Scattering theory on symmmetric spaces and N-body scattering

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New York, February 2006

N-body scattering:

Let (X, g) be a vector space with a translation invariant Riemannian metric, $\mathcal{X} = \{X_a : a \in I\}$ a finite collection of linear subspaces of X closed under intersections, including $X_0 = X$ and $X_1 = \{0\}$. Let $X^a = X_a^\perp$, V_a real valued functions on X^a decaying at infinity, e.g. $V_a \in S^{-\rho}(X^a)$, $\rho > 0$; for instance $V_a(x^a) \sim c|x^a|^{-1}$ (Coulomb decay). Let

$$H = \Delta_g + \sum_{a \in I} (\pi^a)^* V_a,\ \pi^a : X \to X^a \text{ orth. proj.};$$

assume $V_0 = 0$. Write V_a for $(\pi^a)^* V_a$ as well. Note that V_a is constant on translates of X_a, so it does *not* decay at infinity (except V_1) unlike in 2-body problems.

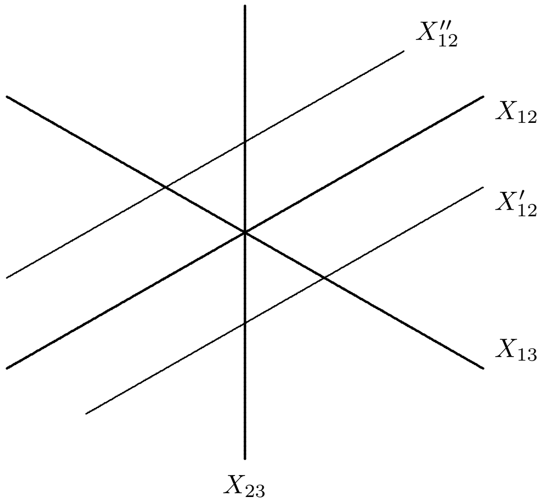
We may replace V_a by formally self-adjoint first or even second order operators on X^a, with decaying coefficients, as long as H remains elliptic in the usual sense.

This is a natural (linear algebra) generalization of the standard N-body Hamiltonians

$$H = \sum_i \frac{\hbar^2}{2m_i}\Delta_{x_i} + \sum_{i<j} V_{ij}(x_i - x_j)$$

where x_i is the position, m_i the mass of the ith particle, and V_{ij} is the interaction between particle i and j.

Example: 3-body scattering (inside $x_1 + x_2 + x_3 = 0$): $X_{ij} = \{x_i = x_j\}$ are the 'collision planes'; X'_{12} and X''_{12} are translates of X_{12}, so V_{12} is constant (does not decay) along these.



Higher rank symmetric spaces:

I describe $\mathsf{SL}(N,\mathbb{R})/\,\mathsf{SO}(N,\mathbb{R})$ in some detail, although other higher rank spaces of non-compact type work similarly. The corresponding rank-one example, $\mathbb{H}^2 = \mathsf{SL}(2)/\,\mathsf{SO}(2)$ is extremely well understood.

If $C \in \mathsf{SL}(N)$, we can write it uniquely as $C = VR$ where $R \in \mathsf{SO}(N)$, V positive definite, $\det V = 1$ – indeed, $V = (CC^t)^{1/2}$. So we may identify $\mathsf{SL}(N)/\,\mathsf{SO}(N)$ with the real analytic manifold of $N \times N$ positive definite matrices of determinant 1.

Then $\mathsf{SL}(N)$ acts on M by

$$\mathsf{SL}(N) \times M \ni (B, A) \mapsto (BA^2B^t)^{1/2}.$$

The Killing form of $\mathsf{SL}(N)$ gives a fiber metric on T_oM, $o = \mathrm{Id}$, via identifying T_oM with symmetric $N \times N$ matrices of trace 0, and if A is such a matrix, its length is

$$(2N)\,\mathrm{Tr}(AA^t) = (2N)\sum a_{ij}^2.$$

This gives a Riemannian metric g on M via the SL(N) action.

We now want to study $H = \Delta_g$.

In fact, because of the SL(N)-action, it suffices to study the action of Δ on $K_o = \mathrm{SO}(N)$-invariant functions. Notice that δ_o is K_o-invariant, hence so is $(\Delta_g - \lambda)^{-1}\delta_o$ for λ outside the spectrum of Δ_g. The main claim is that the radial Laplacian Δ_{rad}, i.e. Δ_g *acting on* SO(N)-*invariant functions*, has a many-body structure ($N-1$-body, to be precise). In particular, the collision planes are (intersections of) the walls of Weyl chambers.

To see this, we diagonalize $A \in M$: $A = O\Lambda O^t$, $O \in \mathrm{SO}(N)$, Λ diagonal, $\det\Lambda = 1$. Neither Λ nor O is completely determined. The diagonal entries of Λ are the eigenvalues of A, so they are determined up to permutations.

If all eigenvalues are distinct, there is only a finite indeterminacy of O. But if two eigenvalues coincide, the indeterminacy is bigger: the eigenspaces, rather than the eigenvectors, of A are well-defined.

