ABSENCE OF SUPER-EXPONENTIALLY DECAYING EIGENFUNCTIONS ON RIEMANNIAN MANIFOLDS WITH PINCHED NEGATIVE CURVATURE

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1. Introduction and statement of results

Let (X, g) be a metrically complete, simply connected Riemannian manifold with bounded geometry and pinched negative curvature, i.e. there are constants $a > b > 0$ such that $-a^2 < K < -b^2$ for all sectional curvatures K. Here bounded geometry is used in the sense of Shubin, [15, Appendix 1], namely that all covariant derivatives of the Riemannian curvature tensor are bounded and the injectivity radius is uniformly bounded below by a positive constant. We show that there are no superexponentially decaying eigenfunctions of Δ on X; here Δ is the positive Laplacian of g. That is, fix some $o \in X$, and let $r(p) = d(p, o)$, $p \in X$, be the distance function. Then:

Theorem 1.1. *Suppose that (X, g) is as above. If $(\Delta-\lambda)\psi = 0$ and $\psi \in e^{-\alpha r}L^2(X)$ for all α, then ψ is identically 0.*

Since the curvature assumptions imply exponential volume growth, and due to elliptic regularity, the L^2 norm may be replaced by any L^p, or indeed Sobolev, norm. This result strengthens Mazzeo's unique continuation theorem at infinity [12] by eliminating the asymptotic curvature assumption (4) there.

As shown below, the negative curvature assumption enters via the strict uniform convexity of the geodesic spheres centered at o, much as in the work of Mazzeo [12]. Thus, as observed by Rafe Mazzeo, the arguments go through equally well if X is replaced by a manifold M which is the union of a 'core' M_0 (not necessarily compact) and a product manifold $M_1 = (1, \infty)_r \times N$, with a Riemannian metric $g = dr^2 + k(r, .)$, k a metric on $N(r) = \{r\} \times N$, M_1 having bounded geometry, provided that the second fundamental form of $N(r)$ is strictly positive, uniformly in r. Indeed, the assumptions on ψ only need to be imposed on M_1, see Remark 2.6.

Following [15, Appendix 1] we remark that an equivalent formulation of the definition of a manifold of bounded geometry is the requirement that the injectivity radius is bounded below by a positive constant r_{inj}, and that the transition functions between intersecting geodesic normal coordinate charts (called canonical coordinates in [15]) of radius $< r_{\mathrm{inj}}/2$, say, are $\mathcal{C}^\infty$ with uniformly bounded derivatives (with bound independent of the base points). We let $\mathrm{Diff}(X)$ denote the algebra of differential operators corresponding to the bounded geometry, called the algebra of $\mathcal{C}^\infty$-bounded differential operators in [15, Appendix 1]. Thus, in

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any canonical coordinates, $A \in \mathrm{Diff}^m(X)$ has the form $\sum_{|\alpha|\leq m} a_\alpha(x) D_x^\alpha$, with $\partial^\beta a_\alpha$ uniformly bounded, with bound $C_{|\beta|}$ independent of the canonical coordinate chart, for all multiindices β. Below we also write $H^k(X)$ for the L^2-based Sobolev spaces, defined by localization, see e.g. [15, Appendix 1, Equation (1.3)] for details.

If E is a vector bundle of bounded geometry, in the sense of [15, Appendix 1], then Theorem 1.1 is also valid for Δ replaced by any second order differential operator $P \in \mathrm{Diff}^2(X, E)$ acting on sections of E with scalar principal symbol equal to that of Δ, i.e. the metric function on T^*X. Indeed, we can even localize at infinity, i.e. assume $Pu = 0$ only near infinity, and obtain the conclusion that u is 0 near infinity.

Theorem 1.2. *Let (X, g) be as above. Suppose $P \in \mathrm{Diff}^2(X, E)$, $\sigma_2(P) = g\,\mathrm{Id}$, where g denotes the metric function on T^*X. If $P\psi \in \mathcal{C}_c^\infty(X, E)$ and $\psi \in e^{-\alpha r} L^2(X, E)$ for all α then $\psi \in \mathcal{C}_c^\infty(X, E)$.*

Remark 1.3. This theorem, together with the standard unique continuation result, [10, Theorem 17.2.1], implies that if $P\psi = 0$ on X then $\psi = 0$ on X, just as in Theorem 1.1.

In addition, the Sobolev order of the assumptions on ψ and $P\psi$ is immaterial: Let us assume $P\psi$ is merely a compactly supported distribution, and that for each α, $\psi \in e^{-\alpha r} H^{k(\alpha)}(X, E)$ for some $k(\alpha) \in \mathbb{R}$. Elliptic regularity allows us to conclude that near infinity, $\psi \in e^{-\alpha r} L^2(X, E)$. Since, as discussed in Remark 2.6, our argument localizes near infinity, we conclude that $\psi = 0$ near infinity, hence is a compactly supported distribution. If $P\psi = 0$, we deduce $\psi = 0$ on X by unique continuation.

