Math 216A. A gluing construction of Proj(S)

1. Some basic definitions

Let $S = \bigoplus_{n \geq 0} S_n$ be an **N**-graded ring (we follows French terminology here, even though outside of France it is commonly accepted that **N** does not include 0). *Morphisms* between **N**-graded rings are understood to respect the grading. The *irrelevant ideal* is

$$S_+ = \bigoplus_{n>0} S_n;$$

keep in mind that we allow S_0 to have a nontrivial ideal theory (that is, it need not be a field). An element $f \in S$ is *homogeneous* if $f \in S_d$ for some d, and then d is unique if $f \neq 0$; we call d the *degree* of f (when $f \neq 0$), and we consider 0 as having arbitrary degree. Note that the equation

$$\deg(fg) = \deg(f) + \deg(g)$$

is valid even if one of f, g, or fg vanishes, using the convention that 0 may be considered to have arbitrary degree. For example, S_+ is exactly the set of elements (including 0) with positive degree.

For a general element $f \in S$, the *homogeneous parts* of f are the projections f_d of f into each S_d (so $f_d = 0$ for all but finitely many d).

An ideal I in S is *homogeneous* if an element $f = \sum_{n \geq 0} f_n$ of S lies in I if and only if each homogeneous part f_n lies in I. It is a simple exercise (inducting on degrees) to check that an ideal generated by homogeneous elements is a homogeneous ideal, and that homogeneous ideal I in S is *prime* if and only if it is a proper ideal and

$$fg \in I \Rightarrow f \in I \text{ or } g \in I$$

for homogeneous $f, g \in S$. It is also clear that the kernel of a morphism of **N**-graded rings is a homogeneous ideal, and that for any homogeneous ideal I of S there is a natural **N**-grading on S/I.

Definition 1.1. Let S be an **N**-graded ring. The topological space Proj(S) has underlying set

$$\mathrm{Proj}(S) = \{\mathfrak{p} \text{ a homogeneous prime such that } S_+ \not\subseteq \mathfrak{p}\},$$

and the closed sets are the loci $V(I) = \{\mathfrak{p} \in \mathrm{Proj}(S) \,|\, I \subseteq \mathfrak{p}\}$ for homogeneous ideals I of S (context will prevent confusion with the analogous "$V(I)$" notation for affine schemes).

It is easy to check that the $V(I)$'s do satisfy the axioms to define the closed sets for a topology on Proj(S) (the empty set is $V(S)$ and $\mathrm{Proj}(S) = V(0)$). A homogeneous prime $\mathfrak{p}$ fails to contain I if and only if there exists a homogeneous element $f \in I$ that does not lie in $\mathfrak{p}$ (here we use crucially that I is *homogeneous*). Thus, a base of open sets for the topology on Proj(S) is given by loci

$$D_+(f) = \{\mathfrak{p} \in \mathrm{Proj}(S) \,|\, f \notin \mathfrak{p}\} = \mathrm{Proj}(S) - V(fS)$$

for homogeneous $f \in S$. A crucial fact is that it is even enough to take f with *positive* degree:

Lemma 1.2. *A base of open sets for the topology on* Proj(S) *is given by loci* $D_+(f)$ *for homogeneous* $f \in S_+$.

Proof. Consider a homogeneous $f \in S$ and a point $\mathfrak{p} \in D_+(f)$. We need to find a homogeneous element $g \in S_+$ such that $\mathfrak{p} \in D_+(g) \subseteq D_+(f)$. Since $\mathfrak{p}$ does not contain S_+ (by the definition of Proj(S)), there exists $h \in S_+$ not in $\mathfrak{p}$. Thus, the condition $f \notin \mathfrak{p}$ implies $fh \notin \mathfrak{p}$ (so $fh \neq 0$), and fh is homogenous with positive degree since both f and h are homogeneous and $\deg h > 0$. We conclude that $\mathfrak{p} \in D_+(fh)$, and clearly $D_+(fh) \subseteq D_+(f)$. ∎

Beware that Proj(S) is generally *not* quasi-compact! For example, $\mathrm{Proj}(k[x_1, x_2, \dots])$ with infinitely many indeterminates of degree 1 is not quasi-compact, as it is covered by opens $D_+(x_i)$ and there is evidently no finite subcover (compare with the non-quasi-compact $\mathrm{Spec}(k[x_1, x_2, \dots]) - \{0\}$ and the quasi-compact $\mathrm{Spec}(k[x_1, x_2, \dots])$). This failure of quasi-compactness is best understood as follows:

Theorem 1.3. *For an* $\mathbf{N}$*-graded ring* S*,* $\mathrm{Proj}(S)$ *is empty if and only if all elements of* S_+ *are nilpotent. More generally, for* positive-degree *homogeneous elements* f *and* $\{f_i\}_{i\in I}$ *in* S*,* $D_+(f) \subseteq \cup D_+(f_i)$ *if and only if some power of* f *lies in the homogeneous ideal generated by the* f_i*'s.*

