Contemporary Mathematics

Geometric optics and the wave equation on manifolds with corners

András Vasy

1. Introduction

According to geometric optics, light propagates in straight lines (in homogeneous media), reflects/refracts from surfaces according to Snell's law: energy and tangential momentum are conserved. Thus, when reflecting from a hypersurface (which has codimension one) one gets the usual law of incident and reflected rays enclosing an equal angle to the normal to the surface. Indeed, conservation of tangential momentum and kinetic energy implies that of the *magnitude* of the normal component. When reflecting from a higher codimension (≥ 2) corner, the law is unchanged (momentum tangential to the corner and energy are conserved) – but now this allows each incident ray to generate a whole cone of reflected rays, see Figures 1-2.

On the other hand, light is a form of electromagnetic radiation, satisfying Maxwell's equations – which in turn implies that each component of the electromagnetic field (in free space) satisfies the wave equation,

$$Pu = 0, \quad Pu = D_t^2 u - \Delta_g u,$$

Δ_g is the Laplacian, so it is $c^2 \sum_{j=1}^n D_{x_j}^2$ in $\mathbb{R}^n$, where c is the speed of light (this corresponds to a Riemannian metric $g = c^{-2} \sum dx_j^2$), $D_{x_j} = \frac{1}{i}\partial_{x_j}$, with suitable boundary conditions.

It is natural to ask how these points of view are related. One way of discussing the relationship between these is that singularities (lack of smoothness) of solutions of $Pu = 0$ follow geometric optics rays. Due to its relevance, this problem has a long history, and has been studied extensively by Keller in the 1940s and 1950s in various special settings, see e.g. [**1, 11**]. The present work (and ongoing projects continuing it, especially joint work with Melrose and Wunsch [**16**], see also [**2, 17**]) can be considered a justification of Keller's work in the general geometric setting (curved edges, variable coefficient metrics, etc). In order to describe this relationship precisely, I discuss an even more general setting.

1991 *Mathematics Subject Classification.* 58J47, 35L20.

This work is partially supported by NSF grant DMS-0201092, a Clay Research Fellowship and a Fellowship from the Alfred P. Sloan Foundation.

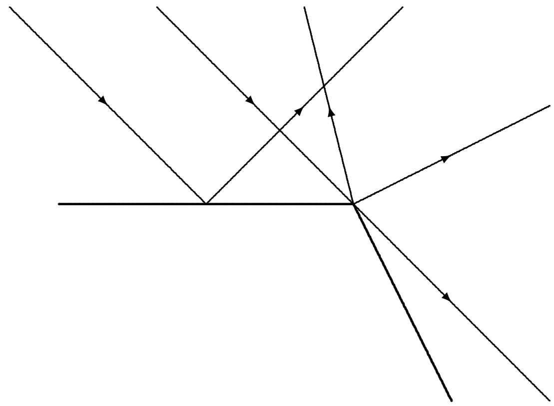

FIGURE 1. Geometric optics rays hitting a surface. The ones hitting a corner generate a whole cone of reflected rays, see also Figure 2.

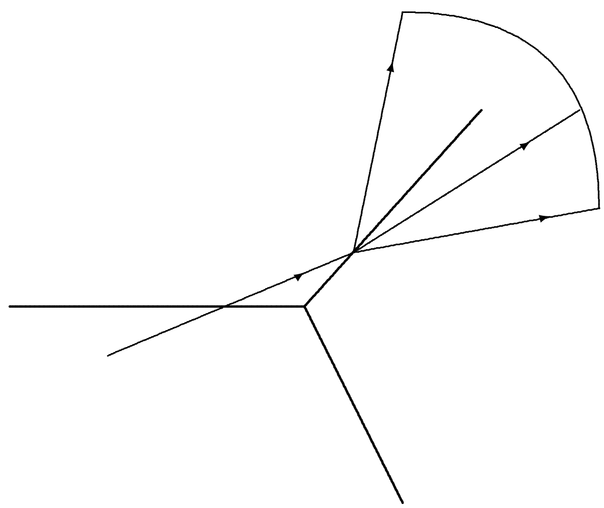

FIGURE 2. Geometric optics rays hitting a surface at a codimension 2, dimension 1, corner (which may be called an edge). The momentum component parallel to the edge is preserved when the edge is hit, as is the magnitude of the normal component, so a single incident ray generates a cone with apex at the point where the edge is hit, axis given by the edge, and angle at the apex given by the angle between the incident ray and the edge.

I should state here that these notes are meant to be expository. The full details of the proof of the main theorem, as well as various claims scattered throughout the notes, are written up in [**26**]. Moreover, [**27**] provides an exposition at an intermediate level: the main technical points are explained there. In the present notes the goal is to explain at least the statement of the results, and also to explain a proof in the simplest boundaryless setting that can be adapted to the more general situation we face here.

The original version of these notes were based on my transparencies and lecture note at the UAB conference, and it was put together somewhat hastily as a deadline approached. I am thus very grateful to the anonymous referee who had some excellent suggestions in making these notes more accessible. I also thank the conference organizers for the invitation, and the University of Alabama, Birmingham, for hosting the meeting so well.

2. PDEs on manifolds without boundary

Let's start with the boundaryless case. So suppose X is a manifold without boundary of dimension n. As outlined above, the basic goal is to connect analytic objects (such as the wave operator) with geometric objects (such as certain curves related to the light rays). This is accomplished by the so-called microlocal, or phase space, analysis. The standard setting for microlocal analysis is the *cotangent bundle* – T^*X *is* the phase space. If z_j are local coordinates on X, and we write one-forms as $\sum \zeta_j\, dz_j$, then (z_j, ζ_j), $j = 1, \ldots, n$, are local coordinates on T^*X. (Actually, they are global on the fibres of $T^*X \to X$.)

For our purposes there are two important structures on T^*X. First, being a vector bundle, T^*X is *equipped with an* $\mathbb{R}^+$*-action* (dilation in the fibers): $\mathbb{R}^+_s \times T^*X \ni (s, z, \zeta) \mapsto (z, s\zeta)$. It is also a *symplectic manifold*, equipped with a canonical symplectic form ω, $\omega = \sum d\zeta_j \wedge dz_j$ in local coordinates.

We can now turn to differential operators. It is useful to recall the multiindex notation: if $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, then $D_z^\alpha = D_{z_1}^{\alpha_1} \ldots D_{z_n}^{\alpha_n}$, with $D_j = D_{z_j} = \frac{1}{i}\partial_{z_j}$ (and $\mathbb{N}$ is the set of non-negative integers).

If P is a differential operator on X, say $P = \sum_{|\alpha| \le m} a_\alpha(z) D_z^\alpha$ in some local coordinates, one can associate a *principal symbol* $p = \sigma_m(P) = \sum_{|\alpha|=m} a_\alpha(z)\zeta^\alpha$ to P; this is homogeneous degree m with respect to the $\mathbb{R}^+$-action (called *positively homogeneous*).

In fact, the same works for a more general class of operators, called *pseudodifferential operators*, or ps.d.o.'s for short. I will give a concrete description of what these are, but one may learn more by listing their properties first. For $P \in \Psi^m_{cl}(X)$, i.e. P is a pseudodifferential operator of order m, then $p = \sigma_m(P)$ is *positively homogeneous* degree m on $T^*X \setminus o$, o denoting the zero section.

From an algebraic point of view, some of the most important properties are that $\Psi^\infty_{cl}(X) = \cup_m \Psi^m_{cl}(X)$ is an order-filtered ring, the space $\Psi^m(X)$ increasing with m, so

$$A \in \Psi^m_{cl}(X),\ B \in \Psi^{m'}_{cl}(X) \Rightarrow AB \in \Psi^{m+m'}_{cl}(X),$$

that the principal symbol is a ring homomorphism, that $\Psi^0_{cl}(X)$ is bounded on $L^2(X)$, $\Psi^m(X)$ (m arbitrary) maps $C^\infty(X)$ (and its dual $\mathcal{D}'(X)$) to itself, and that there is a short exact sequence

$$0 \to \Psi^{m-1}_{cl}(X) \to \Psi^m_{cl}(X) \to S^m_{hom}(T^*X \setminus o) \to 0;$$

where S^m_{hom} stands for C^∞ homogeneous functions of degree m.

