Math 248A. Some basics concerning absolute values

A remarkable unification of congruential and Euclidean-space methods in number theory is achieved through the systematic use of absolute values (and completions thereof) on number fields. These give rise to a vast array of locally compact fields and groups on which Haar measures and Fourier analysis permit the application of powerful topological and analytic tools in the study of arithmetic questions. In these notes, we undertake a general study of the theory of absolute values on arbitrary fields, as this will help to clarify those aspects of the theory that are purely algebraic and those that are specific to the fields of interest in number theory. (Keep in mind that much of the development of the theory of Dedekind domains, including the definition of class groups, was really a part of commutative algebra. Only once we began to prove finiteness theorems could one say we were really studying a situation specific to number theory.)

Let F be a field. An *absolute value* on F is a map $|\cdot| : F \to \mathbf{R}$ satisfying

- $|x| \geq 0$ for all $x \in F$, with equality if and only if $x = 0$,
- $|xy| = |x||y|$ for all $x, y \in F$,
- $|x + y| \leq |x| + |y|$ for all $x, y \in F$.

These conditions force $|1|^2 = |1|$, so $|1| = 1$ because $|1| \neq 0$ (as $1 \neq 0$ in F). Hence, $|1/x| = 1/|x|$ for all $x \in F^\times$, so $|\cdot| : F^\times \to \mathbf{R}_{>0}$ is a group homomorphism. In particular, if $\zeta \in F^\times$ is a root of unity then $|\zeta| = 1$. For example, $|-1| = 1$ and hence $|-x| = |x|$ for all $x \in F$.

It is clear that $d(x, y) = |x - y|$ gives F a structure of metric space, and the resulting topology is the discrete topology if and only if $|x| = 1$ for all $x \neq 0$. (Indeed, if $|x| \neq 1$ for some $x \neq 0$ then by replacing x with $1/x$ if necessary we may find x with $0 < |x| < 1$, so $|x^n| = |x|^n \to 0$. Hence, $\{x^n\}$ is a sequence of nonzero elements converging to 0, so $\{0\}$ is not open and thus the topology is not discrete.) We shall call $|\cdot|$ a *trivial* absolute value if $|x| = 1$ for all $x \neq 0$. In general we shall be interested in non-trivial absolute values.

It is straightforward to adapt the usual arguments from elementary analysis to show that if F is a field equipped with an absolute value $|\cdot|$, then the resulting Hausdorff topology makes F into a *topological field* (that is, the addition and multiplication maps are continuous, and inversion with respect to addition on F and multiplication on $F^\times$ are continuous). In particular, $F^\times$ with its open subspace topology is a topological group. The pairs $(F, |\cdot|)$ of most interest in number theory are those for which $|\cdot|$ is non-trivial and F is locally compact; in Homework 8 such fields will be shown to be complete as metric spaces and they will be explicitly classified. (Strictly speaking, Homework 8 will treat the non-archimedean case; the archimedean case is addresses in Corollary 6.3 below.)

1. Topological equivalence

Let $|\cdot|$ be an absolute value on F. For any $0 < e \leq 1$, it is easy to check that $|\cdot|^e$ is also an absolute value. Indeed, the only issue is the triangle inequality, and for this we need the inequality $(a+b)^e \leq a^e + b^e$ for $a, b \geq 0$ in $\mathbf{R}$. To prove this inequality we may assume $a \geq b > 0$, and then division by b^e reduces us to proving $(t+1)^e \leq t^e + 1$ for all $t \geq 1$; this latter inequality is clear at $t = 1$ (since $0 < e \leq 1$) and the respective rates of growth (derivatives) at $t > 0$ are $e(t+1)^{e-1}$ and et^{e-1} with $(t+1)^{e-1} \leq t^{e-1}$ because $e - 1 \leq 0$ and $t + 1 \geq t > 0$. It is trivial (taking $x = y = 1$) to check that for the usual absolute value $|\cdot|_{\mathbf{R}}$ on $\mathbf{R}$, $|\cdot|_{\mathbf{R}}^e$ does *not* satisfy the triangle inequality for any $e > 1$. However, in §3 we shall see many examples of non-trivial absolute values $|\cdot|$ on $\mathbf{Q}$ such that $|\cdot|^e$ is an absolute value for all $e > 0$.

It is clear that if $|\cdot|$ and $|\cdot|'$ are two absolute values on a field F and one of them is a power of the other via a positive exponent (and so the same relation holds in the other direction, using the reciprocal power) then they define the same topology on F. It is important that the converse is true:

Theorem 1.1. *Let $|\cdot|$ and $|\cdot|'$ be two absolute values on F that induce the same topology on F. There exists $e > 0$ such that $|\cdot|' = |\cdot|^e$.*

Proof. Since the trivial absolute value is the unique one giving rise to the discrete topology, we may suppose that the topology is non-discrete and hence that both absolute values are non-trivial. Pick $c \in F^\times$ such that $0 < |c| < 1$. Hence, $\{c^n\}$ converges to 0 with respect to the common topology, so $|c^n|' \to 0$ and thus

$0 < |c|' < 1$. There is a unique $e > 0$ such that $|c|' = |c|^e$. By switching the roles of $|\cdot|$ and $|\cdot|'$ and replacing e with $1/e$, we may assume $0 < e \leq 1$. Hence, $|\cdot|^e$ is an absolute value and our goal is to prove that it is equal to $|\cdot|'$. Since $|\cdot|^e$ defines the same topology as $|\cdot|$, we may replace $|\cdot|$ with $|\cdot|^e$ to reduce to the case $e = 1$. That is, we have $0 < |c| = |c|' < 1$ for some $c \in F^\times$. Under this condition, we want to prove $|x| = |x|'$ for all $x \in F$, and we may certainly restrict attention to $x \in F^\times$.

We assume $|x| \neq |x|'$ for some $x \in F^\times$, and we seek a contradiction. We can assume $|x| < |x|'$, or equivalently $|x|'/|x| > 1$. By replacing x with x^n for a large n, we may arrange that $|x|'/|x| > 1/|c| > 1$. Hence, there exists $n > 0$ such that $|c^n| = |c|^n$ is strictly between $|x|'$ and $|x|$. Clearly $|c^n| = |c^n|'$, so by renaming c^n as c without loss of generality we arrive at $0 < |x| < |c| = |c|' < |x|'$. Thus, $|x/c| < 1 < |x/c|'$. Hence, $\{(x/c)^n\}$ converges to zero with respect to the metric topology of $|\cdot|$ but not with respect to the metric topology of $|\cdot|'$. This is a contradiction since the two topologies are assumed to coincide. ∎

2. Non-archimedean absolute values

An absolute value $|\cdot|$ on a field is *non-archimedean* if its restriction to the image of $\mathbf{Z}$ in F is bounded, and otherwise (that is, if $\mathbf{Z}$ is unbounded for the metric structure) $|\cdot|$ is *archimedean.* The non-archimedean property is inherited by any absolute value of the form $|\cdot|^e$ with $e > 0$, and so Theorem 1.1 implies that this condition is intrinsic to the underlying topology associated to the absolute value. Obviously the trivial absolute value is non-archimedean, and any absolute value on a field F with positive characteristic must be non-archimedean (as the image of $\mathbf{Z}$ in F consists of 0 and the set $\mathbf{F}_p^\times$ of $(p-1)$th roots of unity in F). Of course, the usual absolute value on $\mathbf{Q}$ is archimedean.

The *non-archimedean triangle inequality* (also called the *ultrametric triangle inequality*) is

$$|x + y| \leq \max(|x|, |y|).$$

This is clearly much stronger than the usual triangle inequality, and it forces $|k| \leq 1$ for all $k \in \mathbf{Z}$ (since $\mathbf{Z}$ is additively generated by 1, and $|\pm 1| = 1$), so $|\cdot|$ is forced to be non-archimedean in such cases. Interestingly, this stronger form of the triangle inequality is also *necessary* for $|\cdot|$ to be non-archimedean, and so the following theorem is often taken as the *definition* of a non-archimedean absolute value:

Theorem 2.1. *An absolute value $|\cdot|$ on a field F is non-archimedean if and only if it satisfies the non-archimedean triangle inequality. In particular, any absolute value on a field with positive characteristic must satisfy the non-archimedean triangle inequality.*

Proof. The sufficiency has already been noted, so the only issue is necessity. Consider the binomial theorem

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

in F for $n \geq 1$. Applying the absolute value to both sides and using the hypothesis that $|\cdot|$ is bounded on the image of $\mathbf{Z}$ in F, say with $|k| \leq C$ for all $k \in \mathbf{Z}$, we get

$$|x + y|^n \leq \sum_{j=0}^{n} C|x|^{n-j}|y|^j \leq (n+1)C \max(|x|, |y|)^n$$

for all $n \geq 1$. Extracting nth roots gives $|x+y| \leq ((n+1)C)^{1/n} \max(|x|, |y|)$ for all $n \geq 1$. As $n \to \infty$ clearly $((n+1)C)^{1/n} \to 1$, so we obtain the non-archimedean triangle inequality. ∎

Corollary 2.2. *If $|\cdot|$ is a non-archimedean absolute value on a field F, then so is $|\cdot|^e$ for all $e > 0$. In particular, $|\cdot|^e$ is an absolute value for all $e > 0$.*

Proof. By the theorem, $|x + y| \leq \max(|x|, |y|)$ for all $x, y \in F$. Raising both sides to the eth power gives the same for $|\cdot|^e$ for any $e > 0$, so in particular $|\cdot|^e$ satisfies the triangle inequality. The rest follows immediately. ∎

Here is an important refinement of the non-archimedean triangle inequality. Suppose that $|\cdot|$ is non-archimedean. We claim that the inequality $|x+y| \le \max(|x|,|y|)$ is an *equality* if $|x| \ne |y|$. Indeed, suppose (by symmetry) $|x| < |y|$. We then want to prove $|x+y| = |y|$. Suppose not, so $|x+y| < |y|$. Hence, $|x|, |x+y| < |y|$, so

$$|y| = |(y+x)-x| \le \max(|y+x|, |-x|) < \max(|y|,|y|) = |y|,$$

a contradiction. This has drastic consequences for the topology on F. For example, if $r > 0$ and $a, a' \in F$ satisfy $|a - a'| \le r$ then $|x-a| \le r$ if and only if $|x - a'| \le r$. Hence, any point in the disc

$$\overline{B}_r(a) = \{x \in F \mid |x-a| \le r\}$$

serves as a "center". More drastically, whereas $\overline{B}_r(a)$ is trivially a closed set in F (as for any metric space), it is also open! Indeed, if $|x_0 - a| \le r$ then the non-archimedean triangle inequality implies that

$$|x - x_0| < r \Rightarrow |x-a| \le r.$$

Thus, $\overline{B}_r(a)$ contains an open disc around any of its points. In contrast with the classical case, a small open disc centered at a point on $|x-a| = r$ (which is *not* the topological boundary of $\overline{B}_r(a)$) lies inside of $\overline{B}_r(a)$. The following result makes explicit how different the non-archimedean topology is in comparison with the topology of $\mathbf{R}$:

Theorem 2.3. *The topological space F is totally disconnected. That is, its only non-empty connected subsets are one-point sets.*

Proof. Let C be a connected set containing a point x_0. If $C \ne \{x_0\}$ then there exists $x_1 \in C$ with $x_1 \ne x_0$. Choose $r > 0$ with $r < |x_1 - x_0|$. We have a disjoint union decomposition

$$C = (C \cap \{|x - x_0| \le r\}) \coprod (C \cap \{|x-x_0| > r\})$$

with both pieces *open* in C (as each is an intersection of C with an open set in F), and each piece is non-empty (the first contains x_0 and the second contains x_1). This contradicts the assumption that C is connected, so no such x_1 can exist. ■

We now analyze the algebraic structure of the non-archimedean case more closely. Let F be a field equipped with a non-archimedean absolute value $|\cdot|$. Let $A = \{x \in F \mid |x| \le 1\}$ and let $M = \{x \in F \mid |x| < 1\}$. Observe that A is a *ring* because of the non-archimedean triangle inequality, and M is an ideal in A. It is clear via multiplicativity of the absolute value that the units in A are precisely the elements $x \in F$ satisfying $|x| = 1$, and these are exactly the elements in $A - M$. Hence, A is a local ring with M as its unique maximal ideal.

Definition 2.4. The ring $A = \{x \in F \mid |x| \le 1\}$ is the *valuation ring* of the non-archimedean absolute value $|\cdot|$ on F. The field $k = A/M$ is the *residue field* attached to the absolute value. (If $|\cdot|$ is trivial then $A = F$ and $M = 0$, but we will never be interested in such cases.)

By Theorem 1.1, the valuation ring (and hence the residue field) only depend on the topological equivalence class of $|\cdot|$. The structure of the valuation ring works out rather nicely when the value group $|F^\times| \subseteq \mathbf{R}^\times$ is a discrete subgroup of $\mathbf{R}^\times$:

Theorem 2.5. *Let $|\cdot|$ be a non-trivial non-archimedean absolute value on F. The value group $|F^\times| \subseteq \mathbf{R}^\times$ is a discrete subgroup if and only if the valuation ring A for $|\cdot|$ is a discrete valuation ring.*

This theorem is the reason for the terminology "discrete valuation ring."