In particular, a collection of $D_+(f_i)$*'s with all* $\deg f_i > 0$ *covers* $\mathrm{Proj}(S)$ *if and only if every element of* S_+ *has some power lying in the homogeneous ideal generated by the* f_i*'s. So when* S_+ *is generated by finitely many homogenous elements (or equivalently, by homogeneity, is finitely generated as an ideal) then* $\mathrm{Proj}(S)$ *is quasi-compact. In particular, quasi-compactness holds whenever* S *is noetherian.*

The contrast with Spec is of course that $\mathrm{Spec}(A_{f_i})$'s cover $\mathrm{Spec}\,A$ if and only if the f_i's generate the unit ideal (and $\mathrm{Spec}(A)$ is *always* quasi-compact, even when A is highly non-noetherian). The interference of S_+ in the analogous covering criterion for Proj, coupled with the possibility that S_+ might not be finitely generated, is the reason why $\mathrm{Proj}(S)$ can fail to be quasi-compact. On the other hand, in most interesting situations the ideal S_+ is finitely generated and hence $\mathrm{Proj}(S)$ is quasi-compact. However, we note that some fundamental constructions of Mumford in the study of moduli of abelian varieties rest crucially on the use of non-quasi-compact Proj's.

Proof. Let I be the homogeneous ideal generated by the f_i's, so the complement of $\cup D_+(f_i)$ is the set of $\mathfrak{p} \in \mathrm{Proj}(S)$ that contain I. Hence, we need to determine when $D_+(f)$ is disjoint from the set of such $\mathfrak{p}$'s, or equivalently when every $\mathfrak{p} \in \mathrm{Proj}(S)$ that contains I also contains f; we want to show that this condition is exactly the condition that a power of f lies in I. Passing to the $\mathbf{N}$-graded S/I, we are reduced to proving that a homogeneous $f \in S_+$ lies in $\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Proj}(S)$ if and only if f is nilpotent; keep in mind that f has *positive* degree. One direction is obvious, and conversely we must prove that if $f \in S_+$ is homogeneous of degree $d > 0$ and f is not nilpotent, then there exists a homogeneous prime $\mathfrak{p}$ such that $f \notin \mathfrak{p}$ (and so $S_+ \not\subseteq \mathfrak{p}$ too, so $\mathfrak{p} \in \mathrm{Proj}(S)$).

We will make use of an auxiliary construction that will play an important role later. Let $S^{(d)} = \bigoplus_{n\geq 0} S_{dn}$ (so $S^{(d)} = S$ if $d = 1$). This is naturally an $\mathbf{N}$-graded ring with vanishing graded pieces in degrees not divisible by d. Consider the localized ring $(S^{(d)})_f$; since $(S^{(d)})_f = S^{(d)}[T]/(1 - Tf)$, by assigning T degree $-d$ we see that $(S^{(d)})_f$ naturally has a $\mathbf{Z}$-grading (with vanishing terms away from degrees divisible by d). For example, s/f^n is assigned degree $\deg(s) - nd$ for homogeneous elements $s \in S^{(d)}$.

Let $(S^{(d)})_{(f)} \subseteq (S^{(d)})_f$ denote the direct summand of degree-0 elements in the $\mathbf{Z}$-graded $(S^{(d)})_f$. This is a ring, and if f is not nilpotent in S then it is not nilpotent in $S^{(d)}$, so then $(S^{(d)})_f \neq 0$ and hence the subring $(S^{(d)})_{(f)}$ is nonzero. It then follows that there exists a prime ideal $\mathfrak{q}$ in $(S^{(d)})_{(f)}$. We will use this to construct a homogeneous prime $\mathfrak{p}$ in $S^{(d)}$ that does not contain f (and so in particular does not contain $S_+^{(d)}$ since $\deg f > 0$); the ideal generated by the *homogeneous* $a \in S$ such that $a^d \in S^{(d)}$ lies in $\mathfrak{p}$ is then readily checked to be a homogeneous prime ideal of S that does not contain f (this rests crucially on the fact that membership in the homogeneous $\mathfrak{p}$ may be checked on component-parts).

Let $\mathfrak{p}$ be the contraction of $\mathfrak{q}(S^{(d)})_f$ under $S^{(d)} \to (S^{(d)})_f$. The ideal $\mathfrak{p}$ of $S^{(d)}$ does not contain f, since otherwise $\mathfrak{q}(S^{(d)})_f$ would contain the degree-0 element 1, which is absurd since $(\mathfrak{q}(S^{(d)})_f) \cap (S^{(d)})_{(f)} = \mathfrak{q}$ is a proper ideal. To check that $\mathfrak{p}$ is homogeneous prime, first observe that (by contstruction) $\mathfrak{q}(S^{(d)})_f$ is a homogeneous ideal of the $\mathbf{Z}$-graded $(S^{(d)})_f$, so $\mathfrak{p}$ is a homogeneous ideal of the $\mathbf{N}$-graded $S^{(d)}$. Hence, to verify primality it is sufficient to work with homogeneous elements. That is, we consider homogeneous $a, a' \in S^{(d)}$ with respective degrees dn and dn' and we assume $aa' \in \mathfrak{p}$. Our goal is to prove $a \in \mathfrak{p}$ or $a' \in \mathfrak{p}$.