On the other hand, for $X = \mathbb{R}^n$, there are explicit maps, called *quantizations*, sending appropriate classes of functions on T^*X to pseudodifferential operators on X. The standard class of such functions to consider is that of symbols: a *symbol* of order m on T^*X ($X = \mathbb{R}^n$) is a C^∞ function with specified behavior as $\xi \to \infty$

(and uniform control as $x \to \infty$, although this is much less relevant here): for all $\alpha, \beta \in \mathbb{N}^n$ there is $C_{\alpha,\beta} > 0$ such that for all $(x, \xi) \in T^*X$,

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-|\beta|}. \tag{1}$$

This generalizes polynomials in ξ (recall that symbols of differential operators are polynomials): the order of a polynomial decreases each time one differentiates it. Note that a smooth homogeneous function of degree m on $T^*X \setminus o$ is in fact a symbol of order m in $|\xi| > 1$ over bounded regions in x, i.e. it satisfies the symbol estimates (1) there – we need to work away from the zero section, $\xi = 0$, for any smooth homogeneous function on all of T^*X is in fact a polynomial. A *polyhomogeneous symbol* a of order m is a symbol of order m for which there exist smooth homogeneous degree $m - j$ functions a_j ($j \in \mathbb{N}$) on $T^*X \setminus o$ such that, for all k, $a - \sum_{j=0}^{k-1} a_j$ is a symbol of order $m - k$ in $|\xi| > 1$. For quantization, for instance, one can take the 'left quantization'

$$(q_L(a)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a(x,\xi) u(y)\, dy\, d\xi,$$

$q_L(a)$ is (by definition) a ps.d.o. of order m is a is a symbol of order m. For general manifolds one can transfer this definition by localization. These quantizations q have the property that $\sigma_m(q(a)) - a$ is a symbol of order $m-1$ – so to leading order $q(a)$ is independent of the choice of q, but there are still many choices.

It should be emphasized that, in the present setting, the relevant region for microlocal analysis is the asymptotic regime as $\xi \to \infty$. Making various objects homogeneous, or conic, is a way of 'bringing infinity to a finite region'. Another way of accomplishing this is to compactifying the fibers of the cotangent bundle – this is the approach taken by Melrose, e.g. in [**13**].

The symplectic form ω turns p, or rather its differential dp, into a vector field H_p (called the *Hamilton vector field* of p) on T^*X via demanding that $\omega(V, H_p) = Vp$ for all vector fields V. Thus,

$$H_p = \sum_j \frac{\partial p}{\partial \zeta_j} \frac{\partial}{\partial z_j} - \frac{\partial p}{\partial z_j} \frac{\partial}{\partial \zeta_j}.$$

Note that H_p is homogeneous of degree $m - 1$.

Definition 1. Suppose that p is homogeneous degree m on $T^*X \setminus o$. The *characteristic set* of p is $\Sigma = p^{-1}(\{0\})$. *Bicharacteristics* are integral curves of H_p inside Σ.

The role that H_p plays in analysis becomes apparent upon noticing that if $P \in \Psi_{cl}^m(X)$, $Q \in \Psi_{cl}^{m'}(X)$ then $[P, Q] = PQ - QP \in \Psi_{cl}^{m+m'-1}(X)$, and

$$\sigma_{m+m'-1}(i[P,Q]) = H_p q.$$

To do analysis, we also need a notion of singularity of a function or distribution u. The roughest notion is that of the *wave front set* $\mathrm{WF}(u)$, which locates at which points and in which direction a function u is not smooth, here meaning C^∞. Immediately from the definition, given below, this is a closed conic subset of $T^*X \setminus o$; u is C^∞ if and only if $\mathrm{WF}(u) = \emptyset$. In fact, for any point $z_0 \in X$, z_0 has a neighborhood in X on which u is C^∞ if and only if $\mathrm{WF}(u) \cap (T^*_{z_0}X \setminus o) = \emptyset$.

One way of defining $\mathrm{WF}(u)$ for distibutions u is the following:

Definition 2. Suppose that $u \in \mathcal{D}'(X)$. We say that $q \in T^*X \setminus o$ is *not* in $\mathrm{WF}(u)$ if there exists $A \in \Psi^0(X)$ such that $\sigma_0(A)(q) \neq 0$ (i.e. A is *elliptic* at q) and $Au \in C^\infty(X)$.

To get a feeling for this, one should think of A as the quantization of a symbol a which is supported in a cone around q, identically 1 on the $\mathbb{R}^+$-orbit through q (at least outside some compact subset of T^*X).

For example, if δ_0 is the delta distribution at the origin, then

$$\mathrm{WF}(\delta_0) = \{(0, \zeta) : \zeta \neq 0\} = N^*\{0\} \setminus o,$$

i.e. δ_0 is singular only at the origin, and it is singular there in every direction – which is quite reasonable. Here we recall that if S is a submanifold of X then N^*S is the *conormal bundle* of S; at a point $p \in S$, the fiber N_p^*S consists of all covectors $\alpha \in T_p^*S$ such that $\alpha(V) = 0$ for all $V \in T_pS$. Another way of looking at N^*S is that the space of its smooth sections is spanned (over $C^\infty(S)$) by da, as a ranges over all elements of $C^\infty(X)$ that vanish on S. As an aside, conormal bundles are *Lagrangian submanifolds* of T^*X, i.e. the symplectic form vanishes when restricted to their tangent space, and are maximal dimensional (i.e. n-dimensional) with this property. Conic Lagrangian submanifolds of $T^*X \setminus o$ play an important role in many parts od microlocal analysis.

A more interesting example is that of a domain Ω with a C^∞ boundary, and χ_Ω the characteristic function of Ω. If locally $\partial\Omega$ is defined by f, i.e. over some open set $O \subset X$, $\partial\Omega \cap O = \{z \in O : \ f(z) = 0\}$ and df never vanishes on $\partial\Omega \cap O$, then, over O, the space of sections of $N^*\partial\Omega$ is spanned by df, so any covector in $N^*\partial\Omega$ has the form αdf. In this case,

$$\mathrm{WF}(\chi_\Omega) = N^*\partial\Omega \setminus o.$$

That is, χ_Ω is smooth both in Ω and in the complement of its closure (after all, it is constant there!), and it is singular at $\partial\Omega$, but it is only singular in the conormal directions: it is smooth when one moves along $\partial\Omega$. (This can be seen directly from the definition of WF: consider differentiating χ_Ω along a vector field tangential to the boundary, and note that the principal symbol of such a vector field vanishes on the conormal bundle!)

One can measure singularities with respect to other spaces: e.g. the Sobolev spaces $H^s_{\mathrm{loc}}(X)$, where we would write $\mathrm{WF}^s(u)$, or with respect to real analytic functions, where we would write $\mathrm{WF}_A(u)$. Indeed, $\mathrm{WF}^s(u)$ plays a role in the proofs of various results stated below; one often proves in an inductive manner that u is microlocally in H^s for every s (hence is C^∞ microlocally), rather than proving directly that u is C^∞ microlocally. We can define $\mathrm{WF}^s(u)$ for $u \in \mathcal{D}'(X)$ by saying that $q \in T^*X \setminus o$ is *not* in $\mathrm{WF}^s(u)$ if there exists $A \in \Psi^0(X)$ such that $\sigma_0(A)(q) \neq 0$ and $Au \in H^s_{\mathrm{loc}}(X)$. Equivalently, one can shift the weight to the ps.d.o. from the function space:

Definition 3. Suppose that $u \in \mathcal{D}'(X)$. We say that $q \in T^*X \setminus o$ is *not* in $\mathrm{WF}^s(u)$ if there exists $A \in \Psi^s(X)$ with $\sigma_0(A)(q) \neq 0$ and $Au \in L^2_{\mathrm{loc}}(X)$.

The main facts about the analysis of P, which in this generality are due to Hörmander and Duistermaat-Hörmander [**7, 4, 9**] are:

(1) Microlocal elliptic regularity. Let $\Sigma(P) = p^{-1}(\{0\})$ be the characteristic set of P. If $u \in \mathcal{D}'(X)$ then $\mathrm{WF}(u) \subset \mathrm{WF}(Pu) \cup \Sigma(P)$. In particular, if $Pu \in C^\infty(X)$ then $\mathrm{WF}(u) \subset \Sigma(P)$.

(2) Propagation of singularities. Suppose that p is real, $Pu \in C^\infty(X)$. Then $\mathrm{WF}(u)$ is a union of maximally extended bicharacteristics in $\Sigma(P)$. That is, if $q \in \mathrm{WF}(u)$ (hence in $\Sigma(P)$) then so is the whole bicharacteristic through q.

