SCATTERING MATRICES IN MANY-BODY SCATTERING

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ABSTRACT. In this paper we show that generalized eigenfunctions of many-body Hamiltonians H with short-range two-body interactions have distributional asymptotics at non-threshold channels. The leading terms of the asymptotics can be used to define a scattering matrix, and we show that this is the same (up to normalization) as that arising from the standard wave-operator approach. We also prove the existence of local distributional asymptotics for locally approximate generalized eigenfunctions in the more general setting of short range perturbations of a scattering metric, defined by Melrose in [13].

1. INTRODUCTION

The purpose of this paper is to show the equivalence of two different definitions of the S-matrices (corresponding to non-threshold channels) in the many-body problem. The standard definition comes from the channel wave operators, but we can also define them by using a distributional asymptotic expansion of generalized eigenfunctions. The existence of such distributional asymptotics can be proved in a very elementary way. One reason for considering the second definition is that it makes sense even when the operator we wish to study, i.e. the Hamiltonian, is not a perturbation (in some sense) of a well-understood global operator such as Δ in the Euclidian setting, see e.g. [13, 21] and the paragraphs after (1.6).

It is well-known that these two definitions coincide (up to normalization) in two-body scattering. In [10] Isozaki obtained the top term of an asymptotic expansion in three-body scattering which sufficed to show that the 2-cluster to 3-cluster S-matrix in three-body scattering can be described by such asymptotic expansions. Then Hassell proved in [6] the equivalence for the free channel to free channel scattering in the three-body problem. In both of these cases the asymptotic expansion is actually smooth – this holds more generally at the free cluster as shown in [20]. Here we present a perhaps somewhat simpler proof (at the cost of not obtaining some of Hassell's other results, see the remark after the statement of the Theorem) which we also generalize to other channels. The main part of the proof consists of relating the asymptotic expansions to Isozaki's representation of the S-matrix [9], see also [11] and [19]. Another alternative is to relate the asymptotics to Yafaev's representation of the S-matrix in [25]; see the remarks at the end of Section 5.

It should be emphasized that we merely give and relate two definitions of the S-matrices which is quite easy; the works of Gérard, Isozaki and Skibsted [3, 4, 9, 12] provide the main external material. The analysis of the structure of the S-matrices is a much harder task; see e.g. [9, 19, 21, 22]. In addition to connecting two approaches to scattering theory, we hope to make this paper a bridge between

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different systems of notations and concepts, namely the traditional ones and those of Melrose [13]. Before we can state the precise definitions, we need to introduce some basic (and mostly standard) notation.

We consider the Euclidian space $\mathbb{R}^n$, and we assume that we are given a (finite) family $\mathcal{X}$ of linear subspaces X_a, $a \in I$, of $\mathbb{R}^n$ which is closed under intersections and includes the subspace $X_1 = \{0\}$ consisting of the origin, and the whole space $X_0 = \mathbb{R}^n$. Let X^a be the orthocomplement of X_a, and let π^a be the orthogonal projection to X^a, π_a to X_a. A many-body Hamiltonian is an operator of the form

$$H = \Delta + \sum_{a \in I} (\pi^a)^* V_a; \tag{1.1}$$

here Δ is the positive Laplacian, $V_0 = 0$, and the V_a are real-valued functions in an appropriate class which we take here to be (not necessarily polyhomogeneous) symbols of order $-\rho < -1$ on the vector space X_a to simplify the problem: $V_a \in S^{-\rho}(X_a)$. Thus, H is a self-adjoint operator on $L^2 = L^2(\mathbb{R}^n)$. We let $R(\lambda) = (H-\lambda)^{-1}$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ be the resolvent of H.

We have a natural partial ordering on I induced by the ordering of X^a by inclusion. (Though the ordering based on inclusion of the X_a would be sometimes more natural, we use the conventional ordering.) Let $I_1 = \{1\}$ (recall that $X_1 = \{0\}$); 1 is the maximal element of I. A maximal element of $I \setminus I_1$ is called a 2-cluster; I_2 denotes the set of 2-clusters. In general, once I_k has been defined for $k = 1, \ldots, m-1$, we let I_m (the set of m-clusters) be the set of maximal elements of $I'_m = I \setminus \cup_{k=1}^{m-1} I_k$, if I'_m is not empty. If $I'_m = \{0\}$ (so I'_{m+1} is empty), we call H an m-body Hamiltonian. For example, if $I \neq \{0,1\}$, and for all $a, b \notin \{0,1\}$ we have $X_a \cap X_b = \{0\}$, then H is a 3-body Hamiltonian. The N-cluster of an N-body Hamiltonian is also called the free cluster, since it corresponds to the particles which are asymptotically free.

It is convenient to compactify these spaces as in [13]. Thus, we let $\mathbb{S}^n_+$ to be the radial compactification of $\mathbb{R}^n$ to a closed hemisphere, i.e. a ball, (using the standard map SP), and $\mathbb{S}^{n-1} = \partial \mathbb{S}^n_+$. We write $w = r\omega$, $\omega \in \mathbb{S}^{n-1}$, for polar coordinates on $\mathbb{R}^n$, and we let $x \in \mathcal{C}^\infty(\mathbb{S}^n_+)$ be such that $x = (\mathrm{SP}^{-1})^*(r^{-1})$ for $r > 1$. Hence, x is a smoothed version of r^{-1} (smoothed at the origin of $\mathbb{R}^n$), and it is a boundary defining function of $\mathbb{S}^n_+$. We usually identify (the interior of) $\mathbb{S}^n_+$ with $\mathbb{R}^n$. Thus, we write $S^{-\rho}(\mathbb{S}^n_+)$ and $S^{-\rho}(\mathbb{R}^n)$ interchangeably and we drop the explicit pull-back notation in the future and simply write $x = r^{-1}$ (for $r > 1$). We recall that under SP we have the following correspondence of weighted Sobolev spaces

$$H^{k,l}_{\mathrm{sc}}(\mathbb{S}^n_+) = H^{k,l} = H^{k,l}(\mathbb{R}^n) = \langle w \rangle^{-l} H^k(\mathbb{R}^n) \tag{1.2}$$

where $\langle w \rangle = (1 + |w|^2)^{1/2}$. Thus, for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the resolvent extends to a map

$$R(\lambda) : H^{k,l}_{\mathrm{sc}}(\mathbb{S}^n_+) \to H^{k+2,l}_{\mathrm{sc}}(\mathbb{S}^n_+). \tag{1.3}$$

Similarly, we let

$$\bar{X}_a = \mathrm{cl}(\mathrm{SP}(X_a)), \quad C_a = \bar{X}_a \cap \partial \mathbb{S}^n_+. \tag{1.4}$$

Hence, C_a is a sphere of dimension $n_a - 1$ where $n_a = \dim X_a$. Again, we write the polar coordinates on X_a (with respect to the induced metric) as $w_a = r_a \omega_a$, $\omega_a \in C_a$, and let $x_a = r_a^{-1}$ (for $r_a > 1$). Note that if a is a 2-cluster then $C_a \cap C_b = \emptyset$ unless $b \leq a$. We also define the 'singular part' of C_a as the set

$$C_{a,\mathrm{sing}} = \cup_{b \not\leq a} (C_b \cap C_a), \tag{1.5}$$

and its 'regular part' as the set

$$C'_a = C_a \setminus \cup_{b \not\leq a} C_b = C_a \setminus C_{a,\mathrm{sing}}. \tag{1.6}$$

For example, if a is a 2-cluster then $C_{a,\mathrm{sing}} = \emptyset$ and $C'_a = C_a$.

Here we would like to remark that the scattering calculus of Melrose [13], described briefly in Section 2, and the corresponding notation such as $H^{k,l}_{\mathrm{sc}}(\mathbb{S}^n_+)$ play only a very minor role in this paper and the arguments can be reworded to eliminate this role completely (at the cost of a slight loss of generality in Section 3). The advantages of the compactified notation would be more apparent when dealing with uniform behavior near infinity (i.e. near $\partial\mathbb{S}^n_+$), such as when constructing symbols of pseudo-differential operators ('observables') for propagation estimates, as in [13, 21]. Thus, keeping in mind the intended role of this paper as a bridge, we will often write out statements in both notations, e.g. as done above in (1.2).

An example of the more general geometric setting, namely a geometric generalization of the three-body problem, i.e. when $\mathbb{S}^n_+$ is replaced by a manifold with boundary X equipped with a 'scattering metric' (see Section 3), C_a by disjoint closed embedded submanifolds of ∂X (C_0 and C_1 are dropped from the notation), $H = \Delta + V$, and the potential V has certain singularities at $\cup_a C_a$, was studied in [21] and the wave front relation of the free-to-free S-matrix was analyzed there.

Corresponding to each cluster a we introduce the cluster Hamiltonian h_a as an operator on $L^2(X^a)$ given by

$$h_a = \Delta + \sum_{b \leq a} V_b, \tag{1.7}$$

Δ being the Laplacian of the induced metric on X^a. Thus, if H is a N-body Hamiltonian and a is a k-cluster, then h_a is a $(N+1-k)$-body Hamiltonian. The L^2 eigenfunctions of h_a play an important role in many-body scattering; we remark that by Froese's and Herbst's result, [1], $\mathrm{spec}_{pp}(h_a) \subset (-\infty, 0]$ (there are no positive eigenvalues). Moreover, $\mathrm{spec}_{pp}(h_a)$ is bounded below since h_a differs from Δ by a bounded operator. Note that $X^0 = \{0\}$, $h_0 = 0$, so the unique eigenvalue of h_0 is 0.

The eigenvalues of h_a can be used to define the set of thresholds of h_b. Namely, we let

$$\Lambda_a = \cup_{b<a} \mathrm{spec}_{pp}(h_b) \tag{1.8}$$

be the set of thresholds of h_a, and we also let

$$\Lambda'_a = \Lambda_a \cup \mathrm{spec}_{pp}(h_a) = \cup_{b \leq a} \mathrm{spec}_{pp}(h_b). \tag{1.9}$$

Thus, $0 \in \Lambda_a$ for $a \neq 0$ and $\Lambda_a \subset (-\infty, 0]$. It follows from the Mourre theory (see e.g. [2, 16]) that Λ_a is closed, countable, and $\mathrm{spec}_{pp}(h_a)$ can only accumulate at Λ_a. Moreover, $R(\lambda)$, considered as an operator on weighted Sobolev spaces, has a limit

$$R(\lambda \pm i0) : H^{k,l}_{\mathrm{sc}}(\mathbb{S}^n_+) \to H^{k+2,l'}_{\mathrm{sc}}(\mathbb{S}^n_+) \tag{1.10}$$

for $l > 1/2$, $l' < -1/2$, from either half of the complex plane away from

$$\Lambda = \Lambda_1 \cup \mathrm{spec}_{pp}(H). \tag{1.11}$$

In addition, L^2 eigenfunctions of h_a with eigenvalues which are not thresholds are necessarily Schwartz functions on X^a (see [1]). We also label the eigenvalues of h_a, counted with multiplicities, by integers m, and we call the pairs $\alpha = (a, m)$ channels.