Since $aa' \in \mathfrak{p}$, the homogenous image of aa' in $(S^{(d)})_f$ is contained in $\mathfrak{q}(S^{(d)})_f$, so $aa' = (x/f^e)f^r$ with $r \in \mathbf{Z}$, $x \in S_{kd}$, and $x/f^e \in \mathfrak{q} \subseteq (S^{(d)})_{(f)}$. Thus, by comparing degrees we get $dn + dn' = dr$, so $n + n' = r$. Hence, $aa'/f^r = (a/f^n)(a'/f^{n'}) \in (S^{(d)})_f$ is a product of terms with degree 0. However,

$$\frac{a}{f^n}\frac{a'}{f^{n'}} = \frac{x}{f^e} \in (S^{(d)})_{(f)} \cap (\mathfrak{q}(S^{(d)})_f) = \mathfrak{q},$$

so by primality of $\mathfrak{q}$ in $(S^{(d)})_{(f)}$ we conclude that at least of the degree-0 elements a/f^n or $a'/f^{n'}$ lies in $\mathfrak{q}$! Hence, either a or a' in $S^{(d)}$ map into $\mathfrak{q}(S^{(d)})_f$ upon inverting f, so by definition either a or a' lie in $\mathfrak{p}$. ■

2. First steps towards a scheme structure

For homogeneous $f \in S_+$, we get an open set $D_+(f) \subseteq \mathrm{Proj}(S)$ consisting of those $\mathfrak{p} \in \mathrm{Proj}(S)$ that do not contain f. These are a base of open sets, and we claim that $D_+(f)$ is naturally homeomorphic to $\mathrm{Spec}\, S_{(f)}$, where $S_{(f)} \subseteq S_f$ is the degree-0 part of the $\mathbf{Z}$-graded localization of S at the homogeneous f.

To define a homeomorphism

$$\varphi : D_+(f) \to \mathrm{Spec}\, S_{(f)},$$

to each $\mathfrak{p} \in D_+(f)$ we associate the prime ideal

$$\mathfrak{p}_{(f)} = (\mathfrak{p}S_f) \cap S_{(f)} \in \mathrm{Spec}\, S_{(f)};$$

this is prime because it is the contraction of the prime $\mathfrak{p}S_f = \mathfrak{p}_f$ of S_f under the ring map $S_{(f)} \hookrightarrow S_f$ (note that $\mathfrak{p}_f$ is prime since $\mathfrak{p}$ is a prime of S not containing f).

Theorem 2.1. *The map $\varphi : D_+(f) \to \mathrm{Spec}(S_{(f)})$ is a homeomorphism.*

Proof. For any homogeneous ideal $\mathfrak{a}$ of S, we generalize the above operation on homogeneous prime ideals by defining

$$\varphi(\mathfrak{a}) = (\mathfrak{a}S_f) \cap S_{(f)}.$$

For any $\mathfrak{p} \in D_+(f)$, we claim

$$\varphi(\mathfrak{a}) \subseteq \varphi(\mathfrak{p}) \Leftrightarrow \mathfrak{a} \subseteq \mathfrak{p}. \tag{1}$$

Once this is proved, it will follow that φ is at least injective. The ($\Leftarrow$) implication is obvious, and for the converse it suffices to prove that if $a \in \mathfrak{a}$ is a homogeneous element then $a \in \mathfrak{p}$.

Let $n = \deg a \geq 0$ and let $d = \deg f > 0$. It follows that

$$\frac{a^d}{f^n} \in \mathfrak{a}S_f \cap S_{(f)} = \varphi(\mathfrak{a}) \subseteq \varphi(\mathfrak{p}) = \mathfrak{p}S_f \cap S_{(f)},$$

so there exists a homogeneous $x \in \mathfrak{p}$ such that $a^d/f^n = x/f^m$ in S_f with $md = \deg(x)$. Thus, for some $e \geq 0$ we have

$$f^e(f^m a^d - f^n x) = 0$$

in S, and since $f \notin \mathfrak{p}$ we must have $f^m a^d - f^n x \in \mathfrak{p}$. However, $x \in \mathfrak{p}$, so $f^m a^d \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, $f \notin \mathfrak{p}$, and d is *positive*, we conclude $a \in \mathfrak{p}$ as desired. This completes the proof of injectivity for φ.

Once we prove φ is surjective, and hence is bijective, (1) implies

$$\varphi(V(\mathfrak{a}) \cap D_+(f)) = V(\varphi(\mathfrak{a})).$$

Hence, for any ideal $\mathfrak{b}$ of $S_{(f)}$, the preimage $\mathfrak{a}$ of $\mathfrak{b}S_f$ in S is a homogeneous ideal satisfying $\varphi(\mathfrak{a}) = \mathfrak{b}$. We may therefore conclude that every closed set $V(\mathfrak{b})$ in $\mathrm{Spec}\, S_{(f)}$ corresponds (under the bijection φ) to a closed set $V(\mathfrak{a}) \cap D_+(f)$ in $D_+(f)$. However, all closed sets in $D_+(f)$ (with the subspace topology from $\mathrm{Proj}(S)$) have such a form for some $\mathfrak{a}$, so we thereby get that φ is a homeomorphism.