For analogy with the manifolds with corners setting, we restate part of these conclusions in a special case:

Theorem 4. *(See Hörmander and Duistermaat-Hörmander [**7, 4, 9**].) Suppose $P \in \Psi^m(X)$, $p = \sigma_m(P)$ is real, $Pu = 0$, $u \in \mathcal{D}'(X)$. Then $\mathrm{WF}(u) \subset \Sigma = \Sigma(P)$, and it is a union of maximally extended bicharacteristics of P.*

Note that (2) may be vacuous; indeed, if H_p is *radial*, i.e. tangent to the orbits of the $\mathbb{R}^+$-action, then it does not give any information on $\mathrm{WF}(u)$, as the latter is already known to be conic. Such points are called radial points, and in recent work with Hassell and Melrose [**6**], they have been extensively analyzed under non-degeneracy assumptions. If P is the wave operator, there are no radial points in $\Sigma = \Sigma(P)$, but such points are very important in scattering theory (where the $\mathbb{R}^+$-action, or its remnants, are in the base variables z).

As an example, consider the wave operator $P = D_t^2 - \Delta_g$, $X = M \times \mathbb{R}$, M a manifold without boundary. Then $p = \sigma_2(P) = \tau^2 - |\xi|_g^2$, where (x, t, ξ, τ) are coordinates on T^*X (so ξ is dual to x, and τ is dual to t), and the projection of bicharacteristics to M are geodesics. If $M \subset \mathbb{R}^n$ and g is the Euclidean metric, then $H_p = 2\tau\partial_t - 2\xi \cdot \partial_x$, and bicharacteristics inside $p = 0$, i.e. $|\tau| = |\xi|$, are straight lines

$$s \mapsto (x_0 - 2s\xi_0, t_0 + 2\tau_0 s, \xi_0, \tau_0),$$

which explains geometric optics in the absence of boundaries.

3. Propagation of singularities on manifolds with corners: the phase space

We can now turn to boundaries and corners. So suppose X is a manifold with corners. Locally this means that X is diffeomorphic to an open subset U of $[0, \infty)^k \times \mathbb{R}^{n-k}$; we denote the corresponding coordinates by (x, y).

Roughly, the results have the same form as in the boundaryless case, but the definitions of wave front set and the bicharacteristics change significantly. In particular, the relevant wave front set is $\mathrm{WF}_b(u)$, introduced by Melrose (see [**18**], [**8**, Section 18.2] for the setting of smooth boundaries, [**19**] for manifolds with corners). Both $\mathrm{WF}_b(u)$ and the image of the (generalized broken) bicharacteristics are subsets of a new phase space, the *b-cotangent bundle* ${}^bT^*X$.

The reason for this is that one cannot microlocalize in T^*X: naively defined ps.d.o's do not act on functions on X in general, and even when they do, they do not preserve boundary conditions. This causes technical complications, for we are interested in the wave operator, $P = D_t^2 - \Delta$, whose principal symbol is a C^∞ function on T^*X, *not* on ${}^bT^*X$ where we microlocalize. In fact, from a PDE point of view, this discrepancy is what causes the diffractive phenomena.

Rather than defining ${}^bT^*X$ directly, I describe its main properties: these can be easily made into a definition as we shortly see. Being a vector bundle, locally in X it is trivial, and in the local coordinate product decomposition above, it will take the form $U_{x,y} \times \mathbb{R}^k_\sigma \times \mathbb{R}^{n-k}_\zeta$, with $U \subset [0, \infty)^k_x \times \mathbb{R}^{n-k}_y$, where σ is the 'b-dual' variable of x and ζ is the b-dual variable of y.

There is a natural map $\pi : T^*X \to {}^bT^*X$, which in these local coordinates takes the form

$$\pi(x, y, \xi, \zeta) = (x, y, x\xi, \zeta), \quad \text{with } x\xi = (x_1\xi_1, \ldots, x_k\xi_k). \tag{2}$$

(That is, $\sigma_j = x_j\xi_j$.) Thus, π is a C^∞ map, but at ∂X, it is not a diffeomorphism.

Over the *interior* X° of X, ${}^bT^*X$ and T^*X are naturally identified via π, and

$$\mathrm{WF}_b(u) \cap {}^bT^*_{X^\circ}X = \pi(\mathrm{WF}(u) \cap T^*_{X^\circ}X).$$

Note that if q is a linear function on each fiber of ${}^bT^*X$, then it has the form

$$q = \sum a_j(x, y)\sigma_j + \sum b_j(x, y)\zeta_j,$$

so

$$\pi^*q = \sum a_j(x, y)x_j\xi_j + \sum b_j(x, y)\zeta_j,$$

which is the principal symbol of

$$Q = \sum a_j(x, y)x_jD_{x_j} + \sum b_j(x, y)D_{y_j}.$$

Vector fields of this form are exactly the vector fields tangent to all boundary faces of X; we denote their space by $\mathcal{V}_b(X)$.

In fact, this indicates how ${}^bTX$ can be defined intrinsically: the set of all smooth vector fields tangent to all boundary faces is the set of all smooth sections of a vector bundle; indeed, x, y, a_j, b_j above give local coordinates (which were denoted by x, y, σ_j, ζ_j beforehand) on ${}^bTX$. Then ${}^bT^*X$ can be defined as the dual vector bundle. However, as long as all considerations are local, and they are mostly such here, it is safe to consider ${}^bT^*X$ a space arising from a singular change of variables on T^*X (given by (2)) – it is for this reason that it is sometimes called the compressed cotangent bundle.

There is a pseudodifferential algebra microlocalizing $\mathcal{V}_b(X)$ and the corresponding algebra of differential operators $\mathrm{Diff}_b(X)$, denoted by $\Psi_b(X)$. There is also a principal symbol for $A \in \Psi_b^m(X)$; this is now a homogeneous degree m function on ${}^bT^*X \setminus o$. $\Psi_b(X)$ has the algebraic properties analogous to $\Psi(X)$ on manifolds without boundary. It can be described quite explicitly; this was done for instance in [**26**] in the corners setting, and in [**8**, Section 18.3] for smooth boundaries.

Now $\mathrm{WF}_b(u)$ can be defined analogously to $\mathrm{WF}(u)$. For simplicity we state this here for $u \in L^2_g(X)$; this is how the main theorem is stated below. The space of 'very nice' functions corresponding to $\mathcal{V}_b(X)$ and $\mathrm{Diff}_b(X)$, replacing $C^\infty(X)$, is the space of L^2 conormal functions to the boundary, i.e. functions $v \in L^2_g(X)$ such that $Qv \in L^2_g(X)$ for every $Q \in \mathrm{Diff}_b(X)$ (of any order). Then $q \in {}^bT^*X \setminus o$ is *not* in $\mathrm{WF}_b(u)$ if there is an $A \in \Psi^0(X)$ such that $\sigma_{b,0}(A)(q) \neq 0$ and Au is L^2-conormal to the boundary. Spelling out the latter explicitly:

Definition 5. Then $q \in {}^bT^*X \setminus o$ is *not* in $\mathrm{WF}_b(u)$ if there is an $A \in \Psi^0(X)$ such that $\sigma_{b,0}(A)(q) \neq 0$ and $QAu \in L^2(X)$ for all $Q \in \mathrm{Diff}_b(X)$.

Note that the definition of WF could be stated in a completely parallel manner: we would require (for X without boundary) $QAu \in L^2(X)$ for all $Q \in \mathrm{Diff}(X)$ – this is equivalent to $Au \in C^\infty(X)$ by the Sobolev embedding theorem.

