary result:

Proposition 7.1. *For $p \in M$, and $\xi \in \mathfrak{a}^*_{\mathbb{C}}$ with $\operatorname{Im}\xi \in (\mathfrak{a}^*)^+$ and $\xi\cdot\xi = \lambda-\lambda_0$, there exists a K_p-invariant eigenfunction $U_\xi \in \mathcal{C}^\infty(M)^{K_p}$ such that when $\alpha < \kappa$,*

$$|U(z) - u_0(z)| \le C\rho_\sharp(z)\rho^\sharp(z)e^{\alpha|z|},\ z \in \overline{\mathfrak{a}^+}. \tag{7.1}$$

Moreover, U_ξ is the unique K_p-invariant eigenfunction in $\mathcal{C}^{-\infty}(\overline{M})$ with

$$U(z) - u_0(z) \in \rho_\sharp(z)\rho^\sharp(z)e^{\alpha|z|}L^\infty(\mathfrak{a}^+).$$

To obtain a finer description of the structure of $U = U_\xi$, we would require a more explicit construction. This *can* be done were we to examine the parametrix construction more carefully. However, we proceed as follows, relying on results from many-body scattering for motivation. In the present setting, this approach is essentially equivalent to the work of Trombi and Varadarajan [18], although it is more flexible since it does not require the precise symmetric structure.

In any case, by analogy with three-body scattering, we expect that U decomposes as $u + v$, where u is a sum of reflections of the initial plane wave u_0 and v is an outgoing spherical wave, i.e. has the same asymptotics as the Green function (5.2). Thus we posit that

$$u(z) = \rho_\sharp(z)\rho^\sharp(z)\sum_{\sigma\in S_3} c_\sigma(z)e^{-i(\sigma\xi)\cdot z}, \tag{7.2}$$

for some choice of coefficients c_σ. We seek them in $\mathcal{C}^\infty(\overline{M})$, but note that this is much stronger than might be expected, since they should presumably lie $\mathcal{C}^\infty(\widetilde{M})$ instead, which is indeed the case in the analogous construction in three-body scattering. We also demand that $c_1 = 1$ at the corner of $H_\sharp \cap H^\sharp$, which ensures that each summand in (7.2) agrees at the corner. The coefficients c_σ are *not* well defined in the interior of the faces $H^\sharp$ and $H_\sharp$. Indeed, if $\sigma \in S_3$ represents a reflection about one of the walls, then for any ξ', the functions $e^{-i(\sigma\xi')\cdot z}$ and $e^{-i\xi'\cdot z}$ have the same asymptotics at that wall.

Despite these concerns, we can determine the c_σ, which are constant along the main face, up to exponentially decreasing terms. The main requirement is that at each wall, the two reflected terms associated to that wall must combine to yield the product of an exponential along the wall and a generalized eigenfunction of $\Delta_{\mathbb{H}^2}$ in the normal direction. The c_σ are all determined thereby from c_1 since reflections generate S_3. We now deduce that $(\Delta - \lambda)u \in e^{\alpha/x}H^\infty_{\mathrm{ee}}(\overline{M})^{K_p}$, where we obtain the improvement in the weight to α by replacing Δ with $L_\sharp$ or $L^\sharp$ in the relevant regions; in fact, we can take $\alpha = \kappa - 1$. Hence if κ is small (so λ is near to the spectrum), then we can apply Theorem 5.2 to get

$$U = \rho_\sharp(z)\rho^\sharp(z) \sum_{\sigma\in S_3} c_\sigma(z)e^{-i(\sigma\xi)\cdot z} + v,$$
$$v \in e^{-i\sqrt{\lambda-\lambda_0}/x}\, x^{1/2}\, \rho_\sharp\, \rho^\sharp\, h, \qquad h \in \mathcal{C}^\infty(\widetilde{M}).$$

It is now apparent how the symmetric space structure plays a very specific role. From Harish-Chandra's work, cf. [9], it is known that along the main face, the spherical wave is absent and U simply equals the main term u. The following theorem is therefore a consequence of both the scattering theory (which gives the structure near the walls) and the algebraic structure (which eliminates the spherical wave).

