Math 216A. Connected and irreducible components, and dimension for schemes

1. Introduction

A noetherian topological space has a finite "irreducible component decomposition". Since irreducible spaces are obviously connected (as all non-empty open subsets are dense, so no separation is possible), we see that noetherian spaces have finitely many connected components, obtained from chains of irreducible components where each touches another. In particular, the connected components are open (as for any "locally connected" topological space). Using commutative algebra, we also set up a reasonable theory of dimension for affine algebraic sets in terms of chains of irreducible closed sets.

The aim of this handout is to provide the appropriate analogous concepts for general schemes, and relations between geometric definitions and commutative algebra. Beware that with general schemes, these notions will not always behave as nicely as in the classical case (much as is the case with dimension theory for general commutative rings, which behaves far better under noetherian and other hypotheses). We will provide some indications about what is true in general or is true under mild hypotheses, and give some "weird counterexamples".

2. Connected components

For any non-empty topological space, by definition the *connected components* are the maximal connected subsets, all of which are closed (since the closure of a connected subset is connected). The connected components provide a partition fo the space, and they are all open precisely when every point has a connected neighborhood.

For example, a locally noetherian topological space (i.e., a space for which every point has a noetherian open neighborhood) has a connected neighborhood of every point since this clearly holds for a noetherian topological space (due to the existence of the *finite* "irreducible component decomposition" of such a space, combined with the connectedness of irreducible spaces). But a general scheme can have non-open connected components, as the following crazy example shows. This doesn't matter too much in practice, since most schemes one works with in a concrete way are locally noetherian.

Example 2.1. Let $R = \prod_{i \in I} k_i$ be an infinite direct product of fields k_i. This is not noetherian since the ideal J of elements that are 0 in all but finitely many entries is not finitely generated (why not?). The space $X = \operatorname{Spec}(R)$ is quasi-compact, as for any commutative ring, and each factor field k_i gives rise to an open and closed point $x_i = \operatorname{Spec}(k_i)$ of $\operatorname{Spec}(R)$ (since direct factor rings always give rise to a separation). This is an infinite collection of open and closed points, so each is a connected component of X.

The infinitude of this collection of open points and the quasi-compactness of X implies that there are other points of X! It is basically impossible to "explicitly write down" such points (e.g., any maximal ideal $\mathfrak{m}$ containing J as defined above is such an exotic closed point, but one can't write down such an $\mathfrak{m}$ explicitly). Since the $\{x_i\}$'s are connected components which don't cover X, there must exist other connected components C of X. By design, C doesn't contain any x_i (since connected components are pairwise disjoint).

Let's show that all such C are *not* open. It suffices to show more generally that any non-empty open subset U of X contains some x_i. A base for the topology of X is given by basic affine opens, so it suffices to show that any non-empty $\mathrm{Spec}(R_r)$ contains some x_i. Since $\mathrm{Spec}(R_r)$ is non-empty, certainly $r \neq 0$. Thus, as an element of $R = \prod k_i$, r has non-vanishing entry in some k_{i_0}. But the projection $R \to k_i$ is exacty the evaluation map $\mathscr{O}(X) \to k(x_i)$ (check!), so it follows that $r(x_{i_0}) \neq 0$. Hence, $x_{i_0} \in \mathrm{Spec}(R_r)$.

3. Irreducible components

For a non-empty scheme X, an *irreducible component* of X is an irreducible closed subset Z of X that is maximal as such (i.e., Z is not strictly contained in another irreducible closed set). The argument used in point-set topology to show that every point of a non-empty topological space lies in a connected component (so in particular, maximal connected subsets exist!) doesn't carry over so easily in geometric language.

So to show the same for irreducible components (in particular, that they exist!) we will turn the problem into a task in ring theory that is solved via Zorn's Lemma (as usual for such exotic generality). As a preliminary step, we want to relate the irreducible closed sets Z of X containing a chosen point $x \in X$ to prime ideals in a ring. Inspired by the classical setting, we prove:

Lemma 3.1. *For $x \in X$, there is an inclusion-reversing bijection between the set of irreducible closed subsets $Z \subset X$ containing x and the set of prime ideals of $\mathscr{O}_{X,x}$.*

Writing $\mathscr{I}_Z \subset \mathscr{O}_X$ for the "radical" ideal sheaf corresponding to Z, the bijection assigns to Z the (prime!) ideal $\mathscr{I}_{Z,x} \subset \mathscr{O}_{X,x}$, and it assigns to any prime ideal $\mathfrak{q} \subset \mathscr{O}_{X,x}$ the closure in X of the point $i_x(\{\mathfrak{q}\})$ where $i_x : \mathrm{Spec}(\mathscr{O}_{X,x}) \to X$ is the natural map.

