CHOW'S LEMMA

Matthew Emerton

The aim of this note is to prove the following form of Chow's Lemma:

Suppose that $f : X \to S$ is a separated finite type morphism of Noetherian schemes. Then (for some sufficiently large n) there exists a diagram of the following type

$$\begin{array}{ccccc} X' & \overset{i}{\hookrightarrow} & \mathbf{P}^n_X & \overset{f'}{\to} & \mathbf{P}^n_S \\ & & \downarrow p' & & \downarrow p \\ & & X & \overset{f}{\to} & S \end{array}$$

in which the right-hand square is the Cartesian diagram exhibiting $\mathbf{P}^n_X$ as the base-change of $\mathbf{P}^n_S$ via the morphism f, and i is a closed immersion, such that the composition $p' \circ i$ is surjective and induces an isomorphism over a dense open subset of X, and such that the composition $f' \circ i$ is an immersion.

In other words, we can write any separated scheme of finite type over S as the image under a birational projective map of a quasi-projective S-scheme.

We argue by induction on the number of irreducible components of X. Let us complete the inductive step first: suppose that X is reducible, and write $X = Y \cup Z$ with each of Y and Z a non-empty proper closed subset of X. Let $U = Y \setminus Z$ and $V = Z \setminus Y$. If we choose Y and Z each to be a union of irreducible components of X, having no irreducible component in common, then we see that U is dense in Y and that V is dense in Z, and that in fact the intersection of U (respectively V) with any dense open subset of Y (respectively Z) is again dense in Y (respectively Z). We assume that we have chosen Y and Z in this fashion. Then each of Y and Z has fewer irreducible components then X.

Let $\mathcal{I}$ be the ideal sheaf in $\mathcal{O}_X$ which cuts out the reduced induced structure on X. Then when we restrict $\mathcal{I}$ to the open subset (of both Y and X) U it is a nil ideal (i.e. every section is locally nilpotent) and thus is nilpotent, since U is open in a Noetherian scheme and thus is itself a Noetherian scheme. Suppose that $\mathcal{I}_U^M = 0$. Then give Y the closed subscheme strucutre corresponding to the sheaf of ideal $\mathcal{I}^M$, and let $j_Y : Y \hookrightarrow X$ the corresponding closed immersion. This choice of scheme structure has the nice property that the open set U has the same scheme structure whether we regard it as an open subset of X or of Y. Similarly, if we write $\mathcal{J}$ for the ideal sheaf cutting out the reduced induced scheme structure on Z, then $\mathcal{J}_V^N = 0$ for some N. We give Z the closed subscheme structure corresponding to the ideal sheaf $\mathcal{J}^N$, and let $j_Z : Z \hookrightarrow X$ denote the corresponding closed immersion. Then V has the same scheme structure whether we regard it as an open subset of X or of Z.

Since closed immersions are finite type and separated, we see that the compositions $f \circ j_Y : Y \to S$ and $f \circ j_Z : Z \to S$ are finite type and separated. By the

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inductive hypothesis, Chow's Lemma holds for each of Y and Z. So we may find diagrams

$$\begin{array}{ccccc} Y' & \overset{i_Y}{\hookrightarrow} & \mathbf{P}^{n_Y}_Y & \overset{(f\circ j_Y)'}{\rightarrow} & \mathbf{P}^{n_Y}_S \\ & & \downarrow p'_Y & & \downarrow p \\ & & Y & \overset{f\circ j_Y}{\rightarrow} & S \end{array}$$

and

$$\begin{array}{ccccc} Z' & \overset{i_Z}{\hookrightarrow} & \mathbf{P}^{n_Z}_Z & \overset{(f\circ j_Z)'}{\rightarrow} & \mathbf{P}^{n_Z}_S \\ & & \downarrow p'_Z & & \downarrow p \\ & & Z & \overset{f\circ j_Z}{\rightarrow} & S \end{array}$$

having the properties listed in the statement of Chow's Lemma. In particular, combining the conclusions of Chow's Lemma with our above observation, we see that there is a dense open subset U' of Y, which we may assume to be contained in U (by intersecting it with U if necessary), over which $p'_Y \circ i_Y$ is an isomorphism, and that there is a dense open subset V' of Z, which we may assume to be contained in V (by intersecting it with V if necessary), over which $p'_Z \circ i_Z$ is an isomorphism.

