Math 249C. Homework 2

1. Let R be a commutative ring, and consider a pair of R-schemes X and Y. By Yoneda's Lemma, to specify a map of R-schemes $X \to Y$ it is equivalent to specify a natural transformation of the corresponding representable functors $\underline{\mathrm{Hom}}_{\mathrm{Spec}\,R}(\cdot, X) \to \underline{\mathrm{Hom}}_{\mathrm{Spec}\,R}(\cdot, Y)$ on the category of R-schemes. This is general nonsense valid in any category. But for the category of R-schemes we can get by with less: show that if we restrict these functors to the category of *affine* R-schemes (so the functors may now fail to be representable, if X or Y is not affine) then a natural transformation between these restricted functors still arises from a unique R-scheme map $X \to Y$.

In other words, an R-scheme map $X \to Y$ amounts to maps of sets $X(R') \to Y(R')$ for R-algebras R' functorially in R'. (Note that $X(R') := \mathrm{Hom}_{\mathrm{Spec}\,R}(\mathrm{Spec}\,R', X)$ depends contravariantly on R'.)

2. Let S be a scheme. An *S-group* (or *group scheme over S*) is a group object in the category of S-schemes. In other words, it is an S-scheme G equipped with an S-map $m : G \times_S G \to G$, an S-map $i : G \to G$, and a section $e : S \to G$ such that the habitual group axioms diagrams commute. Read the first several pages of §11 to see how this goes. It is important to observe that an equivalent definition is to endow $G(S') = \mathrm{Hom}_S(S', G)$ with a group structure functorially in varying S-schemes S'.

(i) Using the Yoneda interpretation, show that if G and H are S-groups and $f : G \to H$ is an S-scheme map that respects the multiplication morphisms (in an evident sense) then it automatically respects the identity sections and inversion maps. Carry over other trivialities from the beginnings of group theory (such as uniqueness of identity section). Can you do all of this by writing huge diagrams and avoiding Yoneda?

(ii) Let $f : G \to H$ be a homomorphism of S-groups. The fiber product $f^{-1}(e_H) = G \times_{H, e_H} S$ is the *scheme-theoretic kernel* of f, denoted $\ker f$. Prove that it is a subscheme of G whose S'-points (for an S-scheme S') is the subgroup $\ker(G(S') \to H(S'))$. The situation for cokernels is far more delicate, much like for quotient sheaves.

(iii) For each of the following group-valued functors on the category of schemes, write down a representing affine scheme and the multiplication, inversion, and identity maps at the level of coordinate rings: $\mathbf{G}_a(S) = \Gamma(S, \mathscr{O}_S)$, $\mathrm{GL}_n(S) = \mathrm{GL}_n(\Gamma(S, \mathscr{O}_S))$, $\mu_m = \ker(t^m : \mathrm{GL}_1 \to \mathrm{GL}_1)$. For a finite group G, do the same for the functor of locally constant G-valued functions (called the *constant* $\mathbf{Z}$*-group* associated to G).

3. Let V be a locally free module of finite rank $n > 0$ over a commutative ring R. Consider the functor on R-algebras defined by $R' \rightsquigarrow \mathrm{Aut}_{R'}(V \otimes_R R')$. Prove in two ways that this is represented by an affine R-group $\mathrm{GL}(V)$ that is Zariski-locally (on $\mathrm{Spec}\,R$) isomorphic to GL_n:

(i) Work Zariski-locally on $\mathrm{Spec}\,R$ and construct the group scheme by gluing.

(ii) Let S be the symmetric algebra of the dual module $\mathrm{End}(V)^* = V^* \otimes V$. Identify $\det : \mathrm{End}(V) \to R$ with a canonical element in S that is homogeneous of degree n. Prove that $\mathrm{Spec}(S[1/\det])$ does the job.

4. Let X and Y be schemes of finite type over a field k, K/k an extension, $\{K_i\}$ a directed system of subfields of K containing k such that $\varinjlim K_i = K$.

(i) Show that any K-map $X_K \to Y_K$ descends (uniquely) to a K_i-map $X_{K_i} \to Y_{K_i}$ for some i.

(ii) Let $f : X \to Y$ be a k-map. Prove that f has property $\mathbf{P}$ if and only if f_K does, where $\mathbf{P}$ is: affine, finite, quasi-finite, closed immersion, surjective, isomorphism, separated, proper, flat. (For affineness, use the cohomological criterion. For properness, use Chow's Lemma in the separated case to reduce to the case when Y is affine and f is quasi-projective.)

(iii) Suppose that K/k is a finite Galois extension. Prove that a K-map $F : X_K \to Y_K$ descends to a k-map $f : X \to Y$ if and only if F is equivariant for the natural action by $\mathrm{Gal}(K/k)$ on X_K and Y_K (over k!). This is Galois descent for morphisms.

5. Let G be a group scheme locally of finite type over a field k, with $m : G \times G \to G$ the multiplication. Prove that the tangent map $\mathrm{d}m_{(e,e)} : \mathrm{T}_e(G) \oplus \mathrm{T}_e(G) \to \mathrm{T}_e(G)$ is addition.