We denote the eigenvalue of the channel α by ϵ_α, write ψ_α for a corresponding normalized eigenfunction, and let e_α be the orthogonal projection to ψ_α in $L^2(X^a)$. We say that a channel α is a non-threshold channel if $\epsilon_\alpha \notin \Lambda_a$. If $\lambda > \epsilon_\alpha$, we also introduce the notation

$$\lambda_\alpha = (\lambda - \epsilon_\alpha)^{1/2}. \tag{1.12}$$

It is also convenient to introduce

$$H_a = \Delta_{X_a} \otimes \mathrm{Id} + \mathrm{Id} \otimes h_a, \tag{1.13}$$

on $L^2(X_a) \otimes L^2(X^a)$, so $H = H_a + \tilde{V}_a$ where $\tilde{V}_a$ is the intercluster interaction

$$\tilde{V}_a = \sum_{b \not\leq a} V_b. \tag{1.14}$$

Note, in particular, that $\tilde{V}_a \in x^\rho S^0(U) = S^{-\rho}(U)$ where $U = \mathbb{S}^n_+ \setminus \cup_{b \not\leq a} C_b$ is a neighborhood of C'_a. We also let

$$E_\alpha = \mathrm{Id} \otimes e_\alpha, \tag{1.15}$$

and $\pi_\alpha : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(X_a)$ is given by

$$\pi_\alpha(u)(f) = u(f \otimes \overline{\psi_\alpha}) \tag{1.16}$$

(we use the real pairing for distributions as usual), so $E_\alpha(u) = \pi_\alpha(u) \otimes \psi_\alpha$. We sometimes write the coordinates on $X_a \oplus X^a$ as (w_a, w^a).

We can now define the wave-operator scattering matrices as follows. The wave operators are given by

$$W^\pm_\alpha = \operatorname*{s-lim}_{t \to \pm\infty} e^{iHt} e^{-iH_a t} J_\alpha, \tag{1.17}$$

where

$$J_\alpha f = ((\pi^a)^* \psi_\alpha)(\pi_a^* f). \tag{1.18}$$

Then the scattering operator is

$$S_{\beta\alpha} = (W^+_\beta)^* (W^-_\alpha). \tag{1.19}$$

Using the unitary Fourier transform,

$$\mathcal{F}_\alpha : L^2(X_a) \to L^2((\epsilon_\alpha, \infty) \times C_a), \tag{1.20}$$

$$\mathcal{F}_\alpha u(\lambda, \omega) = (2\pi)^{-n_a/2} 2^{-1/2} (\lambda - \epsilon_\alpha)^{(n_a-2)/4} \int e^{-i(\lambda-\epsilon_\alpha)^{1/2} \omega \cdot w} u(w)\, dw, \tag{1.21}$$

one shows that the scattering matrix

$$\hat{S}_{\beta\alpha} = \mathcal{F}_\beta S_{\beta\alpha} \mathcal{F}^*_\alpha \tag{1.22}$$

has the form

$$(\hat{S}_{\beta\alpha} f)(\lambda, \omega) = (\hat{S}_{\beta\alpha}(\lambda) f(\lambda, .))(\omega) \tag{1.23}$$

a.e. $\lambda > \epsilon_\alpha$ (see e.g. [17]).

The other definition comes from the stationary theory, more precisely from the asymptotic behavior of generalized eigenfunctions. Namely, we show in Section 2 that for $\lambda \in (\epsilon_\alpha, \infty) \setminus \Lambda$ and $g \in \mathcal{C}^\infty_c(C'_a)$, there is a unique $u \in H^{\infty, -1/2-\epsilon}_{\mathrm{sc}}(\mathbb{S}^n_+)$ (here $\epsilon \in (0, 1/2)$) such that $(H - \lambda) u = 0$, and u has the form

$$u = e^{-i\lambda_\alpha r} r^{-(n_a-1)/2} ((\pi^a)^* \psi_\alpha) v_- + R(\lambda + i0) f, \tag{1.24}$$

where $v_- \in \mathcal{C}^\infty(\mathbb{S}^n_+)$, $v_-|_{C_a} = g$, and $f \in H^{\infty,1/2+\epsilon'}_{\mathrm{sc}}(\mathbb{S}^n_+)$, $\epsilon' > 0$. In addition, in Sections 3 and 4 we show that for this u and for $\lambda > \epsilon_\beta$ the projection $\pi_\beta u$ has the following distributional asymptotic behavior:

(1.25)
$$\int_{C_b} \pi_\beta u(r_b\omega_b)\, h(\omega_b)\, d\omega_b$$
$$= r_b^{-(n-1)/2}(e^{-i\lambda_\beta r_b} Q^0_{\beta,-}(\pi_\beta u, h) + e^{i\lambda_\beta r_b} Q^0_{\beta,+}(\pi_\beta u, h) + r_b^{-\delta} v), \quad h \in \mathcal{C}^\infty_c(C'_b),$$

where $\delta > 0$, $v \in \mathcal{C}^\infty((1,\infty)) \cap L^\infty((1,\infty))$ and $Q^0_{\beta,\pm}(\pi_\beta u, .)$ define distributions in $\mathcal{C}^{-\infty}(C'_b)$ (so $Q^0_{\beta,\pm}(\pi_\beta u, h)$ give a complex number for each h). In fact, $Q^0_{\alpha,-}(\pi_\alpha u, .) = g$, $Q^0_{\beta,-}(\pi_\beta u, .) = 0$ if $\beta \neq \alpha$ are the 'incoming' distributional boundary values, and $Q^0_{\beta,+}(\pi_\beta u, .)$ are the 'outgoing' ones. Since the scattering matrix is exactly supposed to relate these, we define

$$S_{\beta\alpha}(\lambda)(g)(h) = Q^0_{\beta,+}(\pi_\beta u, h),\ u \text{ given by (1.24)}, \tag{1.26}$$

providing a continuous operator $S_{\beta\alpha}(\lambda) : \mathcal{C}^\infty_c(C'_a) \to \mathcal{C}^{-\infty}(C'_b)$. We remark that our normalization is different from that of [13, 20, 21] where this operator would be denoted by $S_{00}(-\sqrt{\lambda})$ for $\lambda > 0$ (in the free-channel setting discussed there).

In this paper we show the distributional asymptotics (1.25), and prove the following theorem:

Theorem. *If α and β are non-threshold channels then for $\lambda > \max(\epsilon_\alpha, \epsilon_\beta)$, $\lambda \notin \Lambda$,*

$$\hat{S}_{\beta\alpha}(\lambda) = c S_{\beta\alpha}(\lambda) R, \qquad c = e^{i\pi(n_a+n_b-2)/4}(\lambda_\alpha/\lambda_\beta)^{1/2}, \tag{1.27}$$

(as maps $\mathcal{C}^\infty_c(C'_a) \to \mathcal{C}^{-\infty}(C'_b)$) where R is pull back by the antipodal map on C_a.

Note that we take $S_{\beta\alpha}(\lambda)$ to be a map from smooth functions on C_a supported away from $C_{a,\mathrm{sing}}$ to distributions on C'_b, i.e. in the dual space of smooth functions on C_b supported away from $C_{b,\mathrm{sing}}$. Now, $C_{b,\mathrm{sing}}$ has measure 0 in C_b, so the L^2 spaces on C_b and C'_b can be identified. On the other hand, there are distributions on C_b which are supported on the singular part $C_{b,\mathrm{sing}}$ which are thus 0 as distributions on C'_b. Correspondingly, there is a possibility of losing information about the 'outgoing' data while proceeding in this manner. Thus, without specifying the behavior of generalized eigenfunctions of H at the singular part (e.g. by showing that the result is in L^2), our description cannot lead to versions of statements of 'asymptotic completeness' (see e.g. [18, 24]).

We remark that the case of the N-cluster to N-cluster scattering matrix is significantly simpler than the general case. Indeed, in this case we have smooth (not only distributional) asymptotics as discussed in [20]. Thus, we first note that given $\lambda > 0$, $g \in \mathcal{C}^\infty_c(C'_0)$ there exists a unique $u \in \mathcal{C}^{-\infty}(\mathbb{S}^n_+)$ which satisfies $(H - \lambda)u = 0$ and which is of the form

$$u = e^{-i\sqrt{\lambda}/x} x^{(n-1)/2} v_- + R(\lambda + i0) f, \qquad v_-|_{\mathbb{S}^{n-1}} = g \tag{1.28}$$

for some $f \in \dot{\mathcal{C}}^\infty(\mathbb{S}^n_+)$, $v_- \in \mathcal{C}^\infty(\mathbb{S}^n_+)$, at least if the V_a are 'classical' symbols (i.e. they have an asymptotic expansion); otherwise v_- is the sum of a function of the angular variable and a negative order symbol (which thus decays at ∞, i.e. at $\partial \bar{X}^a$). (On a manifold with boundary, X, $\dot{\mathcal{C}}^\infty(X)$ denotes the space of smooth functions which vanish at ∂X with all derivatives, so $\dot{\mathcal{C}}^\infty(\mathbb{S}^n_+)$ corresponds to $\mathcal{S}(\mathbb{R}^n)$ under

SP. The dual space of distributions is denoted $\mathcal{C}^{-\infty}(X)$ which thus corresponds to $\mathcal{S}'(\mathbb{R}^n)$ if $X = \mathbb{S}^n_+$.) Moreover, for any $f' \in \dot{\mathcal{C}}^\infty(\mathbb{S}^n_+)$,

$$R(\lambda + i0)f' = e^{i\sqrt{\lambda}/x} x^{(n-1)/2} v_+, \qquad v_+ \in \mathcal{C}^\infty(\mathbb{S}^{n-1} \setminus C_{0,\mathrm{sing}}). \tag{1.29}$$

Hence, with $f' = f$, we see that the asymptotic-expansion scattering matrix, $S(\lambda)$, becomes the map

$$S_{00}(\lambda) : \mathcal{C}^\infty_c(C'_0) \to \mathcal{C}^\infty(C'_0), \tag{1.30}$$

$$S_{00}(\lambda)g = v_+|_{C'_0}. \tag{1.31}$$

Note that $P_{0,+}(\lambda)g$, $g \in \mathcal{C}^\infty_c(C'_0)$, is of the form (1.28)-(1.29), so we have $u = P_{0,+}(\lambda)g$ above. Here $P_{0,+}(\lambda)$ is the Poisson operator discussed in the next section. Even the proof of the equivalence with the wave-operator S-matrix is simple: the boundary pairing of [13, Proposition 13] localizes easily, as discussed in [21, Lemma 19.8] for three-body scattering, and one immediately obtains the formula (5.7). The simple calculation of Theorem 5.1 now proves the equivalence.

