# Math 216A. From modules to sheaves

## 1. The affine case

For a commutative ring $A$ with spectrum $X = \mathrm{Spec}(A)$, in class we defined the functor $M \rightsquigarrow \widetilde{M}$ from $A$-modules to $\mathscr{O}_X$-modules, and we saw that for $x = \mathfrak{p} \in X$ there is an isomorphism $\widetilde{M}_x \simeq M_{\mathfrak{p}}$ of modules over $\widetilde{A}_x = \mathscr{O}_{X,x} = A_{\mathfrak{p}}$ naturally in $M$. In particular, a diagram

$$0 \to M' \to M \to M'' \to 0$$

of $A$-modules is a short exact sequence if and only if the associated diagram of sheaves

$$0 \to \widetilde{M'} \to \widetilde{M} \to \widetilde{M''} \to 0$$

is short exact since in both cases short-exactness is equivalent to the same after compatibly passing to module localizations at primes of $A$ and sheaf stalks at points of $X$.

It was also noted that for any $\mathscr{O}_X$-module $\mathscr{F}$, there is a natural isomorphism

$$\mathrm{Hom}_A(M, \mathscr{F}(X)) \simeq \mathrm{Hom}_{\mathscr{O}_X}(\widetilde{M}, \mathscr{F}). \tag{1}$$

Concretely, this says that any $A$-linear map $M \to \mathscr{F}(X)$ arises as the map of global sections for a unique $\mathscr{O}_X$-linear map $\widetilde{M} \to \mathscr{F}$.

The following result records the basic properties of the $\widetilde{M}$-construction.

**Theorem 1.1.** *Let $f : Y = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = X$ be a map of affine schemes.*

(i) *The functor $M \rightsquigarrow \widetilde{M}$ from the category of $A$-modules to the category of $\mathscr{O}_X$-modules is fully faithful (i.e., "same morphisms") and exact.*

(ii) *For $A$-modules $M$ and $M'$, the natural map $(M \otimes_A M')^{\sim} \to \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{M'}$ (associated to the $A$-linear map $M \otimes_A M' \to \widetilde{M}(X) \otimes_A \widetilde{M'}(X) \to (\widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{M'})(X)$ on global sections) is an isomorphism.*

(iii) *For a directed system $\{M_i\}$ of $A$-modules, the natural map*

$$(\varinjlim M_i)^{\sim} \to \varinjlim \widetilde{M_i}$$

*(associated to the $A$-linear $\varinjlim M_i = \varinjlim \widetilde{M_i}(X) \to (\varinjlim \widetilde{M_i})(X)$ on global sections) is an isomorphism. In particular, the functor $M \rightsquigarrow \widetilde{M}$ naturally commutes with arbtitrary direct sums (as they are a direct limit of finite direct sums).*

(iv) *For any $B$-module $N$, with associated "underlying" $A$-module denoted ${}_AN$, the natural map of $\mathscr{O}_X$-modules $\widetilde{{}_AN} \to f_*(\widetilde{N})$ (associated to the $A$-linear map ${}_AN = N = \Gamma(\mathrm{Spec}(B), \widetilde{N}) = \Gamma(\mathrm{Spec}(A), f_*(\widetilde{N}))$ on global sections) is an isomorphism.*

(v) *For any $A$-module $M$, the natural map of $\mathscr{O}_Y$-modules*

$$(B \otimes_A M)^{\sim} \to f^*(\widetilde{M})$$

*(associated to the $B$-linear map $B \otimes_A M \to B \otimes_A \widetilde{M}(X) = \mathscr{O}_Y(Y) \otimes_{\mathscr{O}_X(X)} \widetilde{M}(X) \to (f^*(\widetilde{M}))(X)$ on global sections) is an isomorphism.*

(vi) *If $M$ and $M'$ are $A$-modules with $M$ finitely presented (i.e., $M$ is a cokernel of a map $A^{\oplus r} \to A^{\oplus s}$ between finite free modules, as is automatic when $A$ is noetherian and $M$ is finitely generated) then the natural map*

$$\mathrm{Hom}_A(M, M')^{\sim} \to \mathscr{H}om_{\mathscr{O}_X}(\widetilde{M}, \widetilde{M'})$$