It remains to check that φ is surjective. A key observation is that the natural map

$$(S^{(d)})_{(f)} \to S_{(f)}$$

is an isomorphism. The basic idea is that a degree-0 element in $S_{(f)}$ must have the form x/f^n with homogeneous x of degree $\deg(x) = nd \in S_{nd}$, so x is in $S^{(d)}$; the straightforward details are left to the reader (hint: equality of subrings of S_f). Via this identification, any prime ideal of $S_{(f)}$ may be considered as a prime ideal in $(S^{(d)})_{(f)}$. However, in the proof of Theorem 1.3 it was proved (check!) that every prime ideal $\mathfrak{q}$ of $(S^{(d)})_{(f)}$ has the form $\varphi(\mathfrak{p})$ for some homogeneous prime $\mathfrak{p}$ of S not containing f (that is, for some $\mathfrak{p} \in D_+(f)$). ■

Let us now write $\varphi_f : D_+(f) \to \mathrm{Spec}(S_{(f)})$ to denote the homeomorphism constructed above, with $f \in S_+$ any *positive-degree* homogeneous element (so $\varphi_f(\mathfrak{p}) = \mathfrak{p}S_f \cap S_{(f)}$). We shall use this homeomorphism to endow $D_+(f)$ with a structure of *affine scheme*, using the structure sheaf on $\mathrm{Spec}(S_{(f)})$. In view of the fact that the $D_+(f)$'s form a base of opens in $\mathrm{Proj}(S)$, the key issue is to identify $S_{(f)}$ as the ring of sections on the open subset $D_+(f) \subseteq \operatorname{Proj}(S)$, and to this end it is useful to note that $S_{(f)}$ may be described entirely in terms of the subset $D_+(f) \subseteq \operatorname{Proj}(S)$ and the ring S without mentioning f:

Theorem 2.2. *For homogeneous $f \in S_+$, let T_f be the multiplicative set of homogeneous elements $g \in S$ such that $g \notin \mathfrak{p}$ for all $\mathfrak{p} \in D_+(f) \subseteq \operatorname{Proj}(S)$ (despite the notation, T_f only depends on $D_+(f)$ and not on f). The natural map*

$$S_{(f)} \to (T_f^{-1}S)_0$$

to the degree-0 part of the $\mathbf{Z}$-graded $T_f^{-1}S$ induced by $S_f \to T_f^{-1}S$ is an isomorphism.

Proof. Let $d = \deg f > 0$. For injectivity, suppose $x \in S$ is homogeneous of degree nd and the degree-0 element $x/f^n \in S_f$ maps to 0 in $T_f^{-1}S$. Hence, there exists $g \in T_f$ such that $gx = 0$ in S. Replacing g with g^d if necessary, we can assume $\deg g = md$. Thus,

$$(g/f^m)(x/f^n) = 0$$

in S_f, and hence this equality holds in $S_{(f)}$. By the definition of T_f, for all $\mathfrak{p} \in D_+(f)$ we have $g \notin \mathfrak{p}$, so g/f^m is not contained in the prime ideal $\varphi_f(\mathfrak{p}) = (\mathfrak{p}S_f) \cap S_{(f)}$ of $S_{(f)}$ (as $f, g \notin \mathfrak{p}$). But φ_f is bijective onto $\operatorname{Spec} S_{(f)}$, so $g/f^m \in S_{(f)}$ is not contained in any primes. It follows that $g/f^m \in S_{(f)}$ is a unit, so the vanishing of $(g/f^m)(x/f^n)$ in $S_{(f)}$ forces $x/f^n = 0$ in $S_{(f)}$. This gives exactly the desired injectivity.

Now choose $g \in T_f$ and $x \in S$ with $\deg(x) = \deg(g)$, so $x/g \in (T_f^{-1}S)_0$ makes sense. We seek a homogeneous $a \in S$ of some degree nd (for some $n \geq 0$) such that $a/f^n \in S_{(f)}$ maps to x/g in $T_f^{-1}S$. We may replace x with $g^{d-1}x$ and g with g^d to get to the case $\deg g = md$ for some $m \geq 0$. Thus, using the definition of T_f and the bijectivity of φ_f we see that $g/f^m \in S_{(f)}$ is not contained in any prime ideals, so it is a unit. In the degree-0 part of the $\mathbf{Z}$-graded $T_f^{-1}S$ we have

$$\frac{x}{g} = \frac{f^m}{g} \cdot \frac{g}{f^m} \cdot \frac{x}{g} = \frac{f^m}{g} \cdot \frac{x}{f^m},$$

so $(g/f^m)^{-1}(x/f^m) \in S_{(f)}$ maps to $x/g \in (T_f^{-1}S)_0$. This proves the desired surjectivity. ∎