I am very grateful to Andrew Hassell, whose work initiated this project, for our numerous very fruitful discussions on the equivalence of the two different definitions of S-matrices and for providing me with early versions of his manuscript [6]. I would also like to thank Richard Melrose, Erik Skibsted and the referees for their helpful comments about this paper.

2. Poisson operators

It is not hard to write down partial spherical wave type generalized eigenfunctions of H corresponding to a non-threshold channel α of a k-cluster, a. Namely, let $P^0_{a,\pm}(\sigma)$ be the Poisson operator for the free Laplacian on X_a at energy σ. Thus, spherical waves on X_a of energy σ are given by

$$u_{a,\pm} = P^0_{a,\pm}(\sigma)g \in H^{\infty,l'}_{\mathrm{sc}}(\mathbb{S}^n_+), \tag{2.1}$$

where $l' < -1/2$, $g \in \mathcal{C}^\infty(C_a)$ (recall that C_a is the sphere at 'infinity' for X_a), and they have the form

$$u_{a,\pm} = e^{-i\sqrt{\sigma} r_a} r_a^{-(n_a-1)/2} v_- + e^{i\sqrt{\sigma} r_a} r_a^{-(n_a-1)/2} v_+, \quad v_\mp|_{\partial \bar{X}_a} = g \tag{2.2}$$

with $v_\pm \in \mathcal{C}^\infty(\bar{X}_a)$. Thus, for $u_{a,+}$ only $v_-|_{\partial\bar{X}_a} = g$ is specified while for $u_{a,-}$ only $v_+|_{\partial\bar{X}_a} = g$ is. Note that the normalization here is the same as in the introduction, i.e. it differs from [13], etc. We also introduce the 'partially free' Poisson operator corresponding to H_a; it is defined as

$$\tilde{P}_{\alpha,\pm}(\lambda) = ((\pi^a)^*\psi_\alpha)\pi_a^* P^0_{a,\pm}(\lambda - \epsilon_\alpha). \tag{2.3}$$

Then $u_0 = \tilde{P}_{\alpha,\pm}(\lambda)g$ satisfies $(H_a - \lambda)u_0 = 0$ (this remains true for $g \in \mathcal{C}^{-\infty}(C_a)$).

Now assume in addition that g is supported in the regular part of C_a: $g \in \mathcal{C}^\infty_c(C'_a)$. Then it follows from the standard properties of the free Poisson operator that u_0 decays rapidly with all derivatives at $C_b \cap C_a$ for $b \not\leq a$. Hence, $(H - \lambda)u_0 = \tilde{V}_a u_0 \in H^{\infty,\rho-1/2-\epsilon}_{\mathrm{sc}}(\mathbb{S}^n_+)$ for all $\epsilon > 0$ since $\psi_\alpha \in \mathcal{S}(X^a)$ (having a non-threshold eigenvalue), so $\tilde{V}_a u_0$ is in particular in $H^{\infty,1/2+\epsilon'}_{\mathrm{sc}}(\mathbb{S}^n_+)$ for sufficiently small $\epsilon' > 0$. Thus, letting

(2.4)

$$u = u_0 - R(\lambda + i0)(H - \lambda)u_0 \in H^{\infty,l'}_{\mathrm{sc}}(\mathbb{S}^n_+), \qquad \lambda \in (\epsilon_\alpha, \infty) \setminus \Lambda,\ l' < -1/2,$$

we obtain a generalized eigenfunction of H which differs from u_0 by a purely 'outgoing' term modulo $H_{\mathrm{sc}}^{\infty,-1/2+\tilde\epsilon}(\mathbb{S}^n_+)$, $0<\tilde\epsilon<\epsilon'$, in the sense of Gérard, Isozaki and Skibsted [3, 4]. We will discuss this statement below more precisely.

Hence, we can define the channel Poisson operator corresponding to the non-threshold channel α at energy λ as

$$P_{\alpha,\pm}(\lambda):\mathcal{C}^\infty_c(C'_a)\to H_{\mathrm{sc}}^{\infty,-1/2-\epsilon}(\mathbb{S}^n_+)\subset\mathcal{C}^{-\infty}(\mathbb{S}^n_+), \tag{2.5}$$

$$P_{\alpha,\pm}(\lambda)=(\mathrm{Id}-R(\lambda\pm i0)(H-\lambda))\tilde P_{\alpha,\pm}(\lambda),\qquad \lambda\in(\epsilon_\alpha,\infty)\setminus\Lambda. \tag{2.6}$$

Thus, $u=P_{\alpha,+}(\lambda)g$ gives the generalized eigenfunction of (1.24). Note that the term $e^{i\lambda_\alpha r_a}r_a^{-(n_a-1)/2}v_+(w_a)\psi_\alpha(w^a)$ in (2.3), which corresponds to the second term in (2.2), is equal to

$$R(\lambda+i0)(H-\lambda)(e^{i\lambda_\alpha r_a}r_a^{-(n_a-1)/2}v_+(w_a)\psi_\alpha(w^a))$$

by Isozaki's uniqueness theorem (as discussed in the following paragraph) since they are both outgoing and their difference is a generalized eigenfunction.

Uniqueness of solutions of the form (1.24) is a consequence of Isozaki's uniqueness theorem [12] together with the aforementioned 'outgoing' property of $R(\lambda+i0)f$. Indeed, the difference $\tilde u$ of two such solutions satisfies $(H-\lambda)\tilde u=0$ and it is of the form $\tilde u_-+R(\lambda+i0)\tilde f$ with $\tilde u_-\in L^2_{\mathrm{sc}}(\mathbb{S}^n_+)$ and $\tilde f\in H_{\mathrm{sc}}^{\infty,1/2+\epsilon'}(\mathbb{S}^n_+)$. Let

$$B=\frac{1}{2i}\sum_{j=1}^n\Big(\frac{w_j}{\langle w\rangle}\partial_{w_j}+\partial_{w_j}\frac{w_j}{\langle w\rangle}\Big) \tag{2.7}$$

and let F_- be a 0th order symbol on $\mathbb{R}$ (i.e. $F_-\in S^0(\mathbb{R})$) satisfying $\mathrm{supp}(F_-)\subset(-\infty,\sqrt{a(\lambda)})$ where for $\lambda\in\mathrm{spec}_c(H)\setminus\Lambda$ we let

$$a(\lambda)=\inf\{\lambda-\mu:\ \mu\in\Lambda_1,\ \mu<\lambda\}>0. \tag{2.8}$$

Note that $a(\lambda)=\lambda$ for $\lambda>0$ as $\Lambda_1\subset(-\infty,0]$. Since $F_-(B)$ is bounded on $L^2_{\mathrm{sc}}(\mathbb{S}^n_+)$, $F_-(B)\tilde u_-\in L^2_{\mathrm{sc}}(\mathbb{S}^n_+)$. Also, by a version of the 'outgoing' property, [12, Theorem 1.4], $F_-(B)R(\lambda+i0)\tilde f\in H_{\mathrm{sc}}^{0,-\alpha}(\mathbb{S}^n_+)$ for some $\alpha\in(0,1/2)$, so $F_-(B)\tilde u\in H_{\mathrm{sc}}^{0,-\alpha}(\mathbb{S}^n_+)$. Hence, by [12, Theorem 1.5], $\tilde u=0$. (Isozaki actually uses a modified vector field, due to Graf [5], to allow local singularities of the potentials; we do not need this modification.)

Now, another version of the 'outgoing' property of $R(\lambda+i0)f$, $f\in H_{\mathrm{sc}}^{\infty,1/2+\epsilon'}(\mathbb{S}^n_+)$, $\epsilon'>0$, is the following. In [3] (see also [23]) it is proved that for pseudo-differential operators T_- arising by a quantization of a symbol $t_-(w,\xi)\in\mathcal{C}^\infty(T^*\mathbb{R}^n)$ satisfying estimates

$$|\partial_w^\beta\partial_\xi^\gamma t_-(w,\xi)|\le C\langle w\rangle^{-|\beta|}\langle\xi\rangle^{-k},\qquad 0\le|\beta|,\ |\gamma|\le k, \tag{2.9}$$

k sufficiently large, supported in $\frac{w\cdot\xi}{|w|}\le\sigma\sqrt{a(\lambda)}$, $-1<\sigma<1$, we have

$$T_-R(\lambda+i0)f\in H_{\mathrm{sc}}^{\infty,-1/2+\tilde\epsilon}(\mathbb{S}^n_+),\quad 0<\tilde\epsilon<\epsilon'. \tag{2.10}$$

We remark that the rather bad behavior of the symbol t_- in ξ is irrelevant in our setting as $u_0\in H_{\mathrm{sc}}^{\infty,-1/2-\epsilon}(\mathbb{S}^n_+)$. The connection between this T_- and $F_-(B)$ is provided by the fact that B is the Weyl quantization of the symbol $\frac{w\cdot\xi}{\langle w\rangle}$.