Often one actually wants to show that geometrically interesting operators, such as Δ, have no L^2-eigenfunctions, or at least no L^2-eigenfunctions inside the continuous spectrum. Previous work on eigenfunctions on negatively curved manifolds has focused on these issues; see Donnelly [5, 6], Donnelly and Xavier [8], and Donnelly and Garofalo [7]. In particular, these authors showed that under limiting curvature assumptions (typically, K goes to a constant at infinity in some sense), the Laplacian has no embedded eigenvalues. To our knowldege, ours is the first result on the absence of eigenfunctions in the general bounded geometry setting (with negative curvature), although we need stronger than L^2 hypotheses on the eigenfunctions, namely superexponential decay.

The gap between L^2 and superexponential decay assumptions is by no means small. In some settings, e.g. when (X, g) is conformally compact (this was studied by Mazzeo in [12]), it is known that all L^2-eigenfunctions decay superexponentially. In general, it is by no means clear when L^2-bounds imply superexponential decay. Indeed, in the setting of quantum N-body Hamiltonians the structure of thresholds determines the rate of exponential decay for eigenfunctions [9, 18]. Thus, the philosophy underlying the present work is that for general manifolds with bounded geometry and pinched negative curvature, superexponential decay is a natural assumption in ruling out decaying eigenfunctions, while the question whether L^2-eigenfunctions decay in such a fashion depends on the precise nature of such manifolds. It should be mentioned here that the counterexample of Donnelly [5, Section 3] for eigenvalues embedded in the continuous spectrum is *not* of bounded geometry, and indeed the eigenfunctions presented there are superexponentially decaying, so at least at this point it is not known whether, when restricted to the purely geometric setting, the L^2 and superexponential decay assumptions are equivalent.

We also remark on the bounded geometry hypotheses, more precisely on the assumptions on covariant derivatives. The results of Anderson and Schoen [1] on harmonic functions on negatively curved spaces are results below the continuous spectrum. This makes them elliptic problems even at infinity, in a rather strong sense—stronger than just the uniform ellipticity on manifolds with bounded geometry discussed below. In particular, the notion of positivity and the maximum principle are available, and can be used to eliminate conditions on covariant derivatives. However, for eigenfunctions embedded in the continuous spectrum such tools are unavailable. Indeed, for λ large, $\Delta-\lambda$ can be seen to lose 'strong' ellipticity (so e.g. it is not Fredholm on $L^2(X)$), and is in many ways (micro)hyperbolic, at least in settings with an additional structure (cf. the discussion in [13] in asymptotically flat spaces). In such a setting commutator estimates are very natural, and have a long tradition in PDEs; this explains the role of the assumption on the covariant derivatives.

We conclude this section with a short summary of the relevant properties of the bounded geometry setting. Due to the prominent role played by r, we work in Riemannian normal coordinates. So let $g = dr^2+k(r,.)$ be the metric on X, where k is the metric on the geodesic sphere of radius r, denoted by $S(r)$, and let $A(r,.)\,dr\wedge\omega$ denote the volume element, ω being the standard volume form on the unit sphere. By the bounded geometry assumptions, $\partial_r \log A = -\Delta r \in \mathcal{C}^\infty_{\mathrm{b}}(X) = \mathrm{Diff}^0(X)$ (see e.g. [20, Lemma 2.3] for the identity), i.e. is uniformly bounded with analogous conditions on the covariant derivatives. Then

$$-\Delta = \partial_r^2 + (\partial_r \log A)\partial_r - \Delta_{S(r)} \tag{1.1}$$

for $r>0$, so $\Delta \in \mathrm{Diff}^2(X)$.

The role of the pinched negative curvature assumption is to ensure that there is $c>0$ such that

$$H_g^2 r \geq ck; \tag{1.2}$$

here H_g denotes the Hamilton vector field of g and the inequality of is quadratic forms on the fibers of T^*X. To analyze (1.2), recall that arclength parameterized geodesics of g are projections to X of the integral curves of $\frac{1}{2}H_g$ inside S^*X, the unit cosphere bundle of X. Thus, (1.2) tells us that r is strictly convex along geodesics tangent to $S(r_0)$ at the point of contact. Equivalently, the Hessian ∇dr, which is the form on the fibers of TX dual to $H_g^2 r$, is strictly positive on $TS(r_0)$, uniformly as $r_0\to\infty$. As $r=r_0$ defines $S(r_0)$, with $|\nabla r|=1$, this Hessian equals the second fundamental form of $S(r_0)$; hence (1.2) is also equivalent to the uniform convexity of the hypersurfaces $S(r_0)$.

Equation (1.2) follows immediately when the sectional curvatures of X are bounded above by a negative constant $-b^2$, since by the Hessian comparison theorem (see e.g. [14, Theorem 1.1]), $H_g^2 r|_{T_{S(r_0)}X} \geq H_{g_0}^2 r|_{T_{S(r_0)}X}$, where g_0 is the metric with constant negative sectional curvature $-b^2$, and the right hand side is $b\coth br \geq b$ (cf. [14, Equation (1.7)]).

