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We prove here a special case of Borel's general structure thm for Hopf algebras.

Let $H = \bigoplus_{i \geq 0} H^i$ be a graded anticomm K-algebra with K a perfect field, $H^0 = K$, and assume that H is finitely generated, say by nonzero homog elements x_i, $\deg(x_i) > 0$. We say $\{x_i\}$ is <u>minimal</u> if

(1) $\deg(x_1) \leq \deg(x_2) \leq \dots$

(2) x_i is not a poly in $\{x_j\}_{j \neq k}$

(3) $ht(x_k) \geq ht(x_k + P(x_1, \dots, x_{k-1}))$ $\forall$ $P \in K[T_1, \dots, T_{k-1}]$,

where for $h \in H$, $ht(h) = \inf\{n \in \mathbb{Z}^+ \mid h^n = 0, h^{n-1} \neq 0\}$, with $h^0 \equiv 1$. We have $ht = \infty$ allowed.

We say H is a <u>Hopf algebra</u> if $\exists$ a map $f: H \to H \otimes_K H$ of graded K-algebras such that $\forall$ homog $h \in H$ nonconstant

$$f(h) = h \otimes 1 + 1 \otimes h + x_1 \otimes y_1 + \dots + x_s \otimes y_s$$

with x_i, y_i homog, $\deg(x_i), \deg(y_i) > 0$, $\deg(x_i) + \deg(y_i) = \deg(h)$.

<u>Thus</u> Let $\{x_i\}_{i=1}^m$ be a min set of generators of H, $h_i = ht(x_i)$ (maybe ∞). Then $x_1^{r_1} \cdots x_m^{r_m}$, $0 \leq r_i < h_i$ are a K-basis of H, so H is the assoc graded anticomm alg. gen'td by $X_1, \dots, X_m$, (with $\deg(X_i) \equiv \deg(x_i)$) modulo relation $X_i^{h_i} = 0$ for $h_i < \infty$

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Pf: Note first that for prime $p>0$,

$$\binom{r}{s} \equiv 0 \mod p \quad \forall\ 0<s<r$$

iff $r=1$ or $r=p^n$ for some $n \geq 1$.

Also, for homog $x \in H$,

$p \neq 2$, $\deg(x)$ odd $\Rightarrow x \cdot x = -x \cdot x \Rightarrow x \cdot x = 0$
$\Rightarrow ht(x) = 2$.

and $p=2$ or $\deg(x)$ even (or $x=0$) $\Rightarrow x$ lies in center of H.

Thus, $(x+y)^p = x^p + y^p \quad \forall\ x,y \in H$ (take cases, noting first by inducting on # of homog components that WLOG x,y are homog, nonzero and $x+y \neq 0$).

Define I_k to be ideal in $H \otimes H$ gen'td by $x_1 \otimes 1, \ldots, x_k \otimes 1$ (so $I_0 = 0$). Since $\{\deg(x_i)\}$ is monotone increasing

$$f(x_k) \equiv x_k \otimes 1 + 1 \otimes x_k \mod I_{k-1}$$
$$f(x_i) \equiv 1 \otimes x_i \mod I_{k-1} \quad \forall\ 1 \leq i < k.$$

~~Define~~

Thus,

$$(*) \quad f(x_k^{r_k} \cdots x_1^{r_1}) = f(x_k)^{r_k} \cdots f(x_1)^{r_1}$$
$$\equiv (x_k \otimes 1 + 1 \otimes x_k)^{r_k} \cdot (1 \otimes (x_{k-1}^{r_{k-1}} \cdots x_1^{r_1})) \mod I_{k-1}$$
$$\equiv x_k^{r_k} \otimes (x_{k-1}^{r_{k-1}} \cdots x_1^{r_1}) + \sum_{i=0}^{r_k - 1} \binom{r_k}{i} x_k^i \otimes x_k^{r_k - i} x_{k-1}^{r_{k-1}} \cdots x_1^{r_1} \mod I_{k-1}$$

We define a <u>normal monomial in H</u> to be an element $x_k^{r_k} \cdots x_1^{r_1}$ with $0 < r_k < h_k$, $0 \le r_i < h_i$ for $1 \le i < k$. This is either 0 or else has degree $\sum_{i=1}^{k} r_i \deg(x_i)$. We define a <u>normal monomial in $H \otimes H$</u> to be an element $a \otimes b$ with $a, b \in H$ normal monomials. We need to show that normal monomials are nonzero and are lin indep over K. Clearly it suffices to consider only those with $\sum_{i=1}^{k} r_i \deg(x_i) = n$ for fixed $n \in \mathbb{Z}^+$, and we'll induct on n. The case $n=1$ is clear from condition (2) in defn of minimality, ~~and case~~ 0

So now we may assume ~~result~~ that result holds in degrees $< n$, $n > 1$.