3. A scheme structure on $\operatorname{Proj}(S)$

By Theorem 2.2, whenever $f, h \in S_+$ are homogeneous elements such that $D_+(h) \subseteq D_+(f)$ inside of $\operatorname{Proj}(S)$ we have (by the definitions!) $T_f \subseteq T_h$ inside S, and so we get a canonical map

$$S_{(f)} = (T_f^{-1}S)_0 \to (T_h^{-1}S)_0 = S_{(h)} \tag{2}$$

on degree-0 parts induced by the map $T_f^{-1}S \to T_h^{-1}S$ of $\mathbf{Z}$-graded localizations. We may therefore consider the diagram of topological spaces

$$\begin{array}{ccc} D_+(f) & \xrightarrow[\simeq]{\varphi_f} & \operatorname{Spec}((T_f^{-1}S)_0) \\ \uparrow & & \uparrow \\ D_+(h) & \xrightarrow[\varphi_h]{\simeq} & \operatorname{Spec}((T_h^{-1}S)_0) \end{array}$$

where the left column is the inclusion within $\operatorname{Proj}(S)$. One readily checks (upon reviewing the definitions of the various maps) that this diagram commutes, with the right side an open embedding, ultimately because the canonical equality

$$(S_{(f)})_{h^{\deg f}/f^{\deg h}} = S_{(fh)} = (S_{(h)})_{f^{\deg h}/h^{\deg f}}$$

inside of S_{fh} (check!) and the fact that $f^{\deg h}/h^{\deg f} \in S_{(h)}^{\times}$ (since $f \in T_f \subseteq T_h$) together imply that (2) induces an isomorphism $(S_{(f)})_{h^{\deg f}/f^{\deg h}} \simeq S_{(h)}$.

Clearly $D_+(f) \cap D_+(g) = D_+(fg)$, and by taking $h = fg$ above we see that this open subset of $D_+(f)$ is carried by φ_f onto the open subset

$$\operatorname{Spec}((S_{(f)})_{g^{\deg f}/f^{\deg g}}) \subseteq \operatorname{Spec}(S_{(f)}).$$

Likewise, as an open subset of $D_+(g)$ it is carried by φ_g onto the open subset

$$\mathrm{Spec}((S_{(g)})_{f^{\deg g}/g^{\deg f}}) \subseteq \mathrm{Spec}(S_{(g)}).$$

We now have put three scheme structures on $D_+(fg)$, namely $\mathrm{Spec}\, S_{(fg)}$ and the two as basic opens in $\mathrm{Spec}\, S_{(f)}$ and in $\mathrm{Spec}\, S_{(g)}$. These three structures are identified by means of the ring isomorphisms

$$(S_{(f)})_{g^{\deg f}/f^{\deg g}} \simeq S_{(fg)} \simeq (S_{(g)})_{f^{\deg g}/g^{\deg f}} \tag{3}$$

that are really *equalities* as subrings of S_{fg}. Consequently, the cocycle condition for gluing is satisfied (it comes down to transitivity for equality among three subrings of $S_{(fgh)}$ for any three homogeneous $f, g, h \in S_+$), so we may glue the structure sheaves $\mathscr{O}_{\mathrm{Spec}(S_{(f)})}$ over the $D_+(f)$'s via (3). That is, we are gluing the $\mathrm{Spec}\, S_{(f)}$'s (as ringed spaces) along the $\mathrm{Spec}\, S_{(fg)}$'s, where the underlying topological space $\mathrm{Proj}(S)$ of the gluing was made at the start.

The glued structure sheaf over $P = \mathrm{Proj}(S)$ will be denoted $\mathscr{O}_P$, and so the ringed space $(P, \mathscr{O}_P)$ is covered by open subspaces

$$(D_+(f), \mathscr{O}_P|_{D_+(f)}) \simeq \mathrm{Spec}(S_{(f)})$$

for homogeneous $f \in S_+$. Hence, $(P, \mathscr{O}_P)$ is a *scheme.*

Definition 3.1. Let S be an $\mathbf{N}$-graded ring. The scheme $\mathrm{Proj}(S)$ is $(P, \mathscr{O}_P)$ where P is the topological space denoted $\mathrm{Proj}(S)$ above and $\mathscr{O}_P$ is the sheaf of rings on P whose restriction to $D_+(f)$ is $\mathscr{O}_{\mathrm{Spec}(S_{(f)})}$ (using φ_f) for all homogeneous $f \in S_+$, with the overlap-gluing isomorphism

$$\mathscr{O}_{\mathrm{Spec}((S_{(f)})_{g^{\deg f}/f^{\deg g}})} = \mathscr{O}_{\mathrm{Spec}(S_{(f)})}|_{D_+(f)\cap D_+(g)} \simeq \mathscr{O}_{\mathrm{Spec}(S_{(g)})}|_{D_+(g)\cap D_+(f)} = \mathscr{O}_{\mathrm{Spec}((S_{(g)})_{f^{\deg g}/g^{\deg f}})}$$

defined by the isomorphism $\mathrm{Spec}((S_{(f)})_{g^{\deg f}/f^{\deg g}}) \simeq \mathrm{Spec}((S_{(g)})_{f^{\deg g}/g^{\deg f}})$ arising from the canonical ring isomorphism in (3) for homogeneous $f, g \in S_+$.