The closed immersion $j_Y : Y \hookrightarrow X$ induces a closed immersion $\mathbf{P}^{n_Y}_Y \overset{j'_Y}{\hookrightarrow} \mathbf{P}^{n_Y}_X$. Similarly the closed immersion $j_Z : Z \hookrightarrow X$ induces a closed immersion $\mathbf{P}^{n_Z}_Z \overset{j'_Z}{\hookrightarrow} \mathbf{P}^{n_Z}_X$. Thus we can factor each of the above diagrams in the following way:

$$\begin{array}{ccccccc} Y' & \overset{i_Y}{\hookrightarrow} & \mathbf{P}^{n_Y}_Y & \overset{j'_Y}{\hookrightarrow} & \mathbf{P}^{n_Y}_X & \overset{f'}{\rightarrow} & \mathbf{P}^{n}_S \\ & & \downarrow p'_Y & & \downarrow p' & & \downarrow p \\ & & Y & \overset{j_Y}{\hookrightarrow} & X & \overset{f}{\rightarrow} & S \end{array}$$

and

$$\begin{array}{ccccccc} Z' & \overset{i_Z}{\hookrightarrow} & \mathbf{P}^{n}_Z & \overset{j'_Z}{\hookrightarrow} & \mathbf{P}^{n_Z}_X & \overset{f'}{\rightarrow} & \mathbf{P}^{n}_S \\ & & \downarrow p'_Z & & \downarrow p' & & \downarrow p \\ & & Z & \overset{j_Z}{\rightarrow} & X & \overset{f}{\rightarrow} & S \ . \end{array}$$

We may take the disjoint unions of the two composite closed immersions $j'_Y \circ i_Y$ and $j'_Z \circ i_Z$ to obtain the following diagram:

$$\begin{array}{ccccc} Y' \coprod Z' & \overset{(j'_Y\circ i_Y)\coprod(j'_Z\circ i_Z)}{\hookrightarrow} & \mathbf{P}^{n_Y}_X \coprod \mathbf{P}^{n_Z}_X & \overset{f'}{\rightarrow} & \mathbf{P}^{n_Y}_S \coprod \mathbf{P}^{n_Z}_S \\ \downarrow & & \downarrow p' & & \downarrow p \\ Y \coprod Z & \overset{j_Y \coprod j_Z}{\rightarrow} & X & \overset{f}{\rightarrow} & S \ . \end{array}$$

Now $(j'_Y \circ i_Y) \coprod (j'_Z \circ i_Z)$ is a disjoint union of closed immersions, and so is a closed immersion, while $f' \circ ((j'_Y \circ i_Y) \coprod (j'_Z \circ i_Z)) = ((f \circ i)' \circ i_Y) \coprod ((f \circ j)' \circ i_Z)$ is the disjoint union of two immersions and so is an immersion. Also, $p' \circ ((j'_Y \circ i_Y) \coprod (j'_Z \circ i_Z)) = j_Y \circ p'_Y \circ i_Y \coprod j_Z \circ p'_Z \circ i_Z$ induces an isomorphism over U' and V'. (It is here that we are using the fact that U and V, and hence U' and V', have the same scheme structure whether we regard them as open subsets of Y and Z or of X. We are of course also using the fact that U and V, and hence U' and V', are disjoint in X.) Since U' is dense in Y and V' is dense in Z, we see that $U' \cup V'$ is dense in X. Thus in order to prove Chow's Lemma for X, it suffices to find a closed immersion

$\mathbf{P}_S^{n_Y} \coprod \mathbf{P}_S^{n_Z} \hookrightarrow \mathbf{P}_S^n$ for some n, for then the following diagram

$$\begin{array}{ccccccc} & & \mathbf{P}_X^n & \xrightarrow{f'} & \mathbf{P}_S^n \\ & & \uparrow & & \uparrow \\ Y' \coprod Z' & \overset{(j'_Y \circ i_Y) \coprod (j'_Z \circ i_Z)}{\hookrightarrow} & \mathbf{P}_X^{n_Y} \coprod \mathbf{P}_X^{n_Z} & \xrightarrow{f'} & \mathbf{P}_S^{n_Y} \coprod \mathbf{P}_S^{n_Z} \\ \downarrow & & \downarrow p' & & \downarrow p \\ Y \coprod Z & \xrightarrow{j_Y \coprod j_Z} & X & \xrightarrow{f} & S \end{array} .$$