So inside the 'flat' $\exp(\mathfrak{a})$ of diagonal matrices of determinant 1 we have a similar picture to N-body scattering. (Here $\mathfrak{a}$ is the space of diagonal matrices with vanishing trace.)

Th metric g reflects this structure. Concretely, for $N = 3$, for $p \in \exp(\mathfrak{a})$, on T_pM, the metric is

$$\begin{aligned} g =& 6\left(\frac{d\lambda_1^2}{\lambda_1^2} + \frac{d\lambda_2^2}{\lambda_2^2} + \frac{d\lambda_3^2}{\lambda_3^2}\right) \\ &+ 3\left(\frac{\lambda_1}{\lambda_2} - \frac{\lambda_2}{\lambda_1}\right)^2 dc_{12}^2 + 3\left(\frac{\lambda_1}{\lambda_3} - \frac{\lambda_3}{\lambda_1}\right)^2 dc_{13}^2 \\ &+ 3\left(\frac{\lambda_2}{\lambda_3} - \frac{\lambda_3}{\lambda_2}\right)^2 dc_{23}^2, \end{aligned}$$

where the c_{ij} are coordinates on SO(3) near Id given by the off-diagonal entries in so(3). Notice the polar coordinate degeneracy at the walls.

To make the formula truly comparable to the Euclidean setting, we need to write $\Lambda = \exp(D)$, $D \in \mathfrak{a}$ (symmetric, trace 0), so it has diagonal entries $x_i = \log \lambda_i$. The metric on D then is 6 times the Euclidean metric. Since in local coordinates z_j on M,

$$\Delta_g = \sum_{i,j} \frac{1}{\sqrt{g}} D_{z_i} g^{ij} \sqrt{g} D_{z_j},$$

and since we can identify K_o-invariant functions on M with W-invariant functions on $\mathfrak{a}$, Δ_{rad} is a 3-body Hamiltonian, apart from 'polar coordinate' singularities. There is nothing special about the walls in any compact region (they are an artifact of using singular coordinates); the structure is only relevant at infinity.

In general, on symmetric spaces of non-compact type G/K, there is a similar expression for the radial Laplacian on $\mathfrak{a}_{\mathrm{reg}}$. To state it, recall that roots arise by considering the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, with $\mathfrak{k}$ the Lie algebra of K, $\mathfrak{p}$ the orthocomplement with respect to the Killing form, so $\mathfrak{p}$ can be identified with T_oM. Now let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. The symmetric homorphisms $\mathrm{ad}\, H$, $H \in \mathfrak{a}$, on $\mathfrak{g}$ commute, hence are simultaneously diagonalizable. A simultaneous eigenvector X satisfies

$$(\mathrm{ad}\, H)(X) = \alpha(H)X$$

for each $H \in \mathfrak{a}$; here α is a linear functional, so $\alpha \in \mathfrak{a}^*$. The set of α's that arise this way is the set Λ of (restricted) roots, and the space of eigenvectors associated to α is the 'root space' $\mathfrak{g}_\alpha$.

Each $\alpha \in \Lambda$ determines the associated Weyl chamber wall $W_\alpha = \alpha^{-1}(0) \subset \mathfrak{a}$;

$$\mathfrak{a}_{\mathrm{reg}} = \mathfrak{a} \setminus \cup_{\alpha \in \Lambda} W_\alpha$$

is the set of regular vectors. The orthogonal reflections across the walls generate the Weyl group W; this is a finite group. Fixing a connected component $\mathfrak{a}^+$ of $\mathfrak{a}_{\mathrm{reg}}$ to be the positive Weyl chamber, we denote by Λ^+ the set of positive roots (i.e. roots that are positive on $\mathfrak{a}^+$).

Then

$$\Delta_{\mathrm{rad}} = \Delta_{\mathfrak{a}} + \frac{1}{2} \sum_{\alpha \in \Lambda} (m_\alpha \coth \alpha)\, H_\alpha,$$

where $\Delta_{\mathfrak{a}}$ is the standard Laplacian on the vector space $\mathfrak{a}$, $m_\alpha = \dim \mathfrak{g}_\alpha$ and H_α is the root vector associated to the root α. Noting that $m_\alpha = m_{-\alpha}$, $\coth(-\alpha) = -\coth \alpha$ and $H_{-\alpha} = -H_\alpha$, we also have

$$\Delta_{\mathrm{rad}} = \Delta_{\mathfrak{a}} + \sum_{\alpha \in \Lambda^+} (m_\alpha \coth \alpha)\, H_\alpha.$$

Note first that $\coth \alpha$ is singular at the wall $\alpha^{-1}(0)$, and it converges (exponentially fast) to the constant 1 as $\alpha(.) \to +\infty$. Thus, far away from the walls but inside $\mathfrak{a}^+$, one can consider Δ_{rad} as a perturbation of the translation-invariant operator $\Delta_{\mathfrak{a}} + 2H_\rho$ on $\mathfrak{a}$, where

$$\rho = \frac{1}{2} \sum_{\alpha \in \Lambda^+} m_\alpha \, \alpha \in \mathfrak{a}^*.$$

In fact, the resulting analysis is the starting point of Harish-Chandra's work on spherical functions: one constructs local 'eigenfunctions' of Δ_{rad} in $\mathfrak{a}^+$ in perturbation series (in the spirit of Cauchy-Kovalevskaya), and then uses group theory to show that these can be combined into genuine global generalized eigenfunctions.