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2. The proofs

To make the argument more transparent, we write up the proof of Theorem 1.1, at each step pointing out any significant changes that are needed to prove Theorem 1.2. The proofs are a version of Carleman estimates (see (2.2) below), at least for self-adjoint operators, but we phrase these somewhat differently, in the spirit of operators with complex symbol and codimension 2 characteristic variety (which in this case is in the semiclassical limit) on which the Poisson bracket of the real and imaginary part of the (in this case, semiclassical) principal symbol is positive. This corresponds most directly to subelliptic estimates with loss of $1/2$ derivative for a conjugated version of P, see [10, Theorem 27.1.11], but it is also related to the non-solvability of the inhomogeneous PDE in the sense of [10, Section 26.4] (cf. the proof of Proposition 27.1.7 there); see also [21] for a recent discussion. Related conditions play a central role in the work of Tataru on unique continuation [16, 17].

The role of the lost $1/2$ derivative in the subelliptic setting is played here by the loss of a factor of $h^{1/2}$, h being a semiclassical parameter which appears as the (inverse) exponent of an exponential weight. The uniformity of this "subelliptic" estimate near spatial infinity is ensured by the condition that $c > 0$ in (1.2). Thus, from a PDE point of view, the two important aspects of the proofs presented here are showing that uniform conditions on the principal symbol of the conjugated version of P imply global subelliptic-type estimates, and showing that these uniform conditions are satisfied in a geometrically interesting setting.

Carleman estimates have played a prominent role in recent research on spaces with various asymptotic geometries, such as asymptotically Euclidean spaces, as initiated by Burq's work [2] which placed the Carleman estimates of Lebeau-Robbiano [11] into this context. In these settings the absence of embedded eigenvalues has been known, and the main direction of research has been to obtain local energy decay for the wave equation, or related high energy resolvent estimates [2, 3, 19, 4]. While one needs global estimates for this analysis, as in our case, these papers assume much stronger types of asymptotic structures (not merely bounded geometry and curvature assumptions); e.g. in [4], in equation (1.2), the derivatives of the effective potential would have no additional decay over the effective potential itself if one only made bounded geometry assumptions.

Turning to the actual proofs, we consider eigenfunctions of Δ that are superexponentially decaying: $(\Delta - \lambda)\psi = 0$, and $\psi \in e^{-\alpha r} L^2(X)$ for all α, $\|\psi\|_{L^2(X)} = 1$ (for convenience). Note that λ is real by the self-adjointness of Δ. Moreover, $\psi \in \mathcal{C}^\infty(X)$ by standard elliptic regularity, and indeed $\psi \in e^{-\alpha r'} H^m(X)$ for all m, where r' is a smoothed version of r, changed only near the origin, where r fails to be smooth. It is convenient to assume that $r' > 0$, so $\inf_X r' > 0$.

For α real, we thus consider

$$P_\alpha = e^{\alpha r'}(\Delta - \lambda)e^{-\alpha r'}.$$

For notational simplicity, to avoid an additional compactly supported error term on almost every line, we will ignore the distinction between r and r', and simply add back a compactly supported error term later on, in (2.5).

Let

$$\operatorname{Re} P_\alpha = \frac{1}{2}(P_\alpha + P_\alpha^*),\ \operatorname{Im} P_\alpha = \frac{1}{2i}(P_\alpha - P_\alpha^*),$$

be the symmetric and skew-symmetric parts of P_α. Thus, $P_\alpha = \operatorname{Re} P_\alpha + i \operatorname{Im} P_\alpha$, and $\operatorname{Re} P_\alpha$, $\operatorname{Im} P_\alpha$ are symmetric. Note also that $P_\alpha \psi_\alpha = 0$ where $\psi_\alpha = e^{\alpha r}\psi$. Thus,

$$0 = \|P_\alpha \psi_\alpha\|^2 = \|\operatorname{Re} P_\alpha \psi_\alpha\|^2 + \|\operatorname{Im} P_\alpha \psi_\alpha\|^2 + \langle i[\operatorname{Re} P_\alpha, \operatorname{Im} P_\alpha]\psi_\alpha, \psi_\alpha\rangle. \tag{2.1}$$

Roughly speaking, this will give a contradiction provided the commutator is positive, although in the presence of error terms one needs to be a little more careful. Note that the local condition of 'subellipticity with loss of $\alpha^{-1/2}$' is exactly a condition on the 'large α principal symbol' (i.e. semiclassical principal symbol, with $h = \alpha^{-1}$) of $i[\operatorname{Re} P_\alpha, \operatorname{Im} P_\alpha]$, namely that it should be non-negative. (See [10, Theorem 27.1.11] for the standard microlocal setting, namely subellipticity with loss of 1/2 derivative.)

Remark 2.1. The argument sketched above parallels the last part of the N-body argument of [18], showing exponential decay and unique continuation results for N-particle Hamiltonians with second order interactions, which in turn placed the work of Froese and Herbst [9] in potential scattering into this framework. However, in [18] (as in [9]) this is the simplest part of the argument; it is much more work to show that L^2-eigenfunctions decay at a rate given by the next threshold above the eigenvalue λ—hence superexponentially in the absence of such thresholds.