4. Propagation of singularities on manifolds with corners: the bicharacteristic geometry

After the general preliminaries, we turn to bicharacteristics. If $P \in \mathrm{Diff}^m(X)$, the characteristic set $\Sigma(P) = p^{-1}(\{0\})$ is a subset of T^*X. Let $\dot{\Sigma} = \pi(\Sigma(P)) \subset {}^bT^*X$ be the *compressed characteristic set*. Below we will be concerned with the wave operator $P = D_t^2 - \Delta_g$ on $X = M \times \mathbb{R}$, but the following definition is useful in many other cases (which we do not discuss, however). Generalized broken bicharacteristics are curves inside $\dot{\Sigma}$, satisfying a Hamilton vector field condition, plus an additional requirement where the boundary is smooth. More precisely:

Definition 6. *Generalized broken bicharacteristics* are continuous maps $\gamma : I \to \dot{\Sigma}$, where I is an interval, satisfying

(1) for all $f \in C^\infty({}^bT^*X)$ real valued,
$$\liminf_{s\to s_0} \frac{(f\circ\gamma)(s) - (f\circ\gamma)(s_0)}{s - s_0}$$
$$\geq \inf\{H_p(\pi^* f)(q) :\ q \in \pi^{-1}(\gamma(s_0)) \cap \Sigma(P)\},$$

(2) and if $q_0 = \gamma(s_0) \in {}^bT^*_{p_0}X$, and p_0 lies in the interior of a boundary hypersurface (i.e. a boundary face which has codimension 1, so near p_0 ∂X is smooth), then in a neighborhood of s_0, γ is a generalized broken bicharacteristic in the sense of Melrose-Sjöstrand [**14**], see also [**8**, Definition 24.3.7].

(1) is a very natural requirement. In the interior of X, we have defined bicharacteristics as integral curves of the Hamilton vector field of p in the characteristic set. Thus, if γ is an bicharacteristic segment over X°, then for all $f \in C^\infty(T^*X)$, the derivative of f along γ at s_0, i.e. $\lim_{s\to s_0} \frac{(f\circ\gamma)(s)-(f\circ\gamma)(s_0)}{s-s_0}$, is equal to $(H_p f)(\gamma(s_0))$. When we go back to the manifold with corners X, H_p is a vector field on T^*X, while the image of γ lies in ${}^bT^*X$. Moreover, π is not one-to-one, even when restricted to $\Sigma(P)$. Thus, the preimage of $\gamma(s_0)$ under π often contains many points (although it is still compact). Hence we cannot expect that f is differentiable along γ, although we can still expects *bounds* for the $\liminf$ (and $\limsup$) of the difference quotients by taking the worst case scenario as we evaluate $H_p(\pi^* f)(q)$ over $q \in \pi^{-1}(\gamma(s_0)) \cap \Sigma(P)$, which explains the infimum. Thus, it is very natural to demand the estimate in the definition above – and conversely, this gives a useful notion of generalized broken bicharacteristics.

We now make some comments about the additional requirement, (2). First, without (2) the propagation theorem below would still hold, but would be weaker, so at first the reader should feel free to ignore it. Second, (2) is there to rule out certain rays tangent to the boundary hypersurface (where the boundary is smooth): it prevents rays gliding along the boundary to enter the shadow of an obstacle. We remark that this strengthening is special to C^∞ singularities; if we were discussing the analytic wave front set, we could not do so. Indeed, our definition, without the strengthening given by (2), is equivalent to Lebeau's [**12**], see Lemma 7 below. Moreover, it is worthwhile pointing out that our definition also works in quantum N-body scattering even in the presence of bound states in subsystems (which have no analogues for the wave equation) – indeed, it originated there, see [**25**].

If $X = M \times \mathbb{R}$, M a manifold with corners, g a Riemannian metric on M, (x, y) are coordinates on M, with the boundary hypersurfaces locally given by the vanishing of the x_j, then coordinates on X are given by (x,y,t), with boundary hypersurfaces still locally given by the vanishing of the x_j. In particular, t plays a completely analogous role to y as far as the basic geometry is concerned – in the notation of Section 3, t *is* one of the y_j. (This is slightly unfortunate notation, but it would have been even worse to drag along a variable t in Section 3 that played an indistinguishable role from the other y variables.)

Now, if $P = D_t^2 - \Delta_g$ is the wave operator, then Snell's law is encoded in the statement that γ is continuous. Indeed, any (locally defined) smooth functions on ${}^bT^*X$, such as $x, y, t, \sigma, \zeta, \tau$, are continuous along γ, i.e. their composition with γ is continuous (since γ is a continuous map into ${}^bT^*X$). However, $\xi_j = x_j^{-1}\sigma_j$ is *not* continuous, so the normal momentum may jump.

In order to better understand the generalized broken bicharacteristics, we divide $\dot\Sigma$ into two subsets. We thus define the *glancing set* $\mathcal{G}$ as the set of points in $\dot\Sigma$ whose preimage under $\hat\pi = \pi|_\Sigma$ consists of a single point, and define the *hyperbolic set* $\mathcal{H}$ as its complement in $\dot\Sigma$.

For the wave operator these can be described very explicitly. First, in local coordinates (x,y) as above on some open set U in M, the metric is of the form

$$g(x,y,\xi,\zeta) = \sum_{i,j} A_{ij}(x,y)\xi_i\xi_j + \sum_{i,j} 2C_{ij}(x,y)\xi_i\zeta_j + \sum_{i,j} B_{ij}(x,y)\zeta_i\zeta_j$$

with A, B, C smooth. Moreover, the coordinates on M can be chosen (i.e. the y_j can be adjusted) so that $C(0,y) = 0$. Then on ${}^bT^*_{U\times\mathbb{R}}X \setminus o$,

$$p|_{x=0} = \tau^2 - \xi\cdot A(y)\xi - \zeta\cdot B(y)\zeta, \tag{3}$$

with A, B positive definite matrices depending smoothly on y. Thus, with $\mathcal{U} = \{x=0\} \cap {}^bT^*_{U\times\mathbb{R}}X \setminus o$, writing local coordinates on ${}^bT^*X$ as $(x,y,t,\sigma,\zeta,\tau)$,

$$\mathcal{G}\cap\mathcal{U} = \{(0,y,t,0,\zeta,\tau) :\ \tau^2 = \zeta\cdot B(y)\zeta,\ (\zeta,\tau)\neq 0\},$$
$$\mathcal{H}\cap\mathcal{U} = \{(0,y,t,0,\zeta,\tau) :\ \tau^2 > \zeta\cdot B(y)\zeta,\ (\zeta,\tau)\neq 0\}.$$

Note that $\dot\Sigma = \pi(\Sigma(P))$ is disjoint from all points $(x,y,t,\sigma,\zeta,\tau)$ with $x=0$ at which either $\sigma\neq 0$ (for $\sigma_j = x_j\xi_j = 0$ for all j) or $\tau^2 < \zeta\cdot B(y)\zeta$.

We can then desribe broken bicharacteristics more concretely:

Lemma 7. *Suppose γ is a generalized broken bicharacteristic. Then*

(1) *If $\gamma(s_0)\in\mathcal{G}$, let q_0 be the unique point in the preimage of $\gamma(s_0)$ under $\hat\pi = \pi|_\Sigma$. Then for all $f\in C^\infty({}^bT^*X)$ real valued, $f\circ\gamma$ is differentiable at s_0, and*
$$\frac{d(f\circ\gamma)}{ds}|_{s=s_0} = H_p\pi^*f(q_0).$$
(2) *If $\gamma(s_0)\in\mathcal{H}$, lying over a corner given in local coordinates by $x=0$, then exists $\epsilon>0$ such that $x(\gamma(s))=0$ for $s\in(s_0-\epsilon, s_0+\epsilon)$ if and only if $s=s_0$. That is, γ does not meet the corner $\{x=0\}$ in a punctured neighborhood of s_0. (Here, as usual, x is considered as a vector valued function, $x=(x_1,\dots,x_k)$.)*

Indeed, this is the route that Lebeau takes in his definition [**12**], and was also the route taken in [**26**]. That is, the converse of this lemma also holds in the sense that any continuous map $\gamma: I\to\dot\Sigma$ satisfying (1), (2) of this lemma, plus part (2) of our definition, is in fact a generalized broken bicharacteristic.

Proof. First consider case (1). Then

$$\liminf_{s\to s_0}\frac{(f\circ\gamma)(s)-(f\circ\gamma)(s_0)}{s-s_0}\geq H_p\pi^*f(q_0)$$

for $f\in C^\infty({}^bT^*X)$ real-valued by our definition of generalized broken bicharacteristics. Applying the definition to $-f$, we also conclude that

$$\limsup_{s\to s_0}\frac{(f\circ\gamma)(s)-(f\circ\gamma)(s_0)}{s-s_0}\leq H_p\pi^*f(q_0).$$

Combining these two estimates shows that $f\circ\gamma$ is differentiable at s_0, with derivative given by $H_p\pi^*f(q_0)$.