*(associated to the $A$-linear map $\mathrm{Hom}_A(M, M') \to \mathrm{Hom}_{\mathscr{O}_X}(\widetilde{M}, \widetilde{M'})$ on global sections arising from functoriality of the $\widetilde{(\cdot)}$ construction) is an isomorphism.*

*Proof.* For (i), the exactness has already been noted via passages to stalks and module localization at primes, and faithfulness (injectivity on Hom's) is immediate via the natural identification $M \simeq \widetilde{M}(X)$. For fullness (surjectivity on Hom's), pick an $\mathscr{O}_X$-linear map $\varphi : \widetilde{M} \to \widetilde{M'}$ and let $h : M \to M'$ be the induced $A$-linear map on global sections. We want $\varphi = \widetilde{h}$. This is a comparison of $\mathscr{O}_X$-linear maps $\widetilde{M} \rightrightarrows \widetilde{M'}$. But such equality of maps holds on global sections by design, so it holds as sheaf maps due to the general equality (1). Thus, (i) is proved.

For (ii), (iii), and (v) we can pass to stalks and thereby verify the desired isomorphism property easily due to the fact that sheafification has no effect on stalks (recall that direct limits and tensor products and sheaf-pullback are made as sheafifications of corresponding presheaf constructions).

For (iv) it suffices to check we get an isomorphism on the sets of sections over a base of opens, such as over each $D(a) \subset \mathrm{Spec}(A)$. If $b \in B$ is the image of $a \in A$ under $f^{\#} : A \to B$ then the map on $D(a)$-sections is $({}_AN)_a \to \Gamma(\mathrm{Spec}(B_b, \widetilde{N}) = N_b$, and this is the natural identification of localizations of $N$ at $a \in A$ as an $A$-module and at $b = f^{\#}(a) \in B$ as a $B$-module (think in terms of equivalence classes of fractions, for example).

Finally, it remains to prove (vi). For this we will use a version of the "finite presentation trick" as arises when showing that $\mathrm{Hom}_A(M, \cdot)$ commutes with localization when $M$ is finitely presented.

The source and target for the map in (vi) commute with *finite* direct sums in $M$, and the case $M = A$ is obvious (the two sides compatibly identify with $M'$ since $\mathrm{Hom}_A(A, \cdot)$ and $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X, \cdot)$ are the respective identity functors on $A$-modules and $\mathscr{O}_X$-modules). Thus, the case $M = A^{\oplus n}$ is verified for all $n$. Now we bootstrap from that via a choice of right-exact presentation

$$A^{\oplus r} \to A^{\oplus s} \to M \to 0 \tag{2}$$

and the associated right-exact (!) sequence

$$\mathscr{O}_X^{\oplus r} \to \mathscr{O}_X^{\oplus s} \to \widetilde{M} \to 0. \tag{3}$$

We apply $\mathrm{Hom}_A(\cdot, M')$ to (2) to get a left-exact sequence of $A$-modules, which yields a left-exact sequence of sheaves after applying the exact functor $\widetilde{(\cdot)}$, and we apply $\mathscr{H}om_{\mathscr{O}_X}(\cdot, \widetilde{M'})$ to (3) to get a left-exact (check!) sequence of $\mathscr{O}_X$-modules. These two resulting right-exact sheaf diagrams fit into a commutative diagram via the naturality of the map under consideration for (vi): the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_A(M, M')^{\sim} \arrow[r] \arrow[d] & \mathrm{Hom}_A(A^{\oplus s}, M')^{\sim} \arrow[r] \arrow[d] & \mathrm{Hom}_A(A^{\oplus r}, M')^{\sim} \arrow[d] \\ 0 \arrow[r] & \mathscr{H}om_{\mathscr{O}_X}(\widetilde{M}, \widetilde{M'}) \arrow[r] & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X^{\oplus s}, \widetilde{M'}) \arrow[r] & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X^{\oplus r}, \widetilde{M'}) \end{tikzcd}$$

commutes. In this diagram, the second and third vertical arrows are isomorphisms by the settled case of $A^{\oplus n}$'s, so the first arrow induced between the kernels along the top and bottom must be an isomorphism (check!). ■