Remark 2.2. In applying the usual Carleman estimates to a self-adjoint operator P_0, one constructs $P_{\pm\alpha}$, with the same notation as above, and computes $\|P_\alpha \psi_\alpha\|^2 \pm \|P_{-\alpha}\psi_\alpha\|^2$. By self-adjointness of P_0, $P_{-\alpha} = P_\alpha^*$, hence

$$\begin{aligned} \|P_\alpha \psi_\alpha\|^2 + \|P_{-\alpha}\psi_\alpha\|^2 &= 2\|\operatorname{Re} P_\alpha \psi_\alpha\|^2 + 2\|\operatorname{Im} P_\alpha \psi_\alpha\|^2, \\ \|P_\alpha \psi_\alpha\|^2 - \|P_{-\alpha}\psi_\alpha\|^2 &= 2\langle i[\operatorname{Re} P_\alpha, \operatorname{Im} P_\alpha]\psi_\alpha, \psi_\alpha\rangle. \end{aligned} \tag{2.2}$$

Thus, the classical Carleman argument breaks up (2.1) into two pieces, and is completely equivalent to (2.1). However, dividing up P_α into its symmetric and skew-symmetric parts makes the calculations below more systematic, in particular making manifest an important double-commutator term in $\operatorname{Re} P_\alpha$. This double commutator, in turn, makes it clear why various terms, which one might expect by expanding out the squares $\|P_{\pm\alpha}\psi_\alpha\|^2$ in terms of P, α, etc., do not appear in the evaluation of $\|P_\alpha \psi_\alpha\|^2 \pm \|P_{-\alpha}\psi_\alpha\|^2$.

A detailed account of the classical theory of Carleman estimates can be found in [10, Chapter 17].

We now turn to the detailed computation of the terms in (2.1). To begin with, we compute

$$\begin{aligned} P_\alpha &= \Delta - \lambda + e^{\alpha r}[\Delta, e^{-\alpha r}], \\ \operatorname{Re} P_\alpha &= \Delta - \lambda + \frac{1}{2}[e^{\alpha r}, [\Delta, e^{-\alpha r}]] \\ \operatorname{Im} P_\alpha &= \frac{1}{2i}(e^{\alpha r}[\Delta, e^{-\alpha r}] + [\Delta, e^{-\alpha r}]e^{\alpha r}). \end{aligned}$$

Here the expressions for $\operatorname{Re} P_\alpha$ and $\operatorname{Im} P_\alpha$ follow directly from the definition of the symmetric and skew-symmetric parts, using that Δ and $e^{\pm\alpha r}$ are symmetric.

In the double commutator in the expression for $\operatorname{Re} P_\alpha$ above, changing Δ by a first order operator would not alter the result, as commutation with a scalar reduces the order by 1. Thus, in view of (1.1), in the double commutator in $\operatorname{Re} P_\alpha$ all terms but ∂_r^2 give vanishing contribution, so we immediately see that

$$\operatorname{Re} P_\alpha = \Delta - \lambda - \alpha^2.$$

We next compute the skew-symmetric part. This is

$$\operatorname{Im} P_\alpha = \frac{1}{i}(2\alpha\partial_r + \alpha(\partial_r \log A)).$$

Thus,

$$i[\operatorname{Re} P_\alpha, \operatorname{Im} P_\alpha] = \alpha[\Delta, 2\partial_r + (\partial_r \log A)].$$

The crucial estimate for this commutator that we need below is that there is $c > 0$ such that

$$[\Delta, 2\partial_r + (\partial_r \log A)] \geq c\Delta_{S(r)} + R,\ R \in \operatorname{Diff}^1(X); \tag{2.3}$$

here R is symmetric and the inequality is understood in the sense of quadratic forms, e.g. with domain $H^2(X)$. Since the commutator is a priori in $\operatorname{Diff}^2(X)$, this means that we merely need to calculate its principal symbol, which in turn only depends on the principal symbols of the commutants. Thus, with H_g denoting the Hamilton vector field of g, and σ the canonical dual variable of r, with respect to the product decomposition $(0, \infty) \times S$ of $X \setminus o$, the principal symbol of ∂_r is $\sigma_1(\partial_r) = i\sigma$, and

$$\sigma_2([\Delta, 2\partial_r]) = 2H_g\sigma.$$

It is convenient to rephrase this by noting that $2\partial_r = -[\Delta, r] + R'$, $R' \in \operatorname{Diff}^0(X)$, so $2i\sigma = \sigma_1(2\partial_r) = iH_g r$, and hence $\sigma_2([\Delta, 2\partial_r]) = H_g^2 r$. This explains the role of (1.2), i.e. the requirement that there exist $c > 0$ such that

$$H_g^2 r \geq ck.$$

Indeed, (1.2) implies (2.3), since for each $x \in X$, both sides of (1.2) are quadratic forms on T^*X, depending smoothly on x, so their difference can be written as $\sum a_{ij}(x)\xi_i\xi_j$ (ξ_i are canonical dual variables of local coordinates x_i), with a_{ij} a non-negative matrix. This in turn is the principal symbol of $\sum_{ij} D_{x_i}^* a_{ij}(x) D_{x_j}$, and

$$\langle \sum_{ij} D_{x_i}^* a_{ij}(x) D_{x_j} v, v\rangle = \int_X \sum_{ij} a_{ij}(x) D_{x_i} v\, \overline{D_{x_j} v}\, dg \geq 0.$$

