Math 249C. Homework 5

1. Let A be an abelian variety over a field k.

(i) Fix a prime $\ell \neq \text{char}(k)$. Prove that the collection of $A[\ell^n]$'s is relatively schematically dense in A in the sense that for any k-scheme S, a closed subscheme of A_S containing $A[\ell^n]$ for all $n \geq 1$ is equal to A_S. (Hint: reduce to the case $k = \overline{k}$ via base change.)

(ii) For any imperfect k, there are affine k-groups G of finite type such that G_{red} is not a k-subgroup, and examples with G reduced but not smooth. However, by using étale torsion levels, this problem does not arise for connected subgroups of abelian varieties. More precisely, let G be a closed k-subgroup scheme of A and let Z be the Zariski closure of the collection of étale k-subgroups $G[n] = G \cap A[n]$ for n not divisible by char(k). (Any closed subscheme of an étale k-scheme is étale.) Prove the formation of Z commutes with extension of the base field, and deduce by passing to $\overline{k}$ that Z is a smooth k-subgroup of G and $Z^0 = G^0_{\text{red}}$.

In particular, if $f : A \to B$ is a k-homomorphism then $(\ker f)^0_{\text{red}}$ is an abelian subvariety of A.

2. Let X and Y be schemes over a scheme S. The *Hom-functor* $\underline{\text{Hom}}(X, Y)$ assigns to any S-scheme T the set $\text{Hom}_T(X_T, Y_T)$. It is a theorem of Grothendieck (using the theory of Hilbert schemes) that if S is locally noetherian, X is proper and S-flat, and X and Y are quasi-projective Zariski-locally over S then this functor is represented by a locally finite type and separated S-scheme H. (That is, there is an H-map $X_H \to Y_H$ that is universal in an evident sense.)

(i) Let $f, g : X \rightrightarrows Y$ be a pair of S-morphisms (corresponding to elements of $H(S)$). Using that H is separated, show that the condition $f_T = g_T$ is represented by a closed subscheme of S.

(ii) Prove that if A and B are abelian varieties over a field k, then the functor $\underline{\text{Hom}}_{\text{gp}}(A, B) : S \rightsquigarrow \text{Hom}_{S-\text{gp}}(A_S, B_S)$ is represented by a locally finite type and separated k-group scheme.

(iii) Let T be a local noetherian k-scheme. Prove that if a T-group map $A_T \to B_T$ (for abelian varieties A and B) vanishes on the special fiber then it vanishes. (Hint: reduce to T complete, then artin local. Then use Exercise 1(i) to reduce to checking that for finite étale k-schemes X and Y, a T-map $X_T \to Y_T$ is uniquely determined by its pullback to the special fiber.)

(iv) Using (iii), prove that the Hom-scheme in (ii) has vanishing tangent space at the origin (i.e., at the zero map), and deduce that it is an étale k-scheme. Conclude the important fact that for any field K/k_s, every K-homomorphism $A_K \to B_K$ is defined over k_s (so all "geometric" homomorphisms are defined over a finite Galois extension of k).

3. Let A be an abelian variety over k, and S any k-scheme.

(i) Prove the Cubical Structure Theorem for line bundles on A_S, not just bundles coming from A. (Hint: reduce to universal case over the reduced $S = \text{Pic}_{A/k}$, and use Grauert's fibral cohomology theorem.)

(ii) Using (i) and Exercise 2(ii), define a natural S-group map $\phi_{\mathscr{L}} : A_S \to \widehat{A}_S$ for any $\mathscr{L}$ on A_S, and show that this defines a map of k-groups $\text{Pic}_{A/k} \to \underline{\text{Hom}}_{\text{gp}}(A, \widehat{A})$.

(iii) Using Exercise 2(iv), deduce a better "topological" proof that $\phi_{\mathscr{L}} = 0$ for $\mathscr{L}$ on A_K arising from $\text{Pic}^0_{A/k}(K)$ for a field K/k.

4. Let $f : A \to B$ be a homomorphism between abelian varieties over k, and $\widehat{f} : \widehat{B} \to \widehat{A}$ the induced map between identity components of Picard schemes (defined functorially by pullback along f). Prove that the formation of $\widehat{f}$ commutes with extension of the base field, and that it is uniquely characterized by the property that $(1_A \times \widehat{f})^*(P_A) \simeq (f \times 1_{\widehat{B}})^*(P_B)$ as line bundles on $A \times \widehat{B}$. (Hint: for the latter, use the Seesaw Theorem and comparison on fibers over $\widehat{B}(\overline{k})$ and $0 \in A(k)$.)

5. Let A and B be abelian varieties over k.

(i) Prove that the natural map $\widehat{A} \times \widehat{B} \to (A \times B)^\wedge$ is an isomorphism. (Hint: Check that the scheme-theoretic kernel is étale by computing on tangent spaces at the origin. Then use the theorem of the cube to prove that the kernel has trivial geometric points, and deduce the result from this.)

(ii) Using the isomorphism in (i), we can identify $(A \times \widehat{A}) \times (B \times \widehat{B})$ with $(A \times B) \times (A \times B)^\wedge$. Show that this carries $P_{A \times B}$ over to $P_A \boxtimes P_B$ (tensor product of pullbacks of P_A and P_B).