Proof. We first assume that A is a discrete valuation ring. Since $|A^\times| = 1$, if π is a uniformizer of A and $c = |\pi|$ then $c \le 1$ and in fact $c < 1$ because $\pi \notin A^\times$. We obtain $|F^\times| = c^{\mathbf{Z}} \subseteq \mathbf{R}_{>0}$, and this is a discrete subgroup by inspection.

Conversely, suppose that $|F^\times| \subseteq \mathbf{R}_{>0}$ is non-trivial and has the discrete topology. Since the logarithm identifies $\mathbf{R}_{>0}$ with $\mathbf{R}$, we see that a discrete non-trivial subgroup of $\mathbf{R}_{>0}$ is infinite cyclic. Hence, $|F^\times| = c^{\mathbf{Z}}$ for some $c > 0$ with $c \ne 1$, and we may take $0 < c < 1$ by passing to $1/c$ if necessary. Choose $\pi \in F$ with $|\pi| = c$, so $\pi \in A - \{0\}$ and π is not a unit in A. For any nonzero $a \in A$ we have $0 < |a| \le 1$, so

$|a| = c^n = |\pi^n|$ for some $n \geq 0$. Hence, $a/\pi^n \in F^\times$ has absolute value 1 and thus lies in $A^\times$. We conclude $aA = (\pi A)^n$, so A is a local PID that is not a field and has principal maximal ideal. This implies that A is a discrete valuation ring. ■

As a partial converse to Theorem 2.5, if A is any discrete valuation ring and F is its fraction field then we claim that there is a unique topological equivalence class of non-archimedean absolute values on F whose valuation ring is A. The uniqueness follows by inspecting the method of proof of Theorem 2.5, and we shall settle existence in the discussion following Theorem 4.1.

3. Ostrowski's theorem

We now consider the case $F = \mathbf{Q}$. We wish to determine all non-trivial absolute values on $\mathbf{Q}$. We shall write $|\cdot|_\infty$ to denote the usual absolute value, so (as one easily sees by working in $\mathbf{R}$) $|\cdot|_\infty^e$ is an absolute value on $\mathbf{Q}$ for $e > 0$ if and only if $e \leq 1$. In view of Theorem 1.1, these are precisely the absolute values on $\mathbf{Q}$ that define the "usual" (totally disconnected) topology on $\mathbf{Q}$, also called the *archimedean topology.* We shall now show that there are infinitely many other absolute values on $\mathbf{Q}$ up to topological equivalence, and these other topological equivalence classes are all non-archimedean and the set of them is in natural bijection with the set of positive primes of $\mathbf{Z}$. In particular, $|\cdot|_\infty$ and its topologically equivalent counterparts $|\cdot|_\infty^e$ $(0 < e \leq 1)$ are the *only* archimedean absolute values on $\mathbf{Q}$.

Let us first introduce the basic examples to be later shown to exhaust all possibilities. For a fixed prime $p > 0$, define $\mathrm{ord}_p : \mathbf{Z} - \{0\} \to \mathbf{Z}$ by the condition $k = p^{\mathrm{ord}_p(k)}k'$ with $p \nmid k'$. By unique factorization into primes, it is easy to check that $\mathrm{ord}_p(k_1k_2) = \mathrm{ord}_p(k_1) + \mathrm{ord}_p(k_2)$, so this is a logarithmic construction. This property also allows us to extend the definition to $\mathbf{Q}^\times$: $\mathrm{ord}_p(q) = \mathrm{ord}_p(n) - \mathrm{ord}_p(m)$ if $q = n/m$ with $n, m \in \mathbf{Z} - \{0\}$; it is clear that this definition on $\mathbf{Q}^\times$ is well-posed (that is, independent of the choice of representative numerator and denominator) because of the "logarithmic" property of ord_p on $\mathbf{Z} - \{0\}$. Concretely, we have $q = p^{\mathrm{ord}_p(q)}q'$ where $\mathrm{ord}_p(q) \in \mathbf{Z}$ and $q' \in \mathbf{Q}^\times$ has no p in its numerator or denominator when written in reduced form.

Definition 3.1. For $c \in (0,1)$, the absolute value $|\cdot|_{p,c}$ on $\mathbf{Q}$ is defined by $|0|_{p,c} = 0$ and $|x|_{p,c} = c^{\mathrm{ord}_p(x)}$ if $x \in \mathbf{Q}^\times$. The (standard) *p-adic absolute value* is defined by taking $c = 1/p$: $|\cdot|_p = |\cdot|_{p,1/p}$. (The reason for taking $c = 1/p$ as the "preferred" normalization will become clear later.)

It is standard (and reasonable, after a moment's reflection) to define $\mathrm{ord}_p(0) = +\infty$, and via the obvious conventions for the symbol $+\infty$ we may then say that ord_p is logarithmic on $\mathbf{Q}$ (not merely on $\mathbf{Q}^\times$) and $|\cdot|_{p,c} = c^{\mathrm{ord}_p}$ on $\mathbf{Q}$ (not merely on $\mathbf{Q}^\times$) for $0 < c < 1$.

Obviously the functions $|\cdot|_{p,c}$ as $c \in (0,1)$ varies are the same as the functions $|\cdot|_p^e$ as $e = -\log_p c > 0$ varies. We claim that these are non-archimedean absolute values, and it suffices to check $|\cdot|_p$. Obviously $|x|_p \geq 0$ with equality if and only if $x = 0$, and the multiplicativity $|xy|_p = |x|_p|y|_p$ is trivial for $x = 0$ or $y = 0$ and it reduces to the logarithmic property of ord_p on $\mathbf{Z} - \{0\}$ if $x, y \neq 0$. To check $|x + y|_p \leq \max(|x|_p, |y|_p)$ for $x, y \in \mathbf{Q}$, we may assume $x, y \neq 0$ and we may multiply through by $|N|_p$ for suitable $N \in \mathbf{Z} - \{0\}$ to get to the case $x, y \in \mathbf{Z} - \{0\}$. The case $x + y = 0$ is trivial, and otherwise we need to prove

$$\mathrm{ord}_p(x + y) \geq \min(\mathrm{ord}_p(x), \mathrm{ord}_p(y)).$$

However, this is clear: if we let m denote this minimum, then $p^m|x$ and $p^m|y$, so certainly $p^m|(x + y)$ and hence $\mathrm{ord}_p(x + y) \geq m$.

Observe that $|p|_{p,c} = c < 1$ but $|n|_{p,c} = 1$ for $n \in \mathbf{Z}$ not divisible by p. In particular, $|\ell|_{p,c} = 1$ for distinct positive primes ℓ and p.

A classical fact is:

Theorem 3.2 (Ostrowski)**.** *The absolute values on* $\mathbf{Q}$ *are the trivial one,* $|\cdot|_\infty^e$ *for* $0 < e \leq 1$*, and* $|\cdot|_p^e$ *for* $e > 0$ *and a positive prime* p*. These families for each varying exponent* e *also form the topological equivalence classes of such absolute values.*

Proof. By Theorem 1.1, there are no unexpected topological equivalences. Thus, it remains to prove that the only archimedean absolute values are powers of $|\cdot|_\infty$ and the only non-trivial non-archimedean absolute values are powers of $|\cdot|_p$ for some positive prime p.

Let us first consider a non-trivial non-archimedean absolute value $|\cdot|$ on $\mathbf{Q}$. As we have seen early in §2, necessarily $|n| \le 1$ for all $n \in \mathbf{Z}$. If $|p| = 1$ for all positive primes p then since $\mathbf{Q}^\times$ is multiplicatively generated by the positive primes and ± 1 we conclude that $|\cdot|$ is trivial on $\mathbf{Q}$. Thus, $|p| < 1$ for some prime p. Such a p is unique because if $|\ell| < 1$ for some other prime ℓ then we have $px + \ell y = 1$ for some $x, y \in \mathbf{Z}$ with $x, y \ne 0$, in which case $1 = |1| = |px + \ell y| \le \max(|p||x|, |\ell||y|) < \max(|x|, |y|) \le 1$, a contradiction. Hence, $|\ell| = 1$ for all positive primes $\ell \ne p$. Since $|\cdot|$ is non-archimedean, $|\cdot|^e$ is an absolute value for all $e > 0$. Thus, since $|p| \in (0, 1)$ by the choice of p, by replacing $|\cdot|$ with $|\cdot|^e$ for some $e > 0$ we may arrange that $|p| = 1/p$. Hence, $|\cdot|$ and $|\cdot|_p$ agree on all primes, and since these together with -1 generate $\mathbf{Q}^\times$ multiplicatively we conclude $|\cdot| = |\cdot|_p$.

Now we suppose $|\cdot|$ is archimedean and we seek to prove $|\cdot| = |\cdot|_\infty^e$ for some $e > 0$ (and so necessarily $e \le 1$). Since $|\cdot|$ is archimedean, so it is unbounded on $\mathbf{Z}$, we must have $|b| > 1$ for some $b \in \mathbf{Z}$. Switching signs if necessary, we can assume $b > 0$ and hence $b > 1$. We take $b \in \mathbf{Z}^+$ to be minimal with $|b| > 1$; at the end of the proof it will follow that $b = 2$, but right now we do not know this to be the case. Choose the unique $e > 0$ such that $|b| = b^e$.

Consider the base-b expansion of an integer $n \ge 1$: $n = a_0 + a_1 b + \cdots + a_s b^s$ with $0 \le a_j < b$, $s \ge 0$, and $a_s \ge 1$. By minimality of b we have $|a_j| \le 1$ for all j, so

$$|n| \le \sum_{j=0}^{s} |a_j||b|^j \le \sum_{j=0}^{s} |b|^j = |b|^s(1 + 1/|b| + \cdots + 1/|b|^s) = \frac{|b|^s}{1 - 1/|b|}.$$

If we let $C = 1/(1 - 1/|b|) > 0$ we have $|n| \le Cb^{es} \le Cn^e$ because $b^s \le n$ and $e > 0$. This says $|k| \le Ck^e$ for all $k \ge 1$, so by fixing k we have $|k^r| \le Ck^{re}$ for all $r \ge 1$. Extracting rth roots gives $|k| \le C^{1/r}k^e$, and taking $r \to \infty$ gives $|k| \le k^e = |k|_\infty^e$ for all $k \ge 1$. Hence, passing to $-k$ gives $|k| \le |k|_\infty^e$ for all $k \in \mathbf{Z}$ (as the case $k = 0$ is trivial).

We now prove the reverse inequality $|k| \ge |k|_\infty^e$ for all $k \in \mathbf{Z}$, and so $|k| = |k|_\infty^e$ holds for all $k \in \mathbf{Z}$, which in turn gives the identity $|\cdot| = |\cdot|_\infty^e$ on $\mathbf{Q}$ as desired. As above, it suffices to prove $|n| \ge C'n^e$ for some $C' > 0$ and all $n > 0$ (as then we can specialize to rth powers, extract rth roots, and take $r \to \infty$). Using notation as above with the base-b expansion of n, we have $b^{s+1} > n \ge b^s$, so

$$b^{e(s+1)} = |b|^{s+1} = |b^{s+1}| = |b^{s+1} - n + n| \le |b^{s+1} - n| + |n| \le (b^{s+1} - n)^e + |n|,$$

where the final step uses the proved inequality $|k| \le k^e$ for $k = b^{s+1} - n > 0$. Hence,

$$|n| \ge b^{(s+1)e} - (b^{s+1} - n)^e = b^{(s+1)e}(1 - (1 - n/b^{s+1})^e) \ge n^e(1 - (1 - 1/b)^e),$$

so taking $C' = 1 - (1 - 1/b)^e > 0$ gives $|n| \ge C'n^e$ for all $n \ge 1$, as required. ■

4. Variants on Ostrowski's theorem

We shall use a similar method to determine all non-trivial absolute values up to topological equivalence on the rational function field $F = k(T)$ when k is a *finite* field, and we will also study fraction fields of more general Dedekind domains. We first focus on $F = k(T)$ with finite k. Observe that if $|\cdot|$ is a non-trivial absolute value on F then its restriction to k is trivial because $k^\times$ consists of roots of unity. Hence, we shall now abandon the finiteness restriction on k and will instead let k be an arbitrary field, but we will only classify (up to topological equivalence) those absolute values on $F = k(T)$ whose restriction to k is trivial; it is equivalent to say that the absolute value in bounded on k. Since the image of $\mathbf{Z}$ in F lands in k, all such absolute values must be non-archimedean. (If k has characteristic 0, then one can construct archimedean absolute values on $k(T)$, necessarily nontrivial on k, if and only if the underlying set for k does not exceed the cardinality of the continuum; this follows from Theorem 6.2.)