Thus, ~~the natural maps of K-vector spaces~~ in H the elements in $\Sigma_n = \left\{ x_k^{r_k} \cdots x_1^{r_1} \;\middle|\; \begin{array}{l} 0 < r_k < h_k,\ 0 \le r_i < h_i\ \forall 1 \le i < k \\ 1 \le k \le m \\ \sum_{i \le k} r_i \deg(x_i) < n \end{array} \right\}$

are K-lin indep and $\Sigma_n^{\otimes} = \{ a \otimes b \in H \otimes H \mid a, b \in \Sigma_n \}$ are K-lin indep too.

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and a <u>normal monomial</u> in $K[T_k,\ldots,T_1]$ is one of form $T_k^{r_k}\ldots T_1^{r_1}$ with $0<r_k<h_k$, $0\le r_i<h_i$.

Now we assume have a relation

$$P(x_k,\ldots,x_1) = \sum_{0\le r_i<h_i}' a_{r_k\ldots r_1} x_k^{r_k}\ldots x_1^{r_1} = 0$$

with $a_{\underline{r}}\in K$ not all zero and have usual lexicographical ordering, with some $r_k\neq 0$ term appearing. We call such an expression a 'normal polynomial' in H. We also assume wrt lex ordering this is a 'minimal' such example (based on 'leading' term). We can write

$$P = T_k^r\, Q(T_{k-1},\ldots,T_1) + R \quad \text{in } K[T_k,\ldots,T_1]$$

with $r = \max_{a_{\underline{r}}\neq 0} r_k > 0$ and $\deg_{T_k}(R) < r$.

With obvious rule for evaluation (ie, write elts in $K[T_k,\ldots,T_1]$ as K-lin comb of $T_k^{e_k}\ldots T_1^{e_1}$), we have

$$0 = P(x_k,\ldots,x_1) = x_k^r\, Q(x_{k-1},\ldots,x_1) + R(x_k,\ldots,x_1). \quad \text{in } H.$$

<u>Claim</u> $Q = \text{const}$ (nec nonzero by defn of r, R) and minimality of P

and $r = \begin{cases} 1 & \text{if } p=0 \\ \text{power of } p \text{ (incl. 1)} & \text{if } p>0. \end{cases}$

<u>Pf:</u> By $(*)$,

$$f(x_k^r Q) = x_k^r\otimes Q + \sum_{0\le i<r}\binom{r}{i} x_k^i\otimes x_k^{r-i} Q \quad \text{mod } I_{k-1}$$

~~From defining property of f and fact that ... having terms of degree~~

Since $P=0$, so $f(P)=0$, we have

$$x_k^r \otimes Q + \sum_{0\le i<r} \binom{r}{i} x_k^i \otimes (x_k^{r-i} Q) \equiv -f(R)$$

in $(H/(x_1,\dots,x_{k-1}))\otimes H$.

By $(*)$ and $\deg_{T_k}(R) < r$, we have that in $(H/(x_1,\dots,x_{k-1}))\otimes H$, $-f(R)$ is a K-linear combination of elements of the form $x_k^i \otimes x_k^j a$ with a normal in $x_1,\dots,x_{k-1}$ and $i+j<r$. Since $\{x_k^e \mid 0\le e \le r\}$ are K-lin indep in $H/(x_1,\dots,x_{k-1})$ if $\underline{\deg Q>0}$ (due to minimality of P and degree considerations), we conclude that if $\underline{\deg Q>0}$ then

$x_k^r \otimes Q = 0$ in $H\otimes H$. But $x_k^r \neq 0$ $\because$ $r<h_k$ and $Q(x_{k-1},\dots,x_1)\neq 0$ by minimality of P, so have $\Rightarrow\Leftarrow$.

$\therefore$ $Q=$ const, so WLOG $Q=1$.

~~Now we have n = r deg(x_k)~~

Now

$$f(X_k) = X_k \otimes 1 + 1 \otimes X_k + \sum a_i \otimes b_i$$

with $0 < \deg a_i, \deg b_i < \deg(x_i)$

and a_i, b_i are normal monomials in $T_1, \ldots, T_{k-1}$ (if any a_i, b_i occur).

Thus,

$$f(X_k^r) = f(X_k)^r = X_k^r \otimes 1 + 1 \otimes X_k^r + \sum_{0<i<r} \binom{r}{i} X_k^i \otimes X_k^{r-i} + \sum \alpha_j \otimes \beta_j$$

$$= (-R) \otimes 1 + 1 \otimes (-R) + \sum_{0<i<r} \binom{r}{i} X_k^i \otimes X_k^{r-i} + \sum \alpha_j \otimes \beta_j$$

where α_j, β_j are normal monomials in $T_k, \ldots, T_1$ with $\deg_{T_k} \alpha_j + \deg_{T_k} \beta_j \underline{\underline{\leq r}}$.

