# Math 249C. Homework 6

1. A map of abelian varieties $f : A \to B$ over $k$ is an *isogeny* if it is surjective with finite kernel on $\overline{k}$-points, or equivalently (by generic flatness and translations) if $f$ is finite and flat.

(i) Using the theorem of Deligne from Exercise 5 in HW4 and the "quotient" property for fpqc homomorphisms between group schemes (as discussed in class), prove that if $\dim A = \dim B$ then $f$ is an isogeny if and only if there exists $g : B \to A$ such that $g \circ f = [n]_A$, in which case $f \circ g = [n]_B$.

(ii) Let $\ell$ be a prime with $\ell \neq \mathrm{char}(k)$. Prove that $f$ is an isogeny if and only if the induced map $\mathrm{T}_\ell(f) : \mathrm{T}_\ell(A) \to \mathrm{T}_\ell(B)$ on $\ell$-adic Tate modules is injective with finite cokernel, and equivalently if and only if $\mathrm{V}_\ell(f) : \mathrm{V}_\ell(A) \to \mathrm{V}_\ell(B)$ is an isomorphism. In such cases, prove that $\deg f$ is not divisible by $\ell$ if and only if $\mathrm{T}_\ell(f)$ is an isomorphism. (There are analogues for $\ell = \mathrm{char}(k) > 0$, using Dieudonné modules.)

(iii) The *isogeny category* of abelian varieties over $k$ has objects the abelian varieties over $k$ and morphisms $\mathrm{Hom}^0(A, B) := \mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{Hom}_k(A, B)$. Explain why this forms a category, prove that the "forgetful" functor from the category of abelian varieties over $k$ to the isogeny category is faithful but not fully faithful, and that a map of abelian varieties is an isomorphism in the isogeny category if and only if it is an isogeny.

2. Let $S$ be a scheme, $X$ an $S$-scheme, and $G$ an $S$-group scheme. Assume there is given a left action map $G \times_S X \to X$. This action is called *free* if $G(T)$ acts freely on $X(T)$ for all $S$-schemes $T$.

(i) Prove that freeness is equivalent to the map $G \times_S X \to X \times_S X$ defined by $(\gamma, x) \mapsto (\gamma.x, x)$ being a monomorphism, and deduce that freeness is insensitive to fpqc base change. (Hint: in a category with fiber products, a map is a monomorphism if and only if its relative diagonal is an isomorphism.)

(ii) Let $X$ be a scheme locally of finite type over an algebraically closed $k$, equipped with an action by a $k$-group $G$ locally of finite type. For each $x \in X(k)$, prove that the functor assigning to any $k$-scheme $S$ the subgroup of $g \in G(S)$ fixing $x_S \in X(S)$ is represented by a closed $k$-subgroup $G_x$, the *isotropy group scheme* at $x$. Explain why $G_x$ naturally acts on the tangent space $\mathrm{T}_x(X)$ (viewed as an affine space over $k$), so the action map $G_x \to \mathrm{GL}(\mathrm{T}_x(X))$ defines a map of Lie algebras $\mathrm{Lie}(G_x) \to \mathfrak{gl}(\mathrm{T}_x(X))$ (i.e., an action in the sense of Lie algebra representations of $\mathrm{Lie}(G_x)$ on $\mathrm{T}_x(X)$).

Prove that the action is free if and only if $G(k)$ acts freely on $X(k)$ and $\mathrm{Lie}(G_x)$ acts freely on $\mathrm{T}_x(X)$ for all $x \in X(k)$ (i.e., nonzero elements of $\mathrm{Lie}(G_x)$ act without nonzero fixed points on $\mathrm{T}_x(X)$).

(iii) Assume $G \to S$ is fpqc and $G$ acts freely on $X$. A *quotient* of $X$ by the $G$-action is a $G$-invariant fpqc map $\pi : X \to \overline{X}$ such that $G \times_S X \to X \times_{\overline{X}} X$ defined by $(g, x) \mapsto (g.x, x)$ is an isomorphism. Prove that such a quotient, if it exists, is unique up to unique isomorphism, is initial among $G$-invariant maps from $X$ to $S$-schemes, and retains the quotient property after base change to any $S$-scheme.

3. Let $A$ be an abelian variety over $k$, and $G$ a *finite* $k$-subgroup scheme of $A$. This exercise proves the existence and uniqueness of a quotient abelian variety $A/G$, and considers an important example.

(i) Prove that up to unique isomorphism there is at most one pair $(\overline{A}, \pi)$ consisting of an abelian variety $\overline{A}$ and a surjective $k$-homomorphism $\pi : A \to \overline{A}$ with $G = \ker \pi$. Prove that if it exists then it is necessarily a quotient in the strong sense of Exercise 2(iii). Conversely, prove that if there is a quotient $A/G$ in the strong sense of Exercise 2(iii) then it is necessarily an abelian variety. (Hint: a noetherian ring is regular if it admits a faithfully flat regular extension, by Theorem 23.7 of Matsumura CRT, and a $k$-algebra is finite type if it admits a faithfully flat extension of finite type over $k$, by Prop. 9.1 in Exposé V of SGA3.)

(ii) Choose $n \in \mathbf{Z} - \{0\}$ killing $G$ (e.g., the order of $G$), and consider the quotient mapping $[n]_A : A \to A$ that identifies $A$ with $A/A[n]$ (in particular, $A/A[n]$ exists and is an abelian variety). Explain how this identifies the problem of existence of $A/G$ in the sense of (i) with the quotient problem from Exercise 2(iii) for the action of the $A$-group $G \times A$ on $A$ viewed as an $A$-scheme via $[n]_A : A \to A$. The existence of quotients of free actions by finite flat group schemes on schemes affine (even finite!) over a noetherian base is solved in general by Theorem 4.1 in Exposé V of SGA3 (you can read §1–§4 there without the earlier exposés.)

(iii) Let $\mathscr{L}$ be an ample line bundle on $A$, so $K(\mathscr{L})$ is a finite subgroup scheme of $A$. Deduce that the dual abelian variety $\widehat{A}$ is naturally identified with the quotient $A/K(\mathscr{L})$. (In Mumford's book, he develops from scratch a good theory of quotients of abelian varieties modulo finite subgroup schemes and then proves directly for ample $\mathscr{L}$ that the quotient $A/K(\mathscr{L})$ satisfies the required properties to be a dual abelian variety. In this way he constructs the theory of the dual abelian variety without using the theory of Picard schemes.)