Math 249C. Homework 7

1. In class we saw that ampleness of a line bundle on an abelian variety only depends on the geometric connected component in which it lies within the Picard scheme. In more geometric terms, if A is an abelian variety over k and S is a connected k-scheme, then a line bundle on A_S is ample on the fiber over one $s \in S$ if and only if for all $s \in S$. In this formulation, we are led to pose a question for general proper flat maps $f : X \to S$ with locally noetherian S: if $\mathscr{L}$ is a line bundle on X and $\mathscr{L}_{s_0}$ is ample for one $s_0 \in S$ then is it so for all points of the connected component of s_0 in S?

(i) Using cohomology and base change, together with generic flatness of coherent sheaves on reduced schemes (applied to higher direct images) via a stratification of S by locally closed reduced subschemes, prove via noetherian induction that the locus of $s \in S$ such that $\mathscr{L}_s$ is ample is locally constructible.

(ii) Using the valuative criterion ("stability under generization") for openness of a locally constructible set, prove that the locus in (i) is open. (There are examples with f smooth of relative dimension 2 for which the locus is *not* closed; i.e., it is not stable under specialization. It is therefore a remarkable fact that when X has a structure of abelian scheme over S, which is to say that it is a proper S-group with smooth connected fibers, then the locus of ample fibers is closed.)

2. Read the proofs of the first two Theorems stated at the start of §16 in Mumford's book (self-contained, given what has been done already, and note the results as stated make sense over any field and can be proved by passing to an algebraically closed base field.) Note in particular that $\deg \phi_{\mathscr{L}}$ is a perfect square whenever $\phi_{\mathscr{L}}$ is an isogeny.

3. Prove the following result that is used in the proof of $\mathbf{Z}$-finiteness of $\mathrm{Hom}_k(A, B)$ for abelian varieties A and B over k: if F is an infinite field and W is a (possibly infinite-dimensional) vector space over F then a function $f : W \to F$ is polynomial of degree at most n (in the evident sense defined on finite-dimensional subspaces using bases) if and only if its restriction to every subspace of W with dimension at most 2 has this property.

4. Let k'/k be a finite Galois extension of degree $d > 1$ with $\Gamma = \mathrm{Gal}(k'/k)$, and let A' be a k'-simple abelian variety over k' that is not isogenous (over k') to any of its Γ-twists. (Such examples can be constructed over number fields and finite fields by using arithmetic arguments.) Let $B' = \prod_{\gamma \in \Gamma} \gamma^*(A')$ viewed as an abelian variety over k'.

Define a natural action by Γ on B' covering its action on k', thereby defining a Galois descent datum. By quasi-projectivity of B', the descent is effective; let B be the descent to k. Prove that B is a k-simple abelian variety, so this is an example of an abelian variety over k that is simple but not absolutely simple.

5. Read §14 in Mumford's book up through the discussion of α_{p^n} as a local-local group scheme to learn about Cartier duality for finite commutative group schemes over a field. Then do the following, with S any scheme.

(i) Check for yourself that those methods work verbatim with the base $\mathrm{Spec}\, k$ replaced by S provided we work with finite commutative group schemes $f : G \to S$ such that f is finite locally free (i.e., $f_*(\mathscr{O}_G)$ is locally free as an $\mathscr{O}_S$-module), which is to say f is flat in the locally noetherian case.

(ii) Consider a complex $0 \to G' \to G \to G'' \to 0$ of finite locally free commutative S-groups. Prove that it is short exact (in the sense that $G \to G''$ is faithfully flat with kernel G') if and only if the same holds on all geometric fibers over S. You will need the fibral flatness criterion (so assume S is locally noetherian if you are only familiar with the criterion in that case).

(iii) Prove that a diagram $0 \to G' \to G \to G'' \to 0$ of finite locally free commutative S-groups is a short exact sequence if and only if the corresponding Cartier dual diagram is a short exact sequence. (Hint: use (ii) to reduce to the case when $S = \mathrm{Spec}\, k$ for a field, and then use the theory of quotients to prove that if a map of finite commutative k-groups is injective on coordinate rings then it is faithfully flat.)