Lecture Notes on Applied Analysis

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Chapter 1

Qualitative Properties of Elliptic Equations

The purpose of these lecture notes is to provide an introduction to some of the results on the qualitative behavior of non-negative solutions of semi-linear elliptic and parabolic PDE's obtained during the last 15 years or so. The main emphasis is on the results relevant to applications to the initial value problems for the reaction-diffusion equations. The choice of the material is absolutely subjective. Most of the proofs will be either skipped completely or presented in every excruciating detail with the hope that this will enable the reader to believe that the omiited ones are correct. We will avoid each and every regularity result to the greatest extent possible. Sometimes this will be impossible. A standard reference to the elliptic regularity theory is the classical book by Gilbarg and Trudinger [15]. A recent book with the lecture notes on the elliptic equations by Han and Lin [17] is a very enjoyable read.

1 The Strong Maximum Principle

This material is absolutely standard and is taken from [17].

Let Ω be a bounded connected domain in $\mathbb{R}^n$. Consider the operator

$$Lu = a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

with $u \in C^2(\Omega) \cap C(\overline{\Omega})$. The functions a_{ij}, b_i and c are always assumed to be continuous in $\overline{\Omega}$ while L is assumed to be uniformly elliptic:

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for all } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n$$

with a positive constant $\lambda > 0$.

The first result is the weak maximum principle.

Theorem 1.1 *(The Weak Maximum Principle) Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $u \geq 0$ and $Lu \geq 0$ in Ω with $c(x) \leq 0$ in Ω. Then u attains its maximum on $\partial\Omega$.*

Proof. The main observation is that if x_0 is an interior maximum then $\nabla u(x_0) = 0$ while the matrix $D^2u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$ is non-positive semi-definite. Hence the left side of $Lu \geq 0$ may not be positive at an interior maximum, as $c \leq 0$ and $u \geq 0$. This would be a contradiction were it not for the possibility $Lu = 0$ that arises if $D^2u(x_0)$ is degenerate – the rest of the proof fights this.

Given $\varepsilon > 0$ define $w(x) = u(x) + \varepsilon e^{\alpha x_1}$ with α to be determined – this will ensure that $D^2w(x_0)$ is non-degenerate. Then we have

$$Lw = Lu + \varepsilon e^{\alpha x_1}\left(a_{11}\alpha^2 + b_1\alpha + c\right).$$

Recall that $a_{11} \geq \lambda > 0$ and $|b_1|, |c| \leq$ const. Thus we may choose $\alpha > 0$ so that

$$a_{11}\alpha^2 + b_1\alpha + c > 0$$

and thus $Lw > 0$ in Ω. Therefore the function w attains its maximum in $\bar{\Omega}$ at the boundary $\partial\Omega$. Indeed, as before, if w attains its (non-negative) maximum at $x_0 \notin \partial\Omega$ then $\nabla w(x_0) = 0$ and the matrix $D_{ij} = \frac{\partial^2 w}{\partial x_i \partial x_j}$ is non-positive definite. Hence we would have

$$Lw(x_0) = a_{ij}(x_0)D^2_{ij}w(x_0) + c(x_0)w(x_0) \leq 0$$

which is a contradiction. Thus $x_0 \in \partial\Omega$ and we obtain

$$\sup_\Omega u \leq \sup_\Omega w \leq \sup_{\partial\Omega} w \leq \sup_{\partial\Omega} u + \varepsilon \sup_{\partial\Omega} e^{\alpha x_1} \leq C\varepsilon + \sup_{\partial\Omega} u$$

with the constant C independent of ε. We let $\varepsilon \to 0$ and finish the proof. □

Corollary 1.2 *The Dirichlet problem*

$$Lu = f \text{ in } \Omega$$
$$u = \phi \text{ on } \partial\Omega$$

with $f \in C(\Omega)$, $\phi \in C(\partial\Omega)$ has a unique solution if $c(x) \leq 0$.

Remark 1.3 We first note that the assumption of non-negativity of u is not needed if $c = 0$. However, non-positivity of $c(x)$ is essential: otherwise the Dirichlet problem may have a non-unique solution, as $u(x, y) = \sin x \sin y$ solves

$$\Delta u + 2u = 0 \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

with $\Omega = [0, \pi] \times [0, \pi] \subset \mathbb{R}^2$. Second, the assumption that Ω is bounded is also essential both for uniqueness and for the maximum principle to hold: the function $u(x) = \log|x|$ solves

$$\Delta u = 0 \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

with $\Omega = \{|x| > 1\}$.

The Hopf lemma guarantees that the point on the boundary where the maximum is attained is not a critical point.

Theorem 1.4 *(The Hopf Lemma) Let B be an open ball in $\mathbb{R}^n$ with $x_0 \in \partial B$. Suppose $u \in C^2(B) \cap C(B \cup x_0)$ satisfies $Lu \geq 0$ in B with $c(x) \leq 0$ in B. Assume in addition that $u(x) < u(x_0)$ for any $x \in B$ and $u(x_0) \geq 0$. Then we have*

$$\liminf_{t \to 0^+} \frac{u(x_0) - u(x_0 - t\mathbf{m})}{t} > 0$$

for each outward direction $\mathbf{m}$: $\mathbf{m} \cdot \mathbf{n}(x_0) > 0$.

