Math 154. Density for the ring of integers

Let K be a number field. We have seen that the image of the ring of integers O_K under the inclusion $\theta_K : K \to \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$ is discrete (i.e., meets each bounded region in just a finite set). What happens if we drop one of the real or complex embeddings from this map? That is, if we project into $\mathbf{R}^{r_1 - 1} \times \mathbf{C}^{r_2}$ by dropping one of the real embeddings (when $r_1 > 0$) or project into $\mathbf{R}^{r_1} \times \mathbf{C}^{r_2 - 1}$ by dropping one of the non-real embeddings (when $r_2 > 0$) then is the image of O_K still discrete? The effect can be quite dramatic: the case $K = \mathbf{Q}(\zeta_5)$ arose in our discussion of the *Lucy and Lily* video game which relies in the fact that any single embedding $\mathbf{Q}(\zeta_5) \to \mathbf{C}$ carries O_K onto a *dense* subset, and we noted as part of that discussion that the same phenomenon happens for $K = \mathbf{Q}(\sqrt{2})$: the image of either embedding $\mathbf{Z}[\sqrt{2}] \to \mathbf{R}$ (i.e., not using both real embeddings) has dense image too.

The aim of this handout is to use Minkowski's Theorem to prove density holds for *any* number field upon dropping even one embedding. This is sometimes called the *strong approximation theorem* for the ring of integers. If $r_1 + r_2 = 1$ (i.e., $K = \mathbf{Q}$ or K is imaginary quadratic) then there is nothing to do since dropping a factor field collapses the target to $\{0\}$. Thus, the interesting case is $r_1 + r_2 > 1$. Our aim is to prove that O_K has *dense* image under projection to $\mathbf{R}^{r_1 - 1} \times \mathbf{C}^{r_2}$ upon dropping a real embedding when $r_1 > 0$ and also under projection to $\mathbf{R}^{r_1} \times \mathbf{C}^{r_2 - 1}$ upon dropping a non-real embedding when $r_2 > 0$.

To streamline the notation, let K_σ denote $\mathbf{R}$ (resp. $\mathbf{C}$) when σ is a real (resp. non-real) embedding of K, so σ is an embedding of K into K_σ. Thus, letting S be the set of embeddings of K into $\mathbf{C}$ underlying θ_K (all real embeddings and exactly one from each conjugate pair of non-real embeddings), we have the map $\theta_K : K \to \prod_{\sigma \in S} K_\sigma$ and our aim is to prove that for any $\sigma_0 \in S$ the composite map

$$K \to \prod_{\sigma \in S - \{\sigma_0\}} K_\sigma$$

carries O_K onto a *dense* subset of the target. Equivalently, if $j_\tau : K_\tau \to \prod_{\sigma \in S} K_\sigma$ is the inclusion of the τ-factor for $\tau \in S$ then we want to show that the subset $\theta_K(O_K) + j_{\sigma_0}(K_{\sigma_0}) \subset \prod_{\sigma \in S} K_\sigma$ is dense. In more concrete terms:

Theorem 0.1. *For any elements $x_\sigma \in K_\sigma$ for all $\sigma \neq \sigma_0$ and $\varepsilon > 0$, there exists $a \in O_K$ such that $|\sigma(a) - x_\sigma| < \varepsilon$ for all $\sigma \neq \sigma_0$.*

This will be proved by a suitable application of Minkowski's convex body theorem in $\mathbf{R}^n = \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$. Let $B_\sigma \subset K_\sigma$ be a closed ball centered at 0 with positive radius r_σ and define $B = \prod_\sigma B_\sigma$. A parallelotope P spanned by a fixed $\mathbf{Z}$-basis of $\theta_K(O_K)$ is contained in B by taking the radii big enough (exercise in using the equivalence of norms defined by any two bases of $\mathbf{R}^n$), so using big enough radii ensures that $\theta_K(O_K) + B = \mathbf{R}^n = \mathbf{R}^{r_1} \times \mathbf{C}^{r_2} = \prod_\sigma K_\sigma$. Hence, each $\xi \in \prod_\sigma K_\sigma$ can be written in the form $\theta_K(\alpha) + b$ for some $\alpha \in O_K$ and $b \in B$.

Now consider $D = \prod_{\sigma \in S} D_\sigma \subset \prod_\sigma K_\sigma = \mathbf{R}^n$ where $D_\sigma \subset K_\sigma$ is the closed ball centered at 0 with radius $r_\sigma^{-1}\varepsilon$ for $\sigma \neq \sigma_0$ and radius $r_{\sigma_0}^{-1} C$ for $\sigma = \sigma_0$ with C large enough so that the volume

$$\mathrm{vol}(D) = \prod_\sigma \mathrm{vol}(D_\sigma)$$

at least $2^n \mathrm{vol}_{\theta_K(O_K)} = 2^{r_1 + r_2}\sqrt{|\mathrm{disc}(K)|}$. (This can certainly be achieved by taking C extremely large!) By Minkowski's Theorem applied to the compact convex symmetric D, it contains $\theta_K(\alpha')$ for some $\alpha' \in O_K - \{0\}$; i.e., $|\sigma(\alpha')| \leq r_\sigma^{-1}\varepsilon$ for $\sigma \neq \sigma_0$ and $|\sigma_0(\alpha')| \leq r_{\sigma_0}^{-1} C$. (We have no real control on $\sigma_0(\alpha')$ since C is huge.)

Defining $x = (x_\sigma) \in \prod_\sigma K_\sigma$ where we let $x_{\sigma_0} = 0$ (and the other x_σ's are as given), the point $\xi := x \cdot \theta_K(1/\alpha') = (x_\sigma / \sigma(\alpha')) \in \prod_\sigma K_\sigma$ can be written as $\theta_K(\alpha) + b$ for some $\alpha \in O_K$ and $b \in B$. Hence,

$$x = \theta_K(\alpha')\xi = \theta_K(\alpha'\alpha) + \theta_K(\alpha')b$$

with $\alpha'\alpha \in O_K$ and

$$|\sigma(\alpha')b_\sigma| \leq r_\sigma^{-1}\varepsilon r_\sigma = \varepsilon$$

for all $\sigma \neq \sigma_0$. (In contrast, $|\sigma_0(\alpha')b_{\sigma_0}|$ is probably really huge, but we don't care.) Since $\sigma(\alpha')b_\sigma$ is the σ-component of $\theta_K(\alpha')b = x - \theta_K(\alpha'\alpha)$, we see that $a := \alpha'\alpha$ does the job.