Math 154. Volume attached to an ideal

Let K be a number field, $d_K = |\mathrm{disc}(K)|$ its absolute discriminant, and O_K its ring of integers. Let $n = [K : \mathbf{Q}]$ and define r_1 and r_2 as usual (so $n = r_1 + 2r_2$). In class we defined the embedding

$$\theta_K : K \hookrightarrow \mathbf{R}^{r_1} \times \mathbf{C}^{r_2} = \mathbf{R}^n$$

by

$$x \mapsto (\sigma_1(x), \dots, \sigma_{r_1}(x), \sigma_{r_1+1}(x), \dots, \sigma_{r_1+r_2}(x))$$

where $\sigma_1, \dots, \sigma_{r_1}$ are the embeddings $K \to \mathbf{R}$ and $\sigma_{r_1+1}, \dots, \sigma_{r_1+r_2} : K \to \mathbf{C}$ are pairwise non-conjugate non-real embeddings. We saw in class that for any finite free $\mathbf{Z}$-module M of rank n in K, the image $\theta_K(M)$ is a lattice in $\mathbf{R}^n$. In particular, if $\mathfrak{b}$ is a nonzero ideal in O_K then $\theta_K(\mathfrak{b})$ is a lattice in $\mathbf{R}^n$. To each lattice L in $\mathbf{R}^n$ we assigned a "volume" $v_L = \mathrm{vol}(P_{\mathbf{v}})$ where $\mathbf{v} = \{v_1, \dots, v_n\}$ is *any* $\mathbf{Z}$-basis of L and $P_{\mathbf{v}}$ is the associated parallelotope $\{\sum t_i v_i \,|\, 0 \le t_i \le 1\}$. The aim of this handout is to prove:

Theorem 0.1. *For any nonzero ideal* $\mathfrak{b}$ *in* O_K, $v_{\theta_K(\mathfrak{b})} = 2^{-r_2}\sqrt{d_K}\mathrm{N}(\mathfrak{b})$.

Proof. Let's first reduce to the case of the unit ideal. Since $\mathfrak{b}$ is a finite-index submodule of the finite free $\mathbf{Z}$-module O_K, by the structure theorem for modules over a PID there is a $\mathbf{Z}$-basis $\{v_1, \dots, v_n\}$ of O_K such that $\mathfrak{b}$ has as a $\mathbf{Z}$-basis $\{c_1 v_1, \dots, c_n v_n\}$ for some integers $c_1, \dots, c_n > 0$. Moreover, as we have used several times in earlier proofs, the index $\mathrm{N}(\mathfrak{b}) = [O_K : \mathfrak{b}] = \#(O_K/\mathfrak{b})$ is $\prod c_j$. But the parallelotope $P_{\mathbf{cv}}$ associated to the basis $\{c_i v_i\}$ of $\mathfrak{b}$ is covered by the translates of the parallelotope $P_{\mathbf{v}}$ by the vectors $\sum r_i v_i$ with $0 \le r_i \le c_i - 1$, and the overlaps are contained in translated hyperplanes, hence of measure zero. Thus, since translation does not affect volume,

$$v_{\theta_K(\mathfrak{b})} = \mathrm{vol}(P_{\mathbf{cv}}) = \sum_{0 \le r_i \le c_i - 1} \mathrm{vol}(P_{\mathbf{v}}) = (\prod c_j)\mathrm{vol}(P_{\mathbf{v}}) = \mathrm{N}(\mathfrak{b}) v_{\theta_K(O_K)}.$$

Thus, if we can handle the case of the ideal O_K then we get the result for $\mathfrak{b}$.

Now we may and do focus on proving that $v_{\theta_K(O_K)} = 2^{-r_2}\sqrt{d_K}$. We'll really prove the equality after squaring both sides (as we may do), which is to say that we'll prove $(v_{\theta_K(O_K)})^2 = 4^{-r_2}|\mathrm{disc}(K)|$. Let $\{e_j\}$ be a $\mathbf{Z}$-basis of O_K, so by definition

$$\mathrm{disc}(K) = \det(\mathrm{Tr}_{K/\mathbf{Q}}(e_i e_j)) = \det(\sum_{\sigma} \sigma(e_i e_j))$$

where σ varies over all embeddings $K \to \mathbf{C}$. Letting M be the matrix whose ith row is the n-tuple of numbers $\sigma(e_i)$ for some (fixed) arrangement of the choices of σ, the matrix $(\sum_{\sigma} \sigma(e_i e_j))$ is the product MM^{t} of M against its transpose. Thus, $\mathrm{disc}(K) = \det(M)\det(M^{\mathrm{t}}) = \det(M)^2$. Hence, $4^{-r_2}|\mathrm{disc}(K)| = (2^{-r_2}|\det(M)|)^2$. It therefore suffices to prove that the lattice $\theta_K(O_K)$ spanned by the vectors $\theta_K(e_i)$ in $\mathbf{R}^{r_1} \times \mathbf{C}^{r_2} = \mathbf{R}^n$ has associated volume $2^{-r_2}|\det(M)|$.

Consider the matrix N whose ith row is the vector $\theta_K(e_i)$. We saw by row and column operations in lecture that $|\det(N)| = 2^{-r_2}|\det(M)|$ because to get from N to M we multiply some columns by $\sqrt{-1}$ (which has no effect on the absolute determinant), multiply r_2 columns by 2 (which is counteracted by a factor of 2^{-r_2} on the determinant), and adding or subtracting some columns from others (which has no effect on the determinant). Thus, our problem is reduced to checking that the volume associated to the lattice $\theta_K(O_K)$ is $|\det(N)|$. But the rows of N are exactly a $\mathbf{Z}$-basis of $\theta_K(O_K)$, so we win! ■