We compute $f(R)$ similarly, and get $f(R) = \sum_i \gamma_j \otimes \delta_j$ with γ_j, δ_j normal monomials in $T_k, \ldots, T_1$ with $\deg_{T_k}(\gamma_j) + \deg_{T_k}(\delta_j) \leq \deg_{T_k}(R) \underline{\underline{\leq r}}$.

Thus, in the equality

$$-f(X_k^r) = f(R)$$

we have expressions for both sides in the form $\sum u_\ell \otimes v_\ell$ with u_ℓ, v_ℓ normal monomials in $T_k, \ldots, T_1$ with $\deg_{T_k}(u_\ell), \deg_{T_k}(v_\ell) < r$.

Thus, by minimality of $P = T_k^r - R$, we conclude that all of these a_ℓ, v_ℓ monomials remain lin indep in H, so can read off lin indep in $H \otimes H$.
(i.e, safe to think in $K[T_k, \dots, T_1]$)

Since $\deg_{T_k} \alpha_j + \deg_{T_k} \beta_j,\ \deg_{T_k}(\gamma_j) + \deg_{T_k}(\delta_j) < r$, we conclude that all terms

$\binom{r}{i} x_k^i \otimes x_k^{r-i}$ for $0 < i < r$ must vanish, which is to say $\binom{r}{i} = 0$ in K $\forall\, 0 < i < r$.

$\therefore$, $r = 1$ if $p = 0$ and $r \in p^{\mathbb{N}}$ if $p > 0$.

QED Claim.

Set $Q = 1$ WLOG.

If $r = 1$, then $\deg_{T_k} R < r = 1$, so $x_k = -R(x_{k-1}, \dots, x_1) = -R(x_{k-1}, \dots, x_1)$, contrary to minimality condition (2).

This settles $p = 0$, so now we may assume $p > 0$ and $r = p^t$, $t \geq 1$,

$$P = T_k^r + R(x_k, \dots, x_1), \qquad \deg_{T_k}(R) < r$$

We'll show that for each of the normal

monomials in R (which can be viewed in H or $K[T_\bullet]$, as lin indep of these is same in each case, as all have $\deg_{T_k} < r$ and P is a 'minimal' counterexample), all exponents are divisible by r and $\deg_{T_k} R = 0$. Since $r = p^t$ and K is perfect, we conclude (using $(a+b)^p = a^p + b^p$ in H) that $R = R_1^r$ for $R_1 \in K[T_{k-1}, \ldots, T_1]$ a sum of normal monomials, whence in H

$$0 = P = X_k^r + R(X_{k-1}, \ldots, X_1)$$
$$= X_k^r + (R_1(X_{k-1}, \ldots, X_1))^r \quad \text{(since } r \text{ is } p^{th} \text{ power and } (a+b)^p = a^p + b^p \text{ in } H\text{)}$$
$$= (X_k + R_1(X_{k-1}, \ldots, X_1))^r.$$

Thus, $ht(X_k) = h_k > r \geq ht(X_k + R_1(X_{k-1}, \ldots, X_1))$, contrary to condition (3) in minimality of $\{x_i\}$. Thus, we'd be done.

So remains to show

<u>Claim</u> $\deg_{T_k} R = 0$ and all (normal) monomials appearing in R are r^{th} powers of (nice normal) monomials.

<u>Pf:</u> We do decreasing induction wrt lexigraphical ordering on the normal monomials appearing in R, so have in $K[T_k, \ldots, T_1]$; ~~in degree~~

$$P = (T_k + S(T_{k-1}, \ldots, T_1))^r + T_j^t Q(T_{j-1}, \ldots, T_1) + U(T_j, \ldots, T_1)$$

with $j \le k$, $\deg_{T_j}(U) < t$, $Q \neq 0$, and all monomials in $T_j^t Q + U$ are less than those in S^r, with S a K-lin comb of normal monomials (maybe 0, at beginning).

In H have <u>same</u> identity, all in degree n (in particular, $S(x_{k-1}, \ldots, x_1) \in H^{\deg(x_k)}$ component).

We have

$$f(x_k + S(x_{k-1}, \ldots, x_1)) = (x_k + S) \otimes 1 + 1 \otimes (x_k + S) + \sum \xi_i \, a_i \otimes b_i,$$

with $\xi_i \in K^\times$, a_i, b_i normal monomials in $x_{k-1}, \ldots, x_k$ with degree $< \deg(x_k)$ (this $<$ is why no x_k's appear).