Consider (2) now. Taking $f=\sum_j\sigma_j$ (in terms of the 'canonical' coordinates on ${}^bT^*X$) in the definition of generalized broken bicharacteristics, we see that $\pi^*f=\sum_j x_j\xi_j$ in canonical coordinates on T^*X. A simple calculation using (3) shows that $H_p\pi^*f$ is positive (bounded below by a positive constant) at all points in $\hat{\pi}^{-1}(\gamma(s_0))$ if $\gamma(s_0)\in\mathcal{H}$. Since $f\circ\gamma$ vanishes at s_0, this implies that it is non-zero for s near s_0 but $s\neq s_0$. As $f=0$ at $\dot{\Sigma}\cap\{x=0\}$, we deduce that, in a neighborhood of s_0, the bicharacteristic is at the corner $x=0$ only at s_0, proving the lemma. □

Part (2) of this lemma indicates the possibility of an iterative description of the bicharacteristics: at $\mathcal{H}$, where we do not know in which direction they will travel, we still know that they will be in a less singular stratum (a lower codimensional corner) in a punctured neighborhood of s_0. Thus, if we already understand bicharacteristics in less singular strata, we can also understand their behavior at the corner under consideration.

In fact, we have an even stronger description of generalized broken bicharacteristics at $\mathcal{H}$ follows, as in Lebeau's paper.

Lemma 8. *(Lebeau,* [**12**, Proposition 1]*) If γ is a generalized broken bicharacteristic, $s_0\in I$, $q_0=\gamma(s_0)$, then there exist unique $\tilde{q}_+,\tilde{q}_-\in\Sigma(P)$ satisfying $\pi(\tilde{q}_\pm)=q_0$ and having the property that if $f\in C^\infty({}^bT^*X)$ then $f\circ\gamma$ is differentiable both from the left and from the right at s_0 and*

$$\left(\frac{d}{ds}\right)(f\circ\gamma)|_{s_0\pm}=H_p\pi^*f(\tilde{q}_\pm).$$

That is, one can associate an incoming and an outgoing point in T^*X, rather than merely in ${}^bT^*X$, into which the curve γ maps – the point being that even incoming and outgoing *normal* momenta are defined, although they can certainly differ. This indicates that, at least away from rays hitting the boundary tangentially, Figures 1-2 give an accurate indication of the bicharacteristic geometry.

5. Propagation of singularities on manifolds with corners: the main theorem

We are now ready to state the main theorem. Recall that $H_0^1(X)$ is the completion of $C_c^\infty(X^\circ)$ in the norm

$$\|u\|^2_{H^1(X)}=\|du\|^2_{L^2(X)}+\|u\|^2_{L^2(X)},$$

and that elements of $H_0^1(X)$ restrict as 0 to ∂X, i.e. $u\in H^1_{0,\mathrm{loc}}(X)$ means that u satisfies the Dirichlet boundary conditions.

THEOREM 9. *Suppose* $Pu = 0$, $u \in H^1_{0,\mathrm{loc}}(X)$. *Then* $\mathrm{WF}_b(u) \subset \dot{\Sigma}$, *and it is a union of maximally extended generalized broken bicharacteristics of* P.

This theorem can be stated in a completely microlocal manner, and one can also measure the regularity modulo Sobolev spaces. In addition, it also holds for Neumann boundary conditions.

It was proved in the real analytic setting by Lebeau [**12**], and in the C^∞ setting with C^∞ boundaries (and no corners) by Melrose, Sjöstrand and Taylor [**14, 15, 23**]. This result is thus the C^∞ version of Lebeau's theorem: the geometry is similar in the real analytic vs. C^∞ settings, but the analysis is quite different.

The main ideas of the proof of the main theorem are the use positive commutator estimates, b-microlocalization, i.e. using elements $A \in \Psi_b^m(X)$ as the commutants, and using the Dirichlet form. The proof is thus different from the Melrose-Sjöstrand-Taylor proof even in the C^∞ boundary case at hyperbolic points. As this is too technical, I quickly go through the positive commutator proof in the boundaryless setting.

6. A simple commutator proof in the boundaryless setting

For this purpose, we also need that operators $A \in \Psi^m(X)$ (recall that X has no boundary for this exposition!) have an *operator wave front set* associated to them,

$$\mathrm{WF}'(A) \subset T^*X \setminus o,$$

which is again closed and conic. Roughly, if $A = a(x, y, D_x, D_y)$, then a point $q = (x, y, \xi, \zeta)$ is not in $\mathrm{WF}'(A)$ if a is a symbol of order $-\infty$, i.e. decays rapidly, in an open cone around q. Then the action of A is microlocal:

$$\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \cap \mathrm{WF}(u).$$

Positive commmutator estimates originated in [**9**], although they are essentially *microlocal energy estimates*, and as such they have a long history. For $P \in \Psi(X)$ (we are suppressing the order of P for now), we thus want to construct $A \in \Psi(X)$ such that $i[A^*A, P]$ is positive, modulo terms we can control. This is useful since

$$\begin{aligned}
&\langle i[A^*A, P]u, u\rangle \\
&\qquad = \langle iA^*APu, u\rangle - \langle iPA^*Au, u\rangle \\
&\qquad = \langle iA^*APu, u\rangle + \langle u, iA^*APu\rangle,
\end{aligned}$$

and the PDE gives information about Pu. (This argument needs to be regularized, but this can be done.) Now, if $i[A^*A, P] = B^*B + E + F$ where F is lower order, hence negligible, we deduce that $\|Bu\|^2$ can be controlled provided we have information about $\langle Eu, u\rangle$, i.e. about u microlocally on $\mathrm{WF}'(E)$.

The shape this takes in the positive commutator estimates is that we need to assume that $\mathrm{WF}(u)$ is disjoint from $\mathrm{WF}'(E)$, and then we *propagate* the regularity of u and conclude that $\mathrm{WF}(u)$ is disjoint from the elliptic set of B, i.e. where $\sigma(B)$ is non-zero. To accomplish this, one needs an iterative argument, showing that u is in the Sobolev space $H^m(X)$ on the elliptic set of B, provided that it is in $H^{m-1/2}(X)$ in a neighborhood, and of course provided that $\mathrm{WF}(u)$ is disjoint from $\mathrm{WF}'(E)$ and that the PDE holds ($Pu = 0$, etc.). For this, we need $B \in \Psi^m(X)$, for then the L^2 norm of Bu is a microlocal H^m norm of u. For that we need in turn $A \in \Psi^{m-1/2}(X)$ (for P second order). Factoring out a weight (which for the wave equation can even just taken to have principal symbol $|\tau|^{m-1/2}$, and in general any

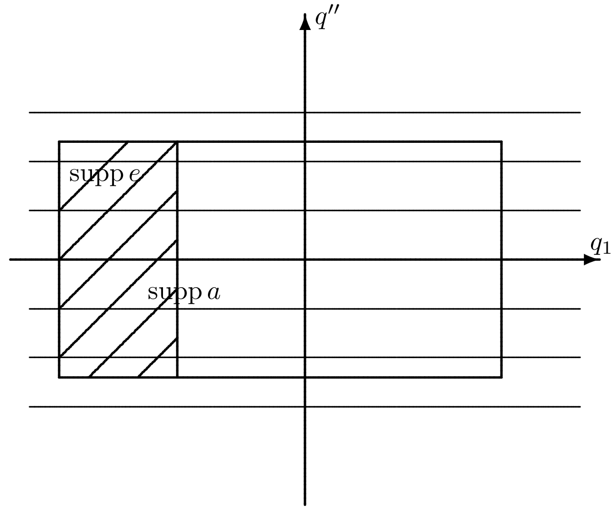

FIGURE 3. supp$a$ in the local coordinates q; supp$e$ is the shaded box on the left. The bicharacteristics are the straight lines with q'' constant.

fixed non-degenerate weight will do), which can be dealt with in a straightforward manner, roughly like the regularization mentioned above, we need to construct the microlocalizing factors, i.e. we can pretend that A is of order 0.

The equation $i[A^*A, P] = B^*B + E + F$ as above is a condition on the principal symbol $a = \sigma(A)$, namely that

$$H_p a = -b^2 + e, \tag{4}$$

with $b = \sigma(B)$ and $e = \sigma(E)$. Thus, we want $a = \sigma(A)$ to be decreasing along bicharacteristics except in a region in which we have a priori information about u. It also needs to have small support.