Remark 1.5 If the normal derivative exists at x_0 then $\dfrac{\partial u}{\partial n}(x_0) < 0$.

Proof. We may assume without loss of generality that B is centered at the origin and has radius r. We may also assume that $u \in C(\bar{B})$ and that $u(x) < u(x_0)$ for all $x \in \bar{B} \backslash \{x_0\}$ – otherwise we would simply consider a smaller ball $B_1 \subset B$ that is tangent to B at x_0.

Consider $w(x) = u(x) + \varepsilon h(x)$ with $h(x) = e^{-\alpha|x|^2} - e^{-\alpha r^2}$ and define a domain $\Sigma = B \cap B(x_0, r/2)$. We observe first that $Lh > 0$ in Σ with an appropriate choice of α:

$$\begin{aligned}
Lh &= e^{-\alpha|x|^2}\left[4\alpha^2 a_{ij}(x)x_i x_j - 2\alpha a_{ii}(x) - 2\alpha b_i(x)x_i + c\right] - ce^{-\alpha r^2} \\
&\geq e^{-\alpha|x|^2}\left[4\alpha^2 a_{ij}(x)x_i x_j - 2\alpha[a_{ii}(x) + b_i(x)x_i]\right] \\
&\geq \alpha e^{-\alpha|x|^2}\left[4\alpha\lambda|x|^2 - 2[a_{ii}(x) + b_i(x)x_i]\right] \\
&\geq \alpha e^{-\alpha|x|^2}\left[4\alpha\lambda\frac{r^2}{4} - 2[a_{ii}(x) + b_i(x)x_i]\right] > 0
\end{aligned}$$

for all $x \in \Sigma$ for a sufficiently large $\alpha > 0$. Hence we have $Lw > 0$ for all $\varepsilon > 0$ and thus w may not attain its non-negative maximum inside Σ. We now show that if $\varepsilon > 0$ is sufficiently small then w attains it non-negative maximum only at x_0. Indeed, we may find δ so that $u(x) < u(x_0) - \delta$ for $x \in \partial\Sigma \cap B$. Take ε so that $\varepsilon h < \delta$ on $\partial\Sigma \cap B$, then $w(x) < u(x_0) = w(x_0)$ for all $x \in \partial\Sigma \cap B$. On the other hand, for $x \in \partial\Sigma \cap \partial B$ we have $h(x) = 0$ and $w(x) = u(x) < u(x_0) = w(x_0)$. We conclude that $w(x)$ attains its non-negative maximum in $\bar{\Sigma}$ at x_0 if ε is sufficiently small. This implies that

$$\frac{w(x_0) - w(x_0 - t\mathbf{m})}{t} \geq 0$$

for any small $t > 0$. Letting $t \to 0$ we obtain

$$\liminf_{t \to 0^+} \frac{u(x_0) - u(x_0 - t\mathbf{m})}{t} \geq -\varepsilon\frac{\partial h}{\partial n}(x_0) = \varepsilon\alpha r e^{-\alpha r^2} > 0.$$

This finishes the proof. □

Theorem 1.6 *(The Strong maximum Principle) Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy $Lu \geq 0$ with $c(x) \leq 0$ in Ω. Then the non-negative maximum of u in Ω may be attained only on $\partial\Omega$ unless u is a constant.*

Proof. Let $M = \sup_{\bar{\Omega}} u(x)$ and define the set $\Sigma = \{x \in \Omega : \; u(x) = M\}$, where the maximum is attained. We need to show that either Σ is empty or $\Sigma = \Omega$. Assume that Σ is non-empty but $\Sigma \neq \Omega$. Choose $p \in \Omega \backslash \Sigma$ so that $d_0 = d(p, \Sigma) < d(p, \partial\Omega)$. Consider the ball $B_0 = B(p, d_0)$ and let $x_0 \in \partial B_0 \cap \partial\Sigma$. Then we have $Lu \geq 0$ in B_0 and $u(x) < u(x_0) = M$, $M \geq 0$ for all $x \in B_0$. The Hopf Lemma implies that $\frac{\partial u}{\partial n}(x_0) > 0$ where n is the normal to B_0 at x_0. However, x_0 is an internal maximum of u in Ω and hence $\nabla u(x_0) = 0$. This is a contradiction. $\square$

Corollary 1.7 *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy $Lu \geq 0$ with $c(x) \leq 0$ in Ω. If $u \leq 0$ on $\partial\Omega$ then either $u = 0$ in Ω or $u < 0$ in Ω.*

Corollary 1.8 *Let Ω have an internal sphere property and $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfy $Lu \geq 0$ with $c(x) \leq 0$ in Ω. Assume that u attains its maximum in $\bar{\Omega}$ at a point x_0. Then $x_0 \in \partial\Omega$ and*

$$\frac{\partial u}{\partial n}(x_0) > 0$$

unless $u = const$ in Ω.