The action of W on $\mathfrak{a}_{\mathrm{reg}}$ leaves Δ_{rad} invariant. The singularities in the coefficients of these first order terms along the Weyl chamber walls might seem to complicate the process of extending this operator to all of $\mathfrak{a}$, and indeed this would be the case if we were to try to let Δ_{rad} act on $C^\infty(\mathfrak{a})$, for example. However, this difficulty disappears if we restrict to W-invariant functions. Indeed, we recall that $C^\infty(M)^K$ is naturally identified with $C^\infty(\mathfrak{a})^W$, and so (tautologically) Δ_{rad} extends to this latter space, and then also to W-invariant distributions, etc.

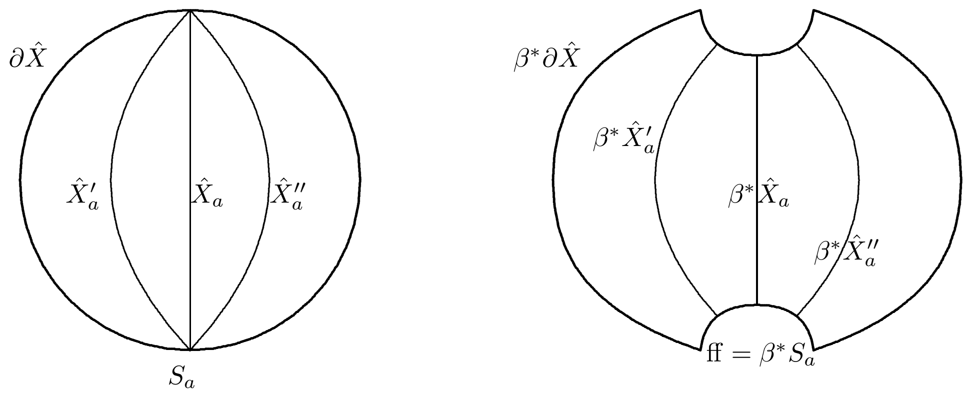
The parametrix point of view: how to construct the resolvent approximately. In N-body scattering, for $\sigma \notin \mathrm{spec}(H)$, one can construct an approximate inverse for $H - \sigma$ by inverting model operators $H_a - \sigma$, $a \in I$, of the various subsystems. Since these are k-body Hamiltonians with $k < N$, these in turn can be inverted by an iterative construction.

The most elegant version of this construction uses non-commutative (operator valued) symbols. However, in the particular cases we consider, a patching construction suffices. (N.B. The patching construction would not suffice if we had second order interactions, or, worse, considered a truly geometric generalization of N-body Hamiltonians.) To describe this, we need a partition of unity and cutoffs adopted to the geometry.

First, we need a way of succintly stating properties of functions near infinity, to avoid making statements analogous to (but more complicated than) '$\phi \in C^\infty(X)$ homogeneous degree zero, with respect to dilations, outside a compact subset of X'. Thus, we let $\hat{X}$ be the radial (or geodesic) compactification of X to a ball. Namely, we add the sphere $\mathbb{S}^{n-1}$ ($n = \dim X$) to X as its boundary. Near $\partial\hat{X}$, $\hat{X}$ is of the form $[0,1)_\rho \times \mathbb{S}^{n-1}_\omega$; we can take the identification of this region with a subset of X to be given by inverse polar coordinates $(\rho, \omega) \mapsto \rho^{-1}\omega \in X$.

Then $\phi \in C^\infty(\hat{X})$ means, in particular, that ϕ has an asymptotic expansion at infinity, namely the Taylor series at the boundary: $\phi(\rho, \omega) \sim \sum_j \rho^j a_j(\omega)$, i.e. ϕ is a (classical) symbol of order 0.

We also let $S_a = \partial X_a$ be the sphere at infinity for the collision plane X_a. We illustrate the geometry of the compactification below on the left hand side. X_a' and X_a'' are translates of the collision plane X_a; the hat denotes their closure in $\hat{X}$.