In our simple setting, H_p can locally be made into a constant vector field $\frac{\partial}{\partial q_1}$ by a change of coordinates on $S^*X = (T^*X \setminus o)/\mathbb{R}^+$. Thus, one works with homogeneous degree zero functions $q = (q_1, q'')$ on $T^*X \setminus o$ which give coordinates on S^*X. In such coordinates it is straightforward to construct such an a as a C^∞ function on S^*X, and then regard it as a homogeneous function on $T^*X \setminus o$: one can take a to be the product of a (compactly supported) function χ_1 of q_1, on the derivative of which we have sign conditions, and an arbitrary (say, non-negative, also compactly supported) function χ_2 of q''. Here the compact support should be small enough so that supp$a$ is a subset of the region of validity of the local coordinates q_j.

On Figure 3 we illustrate the bicharacteristics of H_p, which are now straight lines, and the support of a and e. Here supp$e$ corresponds to the region where χ_1' is positive, i.e. where $H_p a = \chi_1' \chi_2$ is positive. The positive commutator estimates gives us control of a solution of $Pu = 0$ over the rest of supp$a$, provided we have control of u over supp$e$. Since we can choose χ_2 arbitrarily, in particular we may shrink its support to an arbitrarily small set, it is clear in this picture that the control of u propagates along the bicharacteristics.

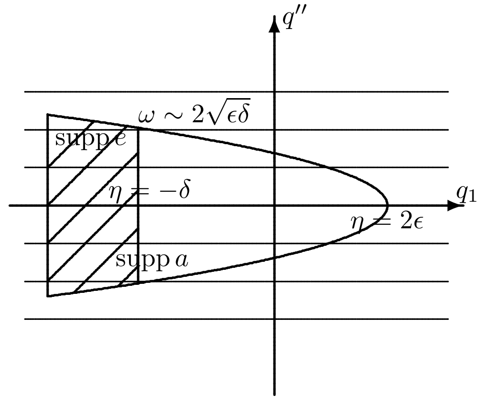

FIGURE 4. supp$a$ in (q_1, q'') coordinates. supp$e$ is again the shaded region on the left. The straight lines with q'' constant need not be bicharacteristics. In this case, $\omega = q_2^2 + \ldots + q_{2n-1}^2$, and $\eta = q_1$.

7. A more flexible proof in the boundaryless setting

In general, at ∂X, the problem is microlocalization: there is no analogue of the functions of q'' – we cannot microlocalize by taking a product of two functions. Instead, one follows a much more robust construction due to Melrose and Sjöstrand [**14**] on manifolds with smooth boundary, which they used to prove propagation of singularities at glancing points (tangential rays). It is somewhat complicated to explain this even in the presence of smooth boundaries. Instead, we illustrate the shape this proof would take in the absence of boundaries (where the above argument is of course simpler). Roughly, the point is that one only has to get the Hamilton vector field 'right' at one point in S^*X, i.e. one can do a construction where the Hamilton vector field only matters at the point at which we wish to get a microlocal estimate, rather than having to put it into a model form (like $\frac{\partial}{\partial q_1}$) in a neighborhood of this point. *Having such flexibility is the key in dealing with complicated situations such as manifolds with boundary, or corners, or N-body scattering.*

The main point is that if we cannot put the Hamilton vector field H_p in a model form, the previous construction will not work. Indeed, unless $H_p\chi_2 = 0$, $H_p(\chi_1\chi_2)$ will always yield a term $\chi_1 H_p\chi_2$, which cannot be controlled by $(H_p\chi_1)\chi_2$: the problem being near the boundary of suppχ_2. So instead use a different form of localization. Before getting into details, we illustrate the support of the commutant a and the error e in this case on Figure 4. This figure is still in local coordinates $q_1, \ldots, q_{2n-1}$, but now the Hamilton vector field need not be $\frac{\partial}{\partial q_1}$ in supp$a$. Note that supp$a$ is a parabolic region, so *if* the Hamilton vector field happens to be $\frac{\partial}{\partial q_1}$ in supp$a$, the estimate is not optimal in the sense it was in Figure 3: there are many bicharacteristics going through supp$e$ (the shaded area) which are used to control u over the rest of supp$a$, even though they do not go through the latter. However, we gain flexibility this way: as long as the bicharacteristics are 'close' to the straight lines, we still obtain a positive commutator estimate. Here, if we want to understand propagation near a point $\bar{q}$, 'close' means that the bicharacteristic through $\bar{q}$ is tangent to the q_1 axis at $\bar{q}$ – for appropriately small values of the parameters ϵ and δ arising below, this gives a positive commutator estimate.

The following more detailed discussion has been adapted from [**28**], where it was described in the scattering setting, where the $\mathbb{R}^+$ action is in the base (X) variables on T^*X; see the concluding remarks of these notes. Here we present it from a somewhat different point of view, and of course we changed the notation corresponding to the different setting. Since notation seems to be one of the main obstacles in understanding certain arguments, we hope that this rewriting proves helpful.

First let $\eta \in C^\infty(S^*X)$ be a function with

$$\eta(\bar{q}) = 0,\ H_p\eta(\bar{q}) > 0.$$

Thus, η measures propagation along bicharacteristics, e.g. $\eta = q_1$ in the above example would work, but so would many other choices. We will use a function ω to localize near putative bicharacteristics. This statement is deliberately vague; at first we only assume that $\omega \in C^\infty(S^*X)$ is the sum of the squares of C^∞ functions σ_j, $j = 1, \ldots, l$, with non-zero differentials at $\bar{q}$ such that $d\eta$ and $d\sigma_j$, $j = 1, \ldots, l$, span $T_{\bar{q}}S^*X$. Such a function ω is non-negative and it vanishes quadratically at $\bar{q}$, i.e. $\omega(\bar{q}) = 0$ and $d\omega(\bar{q}) = 0$. An example is $\omega = q_2^2 + \ldots + q_{2n-1}^2$ with the notation from before, but again there are many other possible choices. We now consider a family symbols, parameterized by constants $\delta \in (0, 1)$, $\epsilon \in (0, \delta]$, of the form

$$a = \chi_0(2 - \frac{\phi}{\epsilon})\chi_1(\frac{\eta + \delta}{\epsilon\delta} + 1),$$

where

$$\phi = \eta + \frac{1}{\epsilon}\omega,$$

$\chi_0(t) = 0$ if $t \leq 0$, $\chi_0(t) = e^{-1/t}$ if $t > 0$, $\chi_1 \in C^\infty(\mathbb{R})$, $\operatorname{supp}\chi_1 \subset [0, +\infty)$, $\operatorname{supp}\chi_1' \subset [0, 1]$. Although we do not do it explicitly here, weights such as $|\tau|^{m-1/2}$ can be accommodated for any $m \in \mathbb{R}$, by replacing the factor $\chi_0(2 - \frac{\phi}{\epsilon})$ by $\chi_0(A_0^{-1}(2 - \frac{\phi}{\epsilon}))$ and taking $A_0 > 0$ large.

We analyze the properties of a step by step. First, note that $\phi(\bar{q}) = 0$, $H_p\phi(\bar{q}) = H_p\eta(\bar{q}) > 0$, and $\chi_1(\frac{\eta+\delta}{\epsilon\delta} + 1)$ is identically 1 near $\bar{q}$, so $H_p a(\bar{q}) < 0$. Thus, $H_p a$ has the correct sign, and is in particular non-zero, at $\bar{q}$.

Next,

$$q \in \operatorname{supp} a \Rightarrow \phi(q) \leq 2\epsilon \text{ and } \eta(q) \geq -\delta - \epsilon\delta.$$

Since $\epsilon < 1$, we deduce that in fact $\eta = \eta(q) \geq -2\delta$. But $\omega \geq 0$, so $\phi = \phi(q) \leq 2\epsilon$ implies that $\eta = \phi - \epsilon^{-1}\omega \leq \phi \leq 2\epsilon$. Hence, $\omega = \omega(q) = \epsilon(\phi - \eta) \leq 4\epsilon\delta$. Since ω vanishes quadratically at $\bar{q}$, it is useful to rewrite the estimate as $\omega^{1/2} \leq 2(\epsilon\delta)^{1/2}$. Combining these, we have seen that on $\operatorname{supp} a$,

$$-\delta - \epsilon\delta \leq \eta \leq 2\epsilon \text{ and } \omega^{1/2} \leq 2(\epsilon\delta)^{1/2}. \tag{5}$$

Moreover, on $\operatorname{supp} a \cap \operatorname{supp}\chi_1'$,

$$-\delta - \epsilon\delta \leq \eta \leq -\delta \text{ and } \omega^{1/2} \leq 2(\epsilon\delta)^{1/2}.$$

Note that given any neighborhood U of $\bar{q}$, we can thus make a supported in U by choosing ϵ and δ sufficiently small. On Figure 4 we illustrate the parabola shaped region given by $\operatorname{supp} a$ in case $\eta = q_1$ and $\omega = q_2^2 + \ldots + q_{2n-1}^2$.