Let us give some examples. If $\pi \in k[T]$ is a monic irreducible polynomial, then we can define ord_π on $k(T)^\times$ much as we defined ord_p on $\mathbf{Q}^\times$ (but we use unique factorization in $k[T]$ to replace unique factorization in $\mathbf{Z}$). We defined $|\cdot|_p = (1/p)^{\mathrm{ord}_p}$ on $\mathbf{Q}^\times$, and as we vary $e > 0$ we may write $|\cdot|_p^e$ in the form c^{ord_p} for varying $c \in (0,1)$. In contrast with that case, there is no "natural" constant c to use in the case of $k(T)$ (unless k is finite, as we shall see later). Hence, we simply consider the family of topologically equivalent non-archimedean absolute values $|\cdot|_{\pi,c} = c^{\mathrm{ord}_\pi}$ on $k(T)$ for $0 < c < 1$ (with $|0|_{\pi,c} = 0$). Note that these absolute values are trivial on k and are bounded on $k[T]$. As in the case of $\mathbf{Q}$, it is easy to check that if π_1 and π_2 are distinct monic irreducibles in $k[T]$ (so they are *not* unit multiples of each other in $k[T]$) then $|\cdot|_{\pi_1,c_1}$ and $|\cdot|_{\pi_2,c_2}$ are not topologically equivalent for any $c_1, c_2 \in (0,1)$ because of Theorem 1.1.

In addition to the absolute values $|\cdot|_{\pi,c}$ on $F = k(T)$ as just defined, all of which are bounded on $k[T]$, there is another family that is to be considered as the analogue of the archimedean absolute values $|\cdot|_\infty^e$ on $\mathbf{Q}$ in the sense that they are all unbounded on $k[T]$ (just as $|\cdot|_\infty^e$ is unbounded on $\mathbf{Z}$ for $0 < e \leq 1$). We simply observe that for $T' = 1/T$ we have $F = k(T')$ as well, and so we define ord_∞ on $F^\times$ using $k[T']$ and its monic irreducible T'; concretely, ord_∞ on $k[T] - \{0\}$ is just the *negated degree* of a polynomial. We define $|\cdot|_{\infty,c} = c^{\mathrm{ord}_\infty}$ on $k(T)^\times = k(T')^\times$ for all $c \in (0,1)$ and $|0|_{\infty,c} = 0$. One should really consider ord_∞ and $|\cdot|_{\infty,c}$ as attached to the k-subalgebra $k[T]$ with fraction field F, rather than as attached to the element T, because for nonzero $f \in k[T]$ we have the formula

$$\mathrm{ord}_\infty(f) = -\deg(f) = -\dim_k k[T]/(f)$$

that is intrinsic to the k-subalgebra $k[T]$. Clearly $|\cdot|_{\infty,c}$ is non-archimedean and trivial on k, and it is unbounded on $k[T]$ so it is not topologically equivalent to any $|\cdot|_{\pi,c}$.

These absolute values "at infinity" may seem *ad hoc* in the sense that perhaps we can be clever and find other PID k-subalgebras of F with fraction field F to make new absolute values. In fact, the above procedure is exhaustive:

Theorem 4.1. *Every non-trivial absolute value on $F = k(T)$ that is trivial on k is equal to some $|\cdot|_{\pi,c}$ for a unique monic irreducible $\pi \in k[T]$ and $c \in (0,1)$ or is equal to some $|\cdot|_{\infty,c}$ with $c \in (0,1)$. These options are mutually exclusive.*

This theorem shows in particular that the absolute values on F that are bounded on $k[T]$ are precisely those of the form $|\cdot|_{\pi,c}$ for monic irreducibles $\pi \in k[T]$ and $c \in (0,1)$ (as an absolute value that is bounded on $k[T]$ is bounded on the field k and hence must be trivial on k). Observe that the concept of $|\cdot|_{\infty,c}$ is dependent on the artificial choice of k-subalgebra $k[T]$ with fraction field F, and so it is *not* intrinsic to the field F as an extension of k. In the language of modern algebraic geometry, which is the only satisfactory method for giving a coordinate-free discussion of such matters, the theorem says that the topological equivalence classes of non-trivial non-archimedean absolute values on F that are trivial on k are in canonical bijection with closed points on the projective line over k. A similar result holds for function fields of arbitrary regular proper connected curves over k, due to the valuative criterion for properness of morphisms of schemes, and one can consider the generic point of the curve (in the sense of schemes) as "corresponding" to the trivial absolute value on the fraction field.

Proof. We have already seen that there are no unexpected topological equivalences among the given list of non-trivial absolute values on F that are trivial on k. Hence, our problem is to show that no other such absolute values exist.

Let $|\cdot|$ be a non-trivial absolute value on F that is trivial on k, so $|\cdot|$ is non-archimedean. As a multiplicative group, $F^\times$ is a product of the group $k^\times$ and the free abelian group generated by the monic irreducibles $\pi \in k[T]$. Thus, $|\cdot| = |\cdot|_{\pi,c}$ for some monic irreducible $\pi \in k[T]$ and some $c \in (0,1)$ if $|\pi| = c < 1$ and $|\Pi| = 1$ for all monic irreducibles $\Pi \neq \pi$.

Since $|\cdot|$ is not identically 1 on $F^\times$ but it is trivial on $k^\times$, there must exist π such that $c \overset{\mathrm{def}}{=} |\pi| \neq 1$. First assume $|T| \leq 1$. In this case, the non-archimedean triangle inequality and the triviality of $|\cdot|$ on k force $|\cdot|$ to be bounded by 1 on $k[T]$ because $k[T]$ is additively generated by k-linear combinations of monomials T^j ($j \geq 0$). Thus, $c = |\pi| < 1$. For any monic irreducible $\Pi \in k[T]$ distinct from π we must have $|\Pi| = 1$ (and hence $|\cdot| = |\cdot|_{\pi,c}$). Indeed, if not then $|\Pi| < 1$ for some such Π and we have an identity $\pi f + \Pi g = 1$ for some $f, g \in k[T]$ that are necessarily nonzero, in which case $1 = |1| \leq \max(|\pi||f|, |\Pi||g|) < \max(|f|, |g|) \leq 1$, a contradiction.

It remains to consider the case $|T| > 1$. Thus, $|c_jT^j| = |T|^j$ is strictly increasing in j (with $c_j \in k^\times$), so (as we saw after Corollary 2.2) the non-archimedean inequality forces $|f| = |T|^{\deg(f)}$ for $f \in k[T] - \{0\}$. If we define $c = 1/|T| \in (0,1)$ then we have $|f| = |f|_{\infty,c}$ for all nonzero $f \in k[T]$. Extending by multiplicativity to $k(T)^\times$, we get $|\cdot| = |\cdot|_{\infty,c}$ on $k(T)$. ■

There is a strong similarity between the arguments used for the fraction fields of $\mathbf{Z}$ and $k[T]$, so let us unify them as follows. Let A be an arbitrary Dedekind domain with fraction field F. For any maximal ideal $\mathfrak{m}$ of A we have seen long ago how to use the fact that $A_\mathfrak{m}$ is a discrete valuation ring to define the *normalized valuation* $\mathrm{ord}_\mathfrak{m} : F^\times \twoheadrightarrow \mathbf{Z}$ that satisfies the logarithmic property. Moreover, if we define $\mathrm{ord}_\mathfrak{m}(0) = +\infty$ then we also have

$$\mathrm{ord}_\mathfrak{m}(x+y) \geq \min(\mathrm{ord}_\mathfrak{m}(x), \mathrm{ord}_\mathfrak{m}(y))$$

for all $x, y \in F$ because the cases $x = 0$ or $y = 0$ or $x + y = 0$ are obvious, and otherwise we may add $\mathrm{ord}_\mathfrak{m}(a)$ to both sides for suitable $a \in A - \{0\}$ such that $ax, ay \in A$ to reduce to the obvious assertion that if $x, y \in A - \{0\}$ and xA and yA are each divisible by $\mathfrak{m}^m$ (that is, $xA, yA \subseteq \mathfrak{m}^m$) then so is $(x + y)A$. Hence, for any $c \in (0,1)$ we may define the non-trivial non-archimedean absolute value

$$|\cdot|_{\mathfrak{m},c} = c^{\mathrm{ord}_\mathfrak{m}}$$

on F that is *bounded* on A. As we vary c for a fixed $\mathfrak{m}$ these are topologically equivalent, but for distinct $\mathfrak{m}$ and $\mathfrak{n}$ the absolute values $|\cdot|_{\mathfrak{m},c}$ and $|\cdot|_{\mathfrak{n},c'}$ are topologically inequivalent for any $c, c' \in (0,1)$ because of Theorem 1.1.

In the special cases $A = \mathbf{Z}$ and $A = k[T]$ for a field k, this procedure has been seen to give *all* non-trivial absolute values on F that are bounded on A (and hence are non-archimedean), but in both cases we found another topological equivalence class of non-trivial absolute values that are *unbounded* on A (with this extra class archimedean for $A = \mathbf{Z}$ but non-archimedean for $A = k[T]$, and in the latter case there are generally many other absolute values on F that are unbounded on A unless $k^\times$ consists of roots of unity). In general, without some further information about A one cannot hope to determine all of the absolute values on F that are unbounded on A, but we shall now see that under a mild condition on A (that is trivially satisfied for PID's, as well as for the Dedekind domains of interest in number theory) the $|\cdot|_{\mathfrak{m},c}$'s exhaust the set of non-trivial absolute values on A that are bounded on A.

Theorem 4.2. *If the class group* $\mathrm{Pic}(A)$ *is torsion* (*for example, if it is finite*) *then any non-trivial absolute value on* F *that is bounded on* A *must have the form* $|\cdot|_{\mathfrak{m},c}$ *for a unique maximal ideal* $\mathfrak{m}$ *of* A *and* $c \in (0,1)$.

This is the best analogue of Ostrowski's theorem that one can hope to prove for "general" Dedekind domains, as it is hopeless to expect to say much about the absolute values that are unbounded on A in such generality. (However, see Corollary 7.3 for a complete answer when F is a number field.) The reader will observe that our proof of Theorem 4.2 is a refinement of the method used above for the special cases $A = \mathbf{Z}$ and $A = k[T]$.

Proof. Let $|\cdot|$ be a non-trivial absolute value on F such that it is bounded on A. We must have $|a| \leq 1$ for all $a \in A$ because if $|a| > 1$ for some a then $\{|a^n|\}$ is unbounded. By non-triviality we conclude that $|a| < 1$ for some nonzero $a \in A$. Since $\mathrm{Pic}(A)$ is torsion, for each maximal ideal $\mathfrak{m}$ we have $\mathfrak{m}^{e_\mathfrak{m}} = x_\mathfrak{m}A$ for some nonzero $x_\mathfrak{m} \in A$ and some $e_\mathfrak{m} \geq 1$. Using the prime factorization $aA = \prod \mathfrak{m}_i^{e_i}$ with $e_i \geq 1$ for all i, we see that by replacing a with a^n for a suitable $n \geq 1$ we may assume $e_{\mathfrak{m}_i} | e_i$ for all i, and hence $a = \prod x_{\mathfrak{m}_i}^{e'_i}$ for suitable $e'_i \geq 1$. Hence, $|x_\mathfrak{m}| < 1$ for some $\mathfrak{m} = \mathfrak{m}_i$. We claim that this $\mathfrak{m}$ is unique. Indeed, suppose $|x_{\mathfrak{m}'}| < 1$ for some $\mathfrak{m}' \neq \mathfrak{m}$. Since $x_\mathfrak{m}A$ and $x_{\mathfrak{m}'}A$ are powers of distinct maximal ideals, and hence are relatively prime ideals, there exist (necessarily nonzero) $y, y' \in A$ such that

$$x_\mathfrak{m}y + x_{\mathfrak{m}'}y' = 1,$$

and hence

$$1 = |1| \leq \max(|x_\mathfrak{m}||y|, |x_{\mathfrak{m}'}||y'|) < \max(|y|, |y'|) \leq 1,$$

a contradiction.