Exercise 1.9 Let Ω be a bounded subset of $\mathbb{R}^n$ that has an internal sphere property and consider the boundary value problem

$$\begin{aligned} &Lu = f \text{ in } \Omega \\ &\frac{\partial u}{\partial n} + \alpha(x) u = \phi \text{ on } \partial\Omega \end{aligned} \tag{1.1}$$

with $f \in C(\bar{\Omega})$ and $\phi \in C(\partial\Omega)$. Assume in addition that $c(x) \leq 0$ and the "heat-loss" parameter $\alpha(x) \geq 0$. Show that (1.1) has a unique solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ provided that either c or α is not identically equal to zero. If both c and α are zero then solution is unique up to an additive constant.

Exercise 1.10 The restriction $c(x) \leq 0$ may be eliminated. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy $Lu \geq 0$ with $u \leq 0$ in Ω, show that then either $u = 0$ in Ω or $u < 0$ in Ω.

The next result is the maximum principle for narrow domains. It is similar in spirit to the much more recent maximum principle for the domains of small volume that we will prove later and which is indispensable for the sliding and moving plane methods.

Theorem 1.11 *(The Maximum Principle for Narrow Domains) Let $\mathbf{e}$ be a unit vector. There exists $d_0 > 0$ that depends on the coercivity constant λ, and the L^∞-norms $\|b_i\|_\infty$ and $\|c^+\|_\infty$ so that if $|(y - x) \cdot \mathbf{e}| < d_0$ for all $(x, y) \in \Omega$ then maximum principle holds for the operator L. That is, if $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies $Lu \geq 0$ and $u \leq 0$ on $\partial\Omega$ then either $u \equiv 0$ or $u < 0$ in Ω.*

The main point is that in a narrow domain we need not assume $c \leq 0$.

Proof. Assume that $\mathbf{e}$ is the unit vector in the direction x_1 so that $\Omega \subset \{0 < x_1 < d\}$, and that $\|b_i\|_\infty, \|c^+\|_\infty \leq N$. Consider the function $w = e^{\alpha d} - e^{\alpha x_1} > 0$ in $\bar{\Omega}$. We compute

$$Lw = -(a_{11}\alpha^2 + b_1\alpha)e^{\alpha x_1} + c(e^{\alpha d} - e^{\alpha x_1}) \leq -\lambda\alpha^2 + N\alpha + Ne^{\alpha d} < 0$$

if we first choose α sufficiently large and then d sufficiently small. We now set $v = u/w$, then v satisfies

$$\tilde{L}v = a_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j} + B_i \frac{\partial v}{\partial x_i} + \left(\frac{Lw}{w}\right) v \geq 0$$

with $B_i = b_i + \frac{2}{w} a_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j}$. We may now apply the usual maximum principle to the operator $\tilde{L}$, as $\tilde{c} = (Lw/w) < 0$ and conclude that $v = u/w$ is non-positive inside Ω and hence so is u. $\square$

2 A Priori Estimates

The most immediate application of the maximum principle is to obtain uniform a priori estimates on the solutions. We still assume that the matrix a_{ij} is uniformly elliptic in $\bar{\Omega}$: $a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$, $\lambda > 0$, and a_{ij}, b_i and c are continuous in $\bar{\Omega}$. We assume in addition that

$$\sup_{\Omega} |a_{ij}| + \sup_{\Omega} |b_i| \leq \Lambda.$$

The first result deals with the Dirichlet boundary conditions.

Theorem 2.1 *Assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies*

$$\begin{aligned} Lu &= f \text{ in } \Omega \\ u &= \phi \text{ on } \partial\Omega \end{aligned}$$

for some $f \in C(\bar{\Omega})$ and $\phi \in C(\partial\Omega)$. There exists a constant $C(\lambda, \Lambda, \mathrm{diam}(\Omega))$ so that

$$|u(x)| \leq \max_{\partial\Omega} |\phi| + C \max_{\Omega} |f| \text{ for all } x \in \Omega \tag{2.1}$$

provided that $c(x) \leq 0$.

Proof. Let us denote $F = \max_{\Omega} |f|$ and $\Phi = \max_{\partial\Omega} |\phi|$ and assume that Ω lies inside a strip $\{0 < x_1 < d\}$. Define $w(x) = \Phi + \left(e^{\alpha d} - e^{\alpha x_1}\right) F$ with $\alpha > 0$ to be chosen so as to ensure

$$\begin{aligned} Lw &\leq -F \text{ in } \Omega \\ w &\geq \Phi \text{ on } \partial\Omega. \end{aligned} \tag{2.2}$$

We calculate $w \geq \Phi$ on $\partial\Omega$ and

$$-Lw = (a_{11}\alpha^2 + b_1\alpha)Fe^{\alpha x_1} - c\Phi - c\left(e^{\alpha d} - e^{\alpha x_1}\right) F \geq (a_{11}\alpha^2 + b_1\alpha)F \geq (\lambda\alpha^2 + b_1\alpha)F \geq F$$

when α is large enough. Hence w satisfies (2.2). The comparison principle implies that $-w \leq u \leq w$ in Ω and in particular

$$\sup_{\Omega} |u| \leq \Phi + \left(e^{\alpha d} - 1\right) F$$

so that (2.1) holds. $\square$

Next we look at the Robin boundary conditions.