Note that as $\epsilon \to 0$, but δ fixed, the parabola becomes very sharply localized at $\omega = 0$. In particular, for very small $\epsilon > 0$ we obtain a picture quite analogous to letting $\operatorname{supp}\chi_2 \to \{0\}$ in Figure 3.

So we have shown that a is supported near $\bar{q}$. We define

$$e = \chi_0(2 - \frac{\phi}{\epsilon})H_p\chi_1(\frac{\eta+\delta}{\epsilon\delta} + 1),$$

so the crucial question in our quest for (4) is whether $H_p\phi \geq 0$ on supp$a$. Note that choosing $\delta_0 \in (0,1)$ and $\epsilon_0 \in (0,\delta_0)$ sufficiently small, one has $H_p\eta \geq c_0 > 0$ where $|\eta| \leq 2\delta_0$, $\omega^{1/2} \leq 2(\epsilon_0\delta_0)^{1/2}$. So $H_p\phi \geq 0$ on supp$a$, provided that $|H_p\omega| \leq \frac{c_0}{2}\epsilon$ there.

But being a sum of squares of functions with non-zero differentials, $H_p\omega$ vanishes at $\omega = 0$ and satisfies $|H_p\omega| \leq C\omega^{1/2}$. Due to (5), we deduce that $|H_p\omega| \leq 2C(\epsilon\delta)^{1/2}$. So $|H_p\omega| \leq \frac{c_0}{2}\epsilon$ holds if $\frac{c_0}{2}\epsilon \geq 2C(\epsilon\delta)^{1/2}$, i.e. if $\epsilon \geq C'\delta$ for some constant $C' > 0$ independent of ϵ, δ. Note that this constraint on ϵ, i.e. that it cannot be too small, gives very rough localization: the width of the parabola at $\eta = -\delta$ is roughly $\omega^{1/2} \sim \delta$, i.e. it is very wide, and in particular insufficient to prove the propagation of singularities along the bicharacteristics. The reason is simple: our localizing function ω has no relation to H_p, so we cannot expect a more precise estimate. Note, however, that the estimate is still non-trivial! Indeed, it shows that singularities propagate in the sense that $\bar{q}$ cannot be an isolated point of $\mathrm{WF}(u)$. (We required $\epsilon \in (0,\delta]$ beforehand, but in fact we could have dealt with $\epsilon \leq \mu\delta$, even if $\mu > 1$, if we localized slightly differently.)

We need to adapt ω to H_p to get a better estimate. If we linearize H_p as above, and take $\omega = q_2^2 + \ldots + q_{2n-1}^2$, then $H_p\omega = 0$ and any $\epsilon > 0$ works. Thus, in this case, we can prove propagation of singularities much like by the previous, simpler, construction.

However, we do not need such a strong relationship to H_p. Suppose instead that we merely get ω 'right' at $\bar{q}$, in the sense that $\omega = \sum \sigma_j^2$ and $H_p\sigma_j(\bar{q}) = 0$. Then $|H_p\sigma_j| \leq C_0(\omega^{1/2} + |\eta|)$, so $|H_p\omega| \leq C\omega^{1/2}(\omega^{1/2} + |\eta|)$. Using (5), we deduce that $|H_p\omega| \leq \frac{c_0}{2}\epsilon$ provided that $\frac{c_0}{2}\epsilon \geq C''(\epsilon\delta)^{1/2}\delta$, i.e. that $\epsilon \geq C'\delta^3$ for some constant C' independent of ϵ, δ. Now the size of the parabola at $\eta = -\delta$ is roughly $\omega^{1/2} \sim \delta^2$, i.e. we have localized along a single direction, namely the direction of H_p at $\bar{q}$. By a relatively simple argument, also due to Melrose and Sjöstrand [**14, 15**] in the case of smooth boundaries (which could easily be used here too), one can piece together such estimates (i.e. where the direction is correct 'to first order') and deduce the propagation of singularities.

We emphasize that the lower bound for ϵ is natural. Indeed, suppose that in some local coordinates $\tilde{q}_j$ on S^*X near $\bar{q}$, $H_p = \frac{\partial}{\partial \tilde{q}_1}$, but we capriciously let $q_1 = \tilde{q}_1$, $q_j = \tilde{q}_j + \tilde{q}_1^2$, $j \geq 2$. The bicharacteristics are $\tilde{q}_j = \text{constant}$, $j \geq 2$, but (with $\omega = q_2^2 + \ldots + q_{2n-1}^2$, $\eta = q_1$) we are localizing near $q_j = \text{constant}$, and at $\eta = -\delta$ these differ by δ^2. So any localization better than $\omega^{1/2} \sim \delta^2$ would in fact contradict the propagation of singularities! This is illustrated above on Figure 5. Note that every bicharacteristic through the unshaded part of supp$a$ also goes through supp$e$ (the shaded part), although the converse is not true, i.e. we use more bicharacteristics than needed to control u in the unshaded region, but we gain because our construction becomes more flexible.

8. Comments on the proof in the setting of manifolds with corners

Both degrees of localization described in the previous section, with ω completely arbitrary, except that it vanishes quadratically at $\bar{q}$, resp. $\omega = \sum \sigma_j^2$ and $H_p\sigma_j(\bar{q}) =$

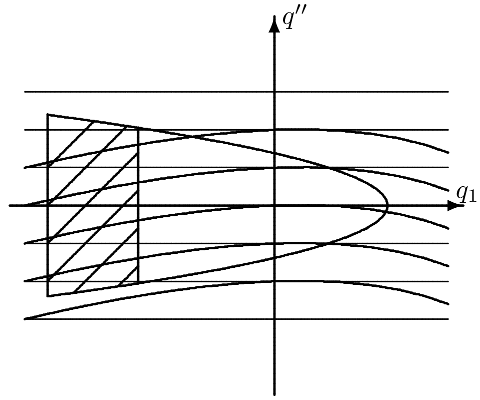

FIGURE 5. Bicharacteristics and supp$a$. The labels from Figure 4 have been removed to make the picture less cluttered. The straight horizontal lines are the lines with q'' constant, while the nearby parabolae are the bicharacteristics.

0, are relevant in the setting of manifolds with corners. While we cannot discuss the details here, the rough picture is that at $\mathcal{G}$, where $\hat{\pi} = \pi|_\Sigma$ is one-to-one, one can use the more precise localization: indeed, the information we need is the knowledge of the Hamilton vector field at one point, which is what is provided by taking the unique point in $\hat{\pi}^{-1}(\gamma(s_0))$. On the other hand, at $\mathcal{H}$ it suffices to prove that singularities cannot stay at $x = 0$; it does not matter in which direction they leave $x = 0$. (One can use an inductive argument: if one already understands what happens in X away from corners of codimension $\geq k$, one can analyze what happens near $\mathcal{H}$ at codimension k corners: bicharacteristics leave the corner instanteneously, and are thus in the previously understood region.) This is accomplished by taking

$$\eta = -\tau^{-1} \sum \sigma_j = -\tau^{-1}(x \cdot \xi); \tag{6}$$

note that this is homogeneous degree zero on ${}^bT^*X$ (and of course its pull-back by π is such on T^*X), so it indeed gives a function on ${}^bS^*X$ (and S^*X). As already indicated when discussing (generalized broken) bicharacteristics, this is strictly increasing along bicharacteristics at points in $\mathcal{H}$, and it vanishes at $x = 0$ inside $\dot{\Sigma}$. Thus, if supp$e$ lies in $\eta < 0$ as in the above construction, we obtain an estimate on u at some point in $x = 0$ without having a priori knowledge of u anywhere else at $x = 0$. In particular, singularities cannot stay at $x = 0$ inside $\mathcal{H}$.