The version of this result stated in [4] is actually more useful for us to prove the equivalence of the S-matrices in Section 5. Thus, let $\zeta_a\in\mathcal{C}^\infty(\mathbb{S}^n_+)$ be supported in a sufficiently small neighborhood of C'_a. We note that this means that ζ_a is a 0th order 'classical' symbol on $\mathbb{R}^n$ with 'cone support' away from $C_{a,\mathrm{sing}} \subset \mathbb{S}^{n-1}$. Also suppose that

$$t_- \in \mathcal{C}^\infty(T^*X_a),\ \operatorname{supp}(t_-) \subset X_a \times K_{\xi_a} \subset T^*X_a,\ K \text{ compact}, \tag{2.11}$$

it satisfies estimates

$$|\partial^\alpha_{w_a}\partial^\beta_{\xi_a} t_-(w_a,\xi_a)| \leq C_{\alpha\beta}\langle w_a\rangle^{-|\alpha|}, \tag{2.12}$$

and its support satisfies

$$\operatorname{supp} t_- \subset \{(w_a,\xi_a) : \frac{w_a \cdot \xi_a}{|w_a|} \leq \sigma\sqrt{a(\lambda)},\ |w_a| \geq 1\} \tag{2.13}$$

for some $\sigma \in (-1,1)$. Let T_- be a quantization of t_- (see the remarks below on the degree of independence from the choice of a quantization map), regarded as an operator on $\mathbb{R}^n$. Then for $f \in H^{\infty,1/2+\epsilon'}_{\mathrm{sc}}(\mathbb{S}^n_+)$,

$$T_-\zeta_a R(\lambda + i0) f \in H^{\infty,-1/2+\tilde{\epsilon}}_{\mathrm{sc}}(\mathbb{S}^n_+). \tag{2.14}$$

Similar results hold if we replace $R(\lambda+i0)$ by $R(\lambda - i0)$, t_- by t_+ where for some $\sigma \in (-1,1)$, t_+ satisfies

$$\operatorname{supp} t_+ \subset \{(w_a,\xi_a) : \frac{w_a \cdot \xi_a}{|w_a|} \geq \sigma\sqrt{a(\lambda)},\ |w_a| \geq 1\}. \tag{2.15}$$

We now connect this statement with Melrose's scattering calculus [13]. Let X be a manifold with boundary. Recall that the set of polyhomogeneous scattering pseudo-differential operators of differential order m and weight l is denoted by $\Psi^{m,l}_{\mathrm{sc}}(X)$. For $X = \mathbb{S}^n_+$ it arises as the 'classical' algebra corresponding to the metric $\langle w\rangle^{-2}\,dw^2 + \langle\xi\rangle^{-2}\,d\xi^2$ and weight $\langle w\rangle^{-l}\langle\xi\rangle^m$ in Hörmander's calculus [7]; in general, it arises from this via a local coordinate identification of X with $\mathbb{S}^n_+$. We write q_L for a left quantization map from symbols on ${}^{\mathrm{sc}}T^*X$ (to be discussed below) to elements of $\Psi_{\mathrm{sc}}(X)$. Thus, operators in $\Psi^{m,l}_{\mathrm{sc}}(\bar{X}_a)$ are given by the quantization of symbols $t \in \mathcal{C}^\infty(T^*X_a)$ satisfying estimates

$$|\partial^\alpha_{w_a}\partial^\beta_{\xi_a} t| \leq C_{\alpha\beta}\langle w_a\rangle^{-l-|\alpha|}\langle\xi_a\rangle^{m-|\beta|} \tag{2.16}$$

and having asymptotic expansions at infinity (both in ξ_a and in w_a). There is also a similar calculus $\Psi_{\mathrm{scc}}(X)$ where asymptotic expansions are not required. We remark that the algebras $\Psi_{\mathrm{sc}}(X)$ and $\Psi_{\mathrm{scc}}(X)$ do not depend on the choice of a (left, right or Weyl) quantization map, and $T \in \Psi^{m,l}_{\mathrm{sc}}(X)$ implies that T is a bounded operator from $H^{m',l'}_{\mathrm{sc}}(X)$ to $H^{m'-m,l'+l}_{\mathrm{sc}}(X)$. Thus, the estimate (2.12) together with the compactness of K imply that $q_L(t_-) \in \Psi^{-\infty,0}_{\mathrm{scc}}(\bar{X}_a)$ and $T_- = q_L(t_-) \otimes \mathrm{Id}$.

To see the role played by the support condition, note that the standard coordinates on the scattering cotangent bundle, ${}^{\mathrm{sc}}T^*X$, near ∂X, are (x,y,τ,μ) where x is a boundary defining function of x and the y_j give coordinates on ∂X. That is, covectors in ${}^{\mathrm{sc}}T^*X$ are written as

$$\tau\,\frac{dx}{x^2} + \mu \cdot \frac{dy}{x}. \tag{2.17}$$

Thus,

$$\tau = -\frac{w\cdot\xi}{|w|},\ \tau^2 + |\mu|^2 = |\xi|^2, \tag{2.18}$$

in terms of the Euclidian coordinates (w,ξ) on $T^*\mathbb{R}^n$, $|\mu|$ denoting the metric length of covectors on $\mathbb{S}^{n-1}$ with respect to the standard metric, and the first term in

(2.17) is just $-\tau\,dr$. Similarly, we write τ_a instead of τ when $\mathbb{S}^n_+$ is replaced by $\bar{X}_a$. Thus, the condition (2.13) means that t_- is supported in $\tau_a \geq \sigma\sqrt{a(\lambda)}$ for some $\sigma \in (-1,1)$, and similarly (2.15) means that on $\operatorname{supp} t_+$, $\tau_a \leq \sigma\sqrt{a(\lambda)}$. Hence, a typical symbol satisfying the conditions mentioned above is a function

$$t = \phi_0(x_a)\chi(\tau_a)\phi(|\xi_a|^2) \tag{2.19}$$

where $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$, $\chi \in \mathcal{C}^\infty(\mathbb{R})$ is supported in $(\sigma\sqrt{a(\lambda)},\infty)$ or $(-\infty,\sigma\sqrt{a(\lambda)})$, and $\phi_0 \in \mathcal{C}^\infty(\mathbb{R})$ is supported near 0. Here ϕ_0 is only used to localize near $x_a = 0$ (i.e. $|w_a| = \infty$) and eliminate the singular behavior of polar coordinates near the origin; we will not indicate this explicitly in the notation from now on.

We remark that the support condition (2.13), and indeed already the support part of (2.11) depend on the choice of the quantization map (left, right, Weyl). That is, if T arises by, say, left quantization of a symbol satisfying (2.11) and (2.13), it may very well happen that it is given by Weyl quantization of a symbol which does not satisfy these estimates. However, the latter symbol still nearly satisfies these properties in the sense that, for example, outside $X_a \times K$ it is given by the restriction of a Schwartz function on T^*X_a, i.e. it is rapidly decaying in both w_a and ξ_a outside $X_a \times K$. Note that rapidly decreasing symbols t' in both w_a and ξ_a give rise to operators $T' \in \Psi^{-\infty,\infty}_{\mathrm{sc}}(\bar{X}_a)$ (using any 'reasonable' quantization map), which thus map $\mathcal{C}^{-\infty}(\bar{X}_a)$ to $\dot{\mathcal{C}}^\infty(\bar{X}_a)$. Correspondingly,

$$(T' \otimes \mathrm{Id})\zeta_a : H^{\infty,k}_{\mathrm{sc}}(\mathbb{S}^n_+) \to \dot{\mathcal{C}}^\infty(\mathbb{S}^n_+) \tag{2.20}$$

for any k with ζ_a as above, since on the support of ζ_a we have $\langle w\rangle \leq c\langle w_a\rangle$ for some $c > 0$. We also remark that if T_- is an operator arising from a symbol t_- satisfying the above support estimates and $A \in \Psi^{k,0}_{\mathrm{sc}}(\mathbb{S}^n_+)$, then the symbols of T_-A and AT_- do not necessarily satisfy the same estimates. However, as a simple consequence of the composition rule in $\Psi_{\mathrm{sc}}(\mathbb{S}^n_+)$ (cf. [7, Theorems 18.1.8 and 18.5.4]), it is still true that the symbols of these are given by a Schwartz function outside $X_a \times K$. The same is true for the commutator $[A, T_-]$. In addition, for commutators the estimate actually improves; the right hand side of (2.12) can be replaced by $C_{\alpha\beta}\langle w_a\rangle^{-1-|\alpha|}$.

3. Distributional asymptotics of locally approximate generalized eigenfunctions

We proceed to define a distributional boundary value for certain locally approximate generalized eigenfunctions of Δ. Since it requires no additional effort and it may even clarify matters, we do this in the geometric setting introduced by Melrose in [13], i.e. we take X to be a compact manifold with boundary, with scattering metric $g = \frac{dx^2}{x^4} + \frac{h'}{x^2}$ where x is a boundary defining function of X and h' is a smooth symmetric 2-cotensor on X which restricts to a metric h on ∂X. We write the induced Riemannian density on the boundary as dy. The Laplacian of g has the form

$$\Delta = (x^2D_x)^2 + i(n-1)x(x^2D_x) + x^2\Delta_0 + P, \qquad P \in x^3\operatorname{Diff}^2_{\mathrm{b}}(X) \cap x^2\operatorname{Diff}^2_{\mathrm{c}}(X), \tag{3.1}$$

on a collar neighborhood $[0,2a)_x \times \partial X$ of ∂X where Δ_0 is (the product coordinate extension of) the Laplacian of the metric on ∂X, see [13, Lemma 3]. An example of this setup is Euclidian space with the standard metric but scattering metrics exist on every compact manifold with boundary, not just on $\mathbb{S}^n_+$. Recall that with $X = \mathbb{S}^n_+$, the Euclidian metric gives the Riemannian density $r^{n-1}\,dr\,d\omega$, $d\omega$ the standard measure on the unit sphere, which in the compactified picture, $x = r^{-1}$, $dy = d\omega$, becomes $\frac{dx\,dy}{x^{n+1}}$, and the Laplacian has the form (3.1) with $P = 0$. Indeed, we have $D_r = -x^2 D_x$, so the polar coordinate expression for the (positive) Euclidian Laplacian,

$$\Delta = D_r^2 - i(n-1)r^{-1}D_r + r^{-2}\Delta_0, \tag{3.2}$$

Δ_0 being the standard Laplacian on the sphere, shows (3.1). In fact, for the many-body problem itself (i.e. in the rest of this paper) we do not need the general geometric setting, so the reader may want to keep the Euclidian picture in mind while reading this section. To facilitate this, we rewrite expressions in the Euclidian notation several times during the arguments of this section.