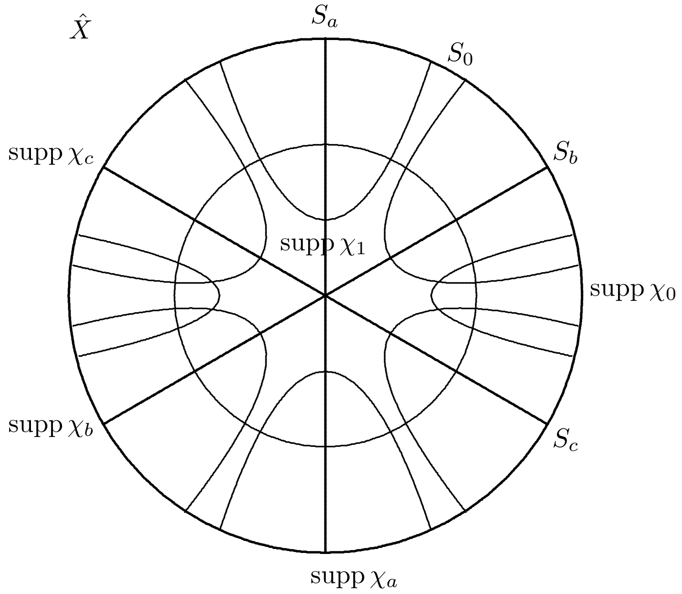
To indicate the relationship of this geometry to N-body potentials, note that V_a is *not* continuous, let alone smooth, on $\hat{X}$; to make it so we need to blow up S_a. The collision planes are really only relevant at infinity, i.e. only the S_a are relevant – the X_a do not play any role in any compact region of X.

Definition: A partition of unity $\{\chi_b : \; b \in I\}$ is $\mathcal{X}$-adapted if

1. each $\chi_b \in C^\infty(\hat{X})$,

2. $\operatorname{supp} \chi_1$ is a compact subset of $X = X_1$,

3. $\operatorname{supp} \chi_b \cap S_c = \emptyset$ unless $S_b \subset S_c$.



There exists such a partition of unity. We can also construct cutoffs ψ_b with the same properties, and in addition with $\psi_b \equiv 1$ on a neighborhood of $\operatorname{supp} \chi_b$.

We also need model operators. Near S_b, $b \neq 1$, the natural model is given by

$$H_b = H^b + \Delta_{X_b},\ \ H^b = \Delta_{X^b} + \sum_{X^a \subset X^b} V_a;$$

here H^b is the subsystem Hamiltonian for the cluster b, which is an operator on (functions on) X^b. This model 'works' (for at most first order V_a with coefficients in $S^{-\rho}$, i.e. decaying like $|x^a|^{-\rho}$) because on $\operatorname{supp} \psi_b$, $\Delta - H_b$ is a first order differential operator with decaying coefficients (in $S^{-\rho}$).

A parametrix for $H-\sigma$ is then given by

$$P(\sigma)=\sum_{b\in I,\ b\neq 0}\psi_b R_b(\sigma)\chi_b+\psi_0 P_0'\chi_0$$

where P_0' is a parametrix in the usual sense for $H-\sigma$. Note that P_0' is sandwiched between compactly supported functions, so the precise sense in which P_0' is constructed (regarding behavior near infinity) does not make much difference.

Then with

$$E(\sigma)=(H-\sigma)P(\sigma)-\mathrm{Id},\ E'(\sigma)=P(\sigma)(H-\sigma)-\mathrm{Id},$$

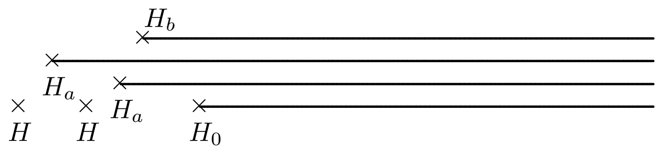
$$\begin{aligned}
&P(\sigma):H^{s,k}(\mathbb{R}^n)\to H^{s+2,k}(\mathbb{R}^n)\\
&E(\sigma):H^{s,k}(\mathbb{R}^n)\to H^{s+1,k+\rho}(\mathbb{R}^n)\\
&E'(\sigma):H^{s,k}(\mathbb{R}^n)\to H^{s+1,k+\rho}(\mathbb{R}^n)
\end{aligned}$$

for all $s,k\in\mathbb{R}$. Here $H^{s,k}(\mathbb{R}^n)$ is the weighted Sobolev space $\langle z\rangle^{-k}H^s(\mathbb{R}^n_z)$. In particular, the error terms $E(\sigma)$, $E'(\sigma)$ are compact on L^2.

An immediate corollary is the HVZ-theorem:

$$\mathrm{spec}_c(H) = \cup_{b\in I,\ b\neq 1}\, \mathrm{spec}(H_b),$$
$$\mathrm{spec}(H_b) = \mathrm{spec}(H^b) + [0, +\infty).$$

In general, the spectrum is quite complicated, as only the continuous spectrum is predicted by the theorem, not the eigenvalues. Moreover, the continuous spectrum of H depends on the whole spectrum of its proper subsystems, i.e. L^2-eigenvalues of the subsystems give rise to branches of continuous spectrum.