Theorem 7.2. *If $p \in M$ and ξ is chosen $\mathfrak{a}^*_{\mathbb{C}}$ with $\operatorname{Im}\xi \in (\mathfrak{a}^*)^+$ and $\xi\cdot\xi = \lambda - \lambda_0$, $\lambda \in \mathbb{C}\setminus[\lambda_0,\infty)$, then spherical function $U = U_\xi$ can be written as*

$$U_\xi(z) = \rho_\sharp(z)\,\rho^\sharp(z) \sum_{\sigma\in S_3} c_\sigma(z)e^{-i(\sigma\xi)\cdot z}, \qquad \textit{where} \quad c_\sigma \in \mathcal{C}^\infty(\overline{M}). \tag{7.3}$$

Remark 7.3. As remarked in the introduction, the fact that $c_\sigma \in \mathcal{C}^\infty(\overline{M})$ means in particular that c_σ has a Taylor expansion at the corner $\mu = \nu = 0$ of the closure of $\exp(\mathfrak{a})$ in $\overline{M}$. In fact, $\overline{M}$ is real analytic, and this Taylor series converges; this corresponds to the sum over the positive root lattice in Harish-Chandra's perturbative construction of the 'components' $\rho_\sharp(z)\,\rho^\sharp(z)\,c_\sigma(z)e^{-i(\sigma\xi)\cdot z}$ of the spherical functions. Thus, Harish-Chandra's infinite sum is 'hidden' in the statement that $c_\sigma \in \mathcal{C}^\infty(\overline{M})$.

As we have already noted, this has been previously proved by purely algebraic methods, cf. Trombi and Varadarajan, [18] and also Anker and Ji [3], Theorem 2.2.8.

Moreover, replacing u_0 by certain eigenfunctions of $L_\sharp$ or $L^\sharp$, we can also let ξ approach the Weyl chamber walls; these special eigenfunctions of the model problems are products of a radial solution of $(\Delta_{\mathbb{H}^2} - \frac{1}{4})u_2 = 0$ (i.e. a spherical function at the threshold eigenvalue 1/4) with the exponential $u_1 = se^{i(-2\log s)\xi}$, $|\xi|^2 = \lambda - \lambda_0$. The precise expression requires an additional factor involving the distance of ξ from the collision plane.

Appendix A. Detailed description of the resolvent kernels

For convenience, we review the definition of the resolvent space from [13] which carries the Schwartz kernel of the resolvent for the operator L_0 considered in §4.

Recall that $X = \mathbb{R}^+ \times \mathbb{H}^2$, and we sometimes write $M_1 = \mathbb{R}^+$, $M_2 = \mathbb{H}^2$, and also that $\overline{\mathbb{R}_+} = I$.

The product 0 double space $\overline{X}^2_{p0} = I^2_0 \times (\overline{\mathbb{H}^2})^2_0$ has six boundary hypersurfaces:

$$\mathrm{ff}_1 \times (\overline{M}_2)^2_0,\ \mathrm{lf}_1 \times (\overline{M}_2)^2_0,\ \mathrm{rf}_1 \times (\overline{M}_2)^2_0,\ (\overline{M}_1)^2_0 \times \mathrm{ff}_2,\ (\overline{M}_1)^2_0 \times \mathrm{lf}_2,\ (\overline{M}_1)^2_0 \times \mathrm{rf}_2;$$

here ff_j is the front face, and lf_j and rf_j the left and right faces of $(\overline{M}_j)^2_0$. We proceed from here by logarithmically blowing up each of the two side faces lf_j and rf_j of each factor, but *not* the front face. In other words, we just introduce the new logarithmic defining functions

$$\mathcal{R}_{\mathrm{lf}_j} = -1/\log \rho_{\mathrm{lf}_j}, \qquad \mathcal{R}_{\mathrm{rf}_j} = -1/\log \rho_{\mathrm{rf}_j}, \qquad j = 1, 2,$$

at these faces. The resulting space is denoted $(\overline{M}_j)^2_{0,\log}$. We next blow up

$$(\overline{M}_1)^2_{0,\log} \times (\mathrm{lf}_2 \cap \mathrm{rf}_2). \tag{A.1}$$

In general we would also need to blow up $(\mathrm{lf}_1 \cap \mathrm{rf}_1) \times (\overline{M}_2)^2_0$ now, but since M_1 is one dimensional, $\mathrm{lf}_1 \cap \mathrm{rf}_1 = \emptyset$.

