Math 249C. Homework 1

1. Let k be a field. An *algebraic torus* over k is a smooth affine k-group scheme T such that $T_{\overline{k}} \simeq \mathrm{GL}_1^n$ as $\overline{k}$-groups for some $n \geq 0$.

(i) Explain how the $\mathbf{R}$-group $G = \{x^2 + y^2 = 1\}$ is naturally a 1-dimensional algebraic torus over $\mathbf{R}$, with $G_{\mathbf{C}} \simeq \mathrm{GL}_1$ defined by $(x, y) \mapsto x + iy$, with x, y viewed over $\mathbf{C}$, not $\mathbf{R}$. Describe the inverse isomorphism. (This example explains the reason for the name "algebraic torus".)

(ii) Generalize to any separable quadratic extension of fields K/k in place of $\mathbf{C}/\mathbf{R}$.

2. Let C be a compact connected Riemann surface. Viewing holomorphic 1-forms on C as smooth $\mathbf{C}$-valued 1-forms on the underlying 2-dimensional $\mathbf{R}$-manifold, we get a natural map $\Omega^1(C) \to \mathrm{H}^1(C, \mathbf{C}) = \mathrm{H}_1(C, \mathbf{C})^*$ via integration along cycles. The natural "complex conjugation" involution on $\mathrm{H}^1(C, \mathbf{C}) = \mathbf{C} \otimes_{\mathbf{Q}} \mathrm{H}^1(C, \mathbf{Q})$ thereby defines a $\mathbf{C}$-linear map $\overline{\Omega}^1(C) \to \mathrm{H}^1(C, \mathbf{C})$ (where the source is the space of "anti-holomorphic" 1-forms, locally given by $\overline{f}(z)\,\mathrm{d}z$ for holomorphic f). Here, V^* is linear dual of a vector space V.

Hodge theory implies that the natural map $\Omega^1(C) \oplus \overline{\Omega}^1(C) \to \mathrm{H}^1(C, \mathbf{C})$ is an isomorphism. Using this, prove that the natural map $\mathrm{H}_1(C, \mathbf{Z}) \to \Omega^1(C)^*$ is a lattice (i.e., $\mathrm{H}_1(C, \mathbf{R}) \to \Omega^1(C)^*$ is an isomorphism).

3. Let X be a scheme locally of finite type over a field k.

(i) If $X(k) \neq \emptyset$ and X is connected then prove that X is geometrically connected over k. (Hint: show that it suffices to prove X_K is connected for K/k finite with X of finite type over k. Then use that $X_K \to X$ is open and closed for such K/k.)

(ii) Assume that k is algebraically closed and X is a group scheme over k. Prove that X_{red} is smooth, and deduce that if X is connected and U and V are non-empty open subschemes then the multiplication map $U \times V \to X$ is surjective. Deduce that for general k, if X is a (locally finite type) group scheme over k then it is connected if and only if it is geometrically irreducible over k, and that such X are of finite type (i.e., quasi-compact) over k.

4. Let X be a proper and geometrically integral scheme over a field k. Assume $X(k) \neq \emptyset$ and choose $e \in X(k)$. Define the functor $\mathrm{Pic}_{X/k}$ on the category of k-schemes to carry a k-scheme S to the group of isomorphism classes of pairs $(\mathscr{L}, i)$ where $\mathscr{L}$ is a line bundle on X_S and $i : e_S^*(\mathscr{L}) \simeq \mathscr{O}_S$ is a trivialization of $\mathscr{L}$ along e_S. It is a theorem of Grothendieck/Oort that this functor is represented by a k-group scheme locally of finite type. In particular, its identity component $\mathrm{Pic}^0_{X/k}$ is a k-scheme of finite type (Exercise 3).

(i) Prove that if X is smooth and projective over k then $\mathrm{Pic}_{X/k}$ satisfies the valuative criterion for properness (so $\mathrm{Pic}^0_{X/k}$ is a proper k-scheme).

(ii) By computing with points valued in the dual numbers, and using Čech theory in degree 1, construct a natural k-linear isomorphism $\mathrm{H}^1(X, \mathscr{O}_X) \simeq \mathrm{T}_0(\mathrm{Pic}^0_{X/k}) = \mathrm{T}_0(\mathrm{Pic}_{X/k})$.

(iii) If X is smooth with dimension 1, prove that $\mathrm{Pic}_{X/k}$ satisfies the infinitesimal smoothness criterion (for schemes locally of finite type over k). Deduce that $\mathrm{Pic}^0_{X/k}$ is an abelian variety of dimension equal to the genus of X.

5. Let C be a compact connected Riemann surface of genus $g > 0$, $c_0 \in C$. Let $J_C = \Omega^1(C)^*/\mathrm{H}_1(C, \mathbf{Z})$, a complex torus by Exercise 2. Prove that the map of sets $i_{c_0} : C \to J_C$ defined by $c \mapsto \int_{c_0}^{c} \bmod \mathrm{H}_1(C, \mathbf{Z})$ is complex-analytic and has smooth image over which C is a finite analytic covering space. Deduce that i_{c_0} is a closed embedding when $g > 1$, and prove that i_{c_0} is an isomorphism when $g = 1$ by identifying $\mathrm{H}_1(i_{c_0}, \mathbf{Z})$ with the identity map when $g = 1$.

6. Let X be a smooth, proper, geometrically connected curve of genus $g > 0$ over a field k, and assume $X(k) \neq \emptyset$. Choose $x_0 \in X(k)$. Prove $X \to \mathrm{Pic}_{X/k}$ defined on R-points (for a k-algebra R) by $x \mapsto \mathscr{O}(x) \otimes \mathscr{O}((x_0)_R)$ (where $\mathscr{O}(x) := \mathscr{I}(x)^{-1}$ for the *invertible* ideal $\mathscr{I}(x)$ of $x : \mathrm{Spec}(R) \hookrightarrow X_R$) is a proper monomorphism, hence a closed immersion. (Hint: use $g > 0$ to prove $R \hookrightarrow \mathrm{H}^0(X_R, \mathscr{O}(x))$ is an equality via base change theorems, and deduce that if $\mathscr{O}(x) \simeq \mathscr{O}(x')$ for $x, x' \in X(R)$ then the inclusion $\mathscr{O}_{X_R} \simeq \mathscr{O}(x) \otimes \mathscr{O}(x')^{-1} \hookrightarrow \mathscr{O}(x)$ is an $R^\times$-multiple of the canonical inclusion; conclude that its cokernel has annihilator ideal $\mathscr{I}(x)$ and $\mathscr{I}(x')$!) Thus, the choice of x_0 defines a closed immersion of X into the abelian variety $\mathrm{Pic}^0_{X/k}$ of dimension g.