We will consider $\alpha \to \infty$, but for the sake of conformity with standard notation it is convenient to work in the semiclassical (rather than the large parameter) setting, i.e. to let $h = \alpha^{-1}$, $h \in (0, 1]$ and let $h \to 0$. So let $\Delta_h = h^2\Delta$, $\Delta_{S(r),h} = h^2\Delta_{S(r)}$, and slightly abuse notation by writing $P_h = h^2 e^{r/h}(\Delta - \lambda)e^{-r/h}$, so

$$\operatorname{Re} P_h = \Delta_h - 1 - h^2\lambda,\ \operatorname{Im} P_h = \frac{1}{i}(2h\partial_r + h(\partial_r \log A)),$$
$$i[\operatorname{Re} P_h, \operatorname{Im} P_h] \geq ch\Delta_{S(r),h} + h^3 R,\ R \in \operatorname{Diff}^1(X).$$

We denote the space of semiclassical differential operators of order m by $\operatorname{Diff}_h^m(X)$. We recall that $A \in \operatorname{Diff}_h^m(X)$ means that, in the usual multiindex notation, $A = \sum_{|\alpha| \leq m} a_\alpha(x)(hD_x)^\alpha$ locally; in our bounded geometry setting we still impose, as for standard differential operators, that for all multiindices β, $\partial^\beta a_\alpha$ be bounded uniformly in all Riemannian normal coordinate charts of radius R (R less than half the injectivity radius, say), with bound only dependent on $|\beta|$. Then, weakening the above statements somewhat, in a way that still suffices below,

$$\operatorname{Re} P_h = \Delta_h - 1 + hR_1',\ \operatorname{Im} P_h = \frac{1}{i}(2h\partial_r + hR_2'),\ i[\operatorname{Re} P_h, \operatorname{Im} P_h] \geq ch\Delta_{S(r),h} + h^2 R_3'. \tag{2.4}$$

with $R_1', R_2' \in \operatorname{Diff}_h^1(X)$, $R_3' \in \operatorname{Diff}_h^2(X)$.

We stated (2.4) in a weakened form to make it only depend on the principal symbol of Δ. Namely, if Δ is replaced by any operator $\Delta + Q$, $Q \in \operatorname{Diff}^1(X, E)$ (not necessarily symmetric), and $P_h' = h^2 e^{r/h}(\Delta + Q - \lambda)e^{-r/h}$, then $P_h' - P_h = h^2 e^{r/h} Q e^{-r/h} \in h\operatorname{Diff}_h^1(X, E)$, so $\operatorname{Re} P_h' - \operatorname{Re} P_h, \operatorname{Im} P_h' - \operatorname{Im} P_h \in h\operatorname{Diff}_h^1(X, E)$, and thus

$$\begin{aligned} &i[\operatorname{Re} P_h', \operatorname{Im} P_h'] - i[\operatorname{Re} P_h, \operatorname{Im} P_h] \\ &= i[\operatorname{Re} P_h' - \operatorname{Re} P_h, \operatorname{Im} P_h'] + i[\operatorname{Re} P_h, \operatorname{Im} P_h' - \operatorname{Im} P_h] \in h^2 \operatorname{Diff}_h^2(X, E), \end{aligned}$$

where we used that $\operatorname{Re} P_h, \operatorname{Im} P_h$ have scalar principal symbols, hence so do $\operatorname{Re} P_h'$ and $\operatorname{Im} P_h'$, giving the extra h (compared to the order of the product) and the lower order in the commutators. In other words, (2.4) still holds for P_h replaced by P_h'.

In the above calculations we ignored a compact subset of X (recall that r should really be replaced by a smooth, nonnegative, globally defined r'), so we need to add a compactly supported error. Thus, we have shown that for some $c > 0$,

$$i[\operatorname{Re} P_h, \operatorname{Im} P_h] \geq ch\Delta_{S(r),h} + h^2 R_0 + hR_0',\ R_0, R_0' \in \operatorname{Diff}_h^2(X), \tag{2.5}$$

R_0' supported in $r \leq r_1$ for some $r_1 > 0$, with the inequality holding in the sense of operators. Since $\Delta_{S,h} = \Delta_h - (hD_r)^2 - h(D_r \log A)hD_r$, this estimate implies

$$\langle i[\operatorname{Re} P_h, \operatorname{Im} P_h]\psi_h, \psi_h\rangle \geq \langle (ch + hR_1 \operatorname{Re} P_h + hR_2 \operatorname{Im} P_h + h^2 R_3 + hR_4)\psi_h, \psi_h\rangle \tag{2.6}$$

with $R_1 \in \operatorname{Diff}_h^0(X)$, $R_2 \in \operatorname{Diff}_h^1(X)$ and $R_3, R_4 \in \operatorname{Diff}_h^2(X)$, R_4 having compact support in $r \leq r_1$. (In fact, for our purposes the compact support assumption is equivalent to assuming that R_4 is $o(1)$ as $r \to \infty$, as such a term can be absorbed in the ch term modulo a compactly supported error term.)