would prove Chow's Lemma for X, by taking $X' = Y \coprod Z$, and taking i to be the closed immersion $i : X' = Y' \coprod Z' \hookrightarrow \mathbf{P}_X^{n_Y} \coprod \mathbf{P}_X^{n_Z} \hookrightarrow \mathbf{P}_X^n$. But if we take $n = n_Y + n_Z + 1$ then we may find a closed immersion $\mathbf{P}_S^{n_Y} \coprod \mathbf{P}_S^{n_Z} \hookrightarrow \mathbf{P}_S^n$ by identifying $\mathbf{P}_S^{n_Y}$ with the linear subspace $T_{n_Y+1} = \cdots = T_n = 0$ and identifying $\mathbf{P}_S^{n_Z}$ with the linear subspace $T_0 = \cdots = T_{n_Y} = 0$. (Here T_i denote the homogeneous coordinates on $\mathbf{P}_S^n$.) Thus by induction, we are reduced to the case in which X is irreducible.

Before continuing, let us remark that if the above argument seems a little complicated, it is only because we have named everything involved. The idea is very simple: we find X' mapping onto X by writing $X = Y \cup Z$, finding Y' mapping onto Y and Z' mapping onto Z, and taking $X' = Y' \coprod Z'$.

Let us now assume that X is irreducible, so that every non-empty open subset of X is dense in X. Since S is Noetherian and X is finite type over S, we may cover S by finitely many affine opens $\operatorname{Spec} A_i$, and then cover the preimage of each $\operatorname{Spec} A_i$ in X by finitely many affine opens $\operatorname{Spec} B_{ij}$, with each B_{ij} a finite type A_i-algebra. Suppose that B_{ij} is a quotient of $A_i[T_1, \dots, T_r]$. (We may take the same r for ever4 B_{ij} with no loss of generality, since the B_{ij} are finite in number.) Then $\operatorname{Spec} B_{ij}$ is a closed subset of $\mathbf{A}^r_{\operatorname{Spec} A_i}$, which is an open subset of $\mathbf{A}^r_S$. To summarize, X can be covered by finitely many affine opens, each of which admits an immersion into $\mathbf{A}^r_S$ for sufficiently large r. Let us now forget the A_i and B_{ij} notation, and simply refer to these affine open subsets of X as $U_1, \dots, U_m$. $\mathbf{A}^r_S$ is an open subset of $\mathbf{P}^r_S$, and thus each U_i immerses into $\mathbf{P}^r_S$. Let P_i denote the scheme theoretic image of U_i in $\mathbf{P}^r_S$. (This scheme-theoretic image exists because we are in the Noetherian case, so this immersion is automatically quasi-compact. The underlying space of P_i is the closure of U_i in $\mathbf{P}^j_S$). Then the map $U_i \to P_i$ is an open immersion.

Let us write $P = P_1 \times_S \cdots \times_S P_m$, and for each i write

$$P^i = P_1 \times_S \cdots \times_S \hat{P}_i \times_S \cdots \times_S P_m$$

(where $\hat{P}_i$ means "omit P_i from the product"). Each P_i is a closed subscheme of a projective space over S and so is proper over S. Thus each of the products P and P^i is proper over S. Let us also write $U = \bigcap_{i=1}^m U_i$. Since X is irreducible, each U_i is dense in X, and so U is dense in X. Let h be the map $U \to X \times_S P$ defined so that the projection onto X is simply the inclusion $U \subset X$, while the projection onto P_i is the open immersion $U \subset U_i \to P_i$. We are going to define X' to be the scheme-theoretic image of h, which is a closed subscheme of $X \times_S P$.

Each P_i is a closed subscheme of a projective space over S, so their product P is a closed subscheme of a product of projective spaces. The Segre embedding realizes a product of projective spaces as a closed subscheme of a projective space of sufficiently large dimension, say n. Thus we obtain a diagram

$$\begin{array}{ccccccc} X' & \hookrightarrow & X \times_S P & \hookrightarrow & \mathbf{P}^n_X & \to & \mathbf{P}^n_S \\ & & \downarrow & & \downarrow & & \downarrow \\ & & P & \to & X & \to & S. \end{array}$$

In order to conclude that Chow's Lemma is true for X, we have to show that the projection $X' \to P$ is an immersion, and that $X' \to X$ is an isomorphism over a dense open set. It will then follow that $X' \to X$ is surjective, because it is closed (being the composition of the closed immersion $X' \to X \times_S P$ and the map $X \times_S P \to X$, which is closed because $P \to S$ is proper) and dominant (being an isomorphism over a dense open subset of X). We now turn to proving these two facts about X'.