This analysis remains valid on symmetric spaces G/K as well, as long as one restricts to K-invariant functions. The only additional requirement is that we make the partition of unity, and the cutoffs, W-invariant, and one also needs to work with weighted Sobolev spaces of K-invariant functions. In this case the coefficients decay exponentially, i.e. one can take $\rho > 0$ arbitrarily large. One slight issue relates to labelling; it is in practice better to work only with the compactified positive Weyl chamber $\hat{\mathfrak{a}}^+$ rather than with $\hat{\mathfrak{a}}$. If X_b^+ is an open face of the closure of $\mathfrak{a}^+$ in $\mathfrak{a}$, there is a unique 'collision plane' (intersection of walls) X_b containing it as an open subset. Let I^+ denote the set of $b \in I$ that arise this way.

Let Λ_b denote the set of roots vanishing at some (hence all) $p \in X_{b,\mathrm{reg}}$,

$$\rho_b = \frac{1}{2} \sum_{\alpha \in \Lambda_b^+} m_\alpha \alpha.$$

Then $H_\rho - H_{\rho_b} \in X_b$, and $H_{\rho_b} \in X^b = \mathfrak{a}^b$.

For such b, the models are now

$$L_b = T_b + \Delta_{b,\mathrm{rad}},$$
$$T_b = \Delta_{X_b} + 2(H_\rho - H_{\rho_b})$$
$$\Delta_{b,\mathrm{rad}} = \Delta_{\mathfrak{a}^b} + 2H_{\rho_b} + \sum_{\alpha\in\Lambda_b^+} m_\alpha(\coth\alpha - 1)H_\alpha.$$

Thus, T_b is a translation-invariant differential operator on X_b, and $\Delta_{b,\mathrm{rad}}$ is the radial part of the Laplacian on a lower rank symmetric space $M_b = G_b/K_b$.

The HVZ-type theorem in this setting is thus

$$\mathrm{spec}_c(\Delta_{\mathrm{rad}})$$
$$= \cup_{b\in I^+,\ b\neq 1}(\mathrm{spec}(\Delta_{b,\mathrm{rad}}) + [|\rho-\rho_b|^2, +\infty)).$$

The first sign of a miracle in this setting is that the spectrum is simple: there are no L^2-eigenvalues at all, so in fact

$$\mathrm{spec}(\Delta_{\mathrm{rad}}) = [\rho^2, +\infty).$$

In general, results are similar in N-body scattering and on symmetric spaces, but there are many special coincidences on symmetric spaces. As an example, consider the analytic continuation of the resolvent through the spectrum.

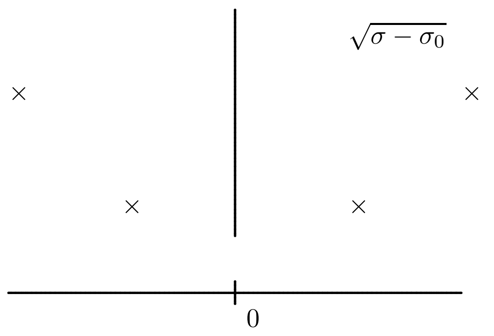
In the N-body setting, under appropriate assumptions for the potentials V_a (basically V_a can be compactly supported on X^a, dilation analytic on X^a with respect to the dilations I will discuss, or more generally dilation analytic near infinity on X^a), $R(\sigma) = (H - \sigma)^{-1}$ continues analytically to a Riemann surface Σ including $\mathbb{C} \setminus \mathrm{spec}(H)$ (e.g. in the sense that its Schwartz kernel, i.e. $R(\sigma)\delta_p$, continues as a distribution). This Riemann surface is usually very complicated, and one may have only be able to extend only a little beyond the continuous spectrum. (This approach started with the work of Aguilar-Balslev-Combes; the stated result is due to Ch. Gérard.)

Normalizing $\log z$ on $\mathbb{C} \setminus [0, +\infty)$ to take values in $(-2\pi, 0)+i\mathbb{R}$, and $\sqrt{z}$ to take values in $\operatorname{Im} z < 0$, on symmetric spaces G/K we have

Theorem 1 *For a suitable constant $L > 0$, the Green function $G_o(\sigma)$ continues analytically as a distribution to the logarithmic plane in $\sigma - \sigma_0$ with the half-lines*

$$\log(\sigma - \sigma_0) \in i(-\pi + 2k\pi) + [2 \log L, +\infty),\ k \in \mathbb{Z} \setminus \{0\},$$

removed, if n is even, and to the Riemann surface of $\sqrt{\sigma - \sigma_0}$, with $\sqrt{\sigma - \sigma_0} \in i[L, +\infty)$ removed, if n is odd.



A weaker version of this, in which one allows ramifications points arising from poles of the analytic continuation of subsystem resolvents, is actually a rather simple result from the N-body perspective.

One considers the scaling $A \mapsto A^w$, which is dilations along geodesics through $o = \mathrm{Id}$. (The Euclidean analogue is $x \mapsto wx$.) For $w = e^\theta > 0$, this is a diffeomorphism Φ_θ, so

$$(U_\theta f)(A) = (\det D_A \Phi_\theta)^{1/2} (\Phi_\theta^* f)(A)$$

defines a unitary operator on L^2. Then $\Delta_\theta = U_\theta \Delta U_\theta^{-1}$ extends analytically to the strip $|\operatorname{Im}\theta| < \frac{\pi}{2}$.