Proof. First we reduce to the case of affine X, where everything becomes more tangible in terms of algebra. Pick an affine open subset $U \subset X$ around x. For any irreducible closed subset $Z \subset X$ passing through $x \in U$, the open subset $Z \cap U$ in the irreducible Z is non-empty (as $x \in Z \cap U$) and hence dense in Z. Thus, the closure of $Z \cap U$ in X is equal to Z, so via the identifications $\mathscr{O}_{X,z} = \mathscr{O}_{U,z}$ and $\mathscr{I}_Z|_U = \mathscr{I}_{Z\cap U}$ inside $\mathscr{O}_X|_U = \mathscr{O}_U$ we see that if the proposed recipes define inverse inclusion-reversing bijections for (U, x) then they do for (X, x). Hence, we can replace X with U so that now $X = \mathrm{Spec}(A)$ is affine.

Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. The irreducible closed sets in $X = \mathrm{Spec}(A)$ are exactly the subsets $V(\mathfrak{q}) = \mathrm{Spec}(A/\mathfrak{q}) \subset \mathrm{Spec}(A)$ for prime ideals $\mathfrak{q} \subset A$, so the irreducible closed sets passing through x are precisely $V(\mathfrak{q})$ for primes $\mathfrak{q}$ satisfying $\{\mathfrak{p}\} \in V(\mathfrak{q})$ or equivalently $\mathfrak{q} \subset \mathfrak{p}$. By the theory of localizations of rings, prime ideals of A contained in $\mathfrak{p}$ (i.e., disjoint from $A - \mathfrak{p}$) correspond exactly to prime ideals of $A_{\mathfrak{p}} = \mathscr{O}_{X,x}$ via $\mathfrak{q} \mapsto qA_{\mathfrak{p}}$. Via the labeling by primes ideals of A contained in $\mathfrak{p}$, we thereby get a bijection between the set of irreducible closed subsets of $\mathrm{Spec}(A)$ passing through $x = \mathfrak{p}$ and the set of prime ideals of $A_{\mathfrak{p}}$ via $V(\mathfrak{q}) \mapsto \mathfrak{q}A_{\mathfrak{p}}$. This is inclusion-reversing since

$$V(\mathfrak{q}) \subset V(\mathfrak{q}') \Leftrightarrow \mathfrak{q}' \subset \mathfrak{q} \Leftrightarrow \mathfrak{q}'A_{\mathfrak{p}} \subset \mathfrak{q}A_{\mathfrak{p}}.$$

Since $i_x : \mathrm{Spec}(A_{\mathfrak{p}}) \to \mathrm{Spec}(A)$ corresponds to the natural ring map $A \to A_{\mathfrak{p}}$ under which the contraction of $\mathfrak{q}A_{\mathfrak{p}}$ is equal to $\mathfrak{q}$ for prime ideals $\mathfrak{q} \subset A$ contained in $\mathfrak{p}$, we have $i_x(\{\mathfrak{q}A_{\mathfrak{p}}\}) = \{\mathfrak{q}\}$ and this has closure $V(\mathfrak{q})$. In the reverse direction, if $Z = V(\mathfrak{q})$ for a prime ideal $\mathfrak{q} \subset A$ contained in $\mathfrak{p}$ and $j : Z \to X$ is the natural closed immersion (with Z given its reduced structure) then $\mathscr{I}_Z := \ker(\mathscr{O}_X \to j_*(\mathscr{O}_Z))$ has x-stalk $\ker(A_{\mathfrak{p}} \to (A/\mathfrak{q})_{\mathfrak{p}}) = \mathfrak{q}_{\mathfrak{p}} =$

$\mathfrak{q}A_{\mathfrak{p}}$. This establishes that the desired recipes in both directions define inverse bijections of sets. ■

The dictionary in Lemma 3.1 can be used to create maximal irreducible closed sets, as follows. By Exercise 2.9 in HW5, every irreducible closed set Z in a scheme has a unique generic point η_Z (i.e., a unique point in Z whose closure in the ambient scheme is equal to Z). Topological spaces with this purely topological property are called *sober* (I have no clue where this terminology comes from, but it is the standard name). The maximality of Z in X can be cleanly characterized in terms of the local ring $\mathscr{O}_{X,\eta_Z}$:

