Math 249C. Homework 9

1. Let G be a group scheme locally of finite type over a field k.

(i) Prove that G^0 is normal in G (i.e., $G^0(T)$ is normal in $G(T)$ for all k-schemes T).

(ii) Assume k is separably closed. Let E be the constant k-group associated to $G(\overline{k})/G^0(\overline{k})$. Prove that the connected components of G are geometrically connected and of finite type, and use this to construct a faithfully flat quasi-compact k-group map $G \to E$ with kernel G^0. Deduce that E serves as G/G^0.

(iii) For any étale k-scheme E, consider $E(k_s)$ as a discrete set with a continuous left action by $\Gamma = \mathrm{Gal}(k_s/k)$. Prove that $E \rightsquigarrow E(k_s)$ is an equivalence between the category of étale k-schemes and the category of "Γ-sets": discrete sets equipped with a continuous left Γ-action. (Hint: reduce to finite k-schemes and finite Γ-sets.) Show this is compatible with products, and deduce the existence of a unique étale k-group Q_G equipped with an identification $Q_G(k_s) = (G_{k_s}/G^0_{k_s})(k_s)$ as Γ-sets.

(iv) Using Galois descent for morphisms (suitably formulated for k_s/k), prove that the natural k_s-group map $G_{k_s} \to G_{k_s}/G^0_{k_s}$ uniquely descends to a k-group map $G \to Q_G$ that is faithfully flat and quasi-compact with kernel G^0. Deduce that Q_G serves as G/G^0. We call this étale k-group the *component group* of G.

(v) Prove $G \to G/G^0$ is initial among k-maps from G to étale k-schemes, and G/G^0 is functorial in G.

2. Let $f : A \to B$ be an isogeny. This exercise interprets the duality pairing $(\cdot,\cdot)_f : \ker f \times \ker \widehat{f} \to \mathbf{G}_m$.

(i) Consider the line bundle $Q = (f \times 1)^*(P_B) \simeq (1 \times \widehat{f})^*(P_A)$ on $A \times \widehat{B}$. Prove that $Q|_{A \times \ker \widehat{f}}$ is trivial; fix a trivializing section σ. For any k-scheme T and points $a \in (\ker f)(T)$ and $y \in (\ker \widehat{f})(T)$, prove that the canonical isomorphism $h_{a,A} : t^*_{(a,0)}(Q) \simeq Q$ over $A \times \widehat{B}$ (via the definition of Q as an $(f \times 1)$-pullback) has pullback along $1 \times y : A_T \to A \times \widehat{B}$ given by $t^*_{(a,0)}(\sigma) \mapsto (a, y)_f \cdot \sigma$.

(ii) For any k-scheme T and points $a \in (\ker f)(T)$ and $\widehat{b} \in (\ker \widehat{f})(T)$ prove the commutativity of the diagram of isomorphisms of line bundles

$$\begin{array}{ccccc}
Q_T & \xleftarrow{h_{A,a}} & t^*_{(a,0)}(Q_T) & \xleftarrow{t^*_{(a,0)}(h_{\widehat{B},\widehat{b}})} & t^*_{(a,0)}t^*_{(0,\widehat{b})}(Q_T) \\
\Vert & & & & \uparrow (a,\widehat{b})_f \\
Q_T & \xleftarrow[h_{\widehat{B},\widehat{b}}]{} & t^*_{(0,\widehat{b})}(Q_T) & \xleftarrow[t^*_{(0,\widehat{b})}(h_{A,a})]{} & t^*_{(0,\widehat{b})}t^*_{(a,0)}(Q_T)
\end{array}$$

over $(A \times \widehat{B})_T$. (Hint: Show it suffices to work over $(A \times \ker \widehat{f})_T$, and that pullback over $A_{T'}$ along any $y \in (\ker \widehat{f})(T')$ for a T-scheme T' makes the lower right map be multiplication against $(a, \widehat{b} + y)_f$ relative to σ. Prove the upper right and lower left maps involve the same multiplier constant, so it "cancels out".)

(iii) Since we may view the Poincaré bundle as identifying the left factor with the dual of the right factor, deduce from the asymmetry of the direction of the right vertical arrow in (ii) that $(\widehat{b}, i_A(a))_{\widehat{f}} = (a, \widehat{b})_f^{-1}$.

(iv) For any $n \geq 1$, $a \in A[n](T)$, and $a' \in \widehat{A}[n](T)$ for a k-scheme T, prove $\langle a, a' \rangle_{A,n}^{-1} = \langle a', i_A(a) \rangle_{\widehat{A},n}$. Using the adjointness of $T_\ell(h)$ and $T_\ell(\widehat{h})$ for any k-homomorphism $h : A \to B$ and $\ell \neq \mathrm{char}(k)$, deduce that a k-homomorphism $h : A \to \widehat{A}$ is symmetric if and only if $e_h : T_\ell(A) \times T_\ell(A) \to \mathbf{Z}_\ell(1)$ is skew-symmetric!

3. This exercise addresses the generality of Mumford's construction of symmetric maps $\phi_{\mathscr{L}}$. Let A be an abelian variety over a field k. Define $\mathrm{NS}(A)$ to be the étale component group of the k-smooth (!) $\mathrm{Pic}_{A/k}$.

(i) Recall that the k-group $\underline{\mathrm{Hom}}(A, \widehat{A})$ is étale. Prove that Mumford's construction $\phi_{\mathscr{L}}$ defines a k-homomorphism between étale k-groups $\mathrm{NS}(A) \to \underline{\mathrm{Hom}}(A, \widehat{A})$. Deduce for any k-homomorphism $f : A \to \widehat{A}$ that if $f_K = \phi_{\mathscr{L}}$ for some K/k and $\mathscr{L}$ on A_K then the same holds with K/k a finite separable extension.

(ii) Let f be as in (i) and define $\mathscr{N}_f = (1, f)^*(P_A)$. Using the behavior of $\phi_{\mathscr{L}}$ with respect to pullback in $\mathscr{L}$, deduce that $\phi_{\mathscr{N}_f} = f + \widehat{f} \circ i_A$. (Hint: make ϕ_{P_A} concrete.) Hence, if f is symmetric then $\phi_{\mathscr{N}_f} = 2f$. (By Theorem 3 in §23 in Mumford's book on abelian varieties, which rests on the list of commutative $\overline{k}$-groups of prime order, this implies $(\mathscr{N}_f)_{\overline{k}} \simeq \mathscr{L}^{\otimes 2}$ for some $\mathscr{L}$ on $A_{\overline{k}}$. Hence, $2f_{\overline{k}} = \phi_{(\mathscr{N}_f)_{\overline{k}}} = 2\phi_{\mathscr{L}}$, so $f_{\overline{k}} = \phi_{\mathscr{L}}$! By (i), $\mathscr{L}$ exists over a finite separable extension of k.)