Theorem 2.2 *Let f and ϕ be as before. Assume that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$\begin{aligned} Lu &= f \text{ in } \Omega \\ \frac{\partial u}{\partial n} + \alpha(x)u &= \phi \text{ on } \partial\Omega \end{aligned} \tag{2.3}$$

where $\mathbf{n}$ is the outward normal to $\partial\Omega$, with $\alpha(x) \geq \alpha_0 > 0$. Then there exists a constant $C = C(\lambda, \Lambda, \mathrm{diam}(\Omega), \alpha_0)$ so that

$$|u(x)| \leq C \left[\max_{\partial\Omega} |\phi| + \max_{\Omega} |f|\right] \text{ for all } x \in \Omega \tag{2.4}$$

provided that $c(x) \leq 0$.

Proof. *Case 1: $c(x) \leq -c_0 < 0$.* We will show that

$$|u(x)| \leq \frac{F}{c_0} + \frac{\Phi}{\alpha_0}. \tag{2.5}$$

We define

$$v^{\pm} = \frac{F}{c_0} + \frac{\Phi}{\alpha_0} \pm u.$$

Then we have

$$Lv^{\pm} = c(x)\left(\frac{F}{c_0} + \frac{\Phi}{\alpha_0}\right) \pm f \leq -F \pm f \leq 0 \text{ in } \Omega$$

and

$$\frac{\partial v^{\pm}}{\partial n} + \alpha(x)v^{\pm} = \alpha(x)\left(\frac{F}{c_0} + \frac{\Phi}{\alpha_0}\right) \pm \phi \geq \Phi \pm f \geq 0 \text{ on } \partial\Omega.$$

If $v^{\pm}$ attains a negative minimum in $\bar{\Omega}$ then it has to be on $\partial\Omega$ by the maximum principle. Then $\frac{\partial v^{\pm}}{\partial n}(x_0) \leq 0$ and hence

$$\frac{\partial v^{\pm}}{\partial n}(x_0) + \alpha(x)v^{\pm}(x_0) \leq \alpha(x)v^{\pm}(x_0) < 0$$

which is a contradiction. Hence $v^{\pm} > 0$ in $\bar{\Omega}$ and thus (2.5) holds.

Case 2: $c(x) \leq 0$ for any $x \in \Omega$. We consider $w(x) = u(x)/z(x)$ and seek $z(x)$ positive to be determined so that Case 1 would apply to $w(x)$. A direct computation shows that w satisfies

$$\begin{aligned} &a_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j} + B_i \frac{\partial w}{\partial x_i} + \left(c + \frac{1}{z}\left[a_{ij}\frac{\partial^2 z}{\partial x_i \partial x_j} + b_i \frac{\partial z}{\partial x_i}\right]\right) w = \frac{f}{z} \text{ in } \Omega \\ &\frac{\partial w}{\partial n} + \left(\alpha(x) + \frac{1}{z}\frac{\partial z}{\partial n}\right) w = \frac{\phi}{z} \text{ on } \partial\Omega. \end{aligned}$$

Hence we need to choose a bounded function $z(x)$ so that

$$c + \frac{1}{z}\left[a_{ij}\frac{\partial^2 z}{\partial x_i \partial x_j} + b_i \frac{\partial z}{\partial x_i}\right] \leq -c_0$$

and

$$\alpha(x) + \frac{1}{z}\frac{\partial z}{\partial n} \geq \beta_0$$

to reduce this case to Case 1. It suffices to require that

$$\frac{1}{z}\left[a_{ij}\frac{\partial^2 z}{\partial x_i \partial x_j} + b_i \frac{\partial z}{\partial x_i}\right] \leq -c_0 < 0$$

and

$$\left|\frac{1}{z}\frac{\partial z}{\partial n}\right| \leq \frac{\alpha_0}{2}.$$

As before we assume that $\Omega \subset \{0 < x_1 < d\}$ and choose $z(x) = A + e^{\beta d} - e^{\beta x_1}$ with $A > 0$ and $\beta > 0$ to be determined. We calculate

$$-\frac{1}{z}\left[a_{ij}\frac{\partial^2 z}{\partial x_i \partial x_j} + b_i \frac{\partial z}{\partial x_i}\right] = \frac{(\beta^2 a_{11} + \beta b_1)e^{\beta x_1}}{A + e^{\beta d} - e^{\beta x_1}} \geq \frac{(\beta^2 a_{11} + \beta b_1)}{A + e^{\beta d}} \geq \frac{1}{A + e^{\beta d}} > 0$$

when β is sufficiently large. Having chosen such β we choose A so large that

$$\left|\frac{1}{z}\frac{\partial z}{\partial n}\right| \leq \frac{\beta e^{\beta d}}{A} \leq \frac{\alpha_0}{2}.$$

Now the problem is reduced to Case 1 for the function w. □

Note that these results may not be extended to the Neumann problem ($\alpha(x) = 0$ in (2.3)) since solution is unique only up to addition of an arbitrary constant.