We now show how to use (2.6) to prove unique continuation at infinity. To be systematic, we set this part up somewhat abstractly. Recall that $P_h \in \operatorname{Diff}_h^2(X, E)$ is elliptic (or more precisely uniformly elliptic, both in X and in h) if there is $C > 0$ such that for all $(x, \xi) \in T^*X \setminus o$, and for all $h \in (0, 1]$, $|\sigma_{2,h}(P_h)(x, \xi)^{-1}| \leq C|\xi|_x^{-2}$, with $|\xi|_x^2 = g_x(\xi, \xi)$ the length of $\xi \in T_x^*X$ with respect to g and $|.|$ is the operator norm of the matrix of $\sigma_{2,h}(P_h)(x, \xi)^{-1}$ in any (bounded geometry) trivialization of E.

Lemma 2.3. *Suppose $P_h \in \operatorname{Diff}_h^2(X, E)$ is elliptic and satisfies* (2.6) *for some $c > 0$. Suppose also that $\psi \in e^{-\alpha r} L^2(X, E)$ for all α. If*

$$P_h \psi_h = 0,\ \psi_h = e^{r/h}\psi, \tag{2.7}$$

then there exists $R > 0$ such that ψ vanishes when $r > R$.

Remark 2.4. To simplify notation, we drop the bundle E below. Its presence would not require any changes, except in the notation.

Proof. Let $\Psi(X)$ the algebra of pseudodifferential operators corresponding to the bounded geometry, with uniform support, see [15, Appendix 1, Definition 3.1-3.2], denoted by $U\Psi(X)$ there. The elements of $\Psi^0(X)$ are bounded on $L^2(X)$, and if

$A \in \Psi^m(X)$ is elliptic, there is $B \in \Psi^{-m}(X)$ such that $AB-\mathrm{Id}, BA-\mathrm{Id} \in \Psi^{-\infty}(X)$, so elliptic regularity statements and estimates work as usual.

We also need the corresponding semiclassical space of operators $\Psi_h(X)$. These can be defined by modifying the definition of $\Psi(X)$ exactly as if X were compact, i.e. defining $\Psi_h^m(X)$ near the diagonal using the semiclassical quantization of symbols a, and globally as the sums of such operators and elements of $\Psi_h^{-\infty}(X)$. The latter space consists of operators with smooth Schwartz kernel that decays rapidly off the diagonal as $h \to 0$. More precisely, for $R \in \Psi_h^{-\infty}(X)$ we require that its Schwartz kernel K satisfy $K \in \mathcal{C}^\infty((0,1] \times X \times X)$, that there is $C_R > 0$ such that $K(x,y) = 0$ if $d(x,y) > C_R$, and for all N there is $C_N > 0$ such that for all α, β with $|\alpha| \leq N$, $|\beta| \leq N$, and for all $h \in (0,1]$,

$$|\partial_x^\alpha \partial_y^\beta K(x,y,h)| \leq C_N h^{-n}(1 + d(x,y)/h)^{-N},$$

in canonical coordinates, with $n = \dim X$. All standard properties of semiclassical ps.d.o's remain valid—indeed here we only require basic elliptic regularity. To simplify such regularity statements we will use the semiclassical Sobolev spaces $H_h^k(X)$ defined by

$$u \in H_h^k(X) \Longleftrightarrow \|(1 + h^2\Delta)^{k/2} u\|_{L^2} \leq C \text{ uniformly in } h \in [0,1).$$

We note that the use of ps.d.o.'s can be eliminated, if desired, by proving the elliptic regularity estimates directly.

Since P_h is an elliptic family, (2.7) and elliptic regularity give

$$\|\psi_h\|_{H_h^2(X)} \leq C_1 \|\psi_h\|_{L^2(X)}, \tag{2.8}$$

C_1 independent of $h \in (0,1]$. Correspondingly, we will not specify below which Sobolev norms we are taking. In general, the letters C, C' will be used to denote constants independent of $h \in (0,1]$, which may vary from line to line.

We first remark that for any $\epsilon > 0$, by the Cauchy-Schwarz inequality, and as $\|R_j^* \psi_h\| \leq C\|\psi_h\|$, $j = 1,2,3,4$,

$$\begin{aligned} &|\langle hR_1 \operatorname{Re} P_h \psi_h, \psi_h\rangle| \leq Ch\|\psi_h\|\|\operatorname{Re} P_h\psi_h\| \leq Ch\epsilon\|\psi_h\|^2 + Ch\epsilon^{-1}\|\operatorname{Re} P_h\psi_h\|^2, \\ &|\langle hR_2 \operatorname{Im} P_h \psi_h, \psi_h\rangle| \leq Ch\|\psi_h\|\|\operatorname{Im} P_h\psi_h\| \leq Ch\epsilon\|\psi_h\|^2 + Ch\epsilon^{-1}\|\operatorname{Im} P_h\psi_h\|^2. \end{aligned} \tag{2.9}$$