Lemma 3.2. *An irreducible closed subset $Z \subset X$ is maximal if and only if $\mathscr{O}_{X,\eta_Z}$ is 0-dimensional, or equivalently if and only if the maximal ideal in $\mathscr{O}_{X,\eta_Z}$ consists of nilpotent elements (i.e., this maximal ideal is the only prime ideal of that local ring). In particular, if $\xi \in X$ is any point and $Y = \overline{\{\xi\}}$ is the associated irreducible closed set in X then it is maximal as such if and only if $\mathscr{O}_{X,\xi}$ is 0-dimensional.*

Proof. A closed subset of X contains Z if and only if it contains the generic point η_Z of Z (why?), so by Lemma 3.1 applied to $x = \eta_Z$ we obtain an inclusion-reversing bijection between the set of irreducible closed subsets of X containing Z and the set of prime ideals of $\mathscr{O}_{X,\eta_Z}$. In particular, maximality of Z in X is equality to minimality in $\mathscr{O}_{X,\eta_Z}$ of the prime ideal corresponding to Z. (A *minimal* prime ideal in a ring is one that doesn't strictly contain another prime ideal.) But this latter prime ideal is the maximal ideal of $\mathscr{O}_{X,\eta_Z}$ (why?), so the maximality of Z in X is equivalent to the maximal ideal of $\mathscr{O}_{X,\eta_Z}$ being a minimal prime ideal, which is to say that it is the only prime ideal of that local ring. This in turn is precisely the 0-dimesionality. ■

For any $x \in X$ and prime ideal $\mathfrak{q}$ of the local ring $\mathscr{O}_{X,x}$ with associated irreducible closure Z in X through x, the local ring $\mathscr{O}_{X,\eta_Z}$ of interest in Lemma 3.2 is exactly $(\mathscr{O}_{X,x})_{\mathfrak{q}}$. Indeed, to check this it suffices to work in an open affine $\mathrm{Spec}(A)$ around x in X, and if $\mathfrak{p} \subset A$ corresponds to x and $\wp$ is the prime ideal of A contained in $\mathfrak{p}$ corresponding to $\mathfrak{q} \subset A_{\mathfrak{p}}$ then the proposed description of $\mathscr{O}_{X,\eta_Z}$ becomes the unique identification $(A_{\mathfrak{p}})_{\mathfrak{q}} = A_{\wp}$ as A-algebras.

We conclude that maximality of Z in X corresponds to 0-dimensionality of $(\mathscr{O}_{X,x})_{\mathfrak{q}}$, which is to say the minimality in $\mathscr{O}_{X,x}$ of the prime ideal $\mathfrak{q}$ corresponding to Z via the recipe in Lemma 3.1. This is interesting since the maximality of Z in X has nothing whatsoever to do with x whereas the minimalty condition on $\mathfrak{q}$ very much involves x since $\mathfrak{q}$ is a prime ideal of the local ring $\mathscr{O}_{X,x}$ at x! Now we have harnessed everything we need to quickly prove:

Theorem 3.3. *Every non-empty scheme X has irreducible components, and every point $x \in X$ is contained in one. Moreover, the set of irreducible components of X passing through x is in bijective correspondence with the set of minimal prime ideals of $\mathscr{O}_{X,x}$.*

Proof. The preceding discussion shows that the set of irreducible components of X passing through a point $x \in X$ is in bijection with the set of minimal primes of the local ring at x, so to complete the proof we just have to show that every nonzero ring A contains minimal prime ideals. This is a standard application of Zorn's Lemma (applied to the non-empty set of prime ideals of A ordered by reverse inclusion). ■

Remark 3.4. For every irreducible closed set Z in X, the irreduble components of X containing Z correspond to the minimal primes of $\mathscr{O}_{X,\eta_Z}$, so in particular there *do* always exist irreducible components of X containing any given irreducible closed subset of X.

In the special case that X is locally noetherian, the set of irreducible components is "locally finite" in the sense that every point $x \in X$ has an open neighborhood $U \subset X$ meeting only finitely many irreducible components of X. Indeed, since a non-empty open subset of an irreducible space is dense we can shrink U to be affine and then it suffices to show for any *noetherian* ring A that there are only finitely many minimal primes. This just expresses the *finite* "irreducible component decomposition" of the noetherian topological space $\mathrm{Spec}(A)$.

Example 3.5. Let's return to the "weird example" $X = \mathrm{Spec}(R)$ from Example 2.1, so $R = \prod k_i$ is a direct product of infinitely many fields k_i. We have seen that X has non-open connected components. Its set of irreducible components is also *not* locally finite. Indeed, if it were locally finite then by quasi-compactness of X there would be only finitely many irreducible components. But there are infinitely many points that are both closed and open (so in particular are certainly irreducible components!), so local finiteness must fail.