In the case of $\mathbb{R}^n$, the similar scaling gives $\Delta_\theta = e^{-2\theta}\Delta$; notice that this rotates the continuous spectrum. Same happens for Δ_{rad}. Then

$$\langle f, (\Delta_{\mathrm{rad}} - \sigma)^{-1} g\rangle = \langle U_{\bar\theta} f, (\Delta_{\theta,\mathrm{rad}} - \sigma)^{-1} U_\theta g\rangle$$

for a dense set of f, g. One can modify the scaling to actually show that for $g \in \mathcal{D}'_c(M)$, $(\Delta - \sigma)^{-1} g \in \mathcal{D}'(M)$.

The symmetric space miracle is that the ramification points do not occur. Note that an innocent looking improvement, namely that one can improve the statement that the subsystems have only poles (no ramification points; one even knows that these poles cannot go to infinity in certain cones) outside ramification points given by their subsystems' poles, to them having no poles at all, gives a huge improvement due to the inductive construction.

Propagation of singularities: this is best stated in terms of a wave front set at infinity, but this would require some background. Roughly, it is a statement that the (appropriate) wave front set, i.e. the singularities in T^*X measuring microlocal decay at infinity in X, of (tempered) generalized eigenfunctions, is a union of maximally extended generalized broken bicharacteristics. Instead, I state the result for 'perturbed plane waves' in a rank 2 setting. For symmetric spaces, these are Harish-Chandra's spherical functions.

First, what are perturbed plane waves coming in from some direction ξ, of energy $\sigma = \xi \cdot \xi$? Note that in the N-body setting one usually considers ξ real. An unperturbed plane wave is $e^{-i\xi \cdot z}$; the phase $-\xi \cdot z$ is smallest, resp. largest, where z is parallel, resp. anti-parallel to ξ. A perturbed plane wave is supposed to have the same asymptotic behavior 'on the incoming side' (to be precise this would be a microlocal statement); thus,

$$u_\xi(z) = \psi(z)e^{-i\xi\cdot z} - R(\sigma+i0)(H-\sigma)(\psi(z)e^{-i\xi\cdot z})$$

is a natural candidate; here $\psi \in C^\infty(\hat{X})$ identically 1 near $\frac{\xi}{|\xi|}$. Note that here ψ is not really needed (one can take $\psi \equiv 1$, and the result is independent of ψ anyway), but its presence makes the comparison with symmetric spaces easier. In fact, such u_ξ can be characterized by the statement that they are the unique (tempered) generalized eigenfunctions of H which microlocally near the 'incoming set' are given by $e^{-i\xi\cdot z}$.

These u_ξ also show up as the Schwartz kernel of the 'Poisson operator', and they contain (in an accessible manner!) the information about the scattering matrices corresponding to free incoming particles, as well as about part of the spectral projector. (One needs to have plane waves corresponding to bound states in general.)

The complex analogue, when ξ is not real, works similarly. Then one would want ψ identically 1 near $\frac{\operatorname{Im}\xi}{|\operatorname{Im}\xi|}$: the real part of the exponent is $\operatorname{Im}\xi \cdot z$, which is largest where z is parallel to $\operatorname{Im}\xi$. Thus, $(H-\sigma)(\psi(z)e^{-i\xi\cdot z})$ is in a weighted L^2 space corresponding to less growth at infinity, and in certain cases, e.g. if $\xi = c\xi_0$ with ξ_0 real, this implies that $R(\sigma)$ can be applied to it. ($c \to 1$, $\operatorname{Im} c > 0$, corresponds to the real limit. It is *different* from $c \to 1$, $\operatorname{Im} c < 0$, which would correspond to fixing the 'outgoing' asymptotics of u_ξ, ξ real.)

For $\xi = c\xi_0$ with ξ_0 real, $\operatorname{Im}\xi \in \mathfrak{a}^+$, one can define the spherical function U_ξ similarly, noting that now $\sigma = \xi\cdot\xi + |\rho|^2$ is the energy, by choosing ψ to be vanishing near the walls, regarding $U_\xi^0(z) = \psi(z)e^{-\rho(z)}e^{-i\xi\cdot z}$, supported in $\mathfrak{a}^+$, as a W-invariant function on $\mathfrak{a}$, or a K-invariant function on M:

$$U_\xi(z) = U_\xi^0(z) - R(\sigma)(\Delta_{\text{rad}} - \sigma)(U_\xi^0(z))$$

In fact, if one makes a better first approximation than $\psi(z)e^{-\rho(z)}e^{-i\xi\cdot z}$, as I discuss below, it is easy to define U_ξ analogously in general.

