Math 216A. Homework 7

"... it seems to me that algebraic geometry fulfills only in the language of schemes that essential requirement of all contemporary mathematics: to state its definitions and theorems in their natural and abstract formal setting in which they can be considered independently of geometric intuition. Moreover, it seems incorrect to assume that any geometric intuition is lost thereby: for example, the underlying variety in an algebraic scheme is rediscovered, and perhaps bettter understood through the concept of geometric points."

David Mumford in *Geometric Invariant Theory*

Read the handout on preservation of properties of morphisms under base change, and [Mat, §10].

Ch. II: 3.8*, 3.10 (for (a), show the natural map $X_y \to X$ is a homeomorphism onto $f^{-1}(y)$), 3.15(a), 3.20(f)*, 3.21*, 3.22 (hint for algebraic solution to (a) [can argue geometrically instead]: for the local map $A = \mathscr{O}_{Y,Y'} \to \mathscr{O}_{X,Z} = B$ at the respective generic points of Y' and Z, show $B/\mathfrak{m}_A B$ is artinian. Deduce B/JB is artinian for any $\mathfrak{m}_A$-primary ideal J of A, and by [Mat, Thm. 13.4] there is such a J with d generators for $d = \dim A$. Conclude $\dim B \leq d$ via Krull's Height Theorem [Mat, Thm. 13.5].)

In 3.15(a) one really should say that X is geometrically irreducible etc. *over* k. For example, if k'/k is a non-trivial separable extension of finite degree, then $\mathrm{Spec}(k')$ is not geometrically irreducible over k but it is geometrically integral over k'! The "ground field" k is nearly always understood from context.

Exercise A. (i) Let $S' \to S$ be a map of schemes, X an S-scheme, $X' = X \times_S S'$. Explain why there is a natural bijection between the set $X(S') = \mathrm{Hom}_S(S', X)$ and $X'(S') = \mathrm{Hom}_{S'}(S', X')$. For example, if X is a scheme over a field k and K/k is an extension field, then $X(K) = \mathrm{Hom}_k(\mathrm{Spec}(K), X)$ can be thought of as the set of K-rational points on the K-scheme $X \times_k K$.

Here, a *K-rational point* on a K-scheme Z is a point $z \in Z$ such that the canonical $K \to \kappa(z)$ is an isomorphism. For example, if $Z = \mathrm{Spec}(K[T_1, \dots, T_n]/(f_1, \dots, f_r))$, then the K-rational points are identified with the simultaneous solutions to $f_i = 0$ in K (the notation $X(S')$ may seem sloppy insofar as it does not mention S, but once it is understood that we work in the category of S-schemes, there is no serious ambiguity; otherwise one should write $X_S(S')$). Reformulating "solutions to equations" as maps is powerful.

(ii) Let $f : X \to Y$ be a *finite type* k-morphism between locally finite type k-schemes, with k algebraically closed. Show that f is surjective (respectively injective) if and only if $X(k) \to Y(k)$ is surjective (respectively injective), and f has open image (respectively closed image) if and only if $T(f_{\mathrm{red}}) : T(X_{\mathrm{red}}) \to T(Y_{\mathrm{red}})$ does.

(iii*) Let $f : X \to Y$ be a *finite type* k-morphism between locally finite type schemes over a field k. Let K be an algebraically closed extension of k. Prove that f is quasi-finite if and only if the map of sets $f : X(K) \to Y(K)$ has finite fibers (hint: to save some grief, first use (i) and some algebra to reduce to the case in which $K = k$ is an algebraically closed field and then X and Y are affine integral schemes). Give an example in which f is locally of finite type, not quasi-compact, and $f : X(K) \to Y(K)$ is a bijection.

Exercise B*. Strengthen [H, Ch. II, Exer. 3.7] as follows. Let $f : X \to Y$ be a dominant generically finite map between integral schemes of finite type over a noetherian base $\mathrm{Spec}(R)$ (so X is noetherian, hence f is quasi-compact and so of finite type). Define $d = [K(X) : K(Y)]_s$. Prove that there is a non-empty open affine $U \subseteq Y$ so that $f^{-1}(U) \to U$ is finite and exhibits the top ring as a finite free module over the bottom ring. Moreover, check that U can be chosen such that the "geometric fiber" $X_y \times_{k(y)} \overline{k(y)}$ has exactly d points for *all* $y \in U$.

Exercise C*. Let k be an algebraically closed field, $f : X \to Y$ be a finite map between smooth affine curves over k, d the degree of $K(X)$ over $K(Y)$. Prove that the fibers X_y over *all* points $y \in Y$ are affine with rings having dimension d over $k(y)$, but the number of actual points in X_y (the "classical" fiber) can fail to be d, even if we just consider closed points $y \in Y$. (Hint: $k[X]$ and $k[Y]$ are Dedekind, by smoothness.)

Exercise D*. Let X be a scheme locally of finite type over a separably closed field k such that X is *geometrically* integral (over k). Show that the set $X(k)$ of k-rational points of X is non-empty, and even dense in X (hint: use [Mat, Thm. 26.2], whose proof is totally self-contained). Give a counterexample if we drop the "geometric" hypothesis or the condition that k is separably closed.