

1. MOTIVATION

Let A be an abelian variety over a field k . For each $n \geq 1$ we have constructed a bi-additive duality pairing

$$\langle \cdot, \cdot \rangle_{A,n} : A[n] \times \widehat{A}[n] \rightarrow \mu_n$$

that is functorial in A . Recall the concrete description of this on T -valued points for a k -scheme T : if $a \in A[n](T)$ is an n -torsion point in $A(T)$ and $a' \in \widehat{A}[n](T)$ corresponds to a line bundle \mathcal{L} on A_T admitting a trivializing section σ for $[n]^*(\mathcal{L})$ over A_T , the *canonical* isomorphism

$$t_a^*([n]^*(\mathcal{L})) \simeq [n]^*(\mathcal{L})$$

carries $t_a^*(\sigma)$ to $\langle a, a' \rangle_{A,n} \cdot \sigma$. (Note that this unit multiplier is *independent* of the choice of σ , as any two choices are related through scaling by a unit on A_T , or equivalently by a unit on T , and units from T pass through t_a^* to cancel out on both sides of the requirement $t_a^*(\sigma) \mapsto u \cdot \sigma$ for a specified unit u on T .)

Here we aim to analyze “functoriality in n ”, by which we mean the relationship between $\langle \cdot, \cdot \rangle_{A,n}$ and $\langle \cdot, \cdot \rangle_{A,mn}$ for any $m, n \geq 1$. Namely, we seek to prove the identity

$$\langle ma, ma' \rangle_{A,n} = \langle a, a' \rangle_{A,mn}^m$$

in $\mu_n(T)$ for $a \in A[mn](T)$ and $a' \in \widehat{A}[mn](T)$ for any k -scheme T (as then taking m and n to be powers of a prime ℓ allows us to pass to the inverse limit to get a perfect pairing between ℓ -adic Tate modules when $\ell \neq \text{char}(k)$ and between Dieudonné modules when $\ell = \text{char}(k)$, the latter being a powerful tool in the study of abelian varieties in positive characteristic).

In case m and n are not divisible by $\text{char}(k)$, it suffices to check the desired identity on geometric points since it involves only finite étale k -group schemes. However, if $\text{char}(k) | mn$ (or if we seek a method that can work more generally for an abelian scheme over a base ring of mixed characteristic) then computing with geometric points is inadequate. In Mumford’s book, the first Proposition of §20 settles the case when n and m are not divisible by $\text{char}(k)$, by computations using the description of such pairings in terms of the classical language of divisors and rational functions.

We want to allow general $n, m \geq 1$ both for aesthetic satisfaction and also because the robust theory of pairings including the p -part in characteristic p is an essential tool in the study of abelian varieties in positive characteristic (through the role of p -divisible groups in deformation theory, the determination by Tate of the local invariants at p -adic places for the endomorphism algebras of simple abelian varieties over finite fields of characteristic p , etc.). So below we adapt Mumford’s method to the framework of line bundles rather than divisors in order that we get the result without divisibility restrictions on m and n .

2. THE COMPUTATION

Choose $a \in A(T)$ that is mn -torsion and choose a line bundle \mathcal{L} on A_T for which $[mn]^*(\mathcal{L})$ admits a trivializing section σ over A_T . The choice of σ (which is implicit when specifying a point in $\text{Pic}_{A/k,e}(T)$, though any two choices are related through a unique automorphism of the line bundle) specifies a trivialization $\sigma(e)$ of \mathcal{L} along the identity section over T , and hence specifies a point $a' \in \widehat{A}[mn](T)$.

Since $[m]_{\widehat{A}} = ([m]_A)^\wedge$, so $\mathcal{L}^{\otimes m} \simeq [m]^*(\mathcal{L})$ (the latter isomorphism being *uniquely determined* by the condition that it carries $\sigma(e)^{\otimes m}$ to $[m]^*(\sigma(e))$ as trivializations along e_T), the n -torsion point $ma' \in \widehat{A}[n](T)$ corresponds to $[m]^*(\mathcal{L})$. In particular, σ may be identified with a

trivializing section of the $[n]$ -pullback of the line bundle $[m]^*(\mathcal{L})$ corresponding to ma' via the canonical isomorphism $[mn]^*(\mathcal{L}) \simeq [n]^*([m]^*(\mathcal{L}))$ (which expresses how multiplication by m on \widehat{A} carries $\widehat{A}[mn](T)$ into $\widehat{A}[n](T)$). Hence, our task is to prove that the canonical isomorphism $t_{ma}^*([n]^*([m]^*(\mathcal{L}))) \simeq [n]^*([m]^*(\mathcal{L}))$ carries $t_{ma}^*(\sigma)$ to $u^m \cdot \sigma$ for the unit $u := \langle a, a' \rangle_{A, mn}$ over T that satisfies $t_a^*(\sigma) \mapsto u\sigma$ via the canonical isomorphism $t_a^*([mn]^*(\mathcal{L})) \simeq [mn]^*(\mathcal{L})$ of line bundles on A_T .