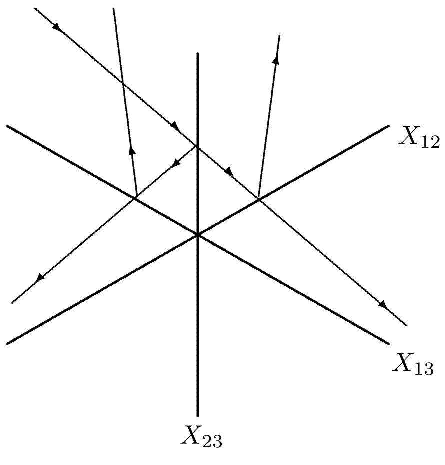
To make the underlying geometry identical, consider 3-body scattering with the collision planes given by the SL(3)-walls. (This corresponds to three one-dimensional particles with equal masses.) The space on which expansions live in this case is the blow-up of $\hat{X}$ along the S_a:

$$\tilde{X} = [\hat{X}; \{S_a : \ a \in I\}].$$

If we in addition assume that no proper subsystem has bound states, e.g. we have three electrons interacting repulsively, then 3-body perturbed plane waves with incoming direction ξ and energy σ (so $\xi \cdot \xi = \sigma$) can be written as a sum

$$u_\xi(z) = \sum_{s \in S_3} c_s(z) e^{-i(s\xi)\cdot z} + \langle z \rangle^{-1/2} e^{i\sqrt{\sigma}\,\langle z \rangle} a(z),$$

with c_s smooth on $\tilde{X}$ away from the direction of $-s\xi$ (where it has a 'conic' singularity), c_s rapidly decreasing in the direction of $s\xi$ for $s \neq 1$, and $c_1 \sim 1$ in the direction of ξ. The sum over S_3 contains the reflections of the incoming wave from the collision planes, corresponding to particles that have collided.



The situation in symmetric spaces is quite similar, except that one should consider the coefficients as either living on $\mathfrak{a}^+$, or being W-invariant on $\mathfrak{a}$, so it is no longer the case that c_s is rapidly decreasing in the direction of $s\xi$, and there is also an additional exponential weight $e^{-\rho}$. Denoting the spherical function by U_ξ, we have the following behavior:

$$U_\xi(z) = \sum_{s \in S_3} c_s(z) e^{-\rho(z)} e^{-i(s\xi)\cdot z},$$

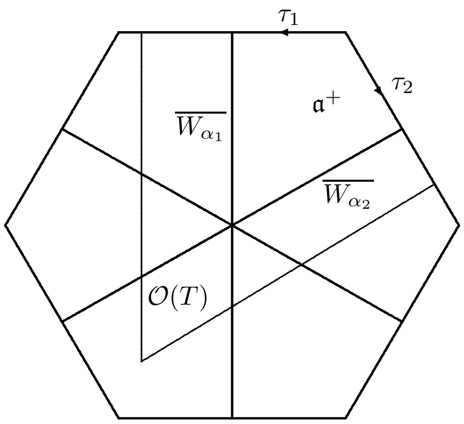
valid in a neighborhood of the closure of $\mathfrak{a}^+$ *in* $\mathfrak{a}$. (From a different point of view, based on Harish-Chandra's approach, such a description is due to Trombi and Varadarajan.)

The miracles in this case (apart from the miraculous geometry, which we already assumed for the purposes of illustration in the 3-body setting) are that $c_s \in C^\infty(\tilde{\mathfrak{a}})^W$ (i.e. there is no 'conic' singularity where waves coming from different directions meet up) and that there is no spherical wave, i.e. $a \equiv 0$. The absence of the conic singularity actually does follow from the construction; it is a combination of geometric coincidence and that the model operators are Laplacians on symmetric spaces, so their scattering matrices have certain symmetries.

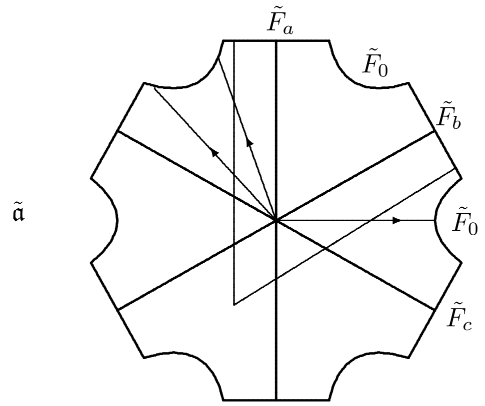
Here, as in the 3-body setting, the coefficients are constructed iteratively, starting with c_1, corresponding to the size of the real part of the exponent in $\mathfrak{a}^+$. (For real ξ, one replaces 'real part of the exponent' by 'phase'.) They are obtained by ensuring that the incident and reflected waves combine to give an approximate generalized eigenfunction at the wall at which the reflection takes place.