Since we have $|x_{\mathfrak{m}}| < 1$ for a unique $\mathfrak{m}$, and $x_{\mathfrak{m}}A = \mathfrak{m}^{e_{\mathfrak{m}}}$, we are led to try to compare $|\cdot|$ and $|\cdot|_{\mathfrak{m},c}$ for $c = |x_{\mathfrak{m}}|^{1/e_{\mathfrak{m}}}$. We want to prove that these coincide on F, and by multiplicativity it suffices to compare on $A - \{0\}$. Moreover, to check if $|a|$ and $|a|_{\mathfrak{m},c}$ coincide for some particular nonzero $a \in A$ it suffices to check with a replaced by a^n for any large n. In view of the torsion hypothesis on $\mathrm{Pic}(A)$ and unique factorization for aA, we may therefore restrict attention to those a such that the prime factorization of aA has all prime factors $\mathfrak{m}_i$ appearing with multiplicity divisible by $e_{\mathfrak{m}_i}$. That is, we may assume a is a unit multiple of a finite product of powers of the $x_{\mathfrak{m}_i}$'s. It is therefore enough to consider the cases $a \in A^\times$ and $a = x_{\mathfrak{n}}$ for a maximal ideal $\mathfrak{n}$ of A. This latter case is trivial, since $|x_{\mathfrak{n}}| = 1 = |x_{\mathfrak{n}}|_{\mathfrak{m},c}$ if $\mathfrak{n} \neq \mathfrak{m}$, whereas $|x_{\mathfrak{m}}| = c^{e_{\mathfrak{m}}} = |x_{\mathfrak{m}}|_{\mathfrak{m},c}$ because $\mathrm{ord}_{\mathfrak{m}}(x_{\mathfrak{m}}) = e_{\mathfrak{m}}$. Meanwhile, for $a \in A^\times$ we clearly have $|a|_{\mathfrak{m},c} = 1$ and we also have $|a| = 1$ because $|a|, |1/a| \leq 1$ due to the fact that $a, 1/a \in A$ and $|\cdot| \leq 1$ on A. ■

5. Completion

Let F be a field and let $|\cdot|$ be an absolute value on F. We say that F is *complete with respect to* $|\cdot|$ (or simply that F is *complete* if $|\cdot|$ is understood from context) if F is complete as a metric space via the metric $d(x,y) = |x-y|$. Clearly the trivial absolute value makes F complete, and the completeness property is inherited by any absolute value of the form $|\cdot|^e$ with $e > 0$. Hence, by Theorem 1.1, the completeness property only depends on the topological equivalence class of $|\cdot|$. In general, we wish to form completions exactly as one completes $\mathbf{Q}$ to get $\mathbf{R}$ in elementary analysis.

Definition 5.1. If $(F, |\cdot|)$ and $(F', |\cdot|')$ are fields equipped with absolute values, an *isometry* $i : F \to F'$ is a map of fields such that $|i(x)|' = |x|$ for all $x \in F$.

Since the non-archimedean property is precisely the condition that the absolute value is bounded on the image of $\mathbf{Z}$ in the field, it is clear that in the preceding definition we must have that $(F, |\cdot|)$ is non-archimedean if and only if $(F', |\cdot|')$ is non-archimedean.

Definition 5.2. A *completion* of $(F, |\cdot|)$ is a triple $(F', |\cdot|', i)$ where $(F', |\cdot|')$ is a complete field and $i : F \to F'$ is an isometry that is initial for isometries from F to complete fields. That is, for any isometry $i_1 : F \to F_1$ to a complete field $(F_1, |\cdot|_1)$ the following universal property holds: there exists a unique isometry $j : F' \to F_1$ such that $j \circ i = i_1$.

It is immediate from the universal property that any two completions of $(F, |\cdot|)$ are *uniquely isomorphic* as such (that is, they are uniquely isomorphic via an isometry that respects the given maps from F to each). Hence, it is justified to speak of *the* completion of F if it exists.

Theorem 5.3. *If $(F, |\cdot|)$ is a field equipped with an absolute value, then a completion $(F', |\cdot|', i)$ exists, and the universal map $i : F \to F'$ has dense image. Moreover, if $(F_1, |\cdot|_1)$ is any complete field and $i_1 : F \to F_1$ is an isometry, then it makes $(F_1, |\cdot|_1, i_1)$ into a completion of F (that is, the unique isometry $j : F' \to F_1$ over F is an isomorphism) if and only if i_1 has dense image.*

The absolute value $|\cdot|$ on F is non-archimedean if and only if the absolute value $|\cdot|'$ on the associated completion F' is non-archimedean, in which case the "value groups" $|F^\times|$ and $|F'^\times|'$ in $\mathbf{R}_{>0}$ coincide.

As an example, this theorem says that the usual inclusion $\mathbf{Q} \to \mathbf{R}$ is an isometry with dense image when we use $|\cdot|_\infty$, and so it identifies $(\mathbf{R}, |\cdot|_{\mathbf{R}})$ as the completion of $(\mathbf{Q}, |\cdot|_\infty)$. Also, the final part concerning the value groups is a sharp dichotomy from the classical completion process on $\mathbf{Q}$ (wherein the value group for $|\cdot|_\infty$ on $\mathbf{Q}^\times$ is much smaller than that for the absolute value $|\cdot|_{\mathbf{R}}$ on the completion $\mathbf{R}$).

Proof. We define F' to be the metric space completion of F, and we define $|\cdot|'$ exactly as one extends metrics to the metric space completion. By considering the construction process via Cauchy sequences (as in the construction of $\mathbf{R}$ from $\mathbf{Q}$ as a field with an absolute value), one readily sees how to give F' a field structure with respect to which $|\cdot|'$ is an absolute value extending $|\cdot|$ on F, and that the universal property holds (by first using the universal property of the metric-space completion and then examining the field structures). The alternative characterization of the completion via isometries with dense image follows immediately from the familiar analogue for metric-space completions. Since the completion is an isometric extension field, it is clear that completion preserves the property of being non-archimedean and the property of being archimedean.

It remains to check that the value group for the absolute value does not change in the non-archimedean case. Suppose that $(F, |\cdot|)$ is non-archimedean, and let $x' \in F'$ be an element in the completion. Thus, $x' = \lim x_n$ for some elements $x_n \in F$, and hence by the usual arguments as in elementary analysis

$$|x'|' = \lim |x_n|' = \lim |x_n|.$$

We may assume $x' \neq 0$, so by dropping some initial terms we have $|x_n| > |x'|'/2 > 0$ and $|x_n - x_m| \leq |x'|'/2$ for all $n, m \geq 1$. Hence, $|x_n - x_m| < \max(|x_n|, |x_m|)$ for all $n, m \geq 1$, so due to the non-archimedean triangle inequality we must have $|x_n| = |x_m|$ for all $n, m \geq 1$. Thus, the sequence $\{|x_n|\}$ is stationary and its limit $|x'|'$ is therefore equal to this stationary value. Hence, $|x'|' \in |F^\times|$. ∎

Definition 5.4. Let $(F, |\cdot|)$ be a field equipped with an absolute value. A *normed vector space* over F is a pair $(V, \|\cdot\|)$ consisting of an F-vector space V and a map $\|\cdot\| : V \to \mathbf{R}$ satisfying

- $\|v\| \geq 0$ with equality if and only if $v = 0$,
- $\|cv\| = |c| \cdot \|v\|$ for all $c \in F$ and $v \in V$,
- $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

The normed vector space is *complete* if the metric $d(v, w) = \|v - w\|$ makes V a complete metric space.

The theory of completion for normed vector spaces goes exactly as in the case of completion for fields, and in particular the completion of a normed vector space over F is uniquely endowed with a compatible structure of normed vector space over the completion of F. We leave it to the reader to work out the straightforward details (including the formulation of a universal property). One big change from the classical case over $\mathbf{R}$ is that in this general setting it may be impossible to scale some nonzero vectors to be unit vectors: in principle, the value group for the norm on V may contain elements not in $|F^\times|$ because this latter value group may not (in contrast with the classical case over $\mathbf{R}$ and $\mathbf{C}$) coincide with $\mathbf{R}_{>0}$.

Theorem 5.5. *If F is complete with respect to an absolute value, and V is a finite-dimensional normed vector space over F, then V is complete and every linear subspace is closed. In fact, if $\{v_1, \ldots, v_n\}$ is an F-basis of V then $\|\cdot\|$ is bounded above and below by positive multiples of the (visibly complete!) sup-norm with respect to this basis.*

Hence, the topology on V defined by the norm is independent of the choice of norm, and if $V \neq 0$ then V has the discrete topology if and only if $|\cdot|$ is trivial on F. Moreover, if F is non-archimedean then there is a constant $C > 0$ (depending on the normed vector space) such that $\|v + w\| \leq C \max(\|v\|, \|w\|)$ for all $v, w \in V$.

Proof. We shall argue by induction on $\dim V$, and the 1-dimensional case is immediate from the axiom $\|cv_0\| = |c|\|v_0\|$ for $c \in F$ and $v_0 \in V$. In particular, the completeness of F ensures completeness of V in the 1-dimensional case. Now assume $n = \dim V > 1$ and that the result is known in the lower-dimensional case. If $W \subseteq V$ is any proper linear subspace, then the induced norm makes it a normed vector space that is necessarily *complete* by induction. Hence, it follows by general nonsense for metric spaces that W must be *closed* in V.

Let H_i be the hyperplane spanned by the vectors v_j for $j \neq i$, so H_i is closed in V. We may therefore give $L_i = V/H_i$ a structure of normed vector space by defining $\|\ell\|_i = \inf\{\|v\| \mid v \bmod H_i = \ell\}$ exactly as in the classical theory of normed vector spaces over $\mathbf{R}$ (the point being that this *is* a norm, with $\|\ell\|_i \neq 0$ for $\ell \neq 0$ because H_i is *closed* in V). For any $v = \sum a_j v_j \in V$ we have $\pi_i(v) = a_i \pi_i(v_i)$ with $\pi_i(v_i) \neq 0$, so

$$|a_i| = \frac{\|\pi_i(v)\|_i}{\|\pi_i(v_i)\|_i} \leq B_i \|v\|$$

with $B_i = 1/\|\pi_i(v_i)\|_i > 0$. Taking $B = \max B_i$, we get that the sup-norm relative to the $\{v_i\}$-basis is bounded above by $B\|\cdot\|$. It is trivial to see that, conversely, $\|\cdot\|$ is bounded above by a positive multiple of this sup-norm via the multiplier $(\dim V)(\max \|v_i\|)$. ∎

6. Extending absolute values in the complete case

A very basic question one can ask is whether an absolute value on a field admits an extension to an absolute value on a given extension field, and if so then in how many ways can this be done? The most important case is that of algebraic extensions, and the answer is particularly nice in the case of complete fields. We begin with a uniqueness result.

Lemma 6.1. *If a field F is complete with respect to an absolute value $|\cdot|$ and F'/F is an algebraic extension, then there exists at most one extension of $|\cdot|$ to an absolute value $|\cdot|'$ on F', and if $[F':F]$ is finite then F' is complete with respect to this absolute value. In fact, if $[F':F]$ is finite then necessarily*

$$|x'|' = |\mathrm{N}_{F'/F}(x')|^{1/[F':F]}.$$

We will see later that the absolute value in this lemma always exists; that is, we will prove that the proposed norm-formula for $|\cdot|'$ is an absolute value on F'. Concretely, the problem is that it is not obvious that this formula satisfies the triangle inequality.

Proof. Any algebraic extension is a directed union of finite subextensions, so it suffices to consider the case when F' is finite over F. Also, any possible $|\cdot|'$ on F' extending $|\cdot|$ on F makes F' into a finite-dimensional normed vector space over F, and hence F' must be complete with respect to such an absolute value by Theorem 5.5. We shall first prove uniqueness, and then derive the necessary explicit formula.

To see uniqueness, suppose $|\cdot|'_1$ and $|\cdot|'_2$ are absolute values on F' extending that on F. These give F' a structure of finite-dimensional normed F-vector space, and so by completeness of F and Theorem 5.5 the resulting topology is the same in either case. In particular, if $|\cdot|$ is trivial then this topology is discrete and hence the absolute values on F' must coincide with the trivial absolute value. In case $|\cdot|$ is non-trivial on F, we may argue as follows. Since $|\cdot|'_1$ and $|\cdot|'_2$ are topologically equivalent absolute values on the field F', by Theorem 1.1 each is a power of the other by a positive exponent, say $|\cdot|'_1 = (|\cdot|'_2)^e$ with $e > 0$. Restricting to F gives $|\cdot| = |\cdot|^e$ on F, and since $|\cdot|$ is not identically 1 on $F^\times$ we conclude $e = 1$. Hence, again $|\cdot|'_1 = |\cdot|'_2$.

With uniqueness proved abstractly, we wish to now verify the correctness of the proposed norm formula. In view of transitivity of norms and the proved uniqueness in general, it is sufficient to work in steps of a tower and so it suffices to treat the separate cases when F'/F is separable and when it is purely inseparable of degree p in characteristic $p > 0$. This second case is trivial because one readily checks that $\mathrm{N}_{F'/F}$ is the pth-power map in such cases. In the separable case, consideration with a Galois closure (and the uniqueness of extensions) easily reduces the problem to the Galois case. If we let G denote the Galois group, then $|g(\cdot)|'$ is an absolute value on F' restricting to $|\cdot|$ on F, so it must also equal $|\cdot|'$. Hence, since $\mathrm{N}_{F'/F}(x') = \prod_{g\in G} g(x')$ we have

$$|\mathrm{N}_{F'/F}(x')| = |\mathrm{N}_{F'/F}(x')|' = \prod_{g\in G} |g(x')|' = \prod_{g\in G} |x'|' = (|x'|')^{[F':F]},$$

so we get the desired formula in this case as well. ■

We now dispose of the archimedean case because it is very special:

Theorem 6.2 (Gelfand–Mazur)**.** *The only complete archimedean fields are $(\mathbf{R}, |\cdot|^e_{\mathbf{R}})$ with $0 < e \leq 1$ and $(\mathbf{C}, |\cdot|^e_{\mathbf{C}})$ with $0 < e \leq 1$, where $|\cdot|_{\mathbf{R}}$ and $|\cdot|_{\mathbf{C}}$ are the usual absolute values on $\mathbf{R}$ and $\mathbf{C}$.*

In particular, if $(F, |\cdot|)$ is an archimedean field then F admits an isometric dense embedding into $\mathbf{R}$ or $\mathbf{C}$ with respect to the eth power of the usual absolute value for some unique $0 < e \leq 1$.