Let $p \in \partial X$, U a product neighborhood of p in X, so $U' = U \cap \partial X$ is a neighborhood of p in ∂X and $U = [0, 2a)_x \times U'$. For an arbitrary $u \in H_{\mathrm{sc}}^{\infty,-1/2-\epsilon}(U)$, $\epsilon \in (0, \frac{1}{2})$, which satisfies $(\Delta - \lambda^2)u \in H_{\mathrm{sc}}^{\infty,1/2+\epsilon'}(U)$, $\epsilon' \in (0, 1-\epsilon]$, we can define a distributional boundary value on U'. Note that we write the spectral parameter as λ^2 as done in [13], this is more convenient in the discussion that follows. Also, the space $H_{\mathrm{sc}}^{k,l}(U)$ denotes the *localized* version of $H_{\mathrm{sc}}^{k,l}(X)$, i.e. if $v \in \mathcal{C}^{-\infty}(U)$, we say that $v \in H_{\mathrm{sc}}^{k,l}(U)$ if $\phi v \in H_{\mathrm{sc}}^{k,l}(X)$ for all $\phi \in \mathcal{C}_c^\infty(U)$. (Perhaps $H_{\mathrm{sc,loc}}^{k,l}(U)$ would be a better notation but it is too cumbersome.) Thus, we evaluate the pairing

$$B : \mathcal{C}^{-\infty}(U) \times \mathcal{C}_c^\infty(U') \to \mathcal{C}^{-\infty}([0, 2a)_x), \tag{3.3}$$

$$B(u, h) = \int_{\partial X} x^{-(n-1)/2} u(x, y)h(y)\,dy. \tag{3.4}$$

In particular, for $X = \mathbb{S}^n_+$, the integral is $\int_{\mathbb{S}^{n-1}} r^{(n-1)/2} u(r, \omega)h(\omega)\,d\omega$. We note that for $h \in \mathcal{C}_c^\infty(U')$, $u \in H_{\mathrm{sc}}^{\infty,k}(U)$, we have

$$B(u, h) \in H_{\mathrm{sc}}^{\infty,k}([0, 2a)) \subset x^{k+1}L^2([0, 2a]; dx) = r^{-k}L^2((\frac{1}{2a}, \infty), dr). \tag{3.5}$$

Thus, $B(u, h) \in \mathcal{C}^\infty((0, a))$. We have the following result.

Proposition 3.1. *For $h \in \mathcal{C}_c^\infty(U')$ and $u \in H_{sc}^{\infty,-1/2-\epsilon}(U)$, $\epsilon \in (0, 1/2)$, satisfying $(\Delta - \lambda^2)u \in H_{sc}^{\infty,1/2+\epsilon'}(U)$, $\epsilon' \in (0, 1-\epsilon]$,*

$$B(u, h) = e^{-i\lambda/x}Q_-(u, h) + e^{i\lambda/x}Q_+(u, h),\ Q_\pm(u, h) \in \mathcal{C}^{0,\alpha}([0, a)_x), \tag{3.6}$$

here $0 < \alpha < \epsilon'$. Moreover, for each u satisfying our hypotheses,

$$Q^0_\pm(u, h) = Q^0_\pm(u)(h) = Q_\pm(u, h)|_{x=0} = Q_\pm(u, h)(0) \tag{3.7}$$

define distributions $Q^0_\pm(u) \in \mathcal{C}^{-\infty}(U')$, called the incoming ($-$) and outgoing ($+$) distributional boundary values of u. If in addition $u \in H_{sc}^{\infty,-1/2}(U)$ then $Q^0_\pm(u, h) = 0$ for all $h \in \mathcal{C}_c^\infty(U')$.

Remark 3.2. In order to avoid confusion due to the notation, we would like to emphasize that $Q_\pm(u, h)$ are functions on the interval $[0, a)_x$ while $Q^0_\pm(u, h)$ are complex numbers, namely the values of $Q_\pm(u, h)$ at $x = 0$.

We prove this proposition directly by an argument similar to those used in [13, 20]. We remark that if $(\Delta - \lambda^2)u = 0$, $u \in \mathcal{C}^{-\infty}(X)$, then $u = P_+(\lambda^2)g$ for some $g \in \mathcal{C}^{-\infty}(\partial X)$ (here $P_+(\lambda^2)$ is the Poisson operator for Δ as in (3.29)), and (3.6) follows with $Q_\pm(u, h) \in \mathcal{C}^\infty([0, a)_x)$ from stationary phase and the construction of

Melrose and Zworski (in the non-Euclidian setting), [15]. A weaker conclusion than that of the proposition, namely the description of $\mathrm{WF}_{\mathrm{sc}}(B(u,h))$, follows from the push-forward theorem proved by Hassell in [6] – of course, that theorem requires only weaker assumptions about u.

Proof. First, under our assumptions

$$B(u,h) \in x^{1/2-\epsilon}L^2(dx) \subset x^{1-\delta}L^1(dx) = r^{1+\delta}L^1(dr), \quad \delta > \epsilon, \tag{3.8}$$

while (recall that $(x^2D_x)^2 - \lambda^2 = D_r^2 - \lambda^2$)

$$\begin{aligned}((x^2D_x)^2 - \lambda^2)B(u,h) \\ = x^{-(n-1)/2}\int_{\partial X}((x^2D_x)^2 + i(n-1)x(x^2D_x) - \frac{(n-1)(n-3)x^2}{4} - \lambda^2)u\,h\,dy\end{aligned} \tag{3.9}$$

where we wrote $L^1(dx)$ for $L^1([0,a];dx)$, etc. Thus, using (3.1),

$$((x^2D_x)^2 - \lambda^2)B(u,h) + x^2x^{-(n-1)/2}\int_{\partial X} u\,\Delta_0 h\,dy \in x^{3/2+\epsilon'}L^2(dx), \tag{3.10}$$

hence

$$((x^2D_x)^2 - \lambda^2)B(u,h) \in x^{3/2+\epsilon'}L^2(dx) \subset x^{2+\delta'}L^1(dx) = r^{-\delta'}L^1(dr), \tag{3.11}$$

$0 < \delta' < \epsilon'$. At this point we are considering a function of one variable, and it makes no difference whether we take it to be x or r. Recall that (x,τ) give coordinates on ${}^{\mathrm{sc}}T^*[0,a)$, i.e. covectors in ${}^{\mathrm{sc}}T^*[0,a)$ have the form $\tau\,\frac{dx}{x^2}$. In the uncompactified notation, $r = x^{-1}$, $-\tau$ is the dual variable of $r \in (a^{-1},\infty)$ in the usual sense. We can divide $B(u,h)$ into two terms by use of a ps.d.o. (really just cutoffs on the Fourier transform side) which is the identity microlocally near $\tau = -\lambda$, vanishes near $\tau = \lambda$. We can thus concentrate on the term, say B_+, which has scattering wave front set near $\tau = -\lambda$, so $\mathcal{F}u$ is smooth away from $\tau = -\lambda$. As $x^2D_x - \lambda$ is elliptic there (it is given by multiplication by $\tau - \lambda$ conjugated by the Fourier transform: $\mathcal{F}^{-1}(\tau - \lambda)\mathcal{F}$), it can be removed, so we conclude that

$$(x^2D_x + \lambda)B_+(u,h) = f \in x^{3/2+\epsilon'}L^2(dx) \subset x^{2+\delta'}L^1(dx) = r^{-\delta'}L^1(dr). \tag{3.12}$$

Multiplying by the integrating factor $e^{-i\lambda/x}$ and integrating gives

$$B_+(u,h)(x) = e^{i\lambda/x}(e^{-i\lambda/x_0}B_+(u,h)(x_0) + \int_{x_0}^{x} ie^{-i\lambda/t}f(t)t^{-2}\,dt). \tag{3.13}$$

Since $f \in x^2L^1(dx)$ (i.e. $f \in L^1(dr)$, note that $-x^{-2}\,dx = d(1/x)$ induces the Lebesgue measure in $r = x^{-1}$), it follows that the integral and hence the factor in parantheses, $Q_+(u,h)$, are continuous functions on $[0,a)_x$. In addition, using this continuity, for $\alpha \in [0,\delta']$,

$$|x^{-\alpha}(Q_+(u,h)(x) - Q_+(u,h)(0))| \le \int_0^x |f(t)|x^{-\alpha}t^{-2}\,dt \le \int_0^x |f(t)|t^{-2-\alpha}\,dt, \tag{3.14}$$

which converges so $Q_+(u,h) \in \mathcal{C}^{0,\alpha}([0,a))$ for all $\alpha \in (0,\epsilon')$. Hence, (3.6) holds. Taking into account the explicit expression for f, it also follows that $Q^0_\pm(u,.)$ are continuous from $\mathcal{C}^\infty_c(U')$ to $\mathbb{C}$, so they define a distribution for each u satisfying our hypotheses.

Finally, if $u \in H_{\mathrm{sc}}^{\infty,-1/2}(U)$, then $B(u,h) \in x^{1/2}L^2(dx)$. Since

$$e^{\pm i\lambda/x}(Q_\pm(u,h) - Q^0_\pm(u,h)) \in x^{1/2}L^2(dx), \tag{3.15}$$

we conclude that

$$e^{-i\lambda/x}Q^0_-(u,h) + e^{i\lambda/x}Q^0_+(u,h) \in x^{1/2}L^2(dx) = r^{1/2}L^2(dr). \tag{3.16}$$

This, however, implies that $Q^0_+(u,h) = Q^0_-(u,h) = 0$. □

Of course, if u has a local asymptotic expansion

$$u = e^{-i\lambda/x}x^{(n-1)/2}v_- + e^{i\lambda/x}x^{(n-1)/2}v_+, \qquad v_\pm \in \mathcal{C}^\infty(U), \tag{3.17}$$

then $Q^0_\pm(u,.) = v_\pm|_{\partial X}$. For example, in the free-to-free scattering discussed in (1.28)-(1.29), $Q^0_\pm(u,.)$ are just the boundary restrictions of $v_\pm$, so for $u = P_{0,+}(\lambda^2)g$, $g \in \mathcal{C}_c^\infty(C_0')$, we have $Q^0_-(u) = g$, $Q^0_+(u) = S_{00}(\lambda^2)g$.

It is very useful to have a different way of calculating the distributional boundary values. The idea is to relate the pairing B to the pairing

$$\langle u_1, (\Delta - \lambda^2)u_2\rangle - \langle(\Delta - \lambda^2)u_1, u_2\rangle, \tag{3.18}$$

here considered under the assumption that $u_j \in H_{\mathrm{sc}}^{\infty,-1/2-\epsilon}(X)$, $(\Delta - \lambda^2)u_j \in H_{\mathrm{sc}}^{\infty,1/2+\epsilon}(X)$, $j = 1, 2$. The connection between this pairing and $Q^0_\pm$ is given by the following proposition. It is convenient to introduce a complex-pairing (linear in the first variable) notation for $Q^0_\pm$ as well (as opposed to the real distributional pairing); we write

$$\langle Q^0_\pm(u), h\rangle = Q^0_\pm(u, \bar{h}). \tag{3.19}$$

Proposition 3.3. *Suppose that* $u_1 = u \in H_{sc}^{\infty,-1/2-\epsilon}(U)$ *satisfies* $(\Delta - \lambda^2)u \in H_{sc}^{\infty,1/2+\epsilon}(U)$, $\epsilon \in (0, 1/2)$, *and*

$$u_2 = u_{2,+} + u_{2,-}, \qquad u_{2,\pm} = e^{\pm i\lambda/x}x^{(n-1)/2}v_\pm, \qquad v_\pm \in \mathcal{C}_c^\infty(U). \tag{3.20}$$

Let $h_\pm = v_\pm|_{\partial X} \in \mathcal{C}_c^\infty(U')$. *Then*

$$\langle Q^0_+(u_1), h_+\rangle - \langle Q^0_-(u_1), h_-\rangle = \frac{1}{2i\lambda}(\langle u_1, (\Delta - \lambda^2)u_2\rangle - \langle(\Delta - \lambda^2)u_1, u_2\rangle). \tag{3.21}$$

Note that

$$(\Delta - \lambda^2)u_{2,+} = e^{i\lambda/x}x^{(n+3)/2}v', \qquad v' \in \mathcal{C}_c^\infty(U), \tag{3.22}$$

so $u_{2,+} \in H_{\mathrm{sc}}^{\infty,-1/2-\epsilon}(X)$, $(\Delta - \lambda^2)u_{2,+} \in H_{\mathrm{sc}}^{\infty,3/2-\epsilon}(X)$, and similarly with $u_{2,-}$. Thus, the right hand side of (3.21) is automatically defined.

