Math 249C. Homework 4

1. Let S be an $\mathbf{F}_p$-scheme, and let $F_S : S \to S$ be the *Frobenius endomorphism*, which is obtained by gluing the Frobenius endomorphisms (p-power map) on each affine open.

(i) Show that F_S makes sense, and that it has the following alternative description: it is the identity map on topological spaces and the p-power map on the structure sheaf. Show that F_S is functorial in S.

(ii) For any S-scheme X, let $X^{(p)} = X \times_{S,F_S} S$. Loosely speaking, this is the scheme defined over S by "raising coefficients of defining equations of X to the p-power". Let $F_{X/S} : X \to X^{(p)}$ be the S-map defined by the commutative diagram expressing the compatibility of F_X and F_S. Loosely speaking, it is "p-power map on coordinates over S".

Prove that the formation of $X^{(p)}$ is compatible with base change on S, and that the formation of $F_{X/S}$ naturally commutes with products over S and with base change on S. In particular, if X is an S-group (so $X^{(p)}$ naturally is too) then $F_{X/S}$ is naturally an S-group map.

(iii) If $S = \operatorname{Spec} \mathbf{F}_p$ then $F_{X/S} = F_X$. Using this, describe $F_{\mathrm{GL}_1/k}$ and $F_{\mathbf{G}_a/k}$ for any field $k/\mathbf{F}_p$, and compute their kernels (the latter denoted as α_p, the "scheme of pth roots of 0").

2. Let X be a proper scheme over an affine noetherian scheme S. Prove that $\mathscr{O}_X$ is ample if and only if X is S-finite.

3. A *left exact* sequence $1 \to G' \to G \to G''$ of locally finite type k-group schemes is a k-homomorphism $G \to G''$ equipped with an isomorphism between its (scheme-theoretic) kernel and G'. Prove that the functor $G \rightsquigarrow \mathrm{T}_e(G)$ from locally finite type k-group schemes to k-vector spaces is left exact; in other words, for a k-homomorphism $f : G \to H$, the subspace $\mathrm{T}_e(\ker f)$ in $\mathrm{T}_e(G)$ coincides with $\ker(\mathrm{T}_e(G) \to \mathrm{T}_e(H))$. (Hint: compute using dual numbers over k)

4. Let X be a smooth geometrically connected scheme of dimension d over a field k of characteristic $p > 0$. Let $K = k(X)$ be the function field.

(i) Prove that $[K : kK^p] = p^d$, so $[K : K^p] = p^d$ when k is perfect. (Hint: express K as a finite separable extension of $k(t_1, \dots, t_d)$)

(ii) Prove that kK^p is the kernel of the natural map $\mathrm{d} : K \to \Omega^1_{K/k}$.

5. Let S be a locally noetherian scheme. The *order* of a finite flat commutative group scheme $f : G \to S$ is the (locally constant) rank of $f_*\mathscr{O}_G$ over $\mathscr{O}_S$. It is a general (elementary but cleverly proved) theorem of Deligne (see the important paper of Oort–Tate "Group schemes of prime order") that G is always killed by its order.

Suppose that G is killed by nm for relatively prime integers $n, m \geq 1$. Using Yoneda and group theory, prove that $G[n] \times G[m] \to G$ is an isomorphism. Deduce that $G[n]$ and $G[m]$ are S-flat. (Hint: prove that if $X \times_S Y$ is S-flat and $X(S) \neq \emptyset$ then Y is S-flat.) Conclude that there is a good theory of "primary decomposition" for finite *flat* commutative S-groups.

6. A locally finite type k-scheme X is *étale* if $X_{\overline{k}}$ is a disjoint union of copies of $\operatorname{Spec} \overline{k}$. Prove that X is étale if and only if X_{k_s} is a disjoint union of copies of $\operatorname{Spec} k_s$, and that it is equivalent to say that every affine open in X has coordinate ring that is a product of finitely many finite separable extensions of k.

7. Read the self-contained exposition of the theory of invariant differentials on group schemes in §4.2, up through the statement and proof of Proposition 2, in the totally awesome book "Néron Models". Then deduce the following refinements for a group scheme G locally of finite type over k.

(i) If G is smooth, prove that a global 1-form $\omega \in \Omega^1_{G/k}(G)$ is left-invariant in the scheme-theoretic sense if and only if ω_{k_s} is invariant under left translation by $G(k_s)$.

(ii) Define $\Omega^{1,\ell}_{G/k}$ to be the k-vector space of left-invariant elements in $\Omega^1_{G/k}(G)$. Prove that the natural k-linear map $\Omega^{1,\ell}_{G/k} \to \mathrm{Cot}_e(G)$ is an isomorphism, and that $\mathscr{O}_G \otimes_k \Omega^{1,\ell}_{G/k} \to \Omega^1_{G/k}$ is an isomorphism. Deduce that the formation of $\Omega^{1,\ell}_{G/k}$ naturally commutes with any extension on k, and compute $\Omega^{1,\ell}_{G/k}$ for $G = \mu_p$ and $G = \alpha_p$ over k of characteristic $p > 0$.