This theorem is very old; the "Mazur" in the attribution is not Barry Mazur, nor is it any relative of his.

Proof. Let F be a field complete with respect to an archimedean absolute value $|\cdot|$. Since absolute values on fields with positive characteristic must be non-archimedean, F must have characteristic 0. Hence, F contains $\mathbf{Q}$ and $|\cdot|$ restricts to an absolute value on $\mathbf{Q}$ that must be archimedean. By Ostrowski's theorem (!), this restriction must be $|\cdot|^e_\infty$ for a unique $0 < e \leq 1$. Since $(\mathbf{R}, |\cdot|^e_{\mathbf{R}})$ is the completion of $(\mathbf{Q}, |\cdot|^e_\infty)$, by the universal property of completion we obtain an isometric injection $\mathbf{R} \to F$ with $|\cdot|$ restricting to $|\cdot|^e_{\mathbf{R}}$. If this is surjective, then we are done. Thus, we can assume that it is not surjective. That is, F is a commutative normed field that strictly contains $\mathbf{R}$ and has absolute value extending the absolute value $|\cdot|_{\mathbf{R}}^e$. We seek to show that F is isometrically isomorphic to the field $\mathbf{C}$ equipped with the absolute value $|\cdot|_{\mathbf{C}}^e$.

Using a little complex analysis, it is shown in Theorem 2.3 and Corollary 2.4 in Ch. XII in Lang's *Algebra* (3rd ed.) that F must be $\mathbf{R}$-isomorphic to $\mathbf{C}$ as a field, and so the only problem is to prove that $|\cdot|_{\mathbf{C}}^e$ is the unique absolute value on $\mathbf{C}$ extending $|\cdot|_{\mathbf{R}}^e$ on $\mathbf{R}$. This follows from the uniqueness aspect that has already been proved in the preceding lemma. ■

Corollary 6.3. *Let $(F, |\cdot|)$ be a field equipped with an archimedean absolute value such that the resulting Hausdorff topology on F is locally compact. The field F must be complete with respect to $|\cdot|$, and so is isometric to $(\mathbf{R}, |\cdot|_{\mathbf{R}}^e)$ or $(\mathbf{C}, |\cdot|_{\mathbf{C}}^e)$ for some $0 < e \leq 1$.*

Proof. By the Gelfand–Mazur theorem, F is isometric to a dense subfield of $\mathbf{R}$ or $\mathbf{C}$ (equipped with the eth power of the standard absolute value for some $0 < e \leq 1$). In this way we consider F as a subfield of $\mathbf{C}$ and we merely have to show that F contains $\mathbf{R}$ (as then we have either $F = \mathbf{R}$ or $F = \mathbf{C}$ equipped with the eth power of the usual absolute value, so we have completeness by inspection). Let U be a compact neighborhood of the origin in F, so $U \cap \mathbf{Q}$ contains $(-c, c) \cap \mathbf{Q}$ for some $c \in \mathbf{Q}_{>0}$ since the induced topology on $\mathbf{Q}$ is the usual archimedean one. Since U is compact, it contains the closure of $(-c, c) \cap \mathbf{Q}$ in $\mathbf{C}$, and so it contains the interval $[-c, c]$ in $\mathbf{R}$. Such an interval additively generates $\mathbf{R}$, so F contains $\mathbf{R}$. ■

Now we address the existence aspect for extending absolute values to algebraic extensions of a complete field.

Theorem 6.4. *Let $(F, |\cdot|)$ be a complete field with a non-trivial absolute value. If F'/F is an algebraic extension then $|\cdot|$ admits a unique extension $|\cdot|'$ to an absolute value on F.*

Proof. We have already proved uniqueness in general, and since algebraic extensions are directed unions of finite subextensions it follows that for existence it is enough to handle the case of finite extensions. The case of archimedean fields is trivial via the Gelfand–Mazur theorem, so we may restrict attention to the non-archimedean case. Motivated by the necessary formula, we *define*

$$|x'|' = |\mathrm{N}_{F'/F}(x')|^{1/[F':F]}$$

for $x' \in F'$. Obviously $|x'|' \geq 0$ with equality if and only if $x' = 0$, $|x|' = |x|$ for $x \in F$, and $|x'y'|' = |x'|'|y'|'$ for all $x', y' \in F'$ (and $|1|' = 1$). The problem is to prove the triangle inequality, and since $|\cdot|$ is non-archimedean it is necessary that $|\cdot|'$ satisfies the non-archimedean triangle inequality. That is, we want $|x' + y'|' \leq \max(|x'|', |y'|')$ for all $x', y' \in F'$. We may assume $|y'|' \geq |x'|'$ with $y' \neq 0$, so by dividing through by $|y'|'$ and using multiplicativity it is equivalent to check $|x' + 1|' \leq 1$ when $|x'|' \leq 1$. That is, if $|\mathrm{N}_{F'/F}(x')| \leq 1$ then we claim that $|\mathrm{N}_{F'/F}(x' + 1)| \leq 1$.

By transitivity of norms it suffices to treat the case $F' = F(x') = F(x' + 1)$. The norm of a primitive element is (up to sign) the constant term in its minimal polynomial over F, and hence our problem is to show that if $f = \sum a_j T^j \in F[T]$ is a monic irreducible polynomial with $|f(0)| \leq 1$ then $|f(-1)| \leq 1$ (since $f(T - 1)$ is the minimal polynomial of $x + 1$ if $f(T)$ is the minimal polynomial of x). In fact, we claim that *all* coefficients a_j of f satisfy $|a_j| \leq 1$. Assuming to the contrary, if some $|a_j| > 1$ then if $|a_{j_0}|$ is maximal (so $0 < j_0 < \deg f$ since $|f(0)| \leq 1$ and f is monic) then $g = (1/a_{j_0})f \in F[T]$ has all coefficients $c_j = a_j/a_{j_0}$ satisfying $|c_j| \leq 1$ yet its leading coefficient and constant term have absolute value < 1 while some intermediate coefficient (in degree j_0) has absolute value 1.

Let $A = \{x \in F \mid |x| \leq 1\}$ be the valuation ring and let k be the associated residue field. The conditions on g imply that $g \in A[T]$ and its reduction $\overline{g} \in k[T]$ is nonzero with constant term zero and degree $< \deg g$, so there is a factorization $\overline{g} = T^r H_0(T)$ where $H_0(0) \neq 0$ (possibly $H_0 \in k^\times$). Since T^r and H_0 are relatively prime in $k[T]$, by Theorem 6.5 below it follows that the factorization of $\overline{g}$ over k lifts reasonably to a factorization of g over A. More precisely, there is a degree-r monic polynomial $h \in A[T]$ with reduction T^r in $k[T]$ and another polynomial $H \in A[T]$ with reduction H_0 in $k[T]$ such that $g = hH$. Since $\deg h = r \leq \deg \overline{g} < \deg g$, necessarily $\deg H > 0$ (and in fact $\deg H > \deg H_0$). Thus, $g \in F[T]$ is reducible, and this is a contradiction because g is an $F^\times$-multiple of the polynomial $f \in F[T]$ that is irreducible. ■

Theorem 6.5 (Hensel's Lemma)**.** *Let F be a field complete with respect to a non-archimedean absolute value $|\cdot|$. Let A be the associated valuation ring, and k the residue field. Let $g \in A[T]$ be a polynomial with nonzero reduction $\overline{g} \in k[T]$, and suppose there is a factorization $\overline{g} = h_0 H_0$ in $k[T]$ with h_0 monic and $\gcd(h_0, H_0) = 1$ in $k[T]$. There exists a factorization $g = hH$ in $A[T]$ with h monic such that $\overline{h} = h_0$ and $\overline{H} = H_0$ in $k[T]$.*

Note the monicity ensures $\deg h = \deg h_0$. Clearly $\deg H > \deg H_0$ if and only if $\deg g > \deg \overline{g}$ (that is, if and only if the leading coefficient of g is not in $A^\times$). The traditional applications are with monic g, in which case $\deg H = \deg H_0$, but in the preceding proof we required the result in a case with g having a leading coefficient that is not a unit in the valuation ring.

Proof. The case of a trivial absolute value is trivial, so we may and do assume that the absolute value is non-trivial. We also may and do assume $\deg h_0 > 0$. The idea of the proof is to construct the desired factorization by successive approximation. Choose monic $h_1 \in A[T]$ reducing to h_0 (so $\deg h_1 = \deg h_0$) and any $H_1 \in A[T]$ lifting H_0 with $\deg H_1 = \deg H_0 = \deg \overline{g} - \deg h_0$, so $\deg H_1 \leq \deg g - \deg h_0$. The relative primality of h_0 and H_0 provides the existence of *nonzero* $r_0, s_0 \in k[T]$ such that

$$h_0 r_0 + H_0 s_0 = 1$$

in $k[T]$. Hence, if we choose $r, s \in A[T]$ lifting $r_0, s_0 \in k[T]$ such that r and s have unit leading coefficients (that is, $\deg r = \deg r_0$ and $\deg s = \deg s_0$), we have

$$h_1(T) r(T) + H_1(T) s(T) = 1 + c\phi(T)$$

with $\phi \in A[T]$ and nonzero $c \in A$ satisfying $|c| < 1$. (Indeed, $hr + Hs - 1 \in A[T]$ has reduction 0 in $k[T]$, so we can take c to be a coefficient with maximal absolute value if not all such coefficients vanish, and otherwise we can take $\phi = 0$ and choose any $c \in F$ with $0 < |c| < 1$ since $|\cdot|$ is non-trivial on F). We also have $g - h_1 H_1 \in A[T]$ with reduction 0 in $k[T]$. If $g - h_1 H_1 = 0$ then there is nothing to do, and otherwise we can pick a nonzero coefficient $c' \in A$ of $g - h_1 H_1$ with maximal absolute value, so $|c'| < 1$. By letting π denote whichever of the nonzero c or c' has a larger absolute value, we clearly have

$$h_1(T) r(T) + H_1(T) s(T) = 1 + \pi\psi(T), \quad g - h_1 H_1 = \pi\theta(T)$$

for $\psi, \theta \in A[T]$ and $\pi \in A$ satisfying $0 < |\pi| < 1$.

Let $I = \{x \in A \,|\, |x| \leq |\pi|\}$; this is a proper nonzero ideal in A. We shall work in A/I^n for increasing $n \geq 1$. For each $n \geq 1$ we will find a sequence of polynomials $h_n \in A[T]$ and $H_n \in A[T]$ (giving h_1 and H_1 as above for $n = 1$) such that all h_n's are monic, $\deg H_n \leq \deg g - \deg h_0$ for all $n \geq 1$, and

$$h_{n+1} \equiv h_n \bmod I^n, \;\; H_{n+1} \equiv H_n \bmod I^n,$$

and $g \equiv h_n H_n \bmod I^n$ for all $n \geq 1$. The monicity forces $\deg h_n = \deg h_1 = \deg h_0$. We have already given the construction for $n = 1$, and we now argue by induction to construct h_{n+1} and H_{n+1} as required.

We seek to take $h_{n+1} = h_n + \pi^n q$ and $H_{n+1} = H_n + \pi^n Q$ for $q, Q \in A[T]$ with $\deg q < \deg h_0$ and $\deg Q \leq \deg g - \deg h_0$ (so h_{n+1} is monic and lifts h_n modulo I^n, and H_{n+1} has degree $\leq \deg g - \deg h_0$ and lifts H_n modulo I^n). To find q and Q, we note that

$$h_{n+1} H_{n+1} \equiv h_n H_n + \pi^n (q H_n + Q h_n) \bmod I^{n+1}$$

because $\pi^{2n} \in I^{2n} \subseteq I^{n+1}$. By hypothesis $h_n H_n \equiv g \bmod I^n$, which is to say $h_n H_n = g + \pi^n \rho_n$ for some $\rho_n \in A[T]$, and clearly we must have $\deg \rho_n \leq \deg g$ because $\deg h_n = \deg h_0$ and $\deg H_n \leq \deg g - \deg h_0$. Thus, it is sufficient to find q and Q with respective degrees $< \deg h_0$ and $\leq \deg g - \deg h_0$ such that

$$q H_n + Q h_n \equiv -\rho_n \bmod I.$$

Note that this final condition on q and Q only uses their images in $(A/I)[T]$, so our problem is really to solve this congruence as an equality in $(A/I)[T]$ with solutions for q and Q in $(A/I)[T]$ that have respective degrees $< \deg h_0$ and $\leq \deg g - \deg h_0$ (allowing for either such polynomial modulo I to be 0, say via the convention $\deg 0 = -\infty$), and then we may choose $q, Q \in A[T]$ to be lifts of these solutions having the same degree.