Informally, the idea is that t_{ma} is the m -fold composition of copies of t_a , so iterating m times the “equality” $t_a^*(\sigma) = u \cdot \sigma$ for a unit u on T should then give the result that $t_{ma}^*(\sigma) = u^m \cdot \sigma$. The mild complication is that we are actually working with an *isomorphism* $t_{ma}^*([mn]^*(\mathcal{L})) \simeq [mn]^*(\mathcal{L})$, so the “equality” of trivializing sections is really expressing a compatibility of sections relative to this isomorphism. Hence, we have to keep track of sections under a composition of isomorphisms among line bundles rather than iterate an actual equality of sections in a single line bundle.

[This highlights a way in which the classical setup with divisors and rational functions is genuinely simpler,: the “ $\mathcal{O}_A(D)$ ” language involves a generic trivialization that turns isomorphisms into actual equalities at the level of rational functions, allowing one to literally carry out the “iterated equality” idea without any need to fuss with isomorphisms.]

For any $x \in A[mn](T)$, let $\theta_x : t_x^*([mn]^*(\mathcal{L})) \simeq [mn]^*(\mathcal{L})$ be the canonical isomorphism expressing the equality $[mn] \circ t_x = [mn]$ of endomorphisms of the k -scheme A . The key point for what follows is that for any $y \in A[mn](T)$, the composition

$$\theta_y \circ t_y^*(\theta_x) : t_y^*(t_x^*([mn]^*(\mathcal{L}))) \simeq t_y^*([mn]^*(\mathcal{L})) \simeq [mn]^*(\mathcal{L})$$

coincides with θ_{x+y} via the natural isomorphism of functors $t_y^* \circ t_x^* \simeq t_{x+y}^*$ on line bundles. (This is readily verified via the equalities of k -morphisms $[mn] \circ t_{x+y} = ([mn] \circ t_x) \circ t_y = [mn] \circ t_y = [mn]$.)

Our aim is to show that for every $i \geq 1$, $\theta_{ia}(t_{ia}^*(\sigma)) = u^i \cdot \sigma$ for the unit u on T satisfying $\theta_a(t_a^*(\sigma)) = u \cdot \sigma$ (as then setting $i = m$ would give what we want). Note that the choice of σ is irrelevant, since any two choices are related through scaling by a unit from the base scheme T . It suffices to show more generally that for any $a, b \in A[mn](T)$, if $\theta_a(t_a^*(\sigma)) = u \cdot \sigma$ for some (equivalently, *any!*) trivializing section σ of $[mn]^*(\mathcal{L})$ and if $\theta_b(t_b^*(\tau)) = v \cdot \tau$ for some (equivalently, *any!*) trivializing section τ of $[mn]^*(\mathcal{L})$, then $\theta_{a+b}(t_{a+b}^*(\sigma)) = uv \cdot \sigma$ for any σ . (Indeed, then setting $b = (i-1)a$ for $i \geq 2$ would allow us to proceed by induction on i .)

In view of the above identity relating θ_{x+y} and $\theta_y \circ t_y^*(\theta_x)$, we are led to consider

$$(\theta_b \circ t_b^*(\theta_a))(t_b^*(t_a^*(\sigma))).$$

This is equal to

$$\theta_b(t_b^*(\theta_a(t_a^*(\sigma)))) = \theta_b(t_b^*(u \cdot \sigma)) = u \cdot \theta_b(t_b^*(\sigma)) = u \cdot (v \cdot \sigma),$$

where the second equality uses crucially that the unit u on A_T comes from the base scheme T and hence passes harmlessly through the pullback functor t_b^* (and certainly passes through the \mathcal{O}_{A_T} -linear isomorphism θ_b). Combining this with the isomorphism of functors $t_{a+b}^* \simeq t_b^* \circ t_a^*$ translates this final computation into the desired equality $\theta_{a+b}(t_{a+b}^*(\sigma)) = uv \cdot \sigma$, as the reader may readily verify.