3 The ABP Maximum Principle

The ABP (Alexandrov, Bakelman and Pucci) maximum principle allows to estimate the L^∞-norm of the solution in terms of the L^n-norm of the right hand side. This is useful in various applications, in particular to the maximum principle for domains of small volume and in the sliding method.

As before we let Ω be a bounded domain in $\mathbb{R}^n$ and consider an elliptic operator

$$L = a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

with continuous coefficients a_{ij}, b_i and c. We let $D = \det a_{ij}$ and denote by $D^* = D^{1/n}$ the geometric mean of the eigenvalues of the matrix $A = \{a_{ij}\}$. We assume in this section that all eigenvalues ω_k of the matrix A are uniformly bounded:

$$0 < \lambda \leq \omega_k \leq \Lambda < +\infty.$$

Given a function $u \in C^2(\Omega)$ we define its upper contact set Γ^+ as

$$\Gamma^+ = \{y \in \Omega : \ u(x) \leq u(y) + \nabla u(y) \cdot (x - y) \text{ for any } x \in \Omega\}.$$

The Hessian matrix D^2u is non-positive on Γ^+. This notion may be extended to continuous functions u as

$$\Gamma^+ = \{y \in \Omega : \ \exists p(y) \text{ so that } u(x) \leq u(y) + p(y) \cdot (x - y) \text{ for any } x \in \Omega\}.$$

Here $z(x) = u(y) + p \cdot (x - y)$ is the supporting hyper-plane at y. Clearly, the function u is concave if and only if $\Gamma^+ = \Omega$, and if $u \in C^1(\Omega)$ then $p(y) = \nabla u(y)$.

Theorem 3.1 *(The ABP Maximum Principle) Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy $Lu \geq f$ in Ω and assume that*

$$\frac{|b|}{D^*}, \ \frac{f}{D^*} \in L^n(\Omega), \quad \text{and } c \leq 0 \text{ in } \Omega.$$

Then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \left\| \frac{f^-}{D^*} \right\|_{L^n(\Gamma^+)}, \tag{3.1}$$

where Γ^+ is the upper contact set of $u(x)$, and C is given explicitly by

$$C = \left\{ \exp\left[\frac{2^{n-2}}{\omega_n n^n} \left(\left\| \frac{b}{D^*} \right\|_{L^n(\Gamma^+)} + 1 \right) \right] - 1 \right\} \mathrm{diam}(\Omega). \tag{3.2}$$

Here ω_n is the volume of the unit ball in $\mathbb{R}^n$.

We will first discuss some applications of this result and then prove it. An important application of the ABP maximum principle is a maximum principle for a domain with a small volume [3]. Despite a simple proof and beautiful applications it has been observed only fairly recently, at least in the West. Consider

$$Lu = a_{ij} D_{ij} u + b_i D_i u + cu$$

where a_{ij} is point-wise positive definite with

$$|b_i| + |c| \leq \Lambda, \quad \mathrm{det} a_{ij} \geq \lambda$$

for some positive constants λ and Λ.

Theorem 3.2 *(The Maximum Principle for Domains of a Small Volume) Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy $Lu \geq 0$ in Ω and assume that $u \leq 0$ on $\partial\Omega$. Then there exists a positive constant $\delta = \delta(n, \lambda, \Lambda, diam(\Omega))$ so that if $|\Omega| \leq \delta$ then $u \leq 0$ in Ω.*

Proof. If $c \leq 0$ then $u \leq 0$ by the usual maximum principle (or by the ABP maximum principle). In general write $c = c^+ - c^-$, then

$$a_{ij} D_{ij} u + b_i D_i u - c^- u \geq -c^+ u.$$

Then the ABP maximum principle implies that

$$\sup_{\Omega} u \leq C \|c^+ u^+\|_{L^n(\Omega)} \leq C \|c^+\|_\infty |\Omega|^{1/n} \sup_{\Omega} u \leq \frac{1}{2} \sup_{\Omega} u$$

with the constant $C = C(n, \lambda, \Lambda, \mathrm{diam}(\Omega))$, when $|\Omega|$ is small. Hence we have $u \leq 0$ in Ω. □

Another application of the ABP Maximum Principle, or, rather, of the next lemma, is to the Monge-Ampere equations. First we need a lemma that we will prove at the end of this section.