The construction of the resolvent double space is completed by blowing up the collection of submanifolds covering

$$\begin{aligned}&(\mathrm{lf}_1 \times (\overline{M}_2)^2_{0,\log}) \cap ((\overline{M}_1)^2_{0,\log} \times \mathrm{lf}_2),\ (\mathrm{lf}_1 \times (\overline{M}_2)^2_{0,\log}) \cap ((\overline{M}_1)^2_{0,\log} \times \mathrm{rf}_2),\\ &(\mathrm{rf}_1 \times (\overline{M}_2)^2_{0,\log}) \cap ((\overline{M}_1)^2_{0,\log} \times \mathrm{lf}_2),\ (\mathrm{rf}_1 \times (\overline{M}_2)^2_{0,\log}) \cap ((\overline{M}_1)^2_{0,\log} \times \mathrm{rf}_2).\end{aligned} \tag{A.2}$$

These are mutually disjoint after the blow-up in the previous step, and so these final blowups may be done in any order. Note that (A.1)-(A.2) blow up *all five* codimension 2 corners coming from the intersections of the four 'side faces' of $\overline{X}^2_{p0}$: $\mathrm{lf}_1 \times (\overline{M}_2)^2_0$, $\mathrm{rf}_1 \times (\overline{M}_2)^2_0$, $(\overline{M}_1)^2_0 \times \mathrm{ff}_2$, $(\overline{M}_1)^2_0 \times \mathrm{lf}_2$, $(\overline{M}_1)^2_0 \times \mathrm{rf}_2$.

Definition A.1. The manifold with corners, $\overline{X}^2_{\mathrm{res}}$, obtained by the series of blow-ups of $\overline{X}^2_{p0}$ described above is called the resolvent compactification of X^2.

As proved in [13], the function

$$S = \frac{\delta_2(z_2, z_2')}{\delta_1(s, s')} \tag{A.3}$$

and its inverse S^{-1} are smooth on those portions of $\overline{X}^2_{\mathrm{res}}$ where they are bounded. In addition,

$$\begin{aligned}\mathcal{R} = \delta(z,z')^{-1} &= \big(\delta_1(s,s')^2 + \delta_2(z_2,z_2')^2\big)^{-1/2}\\ &= \big((\mathcal{R}^{-1}_{\mathrm{lf}_1} + R^{-1}_{\mathrm{rf}_1})^2 + (\mathcal{R}^{-1}_{\mathrm{lf}_2} + \mathcal{R}^{-1}_{\mathrm{rf}_2})^2\big)^{-1/2}\end{aligned}$$

is also smooth on $\overline{X}^2_{\mathrm{res}}$.

Lemma A.2. *If $P \in \mathrm{Diff}^m_{p0}(\overline{X})$ then P lifts to elements $P_L \in \mathrm{Diff}^m_b(\overline{X}^2_{\mathrm{res}})$ and $P_R \in \mathrm{Diff}^m_b(\overline{X}^2_{\mathrm{res}})$ from either the left or the right factors of the projection $\overline{X} \times \overline{X} \to \overline{X}$.*

Theorem 9.2 of [13] describes the structure of $(L_0 - \lambda)^{-1}$:

Theorem A.3. *Suppose that* $\lambda \in \mathbb{C} \setminus \mathrm{spec}(L_0)$. *Then the Schwartz kernel of* $R(\lambda) = (L_0 - \lambda)^{-1}$ *takes the following form:*

$$\begin{aligned} &R(\lambda) = R'(\lambda) + R''(\lambda),\ R'(\lambda) \in \Psi^{-2}_{p0}(\overline{X}), \\ &R''(\lambda) = (\rho_{\mathrm{lf},2}\rho_{\mathrm{rf},2})^{1/2} \exp(-i\sqrt{\lambda-\lambda_0}/\mathcal{R})F(\lambda),\ \lambda_0 = \frac{1}{3}, \end{aligned} \tag{A.4}$$

$F(\lambda)$ *is* $\pi_R^*\Omega_{p0}\overline{X}$*-valued polyhomogeneous on* $\overline{X}^2_{\mathrm{res}}$, *order* 0 *on the lift of the two front faces* $\mathrm{ff}_1 \times (\overline{M}_2)^2_{0,\log}$, $(\overline{M}_1)^2_{0,\log} \times \mathrm{ff}_2$, *order* $3/2$ *on the lift of two of the 'side faces'* $\mathrm{lf}_1 \times (\overline{M}_2)^2_0$, $\mathrm{rf}_1 \times (\overline{M}_2)^2_0$, *as well as on the front face of the blow-up* (A.1), *and order* $1/2$ *on the other two 'side faces'* $(\overline{M}_1)^2_0 \times \mathrm{lf}_2$, $(\overline{M}_1)^2_0 \times \mathrm{rf}_2$, *as well as on the front faces of the four blow-ups* (A.2). *Moreover, the restriction of the leading term of* $F(\lambda)$ *to the boundary, i.e. its principal symbol, is*

