Math 216A. Homework 4

In the textbook exercises, work throughout with abelian sheaves.

Ch. II: 1.6, 1.7 (use universal properties to build the maps in one direction and *then* show the maps to be isomorphisms), 1.8 (the notation $\Gamma(U, \mathscr{F})$ means $\mathscr{F}(U)$), 1.9*, 1.10, 1.11*, 1.12*, 1.15 (define the ρ's!), 1.16 (need Zorn's Lemma for (b)), 1.18, 1.22 (give a universal mapping property of the gluing using the categories of sheaves of abelian groups on X and on the U_i's; your argument should also work for sheaves of sets, rings, and so on).

Remark. For 1.18, check in private that the map is not just a bijection of Hom sets, but also is compatible with the additive group structure on the Hom sets. Also, check in private suitable functoriality for a composition $g \circ f$ using the functorial properties of f^{-1} and f_* with respect to compositions as discussed in class. Finally, if $\varphi : f^{-1}\mathscr{G} \to \mathscr{F}$ corresponds to $\psi : \mathscr{G} \to f_*(\mathscr{F})$, check in private that for all $P \in X$, the diagram

$$\begin{array}{ccc} (f^{-1}\mathscr{G})_P & \xrightarrow{\varphi_P} & \mathscr{F}_p \\ \simeq \downarrow & & \uparrow \\ \mathscr{G}_{f(P)} & \xrightarrow[\psi_{f(P)}]{} & (f_*\mathscr{F})_{f(P)} \end{array}$$

commutes (the left column was defined in class)! It is essential to verify constructions you make have various functorial properties or else they may not be useful.

Check as much as you can think of, and if something is needed later which you have not verified before then check it! Mathematics is not about checking commutative diagrams, but only practice with checking these things at the foundational level will enable you to handle more complicated diagrams that arise later on.

Exercise A. Let Z be an affine algebraic set over an algebraically closed field k, Z^{sm} its open subset of smooth points. This exercise shows that Z^{sm} is always dense in Z.

(a) For irreducible $g \in k[X_1, \dots, X_n]$, show the open $\underline{Z}(g)^{\mathrm{sm}}$ is non-empty (so dense!).

(b) If Z is irreducible, use a separating transcendence basis over k for the "function field" $k(Z) := \mathrm{Frac}(k[Z])$ and the Primitive Element Theorem and some "denominator-chasing" to find a dense affine open U in Z that is also dense open in a hypersurface $\underline{Z}(f) \subset \mathbf{A}^N$ in some affine space (with irreducible $f \in k[y_1, \dots, y_N]$).

(c) Using (a) and (b), show for irreducible Z that the open Z^{sm} is non-empty (so dense!).

(d) For any Z, use basic affine opens and (c) to show the open Z^{sm} is dense in Z.

Some reading (nothing to submit). Read [Mat, Ch. 8] about completions of modules and rings. For any local noetherian ring $(A, \mathfrak{m})$, the noetherian $\mathfrak{m}$-adic completion $\widehat{A}$ has the same finite dimension as A by [Mat, Thm. 13.9] (or by [Mat, Thm. 15.1(b)] applied to $A \to \widehat{A}$ and $P = \widehat{\mathfrak{m}}$), so for any field F the formal power series ring $F[\![x_1, \dots, x_d]\!] = F[x_1, \dots, x_d]^{\wedge}_{(x_1, \dots, x_d)}$ has dimenson d (see [Mat, Thm. 15.4] for a generalization) and hence is *regular*. For the local ring $(B, \mathfrak{n})$ of germs of C^∞-functions near $0 \in \mathbf{R}^d$, by Hadamard's Lemma $\mathfrak{n} = (x_1, \dots, x_d)$ and so the "Taylor expansion" map $f : B \to \mathbf{R}[\![x_1, \dots, x_d]\!]$ is the $\mathfrak{n}$-adic completion (so B is *non-noetherian* by [Mat, Thm. 8.10(i)] since $\ker f \neq 0$!); E. Borel proved f is *surjective*!