Lemma 3.3 *Let $g \geq 0$, $g \in L^1_{loc}(\mathbb{R}^n)$, then for any function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ we have*

$$\int_{B(0;R_u)} g(x)dx \leq \int_{\Gamma^+ \cap \Omega^+} g(\nabla u)|\det D^2 u|. \tag{3.3}$$

Here Γ^+ is the upper contact set of u and $R_u = \frac{1}{\operatorname{diam}(\Omega)} \left(\sup_\Omega u - \sup_{\partial\Omega} u^+ \right)$, while $\Omega^+ = \{u(x) > \sup_{\partial\Omega} u^+\}$.

Note that while (3.3) seems to be an integral inequality, the function $g(u)$ is known in applications. Thus the left side may be evaluated explicitly in terms of R_u – this gives an estimate of $\sup_\Omega u$ in terms of the right side that is an integral quantity and will be estimated in terms of the forcing f from the equation.

We now apply this lemma to Monge-Ampere equations.

Proposition 3.4 *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy a Monge-Ampere equation*

$$\det(D^2 u) = f(x, u, \nabla u) \tag{3.4}$$

with $f \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. Suppose that there exist non-negative functions $g \in L^1_{loc}(\mathbb{R}^n)$ and $h \in L^1(\Omega)$ so that

$$|f(x, z, p)| \leq \frac{h(x)}{g(p)} \text{ for any } (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$$

and

$$\int_\Omega h(x)dx < \int_{\mathbb{R}^n} g(p)dp = g_\infty$$

with g_∞ possibly infinite. Then

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u| + C\operatorname{diam}(\Omega) \tag{3.5}$$

with the constant C depending only on g and h.

Proof. Using (3.3) with the functions g and h as above we get

$$\begin{aligned} &\int_{|p| \leq R_u} g(p)dp \leq \int_{\Gamma^+} g(\nabla u)|\det(D^2 u)| = \int_{\Gamma^+} g(\nabla u)||f(x, u, \nabla u)| \\ &\leq \int_{\Gamma^+} h(x)dx \leq \int_\Omega h(x)dx < \int_{\mathbb{R}^n} g(p)dp. \end{aligned}$$

This implies that there exists a constant C that depends only on the functions h and g so that

$$R_u = \frac{\sup_\Omega u - \sup_{\partial\Omega} u^+}{\operatorname{diam}(\Omega)} \leq C.$$

Thus (3.5) follows. $\square$

In particular, if $f = f(x)$ we may take $h = f$ and $g = 1$. The argument in the proof of Proposition 3.4 implies that

$$\omega_n R_u^n \leq \int_\Omega |f| dx.$$

This means that solution of

$$\det(D^2 u) = f(x)$$

satisfies

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u| + \frac{\operatorname{diam}(\Omega)}{\omega_n^{1/n}} \left(\int_\Omega |f| \right)^{1/n}.$$

We now prove the ABP maximum principle assuming Lemma 3.3.

Proof of Theorem 3.1. First we consider the special case $b_i = c = 0$. Given any two $n \times n$ non-negative semi-definite matrices A and B we have

$$\det(AB^*) \leq \left(\frac{1}{n} \operatorname{Tr}(AB^*) \right)^n,$$

this is just geometric mean-arithmetic mean inequality for the eigenvalues of the matrix AB^*. The latter are all positive, as $AB^*\mathbf{v} = \lambda \mathbf{v}$ implies that $(B^*\mathbf{v}, \mathbf{v}) = \lambda(A^{-1}\mathbf{v}, \mathbf{v})$ so that $\lambda = (B^*\mathbf{v}, \mathbf{v})/(A^{-1}\mathbf{v}, \mathbf{v}) \geq 0$. Using this with $A = (a_{ij})$ and $B = -D^2 u$ on Γ^+, with $[D^*]^n = \det A$, we obtain another form of (3.3):

$$\int_{B(0;R_u)} g(x) dx \leq \int_{\Gamma^+} g(\nabla u) \left(-\frac{a_{ij} D_{ij} u}{n D^*} \right)^n. \tag{3.6}$$

We look at a special case $g = 1$:

$$\omega_n R_u^n \leq \int_{\Gamma^+} |\det(D^2 u)|$$

or, equivalently,

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + \frac{\operatorname{diam}(\Omega)}{\omega_n^{1/n}} \left(\int_{\Gamma^+} |\det(D^2 u)| \right)^{1/n} \leq \sup_{\partial\Omega} u^+ + \frac{\operatorname{diam}(\Omega)}{\omega_n^{1/n}} \left(\int_{\Gamma^+} \left(-\frac{a_{ij} D_{ij} u}{n D^*} \right)^n \right)^{1/n}.$$

This is nothing but (3.1) in the case $b_i = c = 0$.

In the general case, as well as in the case $b_i = c = 0$, we need to choose g appropriately and then apply Lemma 3.3. Note that neglecting f and c we have, roughly "$(-a_{ij} D_{ij} u)^n \leq |b|^n |\nabla u|^n$" which suggests to take $g(p) = 1/p^n$ in (3.6). This function, however, is not integrable at the origin. Hence we take

$$g(p) = (|p|^n + \mu^n)^{-1} \tag{3.7}$$

with $\mu > 0$ to be determined.