Next,

$$|\langle \psi_h, h^2 R_3 \psi_h\rangle| \leq Ch^2\|\psi_h\|^2. \tag{2.10}$$

Since R_4 is supported in $r \leq r_1$, we can take some $\chi \in \mathcal{C}^\infty(\mathbb{R})$ identically 1 on $(-\infty, 3r_1/2)$, supported in $(-\infty, 2r_1)$, and deduce that

$$|\langle \psi_h, hR_4\psi_h\rangle| = |\langle \chi(r)\psi_h, hR_4\chi(r)\psi_h\rangle| \leq h\|\chi(r)\psi_h\|^2_{H_h^1(X)}.$$

Now, for $r \leq 2r_1$, $|\psi_h| = e^{r/h}|\psi| \leq e^{2r_1/h}|\psi|$, with a similar estimate for the semiclassical derivatives, so

$$\begin{aligned} |\langle \psi_h, hR_4\psi_h\rangle| &\leq h\|\chi(r)\psi_h\|^2_{H_h^1(X)} \\ &\leq Che^{4r_1/h}\|\psi\|^2_{H_h^1(X)} \leq Che^{4r_1/h}\|\psi\|^2_{H^1(X)} \leq C'he^{4r_1/h}\|\psi\|^2. \end{aligned} \tag{2.11}$$

Hence, we deduce from (2.1) (with P_α replaced by P_h) and (2.6) that

(2.12)
$$0 \geq (1 - C\epsilon^{-1}h)\|\operatorname{Re} P_h\psi_h\|^2 + (1 - C\epsilon^{-1}h)\|\operatorname{Im} P_h\psi_h\|^2 + h(c - Ch - C\epsilon)\|\psi_h\|^2 - Che^{4r_1/h}\|\psi\|^2.$$

Taking $\epsilon = h^{1/2}$, say, and dropping the first two (positive) terms on the right hand side, we conclude that there exists $h_0 > 0$ such that for $h \in (0, h_0)$,

$$Che^{4r_1/h}\|\psi\|^2 \geq h\frac{c}{2}\|\psi_h\|^2. \tag{2.13}$$

Now suppose that $R > 2r_1$ and $\operatorname{supp}\psi \cap \{r \geq R\}$ is non-empty. Since $e^{2r/h} \geq e^{2R/h}$ for $r \geq R$, we deduce that

$$\|\psi_h\|^2 \geq C'e^{2R/h},\ C' = \|\psi\|^2_{r\geq R} > 0.$$

Thus, we conclude from (2.13) that

$$C\|\psi\|^2 \geq \frac{c}{2}C'e^{2(R-2r_1)/h}. \tag{2.14}$$

But letting $h \to 0$, the right hand side goes to $+\infty$, providing a contradiction.

Thus, ψ vanishes for $r \geq R$. □

The proof of Theorem 1.1 is finished since if ψ vanishes on an open set, it vanishes everywhere on X by the usual Carleman-type unique continuation theorem [10, Theorem 17.2.1].

In fact, it is straightforward to strengthen Lemma 2.3 and allow $P\psi$ to be compactly supported. The following lemma thus completes the proof of Theorem 1.2:

Lemma 2.5. *Suppose $P \in \operatorname{Diff}^2(X; E)$ is elliptic, $\psi \in e^{-\alpha r}L^2(X, E)$ for all α, and there is $r_0 > 0$ such that $P\psi = 0$ for $r > r_0$. Let $P_h = e^{r/h}h^2Pe^{-r/h}$, and suppose that P_h satisfies* (2.6) *for some $c > 0$. Then there exists $R > 0$ such that ψ vanishes when $r > R$.*

Remark 2.6. Note that in this formulation, if X is replaced by a manifold with several ends, one of which is of the product form alluded to in the introduction, our theorem holds locally on this end. That is, if $P\psi$ vanishes on this end and ψ has superexponential decay there, then ψ vanishes on the end—hence globally by the standard unique continuation theorem if $P\psi$ is identically zero. To prove this, we merely multiply by a cutoff function supported on this end, and apply the lemma to the resulting inhomogeneous problem.

Proof. The elliptic regularity estimate now becomes

$$\|\psi_h\|_{H^2_h(X)} \leq C_1(\|\psi_h\|_{L^2(X)} + \|P_h\psi_h\|_{L^2(X)}), \tag{2.15}$$

C_1 independent of $h \in (0, 1]$, and we need to keep track of the second term on the right hand side.