Before giving the details of the argument, let us give the gist of it, by imagining that X, U, and each of the U_i and P_i are simply topological spaces rather than schemes. Assume that U is dense in each U_i, and that each U_i is in turn dense in the respecitve P_i, which we assume are compact Hausdorff (the topological analogue of being proper). We also assume that X is Hausdorff (the topological analogue of being separated). The map h is just the diagonal map

$$u \to h(u) = (u, \cdots, u) \in X \times P_1 \times \cdots \times P_m.$$

We wish to understand X', the closure of the image of h. Thus suppose that $h(u_s)$ is a sequence of points in the image, converging to some point $v = (x, p_1, \dots, p_m)$ in $X \times P_1 \cdots \times P_m = X \times P$. (Since X and each of the P_i is Hausdorff, this limit is uniquely determined.) Since X is covered by the open sets U_i, x must lie in U_i for some i. Then we see that the sequence u_s converges to x in U_i, and so also in P_i (since U_i is a subset of P_i). Since P_i is Hausdorff, the sequence u_s has a unique limit in P_i, and so we see that $x = p_i$.

Suppose that $v' = (x', p'_1, \dots, p'_m)$ is the limit of some other sequence $h(u'_s)$ of points in the image of h such that $(p'_1, \dots, p'_m) = (p_1, \dots, p_m)$. Then in particular $p'_i = p_i$ is in in U_i, and we see that the u'_s converge to p_i in U_i, so that then also $x' = \lim u'_s = p_i = x$, and so $v' = v$. Thus the projection $X \times P \to P$ is a homeomorphism when restricted to X'. Furthermore, $x \in U_i$ if and only if $p_i \in U_i$, in which case $x = p_i$. This easily implies that X' projects homeomorphically onto a locally closed subset of $P_1 \times \cdots \times P_m$ (the topological analogue of being an immersion). For if we let X'_i denote that subset of X' for which x (and thus p_i) is an element of U_i, we see that X'_i maps homeomorphically onto the closure of the diagonal image of U in the open subset $P_1 \times \cdots \times U_i \times \cdots P_m$ of $P_1 \times \cdots P_m$.

In this context, the argument that $X' \to X$ is a surjection can be phrased very simply: let x be an element of X. Then x lies in U_i for some i, and by assumption we can find a sequence u_s of points in U converging to x. Then since each P_i is compact, we may refine this sequence so that it converges in each P_i. Then we see that $h(u_s)$ converges to a point of X' lying over $x \in X$.

Having illustrated our arguments in this simpler context, let us now give the scheme-theoretic arguments for Chow's Lemma. I encourage the reader to consider how the arguments below are simply reformulations in the language of schemes of the arguments just given.

The scheme X is covered by the open sets U_i, and so $X \times_S P$ is covered by the open sets $U_i \times_S P$. Thus X' is covered by its intersections with each open set $U_i \times_S P$. Now since $U \subset U_i$ for every i, we see that $h : U \to X \times_S P$ factors through the open immersion $U_i \times_S P \to X \times_S P$ as

$$U \xrightarrow{h_i} U_i \times_S P \to X \times_S P.$$

Thus the intersection of X' (which is by definition the scheme-theoretic image of h) and $U_i \times_S P$ is precisely the scheme-theoretic image of the map h_i.

Since P_i is projective over S, it is separated over S, and so the graph of the inclusion of S-schemes $U_i \to P_i$ is a closed subscheme of $U_i \times_S P_i$, which maps isomorphically onto U_i via projection onto either the first or second factor (where in the case of projection onto the second factor, we regard U_i as an open subset of P_i). We will denote this graph by Γ_i.

If we base change the closed immersion $\Gamma_i \hookrightarrow U_i \times_S P_i$ via the map $P^i \to S$ we obtain a closed immersion $\Gamma_i \times_S P^i \hookrightarrow U_i \times_S P_i \times_S P^i$. Now

$$P_i \times_S P^i = P_i \times_S P_1 \times_S \cdots \times_S \hat{P}_i \times_S \cdots \times_S P_m \xrightarrow{\sim} P_1 \times_S \cdots \times_S P_i \times_S \cdots \times_S P_m$$

simply by rearranging the factors. (In the subsequent argument we will sometimes use this rearrangement of the factors without explicitly mentioning it.) We define k_i to be the closed immersion which is the composition

$$k_i : \Gamma_i \times_S P^i \hookrightarrow U_i \times_S P_i \times_S P^i \xrightarrow{\sim} U_i \times_S P.$$

By contruction $h_i : U \to U_i \times_S P$ factors through the closed immersion k_i: let us write

$$h_i : U \xrightarrow{h_i'} \Gamma_i \times P^i \xrightarrow{k_i} U_i \times P.$$

Then the scheme-theoretic image of h_i is equal to the scheme-theoretic image of h_i', since k_i is a closed immersion.