This proposition is analogous to the 'boundary pairing' discussed in [13, Proposition 13] and its localized version in [21, Lemma 19.8]. Note also that Isozaki uses essentially the right hand side of (3.21) in [10, Theorem 1.2] to describe the spatial asymptotics of generalized eigenfunctions (coming from a 2-cluster) in three-body scattering in an averaged sense; see also Corollary 3.4.

Proof. We can take $u_2 = u_{2,+}$; the proof for $u_2 = u_{2,-}$ is similar, and in general we just add the two thus derived formulae.

Let $\phi \in \mathcal{C}^\infty([0,a])$ be identically 0 near 0, identically 1 near a, so $\phi(x)$ can be regarded as a smooth function on X which is identically 1 away from a collar neighborhood of ∂X, then (3.18) is given by

(3.23)
$$\lim_{s\to 0}\int (u_1\,\overline{\Delta u_2} - \Delta u_1\,\overline{u_2})\phi(x/s)\,dg = \lim_{s\to 0}\int -u_1\,\overline{2(x^2 D_x\phi(x/s))(x^2 D_x u_2)}\,\frac{dx\,dy}{x^{n+1}}.$$

Here we used that the other terms are of the form $\int f\psi(x/s)\,dg$ with $\psi \in \mathcal{C}_c^\infty((0,1))$ and $f \in L^1(dg)$, and by the dominated convergence theorem these integrals go to 0 as $s \to 0$. Using this remark and the explicit form of u_2, we deduce that (3.18) is given by

$$2i\lambda \lim_{s\to 0}\int u_1\,(\frac{x}{s}\phi'(x/s))e^{-i\lambda/x}\,\overline{v(0,y)}\,x^{-(n+1)/2}\,dx\,dy. \tag{3.24}$$

Letting $h(y) = v(0,y)$, we see that (3.18) is

$$\begin{aligned} &2i\lambda \lim_{s\to 0}\int_0^1 B(u_1,\bar h)\,(\frac{x}{s}\phi'(x/s))e^{-i\lambda/x}\,x^{-1}\,dx\\ &\qquad = 2i\lambda \lim_{s\to 0}\int_0^1 Q^0_+(u_1,\bar h)\,(\frac{x}{s}\phi'(x/s))\,x^{-1}\,dx\\ &\qquad = 2i\lambda Q^0_+(u_1,\bar h)\lim_{s\to 0}\phi(1/s) = 2i\lambda Q^0_+(u_1,\bar h). \end{aligned} \tag{3.25}$$

Here we could replace $Q_\pm(u_1,\bar h)$ by $Q^0_\pm(u_1,\bar h)$ by the dominated convergence theorem applied to the difference

$$\int_0^1 (Q_+(u_1,\bar h) - Q^0_+(u_1,\bar h))\,(\frac{x}{s}\phi'(x/s))\,x^{-1}\,dx, \tag{3.26}$$

noting that

$$x^{-\alpha}(Q_+(u_1,\bar h) - Q^0_+(u_1,\bar h)) \in L^\infty([0,a]), \qquad \alpha \in [0, 1-\epsilon). \tag{3.27}$$

We could also drop the Q^0_- term because it is of the form

$$\begin{aligned} &2i\lambda \lim_{s\to 0}\int_0^1 Q^0_-(u_1,\bar h)\,\frac{x}{s}\phi'(x/s)e^{-2i\lambda/x}\,x^{-1}\,d(\frac{x}{s})\\ &\qquad = 2i\lambda Q^0_-(u_1,\bar h)\lim_{s\to 0} s\int_{\mathbb{R}} e^{2i\lambda t/s}t^{-3}\phi'(t^{-1})\,dt, \end{aligned} \tag{3.28}$$

where the integral is essentially the Fourier transform of a smooth compactly supported function evaluated at $1/s$, which is rapidly decreasing as $s \to 0$. Thus, (3.25), together with the remark at the beginning of the proof, completes the proof of the proposition. □

Let $P_\pm(\lambda^2)$ be the Poisson operator for Δ at energy λ^2, $\lambda > 0$. Thus, for $h \in \mathcal{C}^\infty(\partial X)$, $P_\pm(\lambda^2)h$ has the asymptotic expansion

$$P_\pm(\lambda^2)h = e^{-i\lambda/x}x^{(n-1)/2}v_- + e^{i\lambda/x}x^{(n-1)/2}v_+, \quad v_\mp|_{\partial X} = h, \tag{3.29}$$

$v_\pm \in \mathcal{C}^\infty(X)$. Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ be identically 0 on $(\lambda-\delta,\infty)$, 1 on $(-\infty,-\lambda+\delta)$, $\delta > 0$, and $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ identically 1 near λ^2,

$$T_+ = q_L(\chi(\tau)\phi(\tau^2 + |\mu|_h^2)) \in \Psi^{-\infty,0}_{\mathrm{sc}}(X). \tag{3.30}$$

Note that for $X = \mathbb{S}^n_+$ the second factor is $\phi(|\xi|^2)$ in terms of the Euclidian coordinates, and in case Δ is the Euclidian Laplacian and we use the left quantization, this is the same as $T_+\phi^\flat(\Delta)$ if $\phi^\flat \equiv 1$ on $\operatorname{supp}\phi$ (otherwise the difference is in $\Psi_{\mathrm{sc}}^{-\infty,\infty}(\mathbb{S}^n_+)$). Since $u_2 = u_{2,+} = T_+P_-(\lambda^2)h$ satisfies the hypotheses of the proposition, we immediately deduce the following corollary.

Corollary 3.4. *Suppose that $u \in \mathcal{C}^{-\infty}(X)$ satisfies $u \in H_{sc}^{\infty,-1/2-\epsilon}(U)$ and $(\Delta - \lambda^2)u \in H_{sc}^{\infty,1/2+\epsilon}(U)$, $\epsilon \in (0,1/2)$. For $h \in \mathcal{C}_c^\infty(U')$ we have then*

$$\langle Q_+^0(u), h\rangle = \frac{1}{2i\lambda}(\langle u, (\Delta-\lambda^2)T_+P_-(\lambda^2)h\rangle - \langle(\Delta-\lambda^2)u, T_+P_-(\lambda^2)h\rangle). \tag{3.31}$$

4. Scattering matrices

Motivated by the definition of $S_{00}(\lambda)$, we define the asymptotic expansion scattering matrices at the other k-clusters using distributional boundary values. This is a little more complicated since at the other k-clusters we can have a number of channels, including channels corresponding to adjacent k'-clusters, and since we do not have smooth asymptotic expansions in the sense of (1.28)-(1.29). Thus, we project u to the eigenspace corresponding to a channel $\beta = (b, m)$. As usual, we write $\lambda_\beta = (\lambda - \epsilon_\beta)^{1/2}$. Let $\pi_\beta : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(X_b)$ as in the Introduction, so

$$\pi_\beta : \langle w^b\rangle^s H_{\mathrm{sc}}^{\infty,k}(\mathbb{S}^n_+) \to H_{\mathrm{sc}}^{\infty,k}(\bar{X}_b) \tag{4.1}$$

for any k and s (we can allow a factor of $\langle w^b\rangle^s$ corresponding to the rapid decay of ψ_β). Actually, we need a local version of this, namely that

$$\pi_\beta : \langle w^b\rangle^s H_{\mathrm{sc}}^{\infty,k}(\mathbb{S}^n_+ \setminus C_{b,\mathrm{sing}}) \to H_{\mathrm{sc}}^{\infty,k}(\bar{X}_b \setminus C_{b,\mathrm{sing}}) \tag{4.2}$$

which follows easily from the product decomposition $\mathbb{R}^n = X_b \times X^b$. Thus, if $u \in H_{\mathrm{sc}}^{\infty,-1/2-\epsilon}(\mathbb{S}^n_+)$, $(H-\lambda)u \in \dot{\mathcal{C}}^\infty(\mathbb{S}^n_+)$, then

$$\begin{aligned}(H_b - \lambda)u = (H-\lambda)u - \tilde{V}_b u \in &\langle w^b\rangle^\rho H_{\mathrm{sc}}^{\infty,\rho-1/2-\epsilon}(\mathbb{S}^n_+ \setminus C_{b,\mathrm{sing}})\\ &\subset \langle w^b\rangle^\rho H_{\mathrm{sc}}^{\infty,1/2+\epsilon'}(\mathbb{S}^n_+ \setminus C_{b,\mathrm{sing}})\end{aligned} \tag{4.3}$$

for sufficiently small $\epsilon' > 0$, so

$$(\Delta_{X_b} - (\lambda_\beta)^2)\pi_\beta u \in H_{\mathrm{sc}}^{\infty,1/2+\epsilon'}(U), \quad U = \bar{X}_b \setminus C_{b,\mathrm{sing}}; \tag{4.4}$$

thus U is a neighborhood of C'_b in $\bar{X}_b$. (Recall here that $H_{\mathrm{sc}}^{\infty,1/2+\epsilon'}(U)$ denotes the *localized* weighted Sobolev space!) Hence, the previous analysis applies with $\bar{X}_b$ in place of X, $x_b = |w_b|^{-1}$ in place of x, so we can conclude that for $h \in \mathcal{C}_c^\infty(C'_b)$,

$$B_{\bar{X}_b}(\pi_\beta u, h) = e^{-i\lambda_\beta/x_b}Q_{\beta,-}(\pi_\beta u, h) + e^{i\lambda_\beta/x_b}Q_{\beta,+}(\pi_\beta u, h). \tag{4.5}$$

We let

$$Q^0_{\beta,\pm}\pi_\beta u(h) = Q^0_{\beta,\pm}(\pi_\beta u, h) = Q_{\beta,\pm}(\pi_\beta u, h)|_{x_b=0} \tag{4.6}$$

be the incoming and outgoing distributional boundary values of $\pi_\beta u$ on $\bar{X}_b$.