By the inductive hypothesis $H_n \equiv H_1 \bmod I$ and $h_n \equiv h_1 \bmod I$, so it is equivalent to say

$$qH_1 + Qh_1 \equiv -\rho_n \bmod I$$

with $q, Q \in A[T]$ whose images in $(A/I)[T]$ have respective degrees $< \deg h_0$ and $\leq \deg g - \deg h_0$ (allowing for either reduction modulo I to be 0 as well). Recall that there exist $r, s \in A[T]$ such that

$$h_1 r + H_1 s \equiv 1 \bmod I.$$

Hence, multiplying through by $-\rho_n$ gives *some* $q_n, Q_n \in A[T]$ such that

$$q_n H_1 + Q_n h_1 \equiv -\rho_n \bmod I$$

and we are faced with the problem that the mod-I reductions of q_n and Q_n might have degrees that are too large. Since h_1 is *monic*, we can carry out long division by h_1 in $A[T]$ to write $q_n = h_1 u(T) + q(T)$ for $u, q \in A[T]$ with $\deg q < \deg h_1$ (possibly $q = 0$). Hence, we may replace q_n with q and Q_n with $Q_n + uH_1$ to arrive at a congruence

$$qH_1 + Qh_1 \equiv -\rho_n \bmod I$$

with $\deg q < \deg h_1 = \deg h_0$.

It remains to check that the inequality $\deg(Q \bmod I) \leq \deg g - \deg h_0$ holds. Since h_1 is monic, even though $(A/I)[T]$ may not be a domain we may still compute (via the convention $\deg 0 = -\infty$ if necesary)

$$\deg(Q \bmod I) + \deg(h_1 \bmod I) = \deg(Qh_1 \bmod I) = \deg(-\rho_n - qH_1 \bmod I) \leq \deg g$$

because $\rho_n \in A[T]$ has degree at most $\deg g$ and $qH_1 \in A[T]$ has degree

$$\deg q + \deg H_1 \leq \deg h_0 + \deg H_1 \leq \deg h_0 + (\deg g - \deg h_0) = \deg g.$$

Thus, $\deg(Q \bmod I) \leq \deg g - \deg(h_1 \bmod I) = \deg g - \deg h_1 = \deg g - \deg h_0$.

To summarize for all $n \geq 1$, we have contructed a monic $\widetilde{h}_n \in (A/I^n)[T]$ and an $\widetilde{H}_n \in (A/I_n)[T]$ with degree $\leq \deg g - \deg h_0$ that respectively lift $h_0, H_0 \in k[T]$ such that $\widetilde{h}_{n+1} \bmod I^n = \widetilde{h}_n$, $\widetilde{H}_{n+1} \bmod I^n = \widetilde{H}_n$, and $g \bmod I^n = \widetilde{h}_n \widetilde{H}_n$ in $(A/I^n)[T]$ for all $n \geq 1$. In view of the definition

$$I = \{x \in F \mid |x| \leq |\pi|\}$$

with $0 < |\pi| < 1$ and the *completeness* of F, it follows that the natural map $A \to \varprojlim A/I^n$ is an *isomorphism*. Hence, since the $\widetilde{h}_n$'s are monic (with fixed degree $\deg h_0$) and the $\widetilde{H}_n$'s have bounded degrees ($\leq \deg g - \deg h_0$), we may therefore pass to the inverse limit on coefficients in each fixed degree to find unique monic $h \in A[T]$ with degree $\deg h_0$ and unique $H \in A[T]$ with degree $\leq \deg g - \deg h_0$ such that $h \bmod I^n = \widetilde{h}_n$ and $H \bmod I^n = \widetilde{H}_n$ for all n. Hence, $hH \bmod I^n = \widetilde{h}_n \widetilde{H}_n = g \bmod I^n$ for all n, so $g = hH$ in $A[T]$ as desired. ■

Here is an immediate consequence that is often also called Hensel's Lemma.

Corollary 6.6. *Let F be a field that is complete with respect to a non-archimedean absolute value, and let A and k be the associated valuation ring and residue field. If $f \in A[T]$ is a monic polynomial and its reduction $\overline{f} \in k[T]$ has a simple root $\overline{\alpha} \in k$, then there exists a unique $\alpha \in A$ that is a root of f and reduces to $\overline{\alpha}$, and it is a simple root of f over F.*

Proof. By hypothesis we have a factorization $\overline{f} = (T - \overline{\alpha})h_0$ into relatively prime factors in $k[T]$ with $T - \overline{\alpha}$ monic of degree 1, so Theorem 6.5 lifts this to a factorization $f = (T - \alpha)h$ in $A[T]$ where $\alpha \in A$ lifts $\overline{\alpha}$. Since $h(\alpha) \in A$ reduces to $\overline{h}(\overline{\alpha}) = h_0(\overline{\alpha}) \neq 0$ in k, clearly $h(\alpha) \in A^\times$. Hence, α is a simple root of f. To prove uniqueness of this root lifting $\overline{\alpha}$, suppose $\alpha' \neq \alpha$ is a root of f. Clearly $h(\alpha') = 0$ in A, and hence $h_0(\overline{\alpha}') = 0$ in k. Since h_0 and $T - \overline{\alpha}$ are relatively prime in $k[T]$, it follows that $\overline{\alpha}' \neq \overline{\alpha}$. ■

7. Extending absolute values without assuming completeness

We now prove a very important result when the completeness hypothesis is dropped:

Theorem 7.1. *Let F be a field equipped with an absolute value $|\cdot|$, and let F'/F be an algebraic extension. There exists an absolute value $|\cdot|'$ on F' extending $|\cdot|$, and if F'/F is a Galois extension then the action of* $\mathrm{Gal}(F'/F)$ *on the set of such absolute values extending $|\cdot|$ is transitive.*

Proof. For the existence aspect, let $\widehat{F}$ be the completion of F, and let $K/\widehat{F}$ be an algebraic closure. By Theorem 6.4, the absolute value on $\widehat{F}$ extends uniquely to K. Since K/F is an algebraically closed extension and F'/F is algebraic, there exists an F-embedding of F' into K, and so by using the absolute value on K and a choice of such an embedding we get an absolute value on F' that restricts to $|\cdot|$ on F. This settles existence.

Now we assume that F'/F is Galois, and we pick two absolute values $|\cdot|'_1$ and $|\cdot|'_2$ on F' that extend $|\cdot|$. We seek to prove the existence of $g \in \mathrm{Gal}(F'/F)$ such that $|g(\cdot)|'_1 = |\cdot|'_2$. We first give a routine technical argument to reduce the general case to the finite-degree case. Note that F' is the directed union of its finite Galois subextensions F'_i/F. For each such F'_i, let $\Sigma_i \subseteq \mathrm{Gal}(F'_i/F)$ be the set of elements that carry $|\cdot|'_1$ on F' to $|\cdot|'_2$ on F'. If we can prove that Σ_i is non-empty for every i, then

$$\varprojlim \Sigma_i \subseteq \varprojlim \mathrm{Gal}(F'_i/F) = \mathrm{Gal}(F'/F)$$

is an inverse limit of non-empty finite sets and hence it is non-empty. Any element σ in this set of compatible systems of elements of the Σ_i's is an F-automorphism of F' that carries $|\cdot|'_1$ to an absolute value $|\sigma(\cdot)|'_1$ whose restriction to each F'_i coincides with the restriction of $|\cdot|'_2$, and hence $|\sigma(\cdot)|'_1 = |\cdot|'_2$ as desired.

It remains to handle the transitivity for Galois actions when F' is a finite Galois extension of F. We first need to give a convenient description of the distinct absolute values on F' extending the one on F.

Lemma 7.2. *Let F be a field equipped with an absolute value $|\cdot|$, and let F'/F be a finite separable extension, so F' is a finite étale F-algebra. Let $\widehat{F}$ be the completion of F, and consider the unique decomposition*

$$\widehat{F} \otimes_F F' \simeq \widehat{F}'_1 \times \cdots \times \widehat{F}'_n$$

of the finite étale $\widehat{F}$-algebra $\widehat{F} \otimes_F F'$ into a finite product of finite separable extensions of $\widehat{F}$. Endow each $\widehat{F}'_i$ with its unique absolute value extending that on $\widehat{F}$, and let $|\cdot|'_i$ be its restriction to F'.

The $|\cdot|'_i$'s are topologically inequivalent absolute values on F' extending the one given on F, and the map $F' \to \widehat{F}'_i$ is the completion of F' with respect to $|\cdot|'_i$. Moreover, the set of $|\cdot|'_i$'s is the set of absolute values on F' extending the one given on F.

Proof. The case of the trivial absolute value is trivial (as then $\widehat{F} = F$), so we now assume that $|\cdot|$ is non-trivial. Let us first check that the $|\cdot|'_i$'s are topologically inequivalent on F'. Assuming to the contrary, Theorem 1.1 provides $e > 0$ such that $|\cdot|'_i = (|\cdot|'_j)^e$ on F' with $i \neq j$, and restricting this to F gives $|\cdot| = |\cdot|^e$. Non-triviality of $|\cdot|$ then forces $e = 1$. Thus, $|\cdot|'_i$ and $|\cdot|'_j$ have the same restriction to F'. To deduce a contradiction from this, we need to first work out the completion of F' with respect to each $|\cdot|'_i$ (and this will have no reliance on our assumption of topological equivalence from which we will soon deduce a contradiction).

The map $F' \to \widehat{F}'_i$ is an isometry if we use $|\cdot|'_i$ on F', and to check that this computes the completion for $|\cdot|'_i$ we need to show that the image of F' is dense. Clearly $\widehat{F}'_i$ is a finite-dimensional normed vector space over $\widehat{F}$, and it is a quotient of $\widehat{F} \otimes_F F'$. This tensor product is endowed with a normed vector space structure over $\widehat{F}$ by using the sup-norm with respect to an $\widehat{F}$-basis induced by an F-basis of F', and the quotient map to $\widehat{F}'_i$ is $\widehat{F}$-linear and hence continuous. Thus, the denseness of the image of F' in $\widehat{F}'_i$ is a consequence of the denseness of F' in $\widehat{F} \otimes_F F'$ with respect to the topology described by the sup-norm defined by an F-basis of F'; this latter denseness is obvious by coordinate-chasing because F has dense image in $\widehat{F}$.

Since $\widehat{F}'_i$ has been identified with the $|\cdot|'_i$-completion of F', if $|\cdot|'_i$ and $|\cdot|'_j$ coincide on F' with $i \neq j$ then by uniqueness of completions we obtain a unique isometric isomorphism $\widehat{F}'_i \simeq \widehat{F}'_j$ as F'-algebras, and by denseness considerations it is also an isomorphism as $\widehat{F}$-algebras (over the F-algebra structures), and hence as $\widehat{F} \otimes_F F'$-algebras. But the quotient maps from $\widehat{F} \otimes_F F'$ onto $\widehat{F}'_i$ and $\widehat{F}'_j$ are *distinct* (maximal) ideals of the tensor product, and if R is any commutative ring with distinct ideals I and J then R/I and R/J cannot be isomorphic as R-modules (let alone as R-algebras) due to comparison of their annihilator ideals. This completes the proof that $|\cdot|'_i$ and $|\cdot|'_j$ are topologically inequivalent on F' for $i \neq j$.

Now let $|\cdot|'$ be any absolute value on F' extending the one on F. It remains to prove that $|\cdot|' = |\cdot|'_i$ for some i. We will do this by studying completions. Let $F' \to \widehat{F}'$ be the $|\cdot|'$-completion. By the universal property of completion, there is a unique isometry $\widehat{F} \to \widehat{F}'$ over $F \to F'$, and so we get an $\widehat{F}$-algebra map $\widehat{F} \otimes_F F' \to \widehat{F}'$. The kernel is an $\widehat{F}$-linear subspace, and hence is closed. The image of this linear map is a finite-dimensional normed $\widehat{F}$-vector space using both the quotient topology from $\widehat{F} \otimes_F F'$ (so it is complete) and the subspace topology from $\widehat{F}'$. The two associated norms are each bounded by a positive multiple of the other, due to finite-dimensionality over $\widehat{F}$ and Theorem 5.5. (Beware that we have not yet proved that $[\widehat{F}' : \widehat{F}]$ is finite.) Hence, by completeness via the quotient point of view we conclude that the subspace topology makes this image a complete metric subspace in $\widehat{F}'$. This image is therefore a closed subset and yet it clearly contains the subset F' that is dense in the completion $\widehat{F}'$. We conclude that the map $\widehat{F} \otimes_F F' \to \widehat{F}'$ is *surjective*. In particular, $\widehat{F}'$ has *finite* dimension over $\widehat{F}$.