Correspondingly, $\|R_j^*\psi_h\| \leq C(\|\psi_h\| + \|P_h\psi_h\|)$, $j = 1, 2, 3, 4$. Thus, on the right hand side of (2.9), we need to add $Ch\epsilon\|P_h\psi_h\|^2$ on each line, while on the right hand side of (2.10) we need to add $Ch^2\|P_h\psi_h\|^2$. Similarly, we need to add $C'he^{4r_1/h}\|P\psi\|^2$ to the right hand side of (2.11). Thus, (2.12) becomes

$$(1 + C\epsilon h)\|P_h\psi_h\|^2 \geq (1 - C\epsilon^{-1}h)\|\operatorname{Re} P_h\psi_h\|^2 + (1 - C\epsilon^{-1}h)\|\operatorname{Im} P_h\psi_h\|^2 + h(c - Ch - C\epsilon)\|\psi_h\|^2 - Che^{4r_1/h}(\|\psi\|^2 + \|P\psi\|^2).$$

Since $P_h\psi_h = e^{r/h}h^2 P\psi$, we have $\|P_h\psi_h\| \leq e^{r_0/h}h^2\|P\psi\|$. Let $r_2 = \max(r_0, r_1)$. Thus, taking $\epsilon = h^{1/2}$ again, we see that there exists $h_0 > 0$ such that for $h \in (0, h_0)$,

$$2e^{4r_2/h}\|P\psi\|^2 + Che^{4r_1/h}\|\psi\|^2 \geq h\frac{c}{2}\|\psi_h\|^2. \tag{2.16}$$

Taking $R > 2r_2$, the proof is now finished as in Lemma 2.3, for (2.14) becomes

$$2\|P\psi\|^2 + Ch\|\psi\|^2 \geq \frac{c}{2}C'he^{2(R-2r_2)/h},$$

and the right hand side still goes to $+\infty$, while the left hand side is bounded as $h \to 0$. $\square$

References

[1] M. Anderson and R. Schoen. Positive harmonic functions on complete manifolds of negative curvature. *Ann. of Math*, 121:429–461, 1985.

[2] Nicolas Burq. Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel. *Acta Math.*, 180(1):1–29, 1998.

[3] Nicolas Burq. Lower bounds for shape resonances widths of long range Schrödinger operators. *Amer. J. Math.*, 124(4):677–735, 2002.

[4] F. Cardoso and G. Vodev. Uniform estimates of the resolvent of the Laplace-Beltrami operator on infinite volume Riemannian manifolds. II. *Ann. Henri Poincaré*, 3(4):673–691, 2002.

[5] Harold Donnelly. Eigenvalues embedded in the continuum for negatively curved manifolds. *Michigan Math. J.*, 28(1):53–62, 1981.

[6] Harold Donnelly. Negative curvature and embedded eigenvalues. *Math. Z.*, 203(2):301–308, 1990.

[7] Harold Donnelly and Nicola Garofalo. Riemannian manifolds whose Laplacians have purely continuous spectrum. *Math. Ann.*, 293(1):143–161, 1992.

[8] Harold Donnelly and Frederico Xavier. On the differential form spectrum of negatively curved Riemannian manifolds. *Amer. J. Math.*, 106(1):169–185, 1984.

[9] R. G. Froese and I. Herbst. Exponential bounds and absence of positive eigenvalues of N-body Schrödinger operators. *Commun. Math. Phys.*, 87:429–447, 1982.

[10] L. Hörmander. *The analysis of linear partial differential operators,* vol. 1-4. Springer-Verlag, 1983.

[11] G. Lebeau and L. Robbiano. Contrôle exact de l'équation de la chaleur. *Comm. Partial Differential Equations*, 20(1-2):335–356, 1995.

[12] R. Mazzeo. Unique continuation at infinity and embedded eigenvalues for asymptotically hyperbolic manifolds. *Amer. J. Math.*, 113:25–45, 1991.

[13] R. B. Melrose. *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces.* Marcel Dekker, 1994.

[14] R. Schoen and S.-T. Yau. *Lectures on differential geometry.* International Press, Cambridge, MA, 1994. Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu, Translated from the Chinese by Ding and S. Y. Cheng, Preface translated from the Chinese by Kaising Tso.

[15] M. A. Shubin. Spectral theory of elliptic operators on noncompact manifolds. *Astérisque*, (207):5, 35–108, 1992. Méthodes semi-classiques, Vol. 1 (Nantes, 1991).

[16] Daniel Tataru. Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem. *Comm. Partial Differential Equations*, 20(5-6):855–884, 1995.

[17] Daniel Tataru. Unique continuation for operators with partially analytic coefficients. *J. Math. Pures Appl. (9)*, 78(5):505–521, 1999.

[18] A. Vasy. Exponential decay of eigenfunctions in many-body type scattering with second order perturbations. *J. Func. Anal.*, 209:468–492, 2004.

[19] Georgi Vodev. Exponential bounds of the resolvent for a class of noncompactly supported perturbations of the Laplacian. *Math. Res. Lett.*, 7(2-3):287–298, 2000.

[20] Shunhui Zhu. The comparison geometry of Ricci curvature. In *Comparison geometry (Berkeley, CA, 1993–94)*, volume 30 of *Math. Sci. Res. Inst. Publ.*, pages 221–262. Cambridge Univ. Press, Cambridge, 1997.

[21] Maciej Zworski. Numerical linear algebra and solvability of partial differential equations. *Comm. Math. Phys.*, 229(2):293–307, 2002.

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