4. Dimension

We have now shown that any non-empty scheme contains irreducible components through every point, and we are going to use irreducible closed sets to create a theory of dimension. Motivated by the classical case, for any non-empty scheme X we define

$$\dim X = \sup\{n \geq 0 \,|\, \text{ there exist irreducible closed subsets } Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subset X\}.$$

There is no harm in requiring Z_n to be an irreducible component of X since every irreducible closed set is contained in one (so $\dim X = \sup_i \dim X_i$ for $\{X_i\}$ the set of irreducible components of X; note that each X_i has a natural reduced closed subscheme structure).

But one can't always require Z_0 to be a point since for a general X it isn't actually clear if there are any *closed* points! (Any non-empty affine open has a closed point due to the existence of maximal ideals, but a closed subset of an open subset may not be closed in the ambient space.) The following result provides an abundant supply of closed points in the quasi-compact case, so when X is quasi-compact we can take Z_0 to be a closed point when analyzing $\dim X$.

Proposition 4.1. *Every non-empty closed subset Z of a quasi-compact scheme X has a closed point.*

Beware that there exist integral schemes without any closed point (built as the complement of the closed point in the spectrum of a rather bizarre non-noetherian local domain; of course, removing the closed point doesn't by itself prevent the resulting open subscheme from having its own closed points!). In practice one never has to worry about this.

Proof. We aim to argue topologically for as much as possible, postponing until near the end that X is a scheme (rather than merely a quasi-compact topological space). First we claim via Zorn's Lemma that minimal non-empty closed subsets of Z exist (i.e., closed subsets of Z with no non-empty proper closed subset). Indeed, consider the set Σ of non-empty closed subsets of Z ordered via reverse inclusion. For example, $Z \in \Sigma$. We want to show that Σ has maximal elements, so by Zorn's Lemma it suffices to show that for any collection $\{Z_\alpha\}$ of non-empty closed subsets for which any two have one contained in the other, the intersection $\bigcap_\alpha Z_\alpha$ is non-empty.

Suppose to the contrary that the intersection is empty. Then the open complements $U_\alpha = X - Z_\alpha$ cover X, so by quasi-compactness of X we have that some finite subcollection $U_{\alpha_1}, \dots, U_{\alpha_n}$ covers X. This says $Z_{\alpha_1} \cap \dots \cap Z_{\alpha_n}$ is empty. But for any two Z_α's we are given that one is inside the other, so the intersection of any finite collection of Z_α's is equal to one of them. This forces all finite intersections among the Z_α's to be non-empty (as the Z_α's are all non-empty by design), so we have reached a contradiction. Hence, $\bigcap_\alpha Z_\alpha$ is non-empty as desired, so Zorn's Lemma applies.

Now we can pick a minimal non-empty closed subset Z_0 in Z. We claim that any such minimal Z_0 is a point. Since X is a scheme, we can give Z_0 the reduced scheme structure. Pick a non-empty affine open subset $U \subset Z_0$. The complement $Z_0 - U$ is a *proper* closed subset of Z_0. By the minimality of Z_0, it follows that $Z_0 - U$ is empty, so $Z_0 = U$ is affine. The ring A for Z_0 is reduced and $\mathrm{Spec}(A)$ has no *non-empty* proper closed subset, so for any proper ideal I of A we have $\mathrm{Spec}(A/I) = \mathrm{Spec}(A)$. This says I consists of nilpotent elements, but A is reduced, so $I = (0)$. A ring for which (0) is the only proper ideal is a field (by definition, in effect), so A is a field and hence Z_0 is a point as desired. ∎

By the definition, if $X = \mathrm{Spec}(A)$ is affine then $\dim X$ is the supremum of the "length" of finite chains of distinct prime ideals of A. The latter is the definition of $\dim A$ in commutative algebra (it can be infinite even for a noetherian ring, but Krull proved noetherian *local* rings always have finite dimension; many other classes of non-local noetherian rings also have finite dimension, as one learns with experience). Hence, $\dim(\mathrm{Spec}(A)) = \dim A$. Using Lemma 3.1, one also obtains the formula (please check!)

$$\dim X = \sup_{x \in X} \dim \mathscr{O}_{X,x}$$

(even if some of the local rings have infinite dimension).