By Theorem 1.3, we obtain a useful alternative description:

Corollary 3.2. *Let $\{f_i\}$ be a collection of homogeneous elements in S_+ such that every element of S_+ has some power contained in the ideal generated by the f_i's. The scheme* $\mathrm{Proj}(S)$ *is obtained by gluing the affine schemes* $\mathrm{Spec}(S_{(f_i)})$ *along the open affine overlaps* $\mathrm{Spec}(S_{(f_if_j)}) \hookrightarrow \mathrm{Spec}(S_{(f_i)})$ *defined by the isomorphisms*

$$S_{(f_if_j)} \simeq (S_{(f_i)})_{f_j^{\deg f_i}/f_i^{\deg f_j}}.$$

Remark 3.3. We emphasize that there is content in this construction, namely that the above ring isomorphisms satisfy "triple overlap" compatibility; this is most painlessly seen in terms of a triple equality of *subrings* of $S_{(f_if_jf_k)}$.

Example 3.4. Let $S = A[X_0, \dots, X_n]$ be an $\mathbf{N}$-graded ring by putting A in degree 0 and declaring each X_i to be homogeneous of degree 1. It follows that $\mathrm{Proj}(S)$ is covered by the opens

$$D_+(X_i) = \mathrm{Spec}\, S_{(X_i)} = \mathrm{Spec}\, A[X_0/X_i, \dots, X_n/X_i]$$

for $0 \le i \le n$, and the gluing isomorphism is determined by the isomorphism

$$(S_{(X_i)})_{X_j/X_i} \simeq (S_{(X_j)})_{X_i/X_j}$$

defined by $X_k/X_i \mapsto (X_k/X_j) \cdot (X_i/X_j)^{-1}$ for $k \neq i$. These are exactly the standard formulas that express projective n-space as the gluing of $n+1$ copies of affine n-space along certain open overlaps defined by non-vanishing of various coordinate functions.

Inspired by the above example, for any ring A we define *projective n-space* over A to be

$$\mathbf{P}^n_A = \mathrm{Proj}(A[X_0, \dots, X_n])$$

with the usual grading on $A[X_0, \dots, X_n]$. This is naturally a scheme over $\mathrm{Spec}\, A$ since each basic open affine $D_+(f)$ is naturally an A-scheme (as A has degree 0 in the $\mathbf{N}$-grading being used) and the open-affine gluing data is one of A-algebras (more generally, $\mathrm{Proj}(S)$ is always naturally a scheme over $\mathrm{Spec}\, S_0$). As a particularly degenerate example, we have $\mathbf{P}^0_A = \mathrm{Spec}\, A[X_0]_{(X_0)} = \mathrm{Spec}\, A$.

Example 3.5. If we assign $A[X_0, \dots, X_n]$ an $\mathbf{N}$-graded structure by putting A in degree 0 and assigning X_i some positive degree d_i, the resulting $\mathbf{N}$-graded rings are generally *not* isomorphic as $\mathbf{N}$-graded rings for different n-tuples $\mathbf{d} = (d_0, \dots, d_n)$, and their A-scheme Proj's (called *weighted projective n-spaces over A* with weights $\mathbf{d}$) are generally *not* isomorphic to each other.

These weighted projective spaces over an algebraically closed field are generally not "smooth" (when $n > 1$ and some $d_j > 1$ with $\gcd(d_1, \dots, d_n) = 1$). For example, consider the polynomial ring $S = k[X, Y, Z]$ in which we declare each variable to be homogenous with respective degrees $\deg(X) = 2$, $\deg(Y) = 3$, $\deg(Z) = 4$. Then $\mathrm{Proj}(S)$ contains the affine open $D_+(Z) = \mathrm{Spec}(S_{(Z)})$ with

$$S_{(Z)} = k[X^2/Z, (XY^2)/Z^2, Y^4/Z^3]$$

(please verify this equality of subrings of S_Z via inductive chasing of degrees of numerators in suitable fractions), and if we label the three indicated generators of the k-algebra $S_{(Z)}$ as U, V, W respectively then clearly $UW = V^2$. This defines a surjective k-algebra map $\pi : k[u, v, w]/(uv - w^2) \to S_{(Z)}$ that we'll show is an isomorphism. Hence, $D_+(Z)$ has a singularity at the point $(u, v, w) = (0, 0, 0)$ corresponding to $[0, 0, 1] \in \mathrm{Proj}(S)$.

Let's explain why the map π is between *domains* is an isomorphism. Its source ring has dimension 2 and its target has dimension *at least* 2 (since it contains the elements X^2/Z and Y^4/Z^3 that are readily seen to be algebraically independent over k – why?). Thus, since the kernel of any surjection between domains must be a prime ideal, the good behavior of dimension theory for domains finitely generated over a field forces π to have kernel (0), so π is indeed an isomorphism.

