Math 216A. Homework 2

"... an arithmetic theory of algebraic varieties cannot but be a theory over arbitrary ground fields and not merely the complex numbers."

Zariski (1950)

Ch. I: 3.2, 3.3 (hint for (b): think about elements of the coordinate ring in terms of the concept of regular function), 5.1(a,b), 5.2(a,b), 5.10. Just work with affine algebraic sets in 3.3 and 5.10, but do *not* just work with affine varieties even when [H] says to consider only varieties (except for 3.3(c) where you should assume X and Y are irreducible, and 5.10(a) where you should replace $\dim X$ by $\dim_P X := \dim \mathscr{O}_{X,P}$, the supremum of the dimensions of the irreducible components of X through P). Also, in 3.3(a) show that φ_P^* is a *local* map of local rings and that if $\psi : Y \to Z$ is another morphism of affine algebraic sets, then the composite map $\varphi_P^* \circ \psi_{\varphi(P)}^*$ is equal to $(\psi \circ \varphi)_P^*$ (note how things go "backwards"!).

Exercise A. For an affine algebraic set $Z \subset \mathbf{A}^n$, we know that a base for the topology of Z consists of the open sets Z_f for $f \in k[Z]$. Show that Z_f is isomorphic (in the sense of morphisms between open subsets of affine algebraic sets!) to an affine algebraic set with coordinate ring $k[Z]_f$. One then refers to the open subsets Z_f as "basic affine opens" in Z. (There may be other open subsets of Z that are isomorphic to affine algebraic sets.)

Exercise B. Let $Z \subset \mathbf{A}^n$ be a Zariski-closed subset, and $P \in Z$ a point. Let $\mathfrak{m} \subset k[Z]$ be the maximal ideal corresponding to P (so $\mathfrak{m} = \{g \in k[Z] \,|\, g(P) = 0\}$). For $h \in k[Z]$ with $h(P) \neq 0$, show that the composite k-algebra map $k[Z]_h \to k[Z]_\mathfrak{m} \twoheadrightarrow k[Z]_\mathfrak{m}/\mathfrak{m}k[Z]_\mathfrak{m} = k[Z]/\mathfrak{m} = k$ (where the final equality is inverse to the natural map $k \to k[Z]/\mathfrak{m}$ that is an isomorphism by the Nullstellensatz) is given by $g/h^m \mapsto g(P)/h(P)^m$ for $g \in k[Z]$ and $m \geq 1$. (Hint: reduce to the case $Z = \mathbf{A}^n$ via functoriality considerations.)

Exercise C. Let A be a commutative ring, and S a multiplicative set in A containing 1. For $a, b \in A$, write $b \leq a$ if $b | a^n$ in A for some $n \geq 1$

(a) Prove $\leq$ is a partial order on A for which any two elements are dominated by a third, with $A \to A_a$ factoring (necessarily uniquely) through $A \to A_b$ if and only if $b \leq a$.

(b) By (a), the A-algebras A_a constitute a directed system, so for varying $s \in S$ it makes sense to form $\varinjlim A_s$. Show that this is uniquely isomorphic as an A-algebra to the localization $S^{-1}A$. (Hint: consider the universal property as an A-algebra.)

(c) For prime $\mathfrak{p} \subset A$ show uniquely $\varinjlim A_a \simeq A_\mathfrak{p}$ as A-algebras where a varies through $A - \mathfrak{p}$. For an affine algebraic set Z and *irreducible* closed $Y \subset Z$ corresponding to prime $\mathfrak{q} \subset k[Z]$, show the set of open $U \subset Z$ that *meet* Y is *directed* under reverse inclusion and uniquely $\varinjlim \mathscr{O}(U) \simeq k[Z]_\mathfrak{q}$ as $k[Z]$-algebras.

Exercise D. For an affine *variety* Z and *every* non-empty open $U \subset Z$, show that $\mathscr{O}_Z(U)$ is a domain and the restriction map $k[Z] = \mathscr{O}_Z(Z) \to \mathscr{O}_Z(U)$ is an injection inducing an equality of fraction fields, so $\mathrm{Frac}(\mathscr{O}_Z(U))$ is "independent of U". For non-empty open $U, V \subset Z$, show $\mathscr{O}_Z(U \cup V) = \mathscr{O}_Z(U) \cap \mathscr{O}_Z(V)$ inside the "function field" $k(Z) := \mathrm{Frac}(k[Z])$.

Some reading (nothing to submit). Read about inverse limits of rings and modules: Exer. 10 & 11 in Sec. 7.6 of Dummit & Foote (not only countable index sets) and the end of [Mat, App. A], noting the mapping property. This also work for inverse limits of sets.