This surjective map onto a field thereby identifies $\widehat{F}'$ with one of the factor fields $\widehat{F}'_i$ of $\widehat{F} \otimes_F F'$ as extensions of $\widehat{F}$, and its absolute value through $|\cdot|'$-completion and its absolute value via $|\cdot|'_i$ define the same topology because both give the $\widehat{F}$-vector space $\widehat{F}'$ a structure of normed vector space over the complete field $\widehat{F}$ and this vector space is finite-dimensional. We conclude that the absolute values on $\widehat{F}'$ corresponding to $|\cdot|'$ and $|\cdot|'_i$ define the same topologies, and hence (by Theorem 1.1) there exists $e > 0$ such that $|\cdot|' = (|\cdot|'_i)^e$ via the above identification $\widehat{F}'_i \simeq \widehat{F}'$ as $\widehat{F}$-algebras. By restricting this equality to F we get $|\cdot| = |\cdot|^e$, so by non-triviality of the absolute value on F we conclude $e = 1$. Hence, we have constructed an *isometric* isomorphism $\widehat{F}'_i \simeq \widehat{F}'$, and thus the absolute value $|\cdot|'$ on F' must coincide with one of the $|\cdot|'_i$'s. ■

Returning to the situation of interest, namely a finite Galois extension F'/F, the preceding lemma tells us that the absolute values on F' extending the one given on F are obtained from the factor fields of the finite étale $\widehat{F}$-algebra $\widehat{F} \otimes_F F'$. There is a natural action of $\mathrm{Gal}(F'/F)$ on this $\widehat{F}$-algebra, and this action must permute the set of primitive idempotents, so it permutes the factor fields. In view of the recipe in the lemma for making all absolute values on F' extending the one on F via such factor fields, it follows that $\sigma \in \mathrm{Gal}(F'/F)$ carries $|\cdot|'_i$ to $|\cdot|'_j$ on F' if and only if its action on $\widehat{F} \otimes_F F'$ carries $\widehat{F}'_i$ to $\widehat{F}'_j$. Hence, our problem is translated into that of proving that the action of $\mathrm{Gal}(F'/F)$ on $\widehat{F} \otimes_F F'$ is transitive on the set of primitive idempotents.

We are now reduced to the following purely algebraic problem. Let F'/F be a finite Galois extension with Galois group G and let L/F be an arbitrary extension field. We claim that the G-action on the finite étale L-algebra $L \otimes_F F'$ is transitive on the set of primitive idempotents. If L'/L is an extension field, then since

$$L \otimes_F F' \to L' \otimes_L (L \otimes_F F') = L' \otimes_F F'$$

is faithfully flat (or by using bare-hands methods) it follows (by decomposing into factor fields) that each primitive idempotent of $L \otimes_F F'$ is "induced" by one of $L' \otimes_F F'$ (through projection to a suitable factor field). Hence, to solve the problem for L is it enough to solve the problem for L'. In this way, we may replace L with a finite extension so that there exists an F-embedding $i : F' \hookrightarrow L$. Using this i, we get an L-algebra isomorphism

$$L \otimes_F F' = L \otimes_{F'} (F' \otimes_F F') \simeq L \otimes_{F'} (\prod_{g \in G} F')$$

where the isomorphism $F' \otimes_F F' \simeq \prod_{g \in G} F'$ of left F'-algebras is defined by $a \otimes b \mapsto (ag(b))_g$ (and the F'-structure on $\prod_{g \in G} F'$ for this F'-algebra isomorphism is via the diagonal action). Hence, we get an

F'-algebra isomorphism

$$L \otimes_F F' \simeq \prod_{g \in G} L \otimes_{F', i \circ g^{-1}} F' \simeq \prod_{g \in G} L$$

where the gth factor on the right side is an extension of F' via the F-embedding $i \circ g^{-1}$. Under this identification, the action of $h \in G$ on $L \otimes_F F'$ goes over to the action of h on the set of F-embeddings $i \circ g^{-1}$ via $i \circ g^{-1} \mapsto i \circ (hg)^{-1} = (i \circ g^{-1}) \circ h^{-1}$. In other words, we have the standard left G-action on the non-empty set $\mathrm{Hom}_{F-\mathrm{alg}}(F', L)$, and this is transitive because F'/F is Galois with Galois group G. ■

We are now in position to classify *all* non-trivial absolute values on a number field K. For each maximal ideal $\mathfrak{p}$ of $\mathscr{O}_K$ we define the topological equivalence class $|\cdot|_{\mathfrak{p},c}$ (with $c \in (0,1)$) as after the proof of Theorem 4.1. The role of $\mathrm{ord}_{\mathfrak{p}}$ in the definition implies that the common valuation ring for this equivalence class is the algebraic localization $\mathscr{O}_{K,\mathfrak{p}}$, so the residue field is $\kappa(\mathfrak{p})$. As we shall see later, a preferred choice of c to give a representative $|\cdot|_{\mathfrak{p},c}$ for this equivalence class is $c = 1/\mathrm{N}\mathfrak{p} = 1/\#\kappa(\mathfrak{p})$, and so we define the associated *standard* $\mathfrak{p}$*-adic absolute value* on K:

$$|\cdot|_{\mathfrak{p}} \stackrel{\mathrm{def}}{=} |\cdot|_{\mathfrak{p},1/\#\kappa(\mathfrak{p})} = (1/\#\kappa(\mathfrak{p}))^{\mathrm{ord}_{\mathfrak{p}}};$$

this is a preferred representative for the topological equivalence class. (Beware that if $\mathfrak{p}$ lies over $p\mathbf{Z}$ then $|\cdot|_{\mathfrak{p}}$ restricts to $|\cdot|_p^{e(\mathfrak{p}|p\mathbf{Z})f(\mathfrak{p}|p\mathbf{Z})}$ on $\mathbf{Q}$ and not to $|\cdot|_p$ on $\mathbf{Q}$, as we see through evaluation at p.)

In the special case $K = \mathbf{Q}$ and $\mathfrak{p} = p\mathbf{Z}$ for a positive prime $\mathbf{Z}$, note that the "standard" normalization as above recovers the standard p-adic absolute value $|\cdot|_p$ on $\mathbf{Q}$. You may think it would be better to change the normalization so that $|\cdot|_{\mathfrak{p}}$ restricts to $|\cdot|_p$ on $\mathbf{Q}$, but there are several reasons for selecting $|\cdot|_{\mathfrak{p}}$ as the preferred choice as a standard normalization. Perhaps the most compelling reason for the declaration of $|\cdot|_{\mathfrak{p}}$ as the "standard" representative of the $\mathfrak{p}$-adic topological equivalence class is given through arguments with Haar measures on the associated completion of K, as we shall see later on. A "global" justification will be given in §8.

Corollary 7.3. *Let K be a number field. Every non-trivial non-archimedean absolute value on K is topologically equivalent to $|\cdot|_{\mathfrak{p}}$ for a unique maximal ideal $\mathfrak{p}$ of $\mathscr{O}_K$. Up to topological equivalence, every archimedean absolute value on K is induced by an embedding $K \hookrightarrow \mathbf{C}$, and two such embeddings give rise to topologically equivalent absolute values if and only if they coincide or are related through the unique non-trivial* $\mathbf{R}$*-automorphism of* $\mathbf{C}$ *(complex conjugation).*

Proof. First consider the archimedean case. If $|\cdot|$ is an archimedean absolute value on K then it restricts to one on $\mathbf{Q}$, and hence by Ostrowski's theorem $|\cdot|$ restricts to $|\cdot|_\infty^e$ for some $0 < e \leq 1$. The completion of $(\mathbf{Q}, |\cdot|_\infty^e)$ is $(\mathbf{R}, |\cdot|_{\mathbf{R}}^e)$. By Lemma 7.2, it follows that $|\cdot|$ is induced by mapping K into some *unique* factor field $\widehat{K}$ of the finite étale $\mathbf{R}$-algebra $K \otimes_{\mathbf{Q}} \mathbf{R}$, with $\widehat{K}$ given its unique absolute value extending $|\cdot|_{\mathbf{R}}^e$ on $\mathbf{R}$. We know that the factor fields are $\mathbf{R}$-isomorphic to either $\mathbf{R}$ or $\mathbf{C}$, and that in the second case $|\cdot|_{\mathbf{C}}^e$ is the unique absolute value extending $|\cdot|_{\mathbf{R}}^e$ on $\mathbf{R}$. Hence, we conclude that $\widehat{K}$ is isometrically $\mathbf{R}$-isomorphic to either $(\mathbf{R}, |\cdot|_{\mathbf{R}}^e)$ or $(\mathbf{C}, |\cdot|_{\mathbf{C}}^e)$, and in this second case the isometric $\mathbf{R}$-isomorphism is clearly unique up to $\mathrm{Aut}(\mathbf{C}/\mathbf{R})$ since this automorphism group preserves $|\cdot|_{\mathbf{C}}^e$. To summarize, $|\cdot|$ is induced by an inclusion $K \hookrightarrow \mathbf{C}$ that is *unique up to complex conjugation*, with $\mathbf{C}$ given the absolute value $|\cdot|_{\mathbf{C}}^e$. If we instead use $|\cdot|_{\mathbf{C}}$ then we get $|\cdot|^{1/e}$ as an absolute value on K in the same topological equivalence class. This completes the classification in the archimedean case.

Now we turn to the case when $|\cdot|$ is a non-trivial non-archimedean absolute value on K. To prove that $|\cdot| = |\cdot|_{\mathfrak{p},c}$ for some $c \in (0,1)$ and some maximal ideal $\mathfrak{p}$ of $\mathscr{O}_K$, first note that if K'/K is a finite extension of K then $|\cdot|$ extends to K' by Theorem 7.1 and any $|\cdot|_{\mathfrak{p},c}$ is topologically equivalent to the restriction of any $|\cdot|_{\mathfrak{p}',c'}$ for a maximal ideal $\mathfrak{p}'$ of $\mathscr{O}_{K'}$ because $\mathrm{ord}_{\mathfrak{p}'}$ on K' restricts to $e(\mathfrak{p}'|\mathfrak{p})\mathrm{ord}_{\mathfrak{p}}$ on K. Hence, it suffices to solve the problem upon replacing K with a finite extension, so we may assume that K is Galois over $\mathbf{Q}$.

Since $|\cdot|$ restricts to a non-trivial non-archimedean absolute value on $\mathbf{Q}$, Ostrowski's theorem implies that this restriction is $|\cdot|_{p,b}$ for a unique positive prime p and $b \in (0,1)$. For any prime $\mathfrak{p}$ of $\mathscr{O}_K$ over $p\mathbf{Z}$, any $|\cdot|_{\mathfrak{p}}$ restricts to a non-trivial non-archimedean absolute value on $\mathbf{Q}$ that is non-trivial on p, and hence by Ostrowki's theorem this restriction must be in the p-adic equivalence class. In other words, the restriction of

$|\cdot|$ to $\mathbf{Q}$ is topologically equivalent to the restriction of some $|\cdot|_{\mathfrak{p}}$, and so by replacing the non-archimedean $|\cdot|$ with a suitable power we may suppose that it has the same restriction to $\mathbf{Q}$ as does some $|\cdot|_{\mathfrak{p}}$. By Theorem 7.1, the action of $\mathrm{Gal}(K/\mathbf{Q})$ is transitive on the set of absolute values on K having a common restriction to $\mathbf{Q}$, so there exists $\sigma \in \mathrm{Gal}(K/\mathbf{Q})$ such that $|\cdot| = |\sigma(\cdot)|_{\mathfrak{p}} = |\cdot|_{\sigma^{-1}(\mathfrak{p})}$. ■

8. The product formula

Let K be a number field. As we have explained above, the topological equivalence classes of non-trivial non-archimedean absolute values on K are in natural bijective correspondence with the maximal ideals of $\mathscr{O}_K$, and for each such $\mathfrak{p}$ we have singled out a preferred representative absolute value $|\cdot|_{\mathfrak{p}}$. For each topological equivalence class of archimedean absolute values, we have seen that the associated completion is isomorphic to $\mathbf{R}$ or $\mathbf{C}$ as a topological field (endowed with a power of the standard absolute value), and we may therefore speak of *real* absolute values and *complex* absolute values depending on the associated completion of K. In these respective cases we have seen that there is a *unique* representative absolute value for the topological equivalence class that arises from an embedding $K \hookrightarrow \mathbf{C}$, with $|\cdot|_{\mathbf{C}}$ used on $\mathbf{C}$, and that this embedding is unique up to complex conjugation (so it is unique precisely in the real case). In this way, even for the archimedean case we have singled out a preferred representative from each topological equivalence class by demanding that the completion of K be *isometrically* isomorphic to $(\mathbf{R}, |\cdot|_{\mathbf{R}})$ in the real case and $(\mathbf{C}, |\cdot|_{\mathbf{C}})$ in the complex case.

The preceding paragraph provides a recipe for singling out preferred representatives from every topological equivalence class of non-trivial absolute values on K. This set of absolute values on K is the set of *normalized* non-trivial absolute values on K. It is traditional to write the typical such absolute value in the form $|\cdot|_v$ with an indexing parameter v (for "valuation"), and to write $v|\infty$ if $|\cdot|_v$ is archimedean and $v \nmid \infty$ if $|\cdot|_v$ is non-archimedean. Our goal is to show that the set of $|\cdot|_v$'s is not merely an *ad hoc* set, but that the normalized absolute values are globally linked through an elegant *product formula*. (As we have noted earlier, the "right" way to understand the reason for these preferred choices, at least in the non-archimedean case, is to carry out considerations with Haar measure, but we omit that in these notes.) Observe that for any $x \in K^\times$, $|x|_{\mathfrak{p}} = 1$ for all but finitely many $\mathfrak{p}$ because $\mathrm{ord}_{\mathfrak{p}}(x) = 0$ for all but finitely many $\mathfrak{p}$. Hence, the infinite product $\prod_{v\nmid\infty} |x|_v$ is a finite product and so it makes sense for any $x \in K^\times$.