First, we correct the above rough estimate in $\Omega^+ \cap \Gamma^+$ to

$$0 \leq -a_{ij} D_{ij}^2 u \leq b_i D_i u + cu - f \leq b_i D_i u - f \leq |b| \cdot |\nabla u| + \frac{f^-}{\mu} \cdot \mu$$

$$\leq \left(|b|^n + \frac{(f^-)^n}{\mu^n} \right)^{1/n} (|\nabla u|^q + \mu^q)^{1/q} \leq \left(|b|^n + \frac{(f^-)^n}{\mu^n} \right)^{1/n} (|\nabla u|^n + \mu^n)^{1/n} (1+1)^{(n-2)/n},$$

with $1/p + 1/q = 1$, so that

$$(-a_{ij} D_{ij} u)^n \leq \left(|b|^n + \frac{(f^-)^n}{\mu^n} \right) (|\nabla u|^n + \mu^n)\, 2^{n-2}. \tag{3.8}$$

We now use (3.8) in (3.6) with g as in(3.7) to get

$$\int\limits_{|p| \leq R_u} \frac{dp}{|p|^n + \mu^n} \leq \frac{2^{n-2}}{n^n} \int\limits_{\Gamma^+ \cap \Omega^+} \frac{|b|^n + \mu^{-n}(f^-)^n}{D}.$$

The integral on the left side is easily computed to be

$$\int\limits_{|p| \leq R_u} \frac{dp}{|p|^n + \mu^n} = \omega_n \int_0^{R_u} \frac{r^{n-1} dr}{r^n + \mu^n} = \frac{\omega_n}{n} \log \left(\frac{R_u^n}{\mu^n} + 1 \right).$$

Thus we get

$$R_u \leq \mu \left(\exp \left\{ c_n \left[\left\| \frac{b}{D^*} \right\|^n_{L^n(\Gamma^+ \cap \Omega^+)} + \frac{1}{\mu^n} \left\| \frac{f^-}{D^*} \right\|^n_{L^n(\Gamma^+ \cap \Omega^+)} \right] \right\} - 1 \right)^{1/n}.$$

Now, if $f^- = 0$ we let $\mu \to 0^+$ and get $R_u = 0$, that is,

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+.$$

If $f^- \neq 0$ then we choose

$$\mu = \left\| \frac{f^-}{D^*} \right\|_{L^n(\Gamma^+ \cap \Omega^+)}$$

and get

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C \left\| \frac{f^-}{D^*} \right\|_{L^n(\Gamma^+ \cap \Omega^+)}$$

with the constant C as in (3.2). □

It remains only to prove Lemma 3.3.

Proof of Lemma 3.3. Replacing u by $u - \sup_{\partial\Omega} u^+$ does not change (3.3) and thus we may assume without loss of generality that $u \leq 0$ on $\partial\Omega$. Set $\Omega^+ = \{u > 0\}$, then using the area formula for ∇u in $\Gamma^+ \cap \Omega^+$ we obtain

$$\int\limits_{\nabla u(\Gamma^+ \cap \Omega^+)} g \leq \int\limits_{\Gamma^+ \cap \Omega^+} g(\nabla u) |\det(D^2 u)| \tag{3.9}$$

where $|\det(D^2 u)|$ is the Jacobian of the map $\nabla u : \Omega \to \mathbb{R}^n$. Alternatively, in order to get (3.9) we may consider the map $\chi_\varepsilon(x) = \nabla u(x) - \varepsilon x : \Omega \to \mathbb{R}^n$. The Jacobian matrix $\nabla \chi_\varepsilon = D^2 u - \varepsilon \mathrm{Id}$ is negative-definite on Γ^+ so that $x \to \chi_\varepsilon(x)$ is a legitimate change of variables. Hence the usual change of variables formula implies that

$$\int\limits_{\chi_\varepsilon(\Gamma^+ \cap \Omega^+)} g = \int\limits_{\Gamma^+ \cap \Omega^+} g(\nabla u(x) - \varepsilon x) |\det(D^2 u - \varepsilon \mathrm{Id})| \tag{3.10}$$

that implies (3.9) if we let $\varepsilon \to 0$.

We now claim that $B(0; R_u) \subset \nabla u(\Gamma^+ \cap \Omega^+)$, that is, for any $a \in \mathbb{R}^n$ with $|a| < R_u$ there exists $x \in \Gamma^+ \cap \Omega^+$ so that $a = \nabla u(x)$.