Now the composition $\Gamma_i \times_S P^i \overset{k_i}{\hookrightarrow} U_i \times_S P \to P$ (where the second arrow is the natural projection $U_i \times_S P \to P$) factors as indicated in the following diagram:

$$\begin{array}{ccccc} \Gamma_i \times_S P^i & \hookrightarrow & U_i \times_S P_i \times_S P^i & \xrightarrow{\sim} & U_i \times_S P \\ \downarrow & & \downarrow & & \downarrow \\ U_i \times_S P^i & \to & P_i \times_S P^i & \xrightarrow{\sim} & P. \end{array}$$

The well-definedness of the left-most vertical arrow in this diagram follows from the fact, which we observed when we introduced Γ_i, that the projection onto the second factor $\Gamma_i \to P_i$ induces an isomorphism of Γ_i with the open subset U_i of P_i. In particular, this left-most arrow is the base change of an isomorphism, and so is itself an isomorphism. The left-most arrow of the bottom row is the base change of the open immersion $U_i \to P_i$ and so is an open immersion. Thus the composition $\Gamma_i \times_S P^i \to P$ is the composition of an open immersion and two isomorphisms, and so is an open immersion.

Now the scheme-theoretic image of h_i equals the scheme-theoretic image of h_i' which is a closed subscheme of $\Gamma_i \times_S P^i$. Let us denote this by X_i'. Then the composition $X_i' \hookrightarrow \Gamma_i \times_S P^i \to P$ is the composition of an open and closed immersion, and so is an immersion.

We now wish to conclude that the composition $X' \to X \times_S P \to P$ is an immersion (recall that X' is the scheme-theoretic image of h). We have seen that X' is covered by the open sets X'_i, each of which has image in the open subscheme $U_i \times_S P^i$ of P. Now we claim that in fact X'_i is equal to the inverse image of $U_i \times_S P^i$ in X'. If we knew this, we would see that the projection $X' \to P$ is an immersion locally on P, and so is indeed an immersion.

Let us prove that X'_i is indeed the inverse image of $U_i \times_S P^i$ in X'. The inverse image of $U_i \times_S P^i$ in $X \times_S P$ is isomorphic to $X \times_S U_i \times_S P^i$. The morphism h factors through the open subscheme $X \times_S U_i \times_S P^i$ of $X \times_S P$: let us factor h as

$$U \xrightarrow{h^i} X \times_S U_i \times_S P^i \to X \times_S P.$$

Then the inverse image of $U_i \times_S P^i$ in X' is equal to the scheme-theoretic image of h^i. We wish to show that this is equal to the scheme-theoretic image of h'_i, which is a closed subscheme of $\Gamma_i \times_S P^i$. Thus it suffices to show that $\Gamma_i \times_S P^i$ is a closed subscheme of $X \times_S U_i \times_S P^i$. For this, it suffices to show that the Γ_i is a closed subscheme of $X \times_S U_i$ (since closed immersions are preserved by base change). But the immersion $\Gamma_i \to X \times_S U_i$ is precisely the graph of the inclusion of U_i in X, which is indeed a closed immersion, since X is separated over S. (This is the only point at which the separatedness of X is required, but of course it is the crux of the argument.) This completes the task of showing that $X' \to P$ is an immersion.

It remains to show that $X' \to X$ is an isomorphism over a non-empty open subset of X (necessarily dense, since X is irreducible). We will show that indeed this map is an isomorphism over the open subset U of X. To see this, we must consider the inverse image of U in X'. The inverse image of U in $X \times_S P$ is the open subscheme $U \times P$. The map $h : U \to X \times_S P$ factors as

$$U \xrightarrow{h} U \times_S P.$$

Thus the intersection of X' and $U \times P$, which is the inverse image of U in X', is equal to the scheme-theoretic image of h'. Thus we would be done if we could show that h' is a closed immersion, for then it would equal its own scheme theoretic image, inducing an isomorphism $U \xrightarrow{\sim} X'_{|U}$. But h' is precisely the graph of the morphism $U \to P$ obtained as the fibre product of the morphisms $U \to U_i \to P_i$, and so is a closed immersion, since P is separated over S. This completes the proof of Chow's Lemma.