Math 154. Minkowski step with units

Let K be a number field, and define $n = [K : \mathbf{Q}]$ and r_1, r_2 as usual (so $r_1 + 2r_2 = n$). To prove the Unit Theorem for K, we saw in class that it suffices to find a compact subset Δ' of the "norm-1 hypersurface"

$$\Sigma = \{(v \in \mathbf{R}^{r_1} \times \mathbf{C}^{r_2} \,|\, |\mathcal{N}(v)| = 1\} = \{(x, z) \in \mathbf{R}^{r_1} \times \mathbf{C}^{r_2} \,|\, \prod |x_i| \prod |z_j|^2 = 1\}$$

such that $\Sigma = \cup_{\varepsilon \in O_K^\times} \theta_K(\varepsilon)\Delta'$.

Fix $C \geq (2/\pi)^{r_2}\sqrt{|\mathrm{disc}(K)|}$, and choose $\xi \in \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$ satisfying $|\mathcal{N}(\xi)| = C$. The compact convex symmetric domain

$$D = \{(x, z) \in \mathbf{R}^{r_1} \times \mathbf{C}^{r_2} \,|\, |x_i| \leq |\xi_i|, |z_j| \leq |\xi_j|\}$$

is a direct product of r_1 closed intervals in $\mathbf{R}$ centered at 0 with radii given by the $|\xi_i|$'s and r_2 closed discs in $\mathbf{C}$ centered at 0 with radii given by the $|\xi_j|$'s, so

$$\mathrm{vol}(D) = \prod_i (2|\xi_i|) \cdot \prod_j (\pi|\xi_j|^2) = 2^{r_1}\pi^{r_2}|\mathcal{N}(\xi)| = 2^{r_1}\pi^{r_2}C \geq 2^n \mathrm{vol}_{\theta_K(O_K)}$$

since $\mathrm{vol}_{\theta_K(O_K)} = 2^{-r_2}\sqrt{|\mathrm{disc}(K)|}$ and $n = r_1 + 2r_2$. Thus, by Minkowski's theorem (!), D contains a nonzero point of the lattice $\theta_K(O_K)$. In other words, there exists $a \in \mathscr{O}_K - \{0\}$ such that $\theta_K(a) \in D$.

Letting $\sigma_1, \dots, \sigma_{r_1+r_2}$ be the initially chosen embeddings $K \to \mathbf{C}$ (the first r_1 being $\mathbf{R}$-valued, and the remaining r_2 being non-real embeddings without conjugate repetition), the membership of $\theta_K(a)$ in D says exactly $|\sigma_m(a)| \leq |\xi_m|$ for all $1 \leq m \leq r_1 + r_2$, so

$$|\mathrm{N}_{K/\mathbf{Q}}(a)| = |\prod_i \sigma_i(a) \cdot \prod_j \sigma_j(a)\overline{\sigma}_j(a)| \leq \prod_i |\xi_i| \cdot \prod_j |\xi_j|^2 = C.$$

Equivalently, the nonzero principal ideal $(a) \subset O_K$ has ideal-norm $\mathrm{N}((a))$ at most C.

Now comes the crucial observaton: for *any* $v \in \Sigma$, the multiplicative translate $D \cdot v \subset \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$ is "another such D" attached to the componentwise-product element $\xi v \in \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$ which may have quite different componentwise absolute values but has the *same* absolute norm since $|\mathcal{N}(\xi v)| = |\mathcal{N}(\xi)||\mathcal{N}(v)| = C \cdot 1 = C$. Hence, $\mathrm{vol}(Dv) = 2^{r_1}\pi^{r_2}C$ coincides with $\mathrm{vol}(D)$, so we can run exactly the same Minkowski argument with Dv in place of v to get an element $a_v \in O_K - \{0\}$ such that $\theta_K(a_v) \in Dv$. In particular, the nonzero principal ideal $(a_v) \subset O_K$ has norm at most C regardless of v. But there are *only finitely many* nonzero ideals of O_K (principal or not!) with norm at most a fixed bound (such as C), so there are only finitely many possibilities for the principal ideal (a_v) as we vary v (though the actual *element* $a_v \in O_K - \{0\}$ may vary through infinitely many possibilities as far as we know). Every time $(a_v) = (a_{v'})$ for some $v, v' \in \Sigma$ we get a unit $a_{v'}/a_v \in O_K^\times$! The aim is to show that enough such units are obtained in this way to translate any point $v \in \Sigma$ into some specific compact subset of Σ.

To make this strategy precise, we observe that the assignment $\alpha \mapsto (\alpha)$ defines an injective map of sets

$$\{\alpha \in O_K - \{0\} \,|\, |\mathrm{N}_{K/\mathbf{Q}}(\alpha)| \leq C\}/O_K^\times \to \{\mathfrak{a} \neq 0 \,|\, \mathrm{N}\mathfrak{a} \leq C\}$$

with target a finite set, so the source is also a finite set. Thus, we can pick a finite set of representatives $\alpha_1, \dots, \alpha_m \in O_K - \{0\}$ for the set on the left; i.e., every $\alpha \in O_K - \{0\}$ with absolute norm at most C is an $O_K^\times$-multiple of some α_j. This allows us to finally define the sought-after compact subset of Σ:

Proposition 0.1. *The subset $\Delta' = \bigcup_{j=1}^m (\Sigma \cap (D\theta_K(\alpha_j)^{-1}))$ of Σ works: it is compact and every element of Σ admits a $\theta_K(O_K^\times)$-multiple contained in Δ'.*

Proof. The compactness is obvious since Σ is closed in $\mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$ and D is a compact subset of $\mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$ (so likewise any multiple of D against an element of $(\mathbf{R}^\times)^{r_1} \times (\mathbf{C}^\times)^{r_2}$ is compact). For each $v \in \Sigma$ we have made $a_v \in O_K - \{0\}$ with absolute norm at most C, so $a_v = \alpha_{j_v}\varepsilon_v$ for some $1 \leq j_v \leq m$ and some $\varepsilon_v \in O_K^\times$. The point $\theta_K(a_v) \in Dv$ can be written as $\theta_K(\alpha_{j_v})\theta_K(\varepsilon_v)$, so $v^{-1} \in (D\theta_K(\alpha_{j_v})^{-1})\theta_K(\varepsilon_v^{-1})$.

In particular, some $\theta_K(O_K^\times)$-multiple of v^{-1} belongs to some $D\theta_K(\alpha_j)^{-1}$. But $|\mathcal{N}(v)| = 1$ by design, and multiplication by anything in $\theta_K(O_K^\times) \subset \Sigma$ preserves $|\mathcal{N}|$, so we conclude that some $\theta_K(O_K^\times)$-multiple of v^{-1} belongs to $\Sigma \cap (D\theta_K(\alpha_j)^{-1}) \subset \Delta'$. The hypersurface Σ is stable under componentwise inversion, so this shows that all elements of Σ admits a $\theta_K(O_K^\times)$-multiple belonging to Δ', as desired. ∎