(A.5)

$$a(\lambda, S)\left(P_1^t(\lambda_1^0(S)) \otimes P_2^t(\lambda - \frac{1}{4} - \lambda_1^0(S))\right) \text{ on the front face of the blow-up of}$$

$$(\mathrm{lf}_1 \times (\overline{M}_2)^2_{0,\log}) \cap ((\overline{M}_1)^2_{0,\log} \times \mathrm{lf}_2),$$

$$a'(\lambda)\left(S_1(0) \otimes P_2^t(\lambda - \frac{1}{4})\right) \text{ on the lift of } (\overline{M}_1)^2_0 \times \mathrm{lf}_2,$$

$$a''(\lambda)\left(P_1^t(\lambda - \lambda_0) \otimes S_2(\frac{1}{12})\right) \text{ on the lift of } \mathrm{lf}_1 \times (\overline{M}_2)^2_0,$$

with $\lambda_1^0(S)$ *given by*

$$\lambda_1^0(S) = \frac{\lambda - \lambda_0}{1 + S^2}, \tag{A.6}$$

a, a', a'' *never zero,* S *as in* (A.3).

In fact, we are mostly interested in $R(\lambda)$ acting on SO(2)-invariant functions. Such functions can be regarded as living on

$$\exp(-\mathfrak{b}^+) = \mathbb{R}^+_s \times (0,1)_\mu,\ \mathfrak{b}^+ = \mathbb{R}_{w_1} \times (0,\infty)_{w_2},\ w_1 = -\log s,\ w_2 = -\log \mu;$$

the corresponding measure has a polar coordinate singularity at $\mu = 1$. As indicated by the notation, it is sometimes convenient to think of $\mathfrak{b}^+$ as a subset of

$$\mathfrak{b} = \mathbb{R}_{w_1} \times \mathbb{R}_{w_2},$$

though we always restrict our attention to $w_2 > 0$.

So let $\phi \in \mathcal{C}^\infty_c((0,\infty))$ be identically 1 near 1. Then the kernel of

$$(1 - \phi(\mu))R(\lambda)(1 - \phi(\mu))$$

can be described rather simply on a compactification of $\mathfrak{b} \times \mathfrak{b}$, supported in $\mathfrak{b}^+ \times \mathfrak{b}^+$. More precisely, in complete analogy with the construction above, we can consider the resolved double space $\overline{\mathfrak{b}^+}^2_{\mathrm{res}}$ of

$$\overline{\mathfrak{b}^+} = \overline{\mathbb{R}^+_s} \times [0,1)_\mu,$$

with the advantage that the left and right faces from the same factor never intersect. Then it is straightforward to describe the Schwartz kernel of $(1-\phi(\mu))R(\lambda)(1-\phi(\mu))$ as a distribution on $\overline{\mathfrak{b}^+}^2_{\mathrm{res}}$.

Theorem A.4. *Suppose that $\lambda \in \mathbb{C}\setminus\mathrm{spec}(L_0)$ and $\phi(\mu) \in \mathcal{C}_c^\infty((0,\infty))$ is identically 1 near $\mu = 1$. Let*

$$\overline{\mathfrak{b}+} = \overline{\mathbb{R}_s^+} \times [0,1)_\mu.$$

Then the Schwartz kernel of $(1-\phi(\mu))R(\lambda)(1-\phi(\mu))$ acting on $\mathrm{SO}(2)$-invariant functions takes the following form:

$$R'(\lambda) + R''(\lambda),\ R'(\lambda) \in \Psi_b^{-2}(\overline{\mathfrak{b}+}),$$

$$R''(\lambda) = \mu^{1/2}(\mu')^{-1/2}\tilde{\delta}(z,z')^{-3/2}\tilde{\delta}_2(z_2,z_2')\exp(-i\sqrt{\lambda-\lambda_0}/R)F_0(\lambda),\ \lambda_0 = \frac{1}{3},$$

$F_0(\lambda)$ is $\pi_R^*\Omega_b$-valued smooth on $\overline{\mathfrak{b}+}^2_{\mathrm{res}}$.

Proof. This follows by integrating out the $\mathrm{SO}(2)$ variables in the preceeding theorem. The factor $(\mu')^{-1/2}$ appears as $(\mu')^{1/2}\pi_R^*\Omega_{p0} = (\mu')^{-1/2}\pi_R^*\Omega_b$. However, it is instructive to see this directly, by constructing a parametrix for the restriction of L_0 to $\mathrm{SO}(2)$-invariant functions, which is an element of $\mathrm{Diff}_b^2(\overline{\mathfrak{b}+})$. □

Proof of Theorem 4.3. (4.10)-(4.11) follow immediately from Theorem A.4. □

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