Math 249C. Homework 10

1. Let K be a global field.

(i) By reducing to the special case of elementary matrices, prove that if $T \in \mathrm{PGL}_n(K)$ then the function $h_{K,n} - h_{K,n} \circ T$ on $\mathbf{P}^n_K(\overline{K})$ is bounded.

(ii) Fill in the details for the proof that if $f : X \to \mathbf{P}^n_K$ is a K-map from a proper K-scheme and $\mathscr{L} = f^*(\mathscr{O}(1))$, then $h_{K,\mathscr{L}} \le h_{K,n} \circ f$.

2. (i) Read §19 after Corollary 3 and up through the definition of the "characteristic polynomial" $P_f \in \mathbf{Z}[t]$ for $f \in \mathrm{End}^0_k(A)$ for an abelian variety A over a field k. (This is the "common" characteristic polynomial of $T_\ell(f)$ on $T_\ell(A)$ for all primes $\ell \neq \mathrm{char}(k)$.) For $k = \mathbf{C}$, using integral homology to give an alternative proof that such independence-of-ℓ holds for these characteristic polynomials.

(ii) As an application, consider an abelian scheme A over a complete discrete valuation ring R with fraction field K and residue field k. (This is a smooth proper R-group with connected fibers.) For any integer $n \in R^\times$, show that $A[n]$ is finite étale, and deduce that the natural maps $A[n](R_s) \to A[n](K_s)$ and $A[n](R_s) \to A[n](k_s)$ are bijective, where R_s is the valuation ring of k_s. Use this to construct a canonical $\mathbf{Z}_\ell$-linear isomorphism $T_\ell(A_K) \simeq T_\ell(A_k)$ for any prime $\ell \neq \mathrm{char}(k)$, and deduce that the natural $\mathrm{Gal}(K_s/K)$-action on $T_\ell(A_K)$ is *unramified*. (This is the "easy direction" of the Néron-Ogg-Shafarevich criterion for "good reduction".)

(iii) Pushing (ii) further, consider an abelian variety A over a global field K. For each non-archimedean place v of good reduction (i.e., A_{K_v} extends to an abelian scheme over the valuation ring R_v), prove that the action of Frob_v on the unramified $T_\ell(A_K)$ has characteristic polynomial $P_v \in \mathbf{Z}[t]$ that is independent of ℓ. The reciprocal $1/P_v(q_v^{-s})$ is the *local Euler factor* at v in the definition of $L(s, A/K)$ (where q_v is the size of the finite residue field at v).

3. Read in §20 from the Rosati involution up through the proof of Theorem 3, and then read Theorem 1 and its proof in §21. As an application, read Application II in §21 to see a proof of the Riemann Hypothesis for abelian varieties over finite fields (Theorem 4).

4. As another application of the Rosati involution from Exercise 3, read the statement and proof of Theorem 5 in §21. Deduce that if (A, ϕ) is a polarized abelian variety over a field k then the pair $(A_{\overline{k}}, \phi_{\overline{k}})$ has finite automorphism group.