Math 216A. Homework 5

"Some may ask: If Algebraic Geometry really consists of (at least) 13 Chapters, 2000 pages, all of commutative algebra, then why not give up? The answer is obvious ... to deal with special topics, only portions of the whole work are necessary ... there is enough stuff lying around to fit everyone's taste. Those whose taste allows them to swallow the Elements*, however, will be richly rewarded."*

Serge Lang, Bulletin of AMS book review for EGA (it turned out to be much more than 2000 pages).

Ch. II: 2.3, 2.4, 2.5, 2.6*, 2.8, 2.9, 2.11, 2.13 (sp(X) is [H]'s notation for the underlying topological space of a scheme X; the more standard notation for it is $|X|$), 2.16 (can you formulate a good functoriality property of the isomorphism in (d)?; note that the natural map goes from A_f to $\Gamma(X_f, \mathscr{O}_{X_f})$), 2.17, 2.18*, 2.19* (hint: for a separation of Spec(A), consider the "function" that is 0 on one part and 1 on the other).

For 2.3, prove $j : X_{\mathrm{red}} \to X$ is a closed immersion (i.e., $\mathscr{O}_X \to j_* \mathscr{O}_{X_{\mathrm{red}}}$ is surjective) and do the following:

2.3(d) If $f, g : X \rightrightarrows Y$ are morphisms of schemes with X reduced, f and g coincide on topological spaces, and $f^\#$, $g^\#$ induce the same field maps $k(f(x)) = k(g(x)) \rightrightarrows k(x)$ for all $x \in X$ then $f = g$.

2.3(e) Let Y be a closed subset (let $i : Y \to X$ denote the inclusion). For open U in X, define $\mathscr{I}_Y(U)$ to be the ideal of all $f \in \mathscr{O}_X(U)$ which vanish on $Y \cap U$ (i.e., $f(y) = 0$ in $k(y)$ for all $y \in Y \cap U$). Prove that $\mathscr{I}_Y$ is a sheaf of ideals in $\mathscr{O}_X$ on X and for $\mathscr{O}_Y := i^{-1}(\mathscr{O}_X/\mathscr{I}_Y)$ the pair $(Y, \mathscr{O}_Y)$ is a *reduced scheme*. In particular, every closed *subset* of a scheme X can be realized as the underlying space of a *closed subscheme* of X (the correct analogue in complex-analytic analytic geometry is deep).

2.3(f) For the natural $i^\# : \mathscr{O}_X \to \mathscr{O}_X/\mathscr{I}_Y \simeq i_*(\mathscr{O}_Y)$, check $(i, i^\#)$ is a morphism of schemes $(Y, \mathscr{O}_Y) \to (X, \mathscr{O}_X)$ and that this is universal in the sense that for any *reduced* scheme Z,

$$\mathrm{Hom}(Z, Y) \to \{f \in \mathrm{Hom}(Z, X) \mid f(Z) \subseteq Y \text{ as sets}\}$$

is a bijection. We say that $(Y, \mathscr{O}_Y)$ is the *reduced induced structure* on Y. (If $Y = X$, this is just X_{red}!) This shows that closed subsets of X can be identified with certain "radical" ideal sheaves. We'll later describe precisely which ideal sheaves arise in this way.

For 2.4, conclude in particular that for any scheme X, there is a canonical morphism of schemes $X \to \mathrm{Spec}(\Gamma(X, \mathscr{O}_X))$, suitably functorial in X, and this is an isomorphism if and only if X is an affine scheme. (See the handout for Tate's Prop. 1.8.1 in the "Errata et addenda" list for EGA I at the end of EGA II for an elegant generalization with X any locally ringed space.)

Exercise A. Let X be a scheme, $x \in X$ a point, $\mathscr{O}_x$ the local ring at x, and $(A, \mathfrak{m})$ a local ring. Construct a natural bijection between the set of scheme morphisms $\mathrm{Spec}(A) \to X$ taking the closed point $\mathfrak{m}$ to x and the set of *local* maps $\mathscr{O}_{X,x} \to A$ (hint: reduce to the case where X is affine by checking that if an open set in Spec(A) contains the closed point, then it must be the whole space!).

In private (not for submission), check suitable functoriality in the pair (X, x) and the local ring A. In the special case that A is a field, check in private (not for submission) that this recovers [H, Ch. II, Exer. 2.7].

Exercise B*. This exercise (not for submission) punches holes in a scheme.

Let A be a commutative ring, $X = \mathrm{Spec}(A)$, $U \subseteq X$ an open subscheme. Define

$$S_U = \{a \in A \mid a \notin \mathfrak{p} \text{ for all } \mathfrak{p} \in U\}.$$

(a) Show that this is a multiplicative set and that the natural map $A \simeq \Gamma(X, \mathscr{O}_X) \to \Gamma(U, \mathscr{O}_X)$ sends all elements of S_U to units, so we get a natural ring map $S_U^{-1}A \to \Gamma(U, \mathscr{O}_X)$.

(b) Prove that if A is domain then the ring map in (a) is injective, and that if A is a principal ideal domain then that map is an isomorphism.

(c) Find an example of a *domain* A and an open set U so that the ring map in (a) is not an isomorphism. Note that if we take U to be the empty set then $S_U = A$ contains 0 and so $S_U^{-1}A = 0$.