We can now define the asymptotic expansion scattering matrix from channel α to channel β for $\lambda > \max(\epsilon_\alpha, \epsilon_\beta)$, $\lambda \notin \Lambda$, as the map

$$S_{\beta\alpha}(\lambda) : \mathcal{C}_c^\infty(C'_a) \to \mathcal{C}^{-\infty}(C'_b) \tag{4.7}$$

given by

$$S_{\beta\alpha}(\lambda)g = Q^0_{\beta,+}\pi_\beta P_{\alpha,+}(\lambda)g. \tag{4.8}$$

We also remark that directly from (2.6) and Proposition 3.1, taking into account that $R(\lambda+i0)f$, $f\in H^{\infty,1/2+\epsilon}_{\mathrm{sc}}(\mathbb{S}^n_+)$, $\epsilon>0$, is 'outgoing' modulo $H^{\infty,-1/2+\epsilon'}_{\mathrm{sc}}(\mathbb{S}^n_+)$, $0<\epsilon'<\epsilon$, by [3], we have

$$Q^0_{\alpha,-}\pi_\alpha P_{\alpha,+}(\lambda)g=g,\qquad Q^0_{\beta,-}\pi_\beta P_{\alpha,+}(\lambda)g=0,\ \beta\neq\alpha. \tag{4.9}$$

Here it is actually convenient to use (2.14) with a replaced by b and with a symbol as in (2.19) to deduce that

$$q_L(t_-)\pi_\beta R(\lambda+i0)f\in H^{\infty,-1/2+\epsilon'}_{\mathrm{sc}}(\bar X_b\setminus C_{b,\mathrm{sing}}). \tag{4.10}$$

An application of the push-forward theorem of [6] (or alternatively conjugation by the Fourier transform and standard wave front set analysis, cf. the discussion after (3.11)) is used to show that the vanishing of $Q^0_{\beta,-}(\pi_\beta R(\lambda+i0)f)$ follows from (4.10).

5. Equivalence of scattering matrices

We show in this section that the asymptotic expansion scattering matrices $S_{\beta\alpha}(\lambda)$ agree with the wave-operator ones, $\hat S_{\beta\alpha}(\lambda)$, up to normalization. We apply Corollary 3.4 with $X=\bar X_b$, λ replaced by λ_β, $u_1=\pi_\beta u$, $u=P_{\alpha,+}(\lambda)g$, $g\in\mathcal{C}^\infty_c(C'_a)$. First, we lift the pairing to one on $\mathbb{R}^n$ by replacing $\pi_\beta u$ by $E_\beta u$ and $P^0_{b,-}(\lambda_\beta^2)h$ by $\tilde P_{\beta,-}(\lambda)h$. This amounts to tensoring each factor in each pairing on the right hand side of (3.31) by ψ_β, so using $\|\psi_\beta\|=1$, we deduce that (3.31) becomes

$$\frac{1}{2i\lambda_\beta}(\langle E_\beta u,(\Delta_{X_b}-(\lambda_\beta)^2)T_+\tilde P_{\beta,-}(\lambda)h\rangle-\langle(\Delta_{X_b}-(\lambda_\beta)^2)E_\beta u,T_+\tilde P_{\beta,-}(\lambda)h\rangle). \tag{5.1}$$

Here T_+ denotes the lifted operator, $T_+=q_L(\chi(\tau_b)\phi(\xi_b^2))\otimes\mathrm{Id}$, $\chi\in\mathcal{C}^\infty(\mathbb{R})$, $\chi\equiv1$ on $(-\infty,-\lambda_\beta+\epsilon)$, $\chi\equiv0$ on $(\lambda_\beta-\epsilon,\infty)$, $\epsilon>0$, and $\phi\in\mathcal{C}^\infty_c(\mathbb{R})$, $\phi\equiv1$ near λ_β^2. As $\lambda_\beta^2\geq a(\lambda)$ by (2.8), we may make the stronger assumption that

$$\chi\in\mathcal{C}^\infty(\mathbb{R}),\ \chi\equiv1\text{ on }(-\infty,-\sqrt{a(\lambda)}+\epsilon),\ \chi\equiv0\text{ on }(\sqrt{a(\lambda)}-\epsilon,\infty). \tag{5.2}$$

We note that, due to the presence of E_β above and the remarks following (3.30), T_+ is the same as $T_+\tilde\phi(H_b)$ if $\tilde\phi\in\mathcal{C}^\infty_c(\mathbb{R})$ is identically 1 near $\operatorname{supp}\phi+\epsilon_\beta$. Also, due to the decay in ξ the factor $\tilde\phi(H_b)$ brings, $T_+\tilde\phi(H_b)$ is a very nice operator; it lies in $\Psi^{-\infty,0}_{3\mathrm{sc}}(\mathbb{S}^n_+)$ in the three-body problem where this calculus has been constructed in [21], and an analogous result is expected to hold in general.

Now E_β is symmetric, commutes with Δ_{X_b} and with T_+, $E_\beta\tilde P_{\beta,-}(\lambda)h=\tilde P_{\beta,-}(\lambda)h$, so we can rewrite the above expression as

$$\frac{1}{2i\lambda_\beta}(\langle u,(\Delta_{X_b}-(\lambda_\beta)^2)T_+\tilde P_{\beta,-}(\lambda)h\rangle-\langle(\Delta_{X_b}-(\lambda_\beta)^2)u,T_+\tilde P_{\beta,-}(\lambda)h\rangle). \tag{5.3}$$

Moreover, taking into account that $\psi_\beta\in\mathcal{S}(X^b)$ and $(h_\beta-\epsilon_\beta)\psi_\beta=0$, we can replace $\Delta_{X_b}-(\lambda_\beta)^2$ by $H_b-\lambda$, yielding

$$\frac{1}{2i\lambda_\beta}(\langle u,(H_b-\lambda)T_+\tilde P_{\beta,-}(\lambda)h\rangle-\langle(H_b-\lambda)u,T_+\tilde P_{\beta,-}(\lambda)h\rangle. \tag{5.4}$$

In addition, as $h\in\mathcal{C}^\infty_c(C'_b)$, $\tilde P_{\beta,-}(\lambda)h$ vanishes with all derivatives to infinite order near $C_{b,\mathrm{sing}}$, thus near $\cup_{c\not\leq b}C_c$ due to the rapid decay of ψ_β. Correspondingly, if we replace $H_b - \lambda$ by $\tilde{V}_b$ in the pairing above, it again vanishes, so we can replace $H_b - \lambda$ by $H - \lambda$. We are thus led to conclude that

$$2i\lambda_\beta \langle Q^0_{\beta,+}\pi_\beta u, h\rangle = \langle u, (H-\lambda)T_+\tilde{P}_{\beta,-}(\lambda)h\rangle - \langle (H-\lambda)u, T_+\tilde{P}_{\beta,-}(\lambda)h\rangle. \tag{5.5}$$

Since we have $u = P_{\alpha,+}(\lambda)g$, $g \in \mathcal{C}^\infty_c(C'_a)$, we deduce that

$$\langle S_{\beta\alpha}(\lambda)g, h\rangle = \frac{1}{2i\lambda_\beta}\langle P_{\alpha,+}(\lambda)g, (H-\lambda)T_+\tilde{P}_{\beta,-}(\lambda)h\rangle, \tag{5.6}$$

so

$$S_{\beta\alpha}(\lambda) = \frac{1}{2i\lambda_\beta}((H-\lambda)T_+\tilde{P}_{\beta,-}(\lambda))^* P_{\alpha,+}(\lambda), \tag{5.7}$$

$\lambda > \max(\epsilon_\alpha, \epsilon_\beta)$, $\lambda \notin \Lambda$.

We are now ready to prove the main theorem. Recall that the wave operator scattering matrix is $\hat{S}_{\beta\alpha}(\lambda)$.

Theorem 5.1. *The asymptotic expansion and the wave-operator scattering matrices are equal up to normalization. More precisely, for $\lambda > \max(\epsilon_\alpha, \epsilon_\beta)$, $\lambda \notin \Lambda$,*

$$\hat{S}_{\beta\alpha}(\lambda) = cS_{\beta\alpha}(\lambda)R, \qquad c = e^{i\pi(n_a+n_b-2)/4}\left(\frac{\lambda_\alpha}{\lambda_\beta}\right)^{1/2}, \tag{5.8}$$

(as maps $\mathcal{C}^\infty_c(C'_a) \to \mathcal{C}^{-\infty}(C'_b)$) where R is pull back by the antipodal map on C_a.

Proof. We only deal with $\alpha = \beta$, the other case being extremely similar. Let $\mathcal{F}_\alpha(\lambda)$ be the trace of the Fourier transform (1.21):

$$(\mathcal{F}_\alpha(\lambda)f)(\omega) = (\mathcal{F}_\alpha f)(\lambda, \omega) \tag{5.9}$$

and $T_+ = q_L(\chi(\tau_a)\phi(|\xi_a|^2))$, $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ identically 1 near λ^2_α, χ as in (5.2), as discussed at the beginning of this section. Note that

$$HT_+ - T_+H_a = [H_a, T_+] + \tilde{V}_a T_+. \tag{5.10}$$

If $g \in \mathcal{C}^\infty_c(C'_a)$ then $\mathcal{F}^*_\alpha(\lambda)g$ vanishes to infinite order near $C_{a,\mathrm{sing}}$, so due to the rapid decay of ψ_α, $J_\alpha\mathcal{F}^*_\alpha(\lambda)g$ vanishes to infinite order near $\cup_{c\not\leq a}C_c$. Therefore,

$$\tilde{V}_a T_+ J_\alpha \mathcal{F}^*_\alpha(\lambda)g \in H^{\infty,1/2+\epsilon'}_{\mathrm{sc}}(\mathbb{S}^n_+) \tag{5.11}$$

for sufficiently small ϵ' since $\tilde{V}_a$ decays away from $\cup_{c\not\leq a}C_c$. On the other hand, since $\mathrm{Id}\otimes h_a$ commutes with T_+ and Δ_{X_a} commutes with H_a, we have

$$[H_a, T_+] = [\Delta_{X_a}, q_L(\chi(\tau_a)\phi(|\xi_a|^2))] \otimes \mathrm{Id}\,. \tag{5.12}$$

Since the identity operator commutes with Δ_{X_a} and composition is microlocal (see the comments at the end of Section 2), it follows that $[\Delta_{X_a}, q_L(\chi(\tau_a)\phi(|\xi_a|^2))]$ is, modulo a trivial term in $\Psi^{-\infty,\infty}_{\mathrm{sc}}(\mathbb{S}^n_+)$, of the form $q_L(t)$ where t is supported in $\tau_a \in [-\sqrt{a(\lambda)}+\epsilon, \sqrt{a(\lambda)}-\epsilon]$, so it satisfies (2.13). Indeed, due to commutativity to top order in $\Psi_{\mathrm{sc}}(\bar{X}_a)$, t also decays like $\langle w_a\rangle^{-1}$. Thus, $[H_a, T_+]$ (hence $[H_a, T_+]^*$) is of the form $q_L(t)\otimes\mathrm{Id}$. (These statements could be proved by direct commutation arguments as in [3].) Therefore, for $u \in H^{\infty,l}_{\mathrm{sc}}(\mathbb{S}^n_+)$, $l > 1/2$, we have, with ζ_a as in (2.14),

$$[H_a, T_+]^*\zeta_a R(\lambda + i0)u \in \langle w_a\rangle^{-1}H^{\infty,-1/2+\epsilon'}_{\mathrm{sc}}(\mathbb{S}^n_+) \tag{5.13}$$

for sufficiently small $\epsilon' > 0$.

