Math 216A. Homework 3

1. Let $Z \subset \mathbf{A}^n$ be an affine algebraic set over an algebraically closed field k, with $f_1, \dots, f_m$ a finite set of generators of $\underline{I}(Z)$. Let J be the $n \times m$ "Jacobian matrix" $(\partial f_i / \partial X_j)$, and for $P \in Z$ let $J(P)$ be the $n \times m$ matrix over k whose entries are those of J evaluated at P. In class we saw $\mathrm{rank}(J(P)) \le n - \dim \mathscr{O}_{Z,P}$ with equality if and only if $\mathscr{O}_{Z,P}$ is regular (in which case we say Z is *smooth* at P).

(a) Show that $\mathscr{O}_{Z,P}$ is a domain if and only if P lies in exactly one irreducible component of Z, and that in such cases there exists an open U containing P such that for all $Q \in U$ the local ring $\mathscr{O}_{Z,Q}$ has the same dimension as $\mathscr{O}_{Z,P}$.

(b) It is a theorem [Mat, Thm. 14.3] that regular local noetherian rings are domains. Using this and (a), show that the locus of smooth points in Z is open. (Later we'll see this open locus is always dense; that it is always non-empty for non-empty Z, roughly an algebraic Sard's theorem, is especially non-obvious when $\mathrm{char}(k) > 0$!)

2. Let A be a finitely generated k-algebra for an algebraically closed field k. Let k'/k be an extension field with k' algebraically closed. Let $A' = k' \otimes_k A$, finitely generated over k'.

In this exercise, you'll need to use your commutative algebra skills (e.g., Noether normalization, localizations, etc.). If you aren't aware of the notion of a separating transcendence basis for finitely generated extensions of a perfect field then read up on this (e.g., see §13 in Chapter II of Volume 1 of the classic "Commutative Algebra" by Zariski & Samuel, or the self-contained first few pages of §26 of [Mat]). Try to think geometrically, if possible.

(a) It often is the case that A has a property P if and only if A' has a property P for various properties P. Prove this for P the following properties: non-zero, domain, reduced, of dimension d, has a unique minimal prime, regular (i.e., all localizations at maximal ideals are regular local rings – keep in mind the link to the rank of a Jacobian matrix). This sort of thing is essential in order to pass between algebraic geometry over $\overline{\mathbf{Q}}$ and algebraic geometry over $\mathbf{C}$ (where analytic tools become available).

(b) Prove that if A_1 and A_2 are finitely generated k-algebras, then $A_1 \otimes_k A_2$ is reduced (respectively, is a domain) if A_1 and A_2 are of this type. (Hint: for reducedness, use the *finite* collection of minimal primes to reduce to A_1 being a domain.) Give counterexamples if we drop the hypothesis that k is algebraically closed.

3. Let $f \in K[T_1, \dots, T_n]$ be a polynomial, K a field.

(a) Explain how f being irreducible over K is equivalent to the non-solvability over K of a suitable system of polynomial equations.

(b) Using the Nullstellensatz over K (!), give a formulation in terms of your polynomial constraints for what it means to say f is irreducible over an algebraic closure of K.

(c) Using (b), show if $f \in \mathbf{Z}[T_1, \dots, T_n]$ is irreducible over an algebraic closure of $\mathbf{Q}$ (or equivalently (!), over $\mathbf{C}$) then for all but finitely many primes p and any algebraically closed field k of characteristic p the image of f in $k[T_1, \dots, T_n]$ is irreducible.

In (c) the "geometric" condition of irreducibility over $\overline{\mathbf{Q}}$ is crucial: $X^2 - 57$ is irreducible in $\mathbf{Q}[X]$ (but of course not in $\overline{\mathbf{Q}}[X]$) yet is reducible in $\mathbf{F}_p[X]$ for infinitely many p. This is a prototype for results that "spread out" *geometric* properties to all but finitely many p.