## 2. Globalization

For a general scheme $X$ and $\mathscr{O}_X$-module $\mathscr{F}$, we want to consider the condition that on "enough" or "all" affine opens $U = \mathrm{Spec}(A) \subset X$, the restriction $\mathscr{F}|_U$ arises from an $A$-module. The possible ways to make this precise are equivalent:

**Theorem 2.1.** *The following are equivalent:*

(i) *For some open cover $\{U_i\}$ of $X$, each $\mathscr{F}|_{U_i}$ is a cokernel of a map $\mathscr{O}_{U_i}^{\oplus I_i} \to \mathscr{O}_{U_i}^{\oplus J_i}$ for some index sets $I_i, J_i$.*

(ii) *There is an affine open cover $\{U_i = \mathrm{Spec}(A_i)\}$ of $X$ such that $\mathscr{F}|_{U_i} \simeq \widetilde{M_i}$ for some $A_i$-module $M_i$ for all $i$.*

(iii) *For every affine open $U = \mathrm{Spec}(A) \subset X$, $\mathscr{F}|_U \simeq \widetilde{M}$ for some $A$-module $M$.*

*Proof.* Certainly (iii) implies (i) (take $U_i$ to vary over all affine opens, and build the cokernel presentation from a free module presentation for each $M_i$ as a cokernel of some map $F_i' \to F_i$ between free $A_i$-modules, remembering that the functor $\widetilde{(\cdot)}$ is exact). Also, (i) implies (ii) by refining the cover in (i) to an affine open cover so we arrange each $U_i = \mathrm{Spec}(A_i)$ is affine, and then use the full faithfulness and compatibility with arbitrary direct sums for $\widetilde{(\cdot)}$ to get that the map $\mathscr{O}_{U_i}^{\oplus I_i} \to \mathscr{O}_{U_i}^{\oplus J_i}$ arises from a linear map $A_i^{\oplus I_i} \to A_i^{\oplus J_i}$. The cokernel $M_i$ of this latter map then satisfies $\widetilde{M_i} \simeq \mathscr{F}|_{U_i}$ by exactness of $\widetilde{(\cdot)}$.

Finally, the real work is to show (ii) implies (iii). Since $\widetilde{M_i}|_{D(a_i)} \simeq \widetilde{(M_i)_{a_i}}$ for any $a_i \in A_i$, by the Nike trick and quasi-compactness of $\mathrm{Spec}(A)$ we can reduce to the situation that $X = \mathrm{Spec}(A)$ is affine and covered by finitely many basic affine opens $U_1 = \mathrm{Spec}(A_{a_1}), \dots, U_n = \mathrm{Spec}(A_{a_n})$ with $\mathscr{F}|_{U_j} \simeq \widetilde{M_j}$ for some $A_{a_j}$-module $M_j$ for $j = 1, \dots, n$. The aim is to show $\mathscr{F} \simeq \widetilde{M}$ for some $A$-module $M$. Actually, there is no mystery what $M$ has to be: we have to use $M = \mathscr{F}(X)$ (since $\widetilde{N}(X) = N$ for any $A$-module $N$), and we want to show that the *natural* map

$$\varphi_{\mathscr{F}} : \widetilde{M} = \mathscr{F}(X)^{\sim} \to \mathscr{F} \tag{4}$$

(corresponding to the identity map $M \to \mathscr{F}(X)$ on global sections) is an isomorphism.