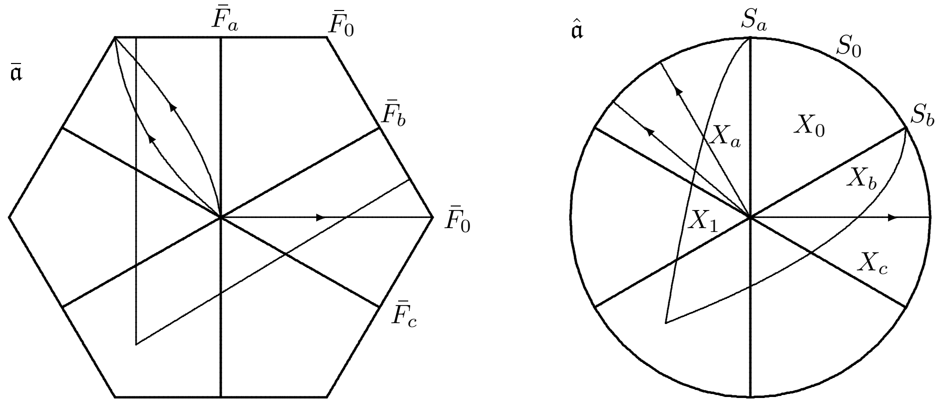
(If we do not assume the absence of bound states in subsystems, the spherical wave is replaced by a more complicated expression.)

In fact, there is one more miracle, the coefficients $c_s(z)$ are actually smooth on a smaller space $\bar{\mathfrak{a}}$ than $\tilde{\mathfrak{a}}$. This space $\bar{\mathfrak{a}}$ is the polyhedral or dual-cell compactification. Roughly, it compactifies $\mathfrak{a}^+$ as a cube, using the positive simple roots $\alpha \in \Lambda^+_{\mathrm{ind}}$. Namely, the latter, denoted by α_j, form a basis of $\mathfrak{a}^*$, hence $(\alpha_1, \ldots, \alpha_n)$ (where $\dim \mathfrak{a} = n$) is a coordinate system on $\mathfrak{a}$. We compactify $\mathfrak{a}^+$, the neighborhood $\mathcal{O}(T) = \cap_{j=1}^n \alpha_j^{-1}((T_j, +\infty))$ by identifying it with $\prod_{j=1}^n (T_j, +\infty)$, and compactifying the latter as $\prod_{j=1}^n [0, e^{-T_j})$ via the map $t_j \mapsto e^{-t_j}$. Thus, we effectively make the $\tau_j = e^{-\alpha_j}$ a coordinate system on a neighborhood of $\mathfrak{a}^+$ in $\bar{\mathfrak{a}}$.



It is worth pointing out now what $\tilde{\mathfrak{a}}$ is in terms of $\bar{\mathfrak{a}}$: first, one takes the logarithmic blowup $\bar{\mathfrak{a}}_{\mathrm{log}}$ (i.e. makes α_j^{-1} the coordinates), then one performs a 'total boundary blow-up', starting with boundary faces of the smallest dimension. Thus, the blow-down map $\beta : \tilde{\mathfrak{a}} \to \bar{\mathfrak{a}}$ is smooth, and the smoothness of $c_s(z)$ on $\bar{\mathfrak{a}}$ near $\mathfrak{a}^+$ is a much stronger statement than its smoothness on $\tilde{\mathfrak{a}}$ in the same region.





The thin lines without arrows show the boundary of $\mathcal{O}(T)$ for $T_1 < 0$, $T_2 < 0$; the thin lines with arrows are geodesic emanating from 0; in particular they bound conic regions. The labels are $a = (23)$, $b = (12)$, $c = (13)$.

However, this miracle *only* holds for the spherical functions, a.k.a. perturbed plane waves, *not* for the asymptotics of the Green's function, a.k.a. spherical waves. In fact:

Theorem 2 *On* $M = \mathrm{SL}(3)/\mathrm{SO}(3)$, *if* $\sigma \notin \mathrm{spec}(\Delta) = [\sigma_0, +\infty)$, $\sigma_0 = |\rho|^2$, *then*

$$R(\sigma)\delta_o = \rho^\sharp \rho_\sharp x^{1/2} x^\sharp x_\sharp \exp(-i\sqrt{\sigma - \sigma_0}/x) g$$

where $g \in C^\infty(\tilde{M} \setminus \{o\})$. *Here* $\tilde{M}$ *is the compactification of* M *analogous to* $\tilde{\mathfrak{a}}$, *etc.,* $x(.) = d(o, .)$ *is the distance function from* o, $\rho_\sharp$ *and* $\rho^\sharp$ *are defining functions of the two boundary hypersurfaces of* $\bar{M}$, *so* $e^{-\rho} \sim \rho_\sharp \rho^\sharp$ *in* $\mathcal{O}(T)$, *and* $x_\sharp$, $x^\sharp$ *are the defining functions of the lifts of these boundary hypersurfaces to* $\tilde{M}$, *i.e. in the* $\tilde{\mathfrak{a}}$ *picture, of* $\tilde{F}_{12}$ *and* $\tilde{F}_{23}$. *Here the leading term* $g|_{\partial \tilde{M}}$ *is strictly positive.*

Note the similarity to 3-body spherical waves! A weaker version of this theorem, with a continuous rather than smooth statment, is due to Anker, Guivarch, Ji and Taylor.