Lemma 8.1. *Let F'/F be a finite separable extension of fields and let $|\cdot|$ be an absolute value on F. Let $|\cdot|'_1, \dots, |\cdot|'_n$ be the set of distinct absolute values on F' extending $|\cdot|$, and let $\widehat{F}'_i$ denote the $|\cdot|'_i$-completion of F' and $\widehat{F}$ denote the $|\cdot|$-completion of F. For any $x' \in F'$,*

$$\prod_i (|x'|'_i)^{[\widehat{F}'_i:\widehat{F}]} = |\mathrm{N}_{F'/F}(x')|.$$

Proof. By Lemma 7.2, we have a canonical $\widehat{F}$-algebra isomorphism $\widehat{F} \otimes_F F' \simeq \widehat{F}'_1 \times \cdots \times \widehat{F}'_n$ with $|\cdot|'_i$ induced by the inclusion $F' \to \widehat{F}'_i$. Hence, by Lemma 6.1 and Theorem 6.4, $|x'|'_i = |\mathrm{N}_{\widehat{F}'_i/\widehat{F}}(x')|^{1/[\widehat{F}'_i:\widehat{F}]}$ where we also write $|\cdot|$ and $|\cdot|'_i$ to denote the absolute values on $\widehat{F}$ and $\widehat{F}'_i$ respectively. Thus, $(|x'|'_i)^{[\widehat{F}'_i:\widehat{F}]} = |\mathrm{N}_{\widehat{F}'_i/\widehat{F}}(x')|$. Taking the product over all i gives

$$\prod_i (|x'|'_i)^{[\widehat{F}'_i:\widehat{F}]} = |\prod_i \mathrm{N}_{\widehat{F}'_i/\widehat{F}}(x')| = |\mathrm{N}_{(\prod_i \widehat{F}'_i)/\widehat{F}}(x')| = |\mathrm{N}_{(\widehat{F}\otimes_F F')/\widehat{F}}(1 \otimes x')|.$$

This final norm in $\widehat{F}$ is exactly $\mathrm{N}_{F'/F}(x') \in F$ by compatibility of norm (as a determinant!) with respect to extension of the base field. Hence, we get the desired equality. ■

Now we specialize to the case when $F = K$ is a number field. Fix a positive prime p and a prime $\mathfrak{p}$ of $\mathscr{O}_K$ over p. We have noted earlier that $|\cdot|_{\mathfrak{p}}$ restricts to $|\cdot|_p^{e_{\mathfrak{p}} f_{\mathfrak{p}}}$ on $\mathbf{Q}$, with $e_{\mathfrak{p}} = e(\mathfrak{p}|p\mathbf{Z})$ and $f_{\mathfrak{p}} = f(\mathfrak{p}|p\mathbf{Z})$. Hence, $|\cdot|'_{\mathfrak{p}} = |\cdot|_{\mathfrak{p}}^{1/e_{\mathfrak{p}} f_{\mathfrak{p}}}$ restricts to $|\cdot|_p$ and as we vary $\mathfrak{p}$ over $p\mathbf{Z}$ the absolute values $|\cdot|'_{\mathfrak{p}}$ vary through all absolute values on K that extend $|\cdot|_p$. Thus, by Lemma 8.1 we have

$$\prod_{\mathfrak{p}|p}(|x|'_{\mathfrak{p}})^{[K_{\mathfrak{p}}:\mathbf{Q}_p]} = |\mathrm{N}_{K/\mathbf{Q}}(x)|_p$$

with $\mathbf{Q}_p$ and $K_{\mathfrak{p}}$ denoting the corresponding completions of $\mathbf{Q}$ and K. Since $|\cdot|_p$ and $|\cdot|_{\mathfrak{p}}$ have discrete value groups on $\mathbf{Q}^\times$ and $K^\times$ respectively, the same holds for the absolute values on the completions $\mathbf{Q}_p$ and $K_{\mathfrak{p}}$, due to Theorem 5.3. Thus, the valuation rings of $\mathbf{Q}_p$ and $K_{\mathfrak{p}}$ are discrete valuation rings, by Theorem 2.5.

Our later study of the structure of complete non-archimedean fields whose valuation ring is a discrete valuation ring (such as $\mathbf{Q}_p$ and $K_{\mathfrak{p}}$) will yield the crucial identity

$$[K_{\mathfrak{p}} : \mathbf{Q}_p] = e_{\mathfrak{p}} f_{\mathfrak{p}},$$

and we accept this identity for present purposes. Using this identity, the degree-term in the exponent of the above product formula at p *cancels* the reciprocal exponent $1/e_{\mathfrak{p}} f_{\mathfrak{p}}$ in the definition of $|\cdot|'_{\mathfrak{p}}$ as a power of $|\cdot|_{\mathfrak{p}}$. Hence, we conclude

$$\prod_{\mathfrak{p}|p} |x|_{\mathfrak{p}} = |\mathrm{N}_{K/\mathbf{Q}}(x)|_p$$

for all $x \in K$. This formula can also be proved directly, thereby eliminating the need to appeal to the identity $[K_{\mathfrak{p}} : \mathbf{Q}_p] = e_{\mathfrak{p}} f_{\mathfrak{p}}$ that we have not yet proved, as follows. We may assume $x \in K^\times$, and by multiplicativity in K it suffices to consider nonzero $x \in \mathscr{O}_K$. In this case the product over $\mathfrak{p}|p$ is exactly the power of p that generates the p-part of the norm ideal $\mathrm{N}_{\mathscr{O}_K/\mathbf{Z}}(x\mathscr{O}_K)$, and the right side is the p-part of the principal ideal $\mathrm{N}_{K/\mathbf{Q}}(x)\mathbf{Z}$. Hence, the desired identity is just the p-part of the general identity $\mathrm{N}_{A'/A}(a'A') = \mathrm{N}_{F'/F}(a')A$ that computes the ideal-norm of a principal ideal (relative to a finite extension of Dedekind domains that is separable on fraction fields) as the principal ideal in the base generated by the norm of a generator of the given principal ideal upstairs.

In the archimedean case there is no analogue for the above considerations with the e's and f's and p-parts (as there are no discrete valuation rings linked with the absolute values), and hence we are simply stuck with the formula

$$\prod_{v|\infty} |x|_v^{[K_v:\mathbf{R}]} = |\mathrm{N}_{K/\mathbf{Q}}(x)|_\infty$$

from Lemma 8.1 for $x \in K$, where the completion K_v is $\mathbf{R}$-isomorphic to $\mathbf{R}$ or $\mathbf{C}$ depending on whether or not v is real or complex (by definition of such terminology). Observe that if there is an $\mathbf{R}$-isomorphism $K_v \simeq \mathbf{C}$ then there are exactly *two* such $\mathbf{R}$-isomorphisms, and hence $|x|_v^{[K_v:\mathbf{R}]} = |x|_v^2 = |\sigma_v(x)|_v |\sigma'_v(x)|_v$ where $\sigma_v, \sigma'_v : K \rightrightarrows \mathbf{C}$ is the pair of non-real conjugate embeddings that define $|\cdot|_v$. Hence, we do have the mildly more appealing expression $\prod_v |x|_v^{[K_v:\mathbf{R}]} = \prod_{j:K\to\mathbf{C}} |j(x)|_{\mathbf{C}}$.

Theorem 8.2 (Product Formula)**.** *Let K be a number field. For $x \in K^\times$,*

$$\prod_{v|\infty} |x|_v^{[K_v:\mathbf{R}]} \cdot \prod_{v\nmid\infty} |x|_v = \prod_{j:K\to\mathbf{C}} |j(x)|_{\mathbf{C}} \cdot \prod_{v\nmid\infty} |x|_v = 1.$$

For many identities in number theory involving all archimedean absolute values, taken up to topological equivalence, one often finds complex absolute values showing up through their squares. This has led some authors to make an *ad hoc* modification to the general definition of "absolute value" (by allowing a coefficient multiplier in the triangle inequality, such as 2) for the sole purpose of being allowed to consider $|\cdot|_{\mathbf{C}}^2$ as an absolute value. This is not a compelling modification of general terminology, and hence the presence of the squares is simply a fact of life on the archimedean side that should not be absorbed into a modification of the general concept of absolute value.

Proof. In view of the preceding calculations, if we let $q = \mathrm{N}_{K/\mathbf{Q}}(x) \in \mathbf{Q}^\times$ then the two infinite products coincide with $|q|_\infty \cdot \prod_p |q|_p$, so we are reduced to the special case $K = \mathbf{Q}$. Since $\mathbf{Q}^\times$ is multiplicatively generated by ± 1 and the positive primes, by multiplicativity it suffices to check that product formula on such elements. The case of roots of unity is trivial, and the case of a positive prime p_0 goes as follows:

$$|p_0|_\infty \cdot \prod_p |p_0|_p = |p_0|_\infty |p_0|_{p_0} = p_0 \cdot (1/p_0) = 1.$$

■

9. Weak approximation

Let F be a field and let $|\cdot|_1, \dots, |\cdot|_n$ be topologically inequivalent non-trivial absolute values on F. Let F_i denote the completion of F with respect to $|\cdot|_i$, so there is a natural map of rings $F \to \prod_i F_i$. It is natural to ask: is the image of this map dense? That is, for any choices of $x_i \in F_i$ and $\varepsilon_i > 0$ for all i does there exist $x \in F$ such that $|x - x_i|_i < \varepsilon_i$ for all i? This is called the *weak approximation property* for the $|\cdot|_i$'s. If F is the fraction field of a Dedekind domain A and the $|\cdot|_i$'s arise from distinct maximal ideals $\mathfrak{m}_i$ of A, then simple arguments with the Chinese Remainder Theorem yield the weak approximation property.

For applications in number theory it is desired to permit the possibility that one or more of the $|\cdot|_i$'s may be archimedean, in which case we have to abandon the methods of commutative algebra. Thus, we are motivated to try to approach the problem from an entirely different perspective that has nothing to do with Dedekind domains and is instead entirely "valuation-theoretic" in spirit.

Theorem 9.1 (Artin–Whaples). *Let F be a field and let $|\cdot|_1, \dots, |\cdot|_n$ be a set of pairwise topologically inequivalent non-trivial absolute values on F. The image of F in $\prod_i F_i$ is dense.*

Proof. Choose elements $x_i \in F_{v_i}$ and $\varepsilon_i > 0$. We seek $x \in F$ such that $|x - x_i|_i < \varepsilon_i$ for all i. Suppose that for each i_0 we can construct elements $c_{i_0} \in F$ such that $|c_{i_0}|_{i_0} > 1$ but $|c_{i_0}|_i < 1$ for all $i \neq i_0$. In this case, the elements $c_{i_0,n} = c_{i_0}^n/(1 + c_{i_0}^n)$ in F tend to 0 in F_i for $i \neq i_0$ and tend to 1 in F_{i_0}. Thus, the elements $c_n = \sum x_i c_{i,n} \in F$ tend to x_{i_0} in F_{i_0} for every i_0, so $x = c_N$ for sufficiently large N does the job.

Our problem now is to construct the elements $c_{i_0} \in F$, and this has nothing to do with the x_i's or ε_i's. By relabelling i_0 as 1, we seek $c \in F$ such that $|c|_1 > 1$ and $|c|_i < 1$ for all $i \neq 1$. The case when the set of places has size $m = 1$ is trivial, so we may assume $m > 1$. By topological inequivalence, there exist $a, a' \in F$ such that $|a|_1 < 1$ with $|a|_m \geq 1$, and $|a'|_1 \geq 1$ with $|a'|_m < 1$ (so $a, a' \neq 0$). Hence, $|a'/a|_1 > 1$ and $|a'/a|_m < 1$. This settles the case $m = 2$, and in general we use induction to suppose that there exists $b \in F$ such that $|b|_1 > 1$ and $|b|_i < 1$ for $2 \leq i < m$. Thus, $|b^n a'/a|_1 > 1$ and $|b^n a'/a|_i < 1$ for $2 \leq i < m$ if we take n to be large enough. If $|b|_m \leq 1$ then $|b^n a'/a|_m < 1$ for any positive n, and so we may take $c = b^n a'/a$ for large n. If $|b|_m > 1$ then $b^n/(1 + b^n)$ tends to 1 in F_m and in F_1, and it tends to 0 in F_i for $2 \leq i < m$, so for $c = (b^n/(1 + b^n))a'/a$ with large n we have $|c|_i < 1$ for $2 \leq i \leq m$ (as $|a'/a|_m < 1$ to handle $i = m$) and $|c|_1 > 1$ because $|a'/a|_1 > 1$. ■