The merit of studying a canonical map like $\varphi_{\mathscr{F}}$ is that we're not trying to build global things but rather prove *properties* about maps we already have in hand; this is generally a simpler thing to do. To show (4) is an isomorphism over $X$, it suffices to show it is an isomorphism of sheaves after restriction over each member $U_j = \mathrm{Spec}(A_{a_j})$ of an open cover of $X$. We know that each $\varphi|_{\mathscr{F}|_{U_j}}$ is an isomorphism since $\mathscr{F}|_{U_j} \simeq \widetilde{M_j}$ actually arises from an $A_j$-module (and the construction in (4) is an isomorphism for the output of the $\widetilde{(\cdot)}$ construction for modules over any ring). Hence, everything comes down to showing that the formation of $\varphi_{\mathscr{F}}$ is suitably compatible with restriction over basic affine opens $D(a)$ for $a \in \{a_1, \ldots, a_n\}$.

To be precise, for any $a \in A$, via the natural map $\mathscr{F}(X) \to \mathscr{F}(D(a))$ over $A \to A_a$, we get a map of $A_a$-modules $\theta_a : \mathscr{F}(X)_a \to \mathscr{F}(D(a))$, and so it makes sense to form the diagram

$$\begin{tikzcd} (\mathscr{F}(X)_a)^{\sim} \arrow[r] \arrow[dr] & \mathscr{F}(D(a))^{\sim} \arrow[d] \\ & \mathscr{F}|_{D(a)} \end{tikzcd}$$

comparing $\varphi_{\mathscr{F}}|_{D(a)}$ and $\varphi_{(\mathscr{F}|_{D(a)})}$. This diagram *commutes* because the two routes from upper-left to lower-right amount to two maps of the form $\widetilde{N} \to \mathscr{F}|_{D(a)}$ for a common $A_a$-module $N$, and checking the equality of such maps can be done on global sections, where it is clear by the way the top horizontal map was *defined*.

Hence, our sheaf isomorphism problem reduces to this: is $\theta_a = \theta_{\mathscr{F},a} : \mathscr{F}(X)_a \to \mathscr{F}(D(a))$ an isomorphism for $a \in \{a_1, \ldots, a_n\}$? In fact, we'll show it is an isomorphism for every $a \in A$. The key is to use the left-exact sequence

$$0 \to \mathscr{F}(X) \to \prod \mathscr{F}(D(a_i)) \to \prod \mathscr{F}(D(a_i a_j))$$

coming from $\mathscr{F}$ being a sheaf. The direct products are over *finite* index sets, so they commute with localization at $a$. Likewise we get the left-exact sequence

$$0 \to \mathscr{F}(X)_a \to \prod_i \mathscr{F}(D(a_i))_a \to \prod_{i,j} \mathscr{F}(D(a_i a_j))_a. \tag{5}$$

For any ring $B$ and $B$-module $N$, the natural map $\widetilde{N}(\mathrm{Spec}(B))_b \to \widetilde{N}(D(b))$ is an isomorphism since both sides are compatibly identified with $N_b$, so applying this to $B = A_i$ and the sheaf $\mathscr{F}|_{D(a_i)} \simeq \widetilde{M_i}$ we see that the natural map $\theta_{\mathscr{F}|_{D(a_i)},a/1} : \mathscr{F}(D(a_i))_a \to \mathscr{F}(D(a_i a))$ is an isomorphism (viewing $a$ as $a/1 \in A_{a_i}$). In this way the right map in (5) is identified with

$$\prod_i \mathscr{F}(D(a_i a)) \to \prod_{i,j} \mathscr{F}(D(a_i a_j a)),$$

which is the "overlap map" associated to the sheaf $\mathscr{F}|_{D(a)}$ and open cover of $D(a) = \mathrm{Spec}(A_a)$ given by the opens $D(a) \cap U_i = D(aa_i)$.

In other words, by the sheaf axioms, the kernel of the right map in (5) is identified with $\mathscr{F}(D(a))$. But we also identified that kernel with $\mathscr{F}(X)_a$. The resulting identification of $\mathscr{F}(X)_a$ and $\mathscr{F}(D(a))$ is *exactly* via the map $\theta_a$ (since it can be checked after we compose back to $\mathscr{F}(X)$ – do it!), so $\theta_a$ is an isomorphism. ■