To prove the theorem, we note that Isozaki's proof in [9, Lemma 3.1] can be repeated nearly verbatim to conclude that for $f, g \in \mathcal{C}_c^\infty(C_a')$

$$\begin{aligned} &\langle (\hat{S}_{\alpha\alpha}(\lambda) - \mathrm{Id}) f, g\rangle \\ &\quad = \langle 2\pi i(-\mathcal{F}_\alpha(\lambda) J_\alpha^* T_+^* \tilde{V}_a + \mathcal{F}_\alpha(\lambda) J_\alpha^* (HT_+ - T_+ H_a)^* R(\lambda + i0)\tilde{V}_a) J_\alpha \mathcal{F}_\alpha^*(\lambda) f, g\rangle. \end{aligned} \tag{5.14}$$

Here Id appears since the incoming and outgoing channels coincide, so we have an additional term $(W_\alpha^+)^* W_\alpha^+ = \mathrm{Id}$ in Isozaki's argument. The pairing in (5.14) makes sense due to (5.10)-(5.13), the rapid decay of ψ_α (present as J_α^*) in w^α and that of $\mathcal{F}_\alpha^*(\lambda) g$ near $C_{a,\mathrm{sing}}$ which also allow us to deal with the $1 - \zeta_a$ term. Thus, as a map $\mathcal{C}_c^\infty(C_a') \to \mathcal{C}^{-\infty}(C_a')$ we have

$$\begin{aligned} &\hat{S}_{\alpha\alpha}(\lambda) - \mathrm{Id} \\ &\quad = 2\pi i(-\mathcal{F}_\alpha(\lambda) J_\alpha^* T_+^* \tilde{V}_a + \mathcal{F}_\alpha(\lambda) J_\alpha^* (HT_+ - T_+ H_a)^* R(\lambda + i0)\tilde{V}_a) J_\alpha \mathcal{F}_\alpha^*(\lambda). \end{aligned} \tag{5.15}$$

Moreover, the Poisson operator satisfies

$$J_\alpha \mathcal{F}_\alpha(\lambda)^* = C\tilde{P}_{\alpha,-}(\lambda) = C'\tilde{P}_{\alpha,+}(\lambda) R, \tag{5.16}$$

$$C = (4\pi\lambda_\alpha)^{-1/2} e^{-i\pi(n_a - 1)/4}, \qquad C' = C^*, \tag{5.17}$$

see [14] (this simply comes from stationary phase methods), so

$$\begin{aligned} &\hat{S}_{\alpha\alpha}(\lambda) - \mathrm{Id} \\ &\quad = C''(-\tilde{P}_{\alpha,-}(\lambda)^* T_+^* \tilde{V}_a + \tilde{P}_{\alpha,-}(\lambda)^* (HT_+ - T_+ H_a)^* R(\lambda + i0)\tilde{V}_a)\tilde{P}_{\alpha,+}(\lambda) R \end{aligned} \tag{5.18}$$

with $C'' = \frac{i^{n_a}}{2\lambda_\alpha}$. Now, from the argument of the previous paragraphs we know that

$$\frac{1}{2i\lambda_\alpha}((H_a - \lambda) T_+ \tilde{P}_{\alpha,-}(\lambda))^* \tilde{P}_{\alpha,+}(\lambda) = i^{-(n_a - 1)} R \tag{5.19}$$

since the left hand side is just the asymptotic expansion scattering matrix of H_a at λ, which coincides with that of Δ_{X_a} at λ_α^2, which is $i^{-(n_a-1)} R$ (see [14]; this is again from stationary phase). The left hand side can be rewritten as

$$\frac{1}{2i\lambda_\alpha}([H_a, T_+]\tilde{P}_{\alpha,-}(\lambda))^* \tilde{P}_{\alpha,+}(\lambda), \tag{5.20}$$

so together with (5.15) we see that

$$\hat{S}_{\alpha\alpha}(\lambda) = -C''((HT_+ - T_+ H_a)\tilde{P}_{\alpha,-}(\lambda))^* (\mathrm{Id} - R(\lambda + i0)\tilde{V}_a)\tilde{P}_{\alpha,+}(\lambda) R. \tag{5.21}$$

But then (2.6) and $(H_a - \lambda)\tilde{P}_{\alpha,+}(\lambda) = 0$ show that

$$\hat{S}_{\alpha\alpha}(\lambda) = -C''((H - \lambda) T_+ \tilde{P}_{\alpha,-}(\lambda))^* P_{\alpha,+}(\lambda) R = i^{n_a - 1} S_{\alpha\alpha}(\lambda) R \tag{5.22}$$

where we used (5.7) in the last step. The proof for $\alpha \neq \beta$ is very similar. □

Remark 5.2. An alternative approach to proving the equivalence of the S-matrices is to use Yafaev's representation, [25]. In our notation this would mean that we replace Corollary 3.4 by another result. Namely, we would want to understand $Q^0_{\beta,+}(\pi_\beta R(\lambda + i0) f)$ where $f = (H - \lambda)\tilde{P}_{\alpha,+}(\lambda) g$, $g \in \mathcal{C}_c^\infty(C_a')$. Since $R(\lambda + i0) f$, and hence $u = \pi_\beta R(\lambda + i0) f$ is 'outgoing' (modulo $H_{\mathrm{sc}}^{\infty,1/2+\epsilon'}(\bar{X}_b)$, $\epsilon' > 0$ is sufficiently small) by [3, 4], we can use the push-forward theorem of [6] to conclude that $Q^0_-(u) = 0$. Hence, Propositon 3.3 implies that

$$\langle Q^0_+(u), h\rangle = \frac{-1}{2i\lambda_\beta}\langle(\Delta - \lambda_\beta^2)u, P^0_{b,-}(\lambda_\beta^2)h\rangle. \tag{5.23}$$

Yafaev's representation follows by rewriting this equation and using the corresponding formula for the Euclidian scattering matrix, much like it was done in the proof of the Theorem.

References

[1] R. G. Froese and I. Herbst. Exponential bounds and absence of positive eigenvalues of N-body Schrödinger operators. *Commun. Math. Phys.*, 87:429–447, 1982.
[2] R. G. Froese and I. Herbst. A new proof of the Mourre estimate. *Duke Math. J.*, 49:1075–1085, 1982.
[3] C. Gérard, H. Isozaki, and E. Skibsted. *Commutator algebra and resolvent estimates*, volume 23 of *Advanced studies in pure mathematics*, pages 69–82. 1994.
[4] C. Gérard, H. Isozaki, and E. Skibsted. N-body resolvent estimates. *J. Math. Soc. Japan*, 48:135–160, 1996.
[5] G. M. Graf. Asymptotic completeness for N-body short range systems: a new proof. *Commun. Math. Phys.*, 132:73–101, 1990.
[6] A. Hassell. Scattering matrices for the quantum 3-body problem. *Preprint*, 1997.
[7] L. Hörmander. *The analysis of linear partial differential operators,* vol. 1-4. Springer-Verlag, 1983.
[8] M. Ikawa, editor. *Spectral and scattering theory.* Marcel Dekker, 1994.
[9] H. Isozaki. Structures of S-matrices for three body Schrödinger operators. *Commun. Math. Phys.*, 146:241–258, 1992.
[10] H. Isozaki. Asymptotic properties of generalized eigenfunctions for three body Schrödinger operators. *Commun. Math. Phys.*, 153:1–21, 1993.
[11] H. Isozaki. On N-body Schrödinger operators. *Proc. Indian Acad. Sci. Math. Sci.*, 104:667–703, 1993.
[12] H. Isozaki. *A uniqueness theorem for the N-body Schrödinger equation and its applications.* In Ikawa [8], 1994.
[13] R. B. Melrose. *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces.* In Ikawa [8], 1994.
[14] R. B. Melrose. *Geometric scattering theory.* Cambridge University Press, 1995.
[15] R. B. Melrose and M. Zworski. Scattering metrics and geodesic flow at infinity. *Inventiones Mathematicae*, 124:389–436, 1996.
[16] P. Perry, I. M. Sigal, and B. Simon. Spectral analysis of N-body Schrödinger operators. *Ann. Math.*, 114:519–567, 1981.
[17] M. Reed and B. Simon. *Methods of modern mathematical physics.* Academic Press, 1979.
[18] I. M. Sigal and A. Soffer. N-particle scattering problem: asymptotic completeness for short range systems. *Ann. Math.*, 125:35–108, 1987.
[19] E. Skibsted. Smoothness of N-body scattering amplitudes. *Reviews in Math. Phys.*, 4:619–658, 1992.
[20] A. Vasy. Asymptotic behavior of generalized eigenfunctions in N-body scattering. *J. Func. Anal.*, 148:170–184, 1997.
[21] A. Vasy. *Propagation of singularities in three-body scattering.* PhD thesis, Massachusetts Institute of Technology, 1997.
[22] A. Vasy. Structure of the resolvent for three-body potentials. *Duke Math. J.*, 90:379–434, 1997.
[23] X. P. Wang. Microlocal estimates for N-body Schrödinger operators. *J. Fac. Sci. Univ. Tokyo Sect. IA, Math.*, 40:337–385, 1993.
[24] D. Yafaev. Radiation conditions and scattering theory for N-particle Hamiltonians. *Commun. Math. Phys.*, 154:523–554, 1993.
[25] D. Yafaev. Resolvent estimates and scattering matrix for N-particle Hamiltonians. *Integr. Equat. Oper. Th.*, 21:93–126, 1995.