$\therefore f((x_k + S)^r) = (x_k + S)^r \otimes 1 + 1 \otimes (x_k + S)^r + \sum \xi_i^r a_i^r \otimes b_i^r$.

~~We can (and do)~~

~~We can (and do) drop all terms with a_i^r or $b_i^r = 0$.~~

~~Note $(x_k + S)^r \neq 0$, since otherwise ht$(e_k) = h_k > r \geq$ ht$(x_k + S(x_{k-1}, \ldots, x_1))$ violates condition (3) in minimality.~~

Since $T_j^t Q + U$ is 'less' than the 'minimal' counterexample P, all (necc normal) monomials in it remain lin indep when viewed in H. Assume $Q \neq$ const, so we can let a be the 'biggest' normal monomial in Q wrt lexicographical ordering; so when we compute $f(x_j^t Q + U) \in H \otimes H$ in form $\sum \xi_j \tilde{a}_j \otimes \tilde{b}_j$ with $\{\tilde{a}_j\}, \{\tilde{b}_j\}$ lin indep sets of normal monomials, we have one term of $\tilde{\xi}\, x_j^t \otimes a$, with $\tilde{\xi} \in K^\times$ the coeff of a in Q.

Since $f(x_j^t Q + U) = -f((x_k + S)^r)$ (as $f(P) = 0$),

~~comparing with~~

$$= -(x_k+S)^r \otimes 1 - 1 \otimes (x_k+S)^r \pm \sum \xi_i^r a_i^r \otimes b_i^r$$

~~and no normal monomials~~

$$= -(x_k^r + S^r) \otimes 1 - 1 \otimes (x_k^r + S^r) - \sum_i \xi_i^r a_i \otimes b_i^r$$

is another expression in terms of normal monomials (w/some repetitions perhaps) and these are K-indep due to minimality of P (note $T_j^t Q + U \neq 0$ by hypothesis), when we collect like terms. — Note indeed that $a_i(x_{k-1}, \dots, x_1)$, $b_i(x_{k-1}, \dots, x_1)$ have degree $< \deg(x_k)$, so r^{th} powers have degree $< r \cdot \deg(x_k) = n$ (and in particular $a_i^r, b_i^r \neq 0$).

The only possible contribution of $x_j^t \otimes a$ has to be from some $a_i^r \otimes b_i^r$ since either $j < k$ or, if $j = k$, then $t < r$ ($\because \deg_{T_k} R < r$).

$\therefore$, for some i have

$$x_j^t = a_i^r, \quad a = b_i^r$$

so $T_j^t a = (a_i b_i)^r$ is r^{th} power of a (normal) monomial in $K[T_{k+1}, \dots, T_1]$ (recall a_i, b_i have no x_k contribution).

Suck this $a_i b_i$ into S.

Continuing in this way, it remains to consider the possibility that $Q = c \in K$ nec nonzero ($\because Q \neq 0$, WLOG, as noted at beginning). Thus, $j < k$ and

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$t\deg(x_j) = r\deg(x_k) \geq r\deg(x_j) \Rightarrow r \leq t$.

If t is a power of p, then $r \mid t$, so x_j^t is r^{th} power and so is coeff $c \in K^\times$ ($\because K$ is perfect), so can suck this into S.

Next, if t is not a power of p, then $f(c \cdot x_j^t)$ has nonzero term of form

$$c \cdot \binom{t}{q} x_j^q \otimes x_j^{t-q} \quad (0 < q < t)$$

relative to usual normal monomial 'basis' (important that $j < k$ so lin indep is not of problem in H for expansion of $f(c \cdot x_j^t)$).

Since $f(U)$ has expansion relative to this 'basis' and contributes no terms like $x_j^q \otimes x_j^{t-q}$ with total degree t in x_j, we conclude that $x_j^q \otimes x_j^{t-q}$ is one of the 'basis' terms appearing in $f((x_j+S)^r) = -f(c \cdot x_j^t + U)$. Since $j < k$, we ~~again~~ have (by reasoning as earlier) that for some i, $x_j^q = a_i^r$, $x_j^{t-q} = b_i^r$, so in this case

$$x_j^t = x_j^q \cdot x_j^{t-q} = (a_i b_i)^r$$

is an r^{th} power of a monomial in $x_{k-1}, \ldots, x_1$ after all (this actually forces $t \geq r$, ~~but~~ hence $a \Rightarrow \in$, but that doesn't matter). Suck this into S.

Keep going.

QED Claim

QED Thm.

<u>Remark</u> The pf doesn't require H to be a finite type, but even just countably generated over K.