4. Functoriality, and lack thereof

The condition in the definition of $\mathrm{Proj}(S)$ that the prime ideal doesn't contain the irrelevant ideal causes some complications when one tries to make the scheme $\mathrm{Proj}(S)$ (or even its underlying set) be reasonably functorial in the $\mathbf{N}$-graded ring S. To see the difficulty, suppose $\varphi : S' \to S$ is a map of $\mathbf{N}$-graded rings and $\mathfrak{p} \in \mathrm{Proj}(S)$. The prime ideal $\varphi^{-1}(\mathfrak{p}) \subset S'$ is certainly homogenous, but does it correspond to a point in $\mathrm{Proj}(S')$? The issue is that perhaps $S'_+ \subseteq \varphi^{-1}(\mathfrak{p})$, or equivalently $\varphi(S'_+) \subseteq \mathfrak{p}$. By hypothesis $\mathfrak{p}$ doesn't contain S_+, but perhaps S_+ is so much larger than the (homogeneous!) ideal generated by $\varphi(S'_+)$ that there can exist homogenous primes $\mathfrak{p}$ of S containing $\varphi(S'_+)$ (or equivalently $\varphi(S'_+)S$ but not S_+. The work in this section takes care of Exercises 2.14 and 3.12 in [H, Ch. II].

Define the open set $U = \mathrm{Proj}(S) - V(\varphi(S'_+)S) \subseteq \mathrm{Proj}(S)$, so when $\mathfrak{p} \in U$ the above difficulty does not arise. Hence, we get a well-defined map of sets $f : U \to \mathrm{Proj}(S')$ via $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$. This is continuous:

Lemma 4.1. *The natural map $f : U \to \mathrm{Proj}(S')$ satisfies $f^{-1}(D_+(s')) = U \cap D_+(\varphi(s'))$ for homogeneous $s' \in S'_+$. In particular, f is continuous.*

Proof. For $\mathfrak{p} \in U$, we have $f(\mathfrak{p}) \in D_+(s')$ if and only if $s' \notin f(\mathfrak{p}) := \varphi^{-1}(\mathfrak{p})$, which is equivalent to saying $\varphi(s') \notin \mathfrak{p}$. This final condition says $\mathfrak{p} \in D_+(\varphi(s'))$. ■

We will enhance f to a scheme morphism shortly but let's first discuss some cases when $U = \mathrm{Proj}(S)$:

Lemma 4.2. *If there exists an integer $d \geq 1$ so that $S'_n \to S_n$ is surjective for all sufficiently large $n \in d\mathbf{N}$ then $U = \mathrm{Proj}(S)$. If $S'_n \to S_n$ is even bijective for all sufficiently large $n \in d\mathbf{N}$ then the map $f : \mathrm{Proj}(S) = U \to \mathrm{Proj}(S')$ is a homeomorphism.*

The hypothesis of the first part of this lemma holds when $S'_n \to S$ is surjective for all large n (the case $d = 1$) and also when $S' \subset S$ is the $\mathbf{N}$-graded subring $S^{(d)} := \bigoplus_{r \geq 0} S_{rd}$.

Proof. It suffices to show that if $\mathfrak{p}$ is a homogeneous prime ideal of S containing $\varphi(S'_+)$ then $\mathfrak{p}$ contains S_+ (so if $\mathfrak{p} \in \mathrm{Proj}(S)$ then $\varphi^{-1}(\mathfrak{p})$ doesn't contain S'_+). Under the hypothesis on φ, $\mathfrak{p}$ contains S_{rd} for all large r. For any homogenous $a \in S_+$, say with degree $n \geq 1$, we have $a^{rd} \in S_{nrd}$ for all $r \geq 1$, so by taking r to be large we have $a^{rd} \in S_{nrd} \subset \mathfrak{p}$. By primality of $\mathfrak{p}$, it follows that $a \in \mathfrak{p}$.

Now suppose $S'_n \to S_n$ is bijective for all large $n \in d\mathbf{N}$. To show that $\mathrm{Proj}(S) \to \mathrm{Proj}(S')$ is a homeomorphism, it suffices to first treat the case of $S' = S^{(d)}$, and then applying that to ${S'}^{(d)} \to S'$ and $S^{(d)} \to S$

would reduce our task to the case $d = 1$ upon dividing degrees by d for $S^{(d)}$ and ${S'}^{(d)}$ (i.e., φ is bijective in all large degrees). For the case $S' = S^{(d)}$, consider general homogeneous $b \in S^+$, say with degree r. Then we get $b^d \in S_{rd} = (S^{(d)})_{rd}$, and these homogeneous elements of $S_+^{(d)}$ may not generate the irrelevant ideal of $S^{(d)}$ but the associated affine opens $\mathrm{Spec}((S^{(d)})_{(f^d)})$ *cover* $\mathrm{Proj}(S^{(d)})$ because of Corollary 3.2. Moreover, we have subrings

$$S^{(d)}_{(b^d)} \subset S^{(d)}_{b^d} \subset S_{b^d} = S_b, \;\; S_{(b)} \subset S_b$$

which *coincide* as subrings: it is clear that $S^{(d)}_{(b^d)} \subset S_{(b)}$, and the reverse inclusion amounts to writing any "degree 0" fraction a/b^m with homogenous a (of degree mr) as $(ab^{dn-m})/b^{dn}$ for any multiple $dn \geq m$. But a direct check via working with fractions (please do it!) shows that the map $\mathrm{Proj}(S) \to \mathrm{Proj}(S')$ carries $D_+(b)$ into $D_+(b^d)$ via the map of affine schemes arising from the ring isomorphism $S^{(d)}_{(b^d)} \simeq S_{(b)}$. But the *entire preimage* of $D_+(b^d)$ is exactly $D_+(b)$ since a homogeneous prime $\mathfrak{p}$ of S fails to contain b precisely when it fails to contain b^d, and that in turn says that the homoegenous prime $\mathfrak{p} \cap S^{(d)}$ of $S^{(d)}$ fails to contains b^d. Hence, $\mathrm{Proj}(S) \to \mathrm{Proj}(S')$ is a homeomorphism since it restricts to one over each member $D_+(b^d)$ of an open cover of $\mathrm{Proj}(S')$.