An interesting technical point is that in the presence of corners the iterative argument gains b-regularity for u, 1/2 a b-derivative at a time, relative to $H^1_{\mathrm{loc}}(X)$. This corresponds to the quadratic form domain of Δ, as dictated by the use of the Dirichlet form. Thus, we work with the spaces $H^{1,m}_{b,\mathrm{loc}}(X)$, consisting of functions u such that $Lu \in H^1_{\mathrm{loc}}(X)$ for $L \in \Psi^m_b(X)$. We can define a wave front set relative to $H^{1,m}_{b,\mathrm{loc}}(X)$, just as before. Note that this is not quite the same as the L^2-based b-wave front set mentioned before (as it is H^1-based), but for the solutions of the wave equation the order $m+1$, L^2-based wave front set is the same as the order m, H^1-based wave front set: roughly, $D_t u$ gives control of the other first derivatives needed for H^1-control of u. We refer to [**27**] for a more detailed discussion, and for [**26**] for complete details.

9. Comparison with N-body scattering

There is a close connection between the wave equation on manifolds with corners and N-body scattering; this has been discussed in some detail in [**28**], approaching the connection from the N-body point of view. (At the time [**28**] was written, the propagation theorem for manifolds with corners was only proved, by Lebeau, in the analytic setting.) Here we emphasize that the proofs in these settings have much in common, especially if one restricts one's attention to N-body systems without bound states in any subsystems.

The relevant scaling in scattering theory on $\mathbb{R}^n$ is in dilations in x rather than ξ, where x is the variable on $\mathbb{R}^n$ and ξ is its canonical dual variable, so (x, ξ) are coordinates on $T^*\mathbb{R}^n$. Namely, one works with finite energy solutions, which roughly means that ξ is finite, and one wants to understand the asymptotics of solutions near spatial infinity. Thus, the roles of x and ξ are interchanged, but apart from this the two settings are quite analogous. This is analogy can be made explicit when comparing manifolds without boundary and scattering of a single particle in an external potential (or reduced two-body scattering): in fact, the Fourier transform intertwines these two settings (so in fact standard microlocal analysis is applicable in this scattering setting!). When comparing N-body scattering to the wave equation on domains with corners, the results *and many of the methods* are similar, but one does not have an explicit correspondence, so the results are certainly *not* equivalent.

In particular, an analogue of η at $\mathcal{H}$ (cf. (6)) plays a major role in N-body scattering: there one would consider $|x|^{-1}(x \cdot \xi)$ on $T^*\mathbb{R}^n$: this is the radial component of the momentum (if one introduces polar coordinates). Note that this is homogeneous of degree zero in x, which is the relevant scaling as discussed above. In N-body scattering the *global* commutator estimate (corresponding to $|x|^{-1}(x \cdot \xi)$ without the weight $|x|^{-1}$) is called the Mourre estimate, see [**20, 21**], and it can be used to show the absence of singular continuous spectrum, etc. – eventually even asymptotic completeness.

Partially microlocal results, in which one only localizes in the radial momentum form the backbone of now-classical analysis of N-body scattering, see e.g. [**22, 5, 10**] and the monograph [**3**]. The author's previous works describing propagation of singularities in the N-body setting [**24, 25**] are fully microlocal, in the sense that they are microlocal to the fullest extent compatible with the structure of N-body Hamiltonians. That is, one can localize in the momentum tangential to the collision planes but not in the normal momenta. (This means that at any point $x \in \mathbb{R}^n$ one needs to consider the collision plane through x of the lowest dimension, i.e. highest codimension, much like considering the boundary face of highest codimension in which a point p lies when analyzing the wave equation.) If there are no bound states in any subsystem, the analogy with the wave equation is rather complete: kinetic energy is conserved, and we can distinguish glancing and hyperbolic points depending on whether the kinetic energy corresponding to the tangential momentum at the point in question is equal to the total kinetic energy or not, i.e. whether there is normal motion or not. If there are bound states in subsytems, the situation is more complicated as kinetic energy is not conserved: so a point in phase space may be hyperbolic with respect to a channel in some subsystem and glancing with respect to another. If there are no bound states in any subsystem, the proofs are also quite analogous at least at the level of construction of the commutant (the technical details differ – any many ways they are more cumbersome for the wave equation as one needs to deal with two different types of operators) – it is instructive to look at [**24**, Section 10] for comparison. In particular, at the hyperbolic set one uses the radial momentum as a propagation variable (i.e. as η in the notation of Section 7) – the main additional ingredient there is to microlocalize using a construction completely parallel to that of Section 7.

References

[1] A. Blank and J. B. Keller. Diffraction and reflection of pulses by wedges and corners. *Comm. Pure Appl. Math.*, 4:75–94, 1951.

[2] J. Cheeger and M. Taylor. Diffraction by conical singularities, I, II. *Comm. Pure Applied Math.*, 35:275–331, 487–529, 1982.

[3] J. Dereziński and C. Gérard. *Scattering theory of classical and quantum N-particle systems.* Springer, 1997.

[4] J. J. Duistermaat and L. Hörmander. Fourier integral operators, II. *Acta Mathematica*, 128:183–269, 1972.

[5] C. Gérard, H. Isozaki, and E. Skibsted. *Commutator algebra and resolvent estimates*, volume 23 of *Advanced studies in pure mathematics*, pages 69–82. 1994.

[6] A. Hassell, R. B. Melrose, and A. Vasy. Scattering for symbolic potentials of order zero and microlocal propagation near radial points. *Preprint*, 2005.

[7] L. Hörmander. Fourier integral operators, I. *Acta Mathematica*, 127:79–183, 1971.

[8] L. Hörmander. *The analysis of linear partial differential operators,* vol. 1-4. Springer-Verlag, 1983.

[9] L. Hörmander. On the existence and the regularity of solutions of linear pseudo-differential equations. *Enseignement Math. (2)*, 17:99–163, 1971.

[10] H. Isozaki. A generalization of the radiation condition of Sommerfeld for N-body Schrödinger operators. *Duke Math. J.*, 74:557–584, 1994.

[11] J. B. Keller. Diffraction of a shock or an electromagnetic pulse by a right-angled wedge. *J. Appl. Phys.*, 23:1267–1268, 1952.

[12] G. Lebeau. Propagation des ondes dans les variétés à coins. *Ann. Scient. Éc. Norm. Sup.*, 30:429–497, 1997.

[13] R. B. Melrose. *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces.* Marcel Dekker, 1994.

[14] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. I. *Comm. Pure Appl. Math*, 31:593–617, 1978.

[15] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. II. *Comm. Pure Appl. Math*, 35:129–168, 1982.

[16] R. B. Melrose, A. Vasy, and J. Wunsch. Propagation of singularities for the wave equation on manifolds with edges. In preparation.

[17] R. Melrose and J. Wunsch. Propagation of singularities for the wave equation on conic manifolds. *Invent. Math.*, 156(2):235–299, 2004.

[18] R. B. Melrose. Transformation of boundary problems. *Acta Math.*, 147(3-4):149–236, 1981.

[19] R. B. Melrose and P. Piazza. Analytic K-theory on manifolds with corners. *Adv. Math.*, 92(1):1–26, 1992.

[20] E. Mourre. Operateurs conjugés et propriétés de propagation. *Commun. Math. Phys.*, 91:279–300, 1983.

[21] P. Perry, I. M. Sigal, and B. Simon. Spectral analysis of N-body Schrödinger operators. *Ann. Math.*, 114:519–567, 1981.

[22] I. M. Sigal and A. Soffer. N-particle scattering problem: asymptotic completeness for short range systems. *Ann. Math.*, 125:35–108, 1987.

[23] M. Taylor. Grazing rays and reflection of singularities of solutions to wave equations. *Comm. Pure Appl. Math.*, 29:1–38, 1976.

[24] A. Vasy. Propagation of singularities in many-body scattering. *Ann. Sci. École Norm. Sup. (4)*, 34:313–402, 2001.
[25] A. Vasy. Propagation of singularities in many-body scattering in the presence of bound states. *J. Func. Anal.*, 184:177–272, 2001.
[26] A. Vasy. Propagation of singularities for the wave equation on manifolds with corners. *Preprint*, 2004.
[27] A. Vasy. Propagation of singularities for the wave equation on manifolds with corners. In *Séminaire: Équations aux Dérivées Partielles, 2004–2005*, Sémin. Équ. Dériv. Partielles, pages Exp. No. XIII, 21. École Polytech., Palaiseau, 2005.
[28] A. Vasy. Geometry and analysis in many-body scattering. In *Inside out: inverse problems and applications*, volume 47 of *Math. Sci. Res. Inst. Publ.*, pages 333–379. Cambridge Univ. Press, Cambridge, 2003.

Department of Mathematics, Massachusetts Institute of Technology and Northwestern University

Current address: Department of Mathematics, Stanford University

E-mail address: andras@math.mit.edu