There is also a good notion of codimension, but we have to set it up correctly so it makes sense even in the infinite-dimensional case. That is, we cannot define the codimension in X of a closed subset Y to be $\dim X - \dim Y$ since it could happen that both of those dimensons are infinite. Instead, we proceed as follows. For an *irreducible* closed subset $Y \subset X$, we define $\mathrm{codim}(Y, X)$ to be

$$\sup\{n \geq 0 \,|\, \text{ there exist irreducible closed subsets } Y = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subset X\}$$

(this can be infinite). For example, if Y is an irreducible component of X then $\mathrm{codim}(Y, X) = 0$ even if X has irreducible components with bigger dimension than Y. Think about it.

Note that when making this definition, there is no harm in limiting attention to chains of irreducibles with Z_n an irreducible component of X (since every irreducible closed subset of X is contained in one). For general non-empty closed $Y \subset X$ with $\{Y_i\}$ its set of irreducible components (we view Y as a non-empty scheme via the reduced structure, so the preceding work on irreducible components is applicable to the topological space Y), we define

$$\mathrm{codim}(Y, X) = \inf_i \mathrm{codim}(Y_i, X)$$

(this is infinite only when all $\mathrm{codim}(Y_i, X)$ are infinite). It isn't quite clear what is a good definition for $\mathrm{codim}(\emptyset, X)$; if the preceding "inf" definition is applied to $Y = \emptyset$ (so its set of irreducible components is empty) then according to some conventions in logic we have $\mathrm{codim}(\emptyset, X) = \infty$ (this is also noted in [EGA 0_{IV}, §14.2]). Fortunately one never really has to bother with weirdness such as the codimension of the empty set.

Example 4.2. If X is *irreducible* of finite type over a field k and Y is any closed subset then by Exercise 3.20(d) in HW6 (not due for submission) we have $\dim Y + \mathrm{codim}(Y, X) = \dim X$; in other words, $\mathrm{codim}(Y, X) = \dim X - \dim Y$ for irreducible X. This expresses in geometric terms some general facts concerning chains of prime ideals in polynomial rings over a field. As an illustration, if k is a field and $Y \subset X := \mathbf{A}_k^3$ is a reduced closed subset with irreducible components given by an irreducible surface S and an irreducible curve C then

$$\begin{aligned} \mathrm{codim}(Y, X) = \inf(\mathrm{codim}(S, \mathbf{A}_k^3), \mathrm{codim}(C, \mathbf{A}_k^3)) &= \inf(3-2, 3-1) \\ &= 3-2 \\ &= \dim(X) - \dim(Y). \end{aligned}$$

In contrast, with the reducible Y and its irreducible closed subset C we have $\mathrm{codim}(C, Y) = 0$ because C is an irreducible component of Y yet $\dim Y - \dim C = 2 - 1 = 1$. Thus, *be careful when working with codimension in the presence of reducible schemes* (or at least with a non-empty closed subset whose irreducible components have varying codimensions).

Example 4.3. If Y is an irreducible closed subset of X with generic point η_Y then the study of $\mathrm{codim}(Y, X)$ is completely controlled by the prime ideal structure of $\mathscr{O}_{X,\eta_Y}$ due to Lemma 3.1. More specifically, Lemma 3.1 implies $\mathrm{codim}(Y, X) = \dim \mathscr{O}_{X,\eta_Y}$, so for a locally noetherian scheme X this codimension is always finite (in view of Krull's theorem that *local* noetherian rings have finite dimension) even though $\dim X$ may be infinite for noetherian X.

For affine algebraic sets X and irreducible closed subsets Y we saw that there exists a chain of irreducible closed sets $Y = Z_0 \subsetneq \cdots \subsetneq Z_n \subset X$ with $n = \mathrm{codim}(Y, X)$ for which any two desired irreducible closed subset $Z \subset Z'$ containing Y appears among the Z_j's in such a chain. By Example 4.3, whether or not such good behavior for codimension holds for a given locally noetherian scheme X is completely controlled by how well-behaved dimension theory is for the local rings of X: for each $x \in X$, can one find a chain of primes in $\mathscr{O}_{X,x}$ with length $\dim \mathscr{O}_{X,x}$ for which any two prime ideals $\mathfrak{p} \subset \mathfrak{q}$ occur in the chain? When this holds we say X is *catenary*.

We know from the algebraic part of the earlier codimension handout that schemes locally of finite type over a field are catenary. Hard work in commutative algebra shows that nearly all noetherian rings one encounters are catenary (e.g., every ring finitely generated over a Dedekind domain or over a complete local noetherian ring is catenary, as is the local ring of convergent power series in n variables over any field complete for a non-trivial absolute value). But catenarity can fail for some noetherian rings. These matters are discussed quite thoroughly in [Mat, §31].