As explained already, it now remains to treat the case $d = 1$: $S'_n \to S_n$ is bijective for all $n \geq n_0$. This case goes almost exactly like the case of $S^{(d)} \hookrightarrow S$ just treated, but we don't need to raise to any powers: instead, we take the denominators to come from homogeneous elements with degree at least n_0 (the associated affine open subschemes of each Proj provide an open cover, again by Corollary 3.2). ∎

Example 4.3. If $I \subset S$ is a homogenous ideal, then $S \to S/I$ is surjective in all degrees and so we get a well-defined map of sets $\mathrm{Proj}(S/I) \to \mathrm{Proj}(S)$. If we pick an integer $m > 0$ and let $J = \bigoplus_{n \geq m} I_n$ be the ideal inside I generated by homogeneous parts in degree at least m then $S/J \to S/I$ is an isomorphism in all degrees at least m; hence, $\mathrm{Proj}(S/I) \to \mathrm{Proj}(S/J)$ is a homeomorphism. Likewise, the $\mathbf{N}$-graded subring $S^{(d)} \subseteq S$ has the same degree-rd part for all r, so we get a homeomorphism $\mathrm{Proj}(S) \to \mathrm{Proj}(S^{(d)})$.

Let's now upgrade to ringed spaces. For any map of $\mathbf{N}$-graded rings $\varphi : S' \to S$, we want to promote the continuous map $f : U = \mathrm{Proj}(S) - V(\varphi(S'_+)S) \to \mathrm{Proj}(S')$ to a map of ringed spaces. The open subset U is covered by open subsets $D_+(\varphi(b'))$ for homogeneous $b' \in S'_+$ (check!), and for such b' there is a natural map of affine schemes

$$D_+(\varphi(b')) = \mathrm{Spec}(S_{(\varphi(b'))}) \to \mathrm{Spec}(S'_{b'}) = D_+(b')$$

corresponding to the ring map $S'_{(b')} \to S_{(\varphi(b')}$ induced on degree-0 parts by the $\mathbf{Z}$-graded map $\varphi : S'_{b'} \to S_{\varphi(b')}$. These morphisms agree on overlaps $D_+(\varphi(b')) \cap D_+(\varphi(c')) = D_+(\varphi(b')\varphi(c')) = D_+(\varphi(b'c'))$ ultimately because the ring maps $S'_{b'} \to S_{\varphi(b')}$ and $S'_{c'} \to S_{\varphi(c')}$ arising from $\varphi : S' \to S$ each induce upon further localization the *same* ring map $S'_{b'c'} \to S_{\varphi(b'c')}$ arising from φ. Hence, they glue to define a morphism $f : U \to \mathrm{Proj}(S')$, and on underlying sets this really is $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$, ultimately because the map $\mathrm{Spec}(\varphi) : \mathrm{Spec}(S_{(\varphi(b')}) \to \mathrm{Spec}(S'_{(b')})$ is also given on underlying sets by the usual recipe of preimage of primes under the ring map $S'_{(b')} \to S_{(\varphi(b')}$ induced in degree 0 by the $\mathbf{Z}$-graded ring map $\varphi : S'_{b'} \to S_{\varphi(b')}$.

In the setting of Lemma 4.2 we have a morphism $f : \mathrm{Proj}(S) = U \to \mathrm{Proj}(S')$ whose restriction over each open affine $D_+(b')$ for homogeneous $b' \in S'_+$ is exactly $D_+(\varphi(b')) = f^{-1}(D_+(b')) \to D_+(b')$ corresponding to $\varphi : S'_{(b')} \to S_{(\varphi(b')}$. The ring map is surjective (resp. an isomorphism) when $S'_n \to S_n$ is surjective (resp. an isomorphism) in all large degrees divisible by d. To summarize, we have shown:

Theorem 4.4. *Under the hypotheses of Lemma* 4.2*, the associated morphism* $\mathrm{Proj}(S) \to \mathrm{Proj}(S')$ *is a closed immersion. It is an isomorphism when* $S'_n \to S_n$ *is bijective for all large* $n \in d\mathbf{N}$*.*

In the setting of Example 4.3 we have the surjection $S/J \to S/I$ that is bijective in all large degrees, so the Theorem gives that the associated morphism $\mathrm{Proj}(S/I) \to \mathrm{Proj}(S/J)$ is an isomorphism of schemes. Likewise, for any $d \geq 1$, the associated morphism $\mathrm{Proj}(S) \to \mathrm{Proj}(S^{(d)})$ is an isomorphism of schemes (a big improvement on the homeomorphism conclude at the end of Example 4.3).