If $m = \max_\Omega u$ is non-positive, then $R_u = 0$ and there is nothing to prove. Let us assume that $m > 0$, then u has to attain its maximum at $x_0 \in \Omega$, as we assumed that $\sup_{\partial\Omega} u^+ \le 0$. Let $|a| < R_u = m/(\text{diam}(\Omega))$ and consider an affine function $L(x) = m + a \cdot (x - x_0)$. First, if $a = 0$, then $a = \nabla u(x_0) \in \nabla u(\Gamma^+ \cap \Omega^+)$ so we may assume $a \neq 0$. Then $L(x) > 0$ for any $x \in \Omega$ and $L(x_0) = u(x_0) = m$. Moreover, $\nabla u(x_0) = 0$, while $|\nabla L(x_0)| = |a| > 0$ which implies that there exists an x_1 close to x_0 so that $u(x_1) \ge L(x_1) > 0$. Note that $u(x) \le 0 < L(x)$ on $\partial\Omega$. Hence the function $u(x) - L(x)$ attains its non-negative maximum inside Ω and there exists $x' \in \Omega$ so that $\nabla u(x') = \nabla L(x') = a$. We may now consider the plane

$$L'(x) = L(x) + u(x') - L(x'),$$

then $L'(x') = u(x')$ and $u(x) - L'(x)$ still attains its maximum at x'. That means that the graph $y = u(x)$ lies below the supporting hyperplane $y = L'(x)$ and thus x' belongs to the upper contact set Γ^+. Moreover, $u(x') = L(x') + (u(x') - L(x'))$ is clearly positive, so that $x' \in \Gamma^+ \cap \Omega^+$ and $a = \nabla u(x') \in \nabla u(\Gamma^+ \cap \Omega^+)$. $\square$

Remark 3.5 It is clear from the proof that the set Γ^+ may be replaced by $\Gamma^+ \cap \{u(x) > \sup_{\partial\Omega} u^+\}$, as we have replaced u by $u - \sup_{\partial\Omega} u^+$.

4 The Moving Plane Method

The following result on the radial symmetry of non-negative solutions was first proved by Gidas, Ni and Nirenberg. It is one of the first examples of a general phenomenon that positive solutions of elliptic equations tend to be monotonic in one form or other: another example we will treat in detail below is a result of Berestycki, Caffarelli and Nirenberg on monotonicity of solutions in unbounded domains, monotonicity will re-appear in the travelling waves, pulsating fronts in reaction-diffusion equations. It has been even shown that solutions of initial value reaction-diffusion problems become monotonic in a finite time [26]. We present a simpler proof of the Gidas-Ni-Nirenberg theorem from [8]. The proof uses the moving plane method combined with the maximum principle for domains of a small volume.

Theorem 4.1 *Let $u \in C(\bar{B}_1) \cap C^2(B_1)$ be a positive solution of*

$$\begin{aligned} &\Delta u + f(u) = 0 \quad \text{in } B_1 \\ &u = 0 \ \text{ on } \partial B_1 \end{aligned} \tag{4.1}$$

with the function f that is locally Lipschitz in $\mathbb{R}$. Then u is radially symmetric in B_1 and $\dfrac{\partial u}{\partial r}(x) < 0$ for $x \neq 0$.

The proof is based on the following lemma.

Lemma 4.2 *Let Ω be a bounded domain that is convex in the x_1-direction and symmetric with respect to the plane $\{x_1 = 0\}$. Let $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ be a positive solution of*

$$\begin{aligned} &\Delta u + f(u) = 0 \quad \textit{in } \Omega \\ &u = 0 \ \textit{on } \partial\Omega \end{aligned} \tag{4.2}$$

with the function f that is locally Lipschitz in $\mathbb{R}$. Then u is symmetric with respect to x_1 and $\dfrac{\partial u}{\partial x_1}(x) < 0$ for any $x \in \Omega$ with $x_1 > 0$.

Proof of Theorem 4.1. Theorem 4.1 follows immediately from Lemma 4.2. Indeed, first, Lemma 4.2 implies that $u(x)$ is decreasing in any given radial direction. Second, it follows from the same lemma that $u(x)$ is symmetric with respect to reflection with respect to any radial direction – this trivially implies that u is radially symmetric. □

Proof of Lemma 4.2. We write $x = (x_1, y) \in \Omega$ with $y \in \mathbb{R}^{n-1}$. We will prove that

$$u(x_1, y) < u(x_1^*, y) \tag{4.3}$$

for all $x_1 > 0$ and $-x_1 < x_1^* < x_1$. This implies monotonicity for $x_1 > 0$. Next, letting $x_1^* \to -x_1$ we get $u(x_1, y) \le u(-x_1, y)$ for any $x_1 > 0$. Changing direction we also get the symmetry.

Given any $\lambda \in (0, a)$, with $a = \sup_\Omega x_1$, define

$$\begin{aligned} &\Sigma_\lambda = \{x \in \Omega : \ x_1 > \lambda\} \\ &T_\lambda = \{x_1 = \lambda\} \\ &\Sigma'_\lambda = \text{the reflection of } \Sigma_\lambda \text{ with respect to } T_\lambda \\ &x_\lambda = (2\lambda - x_1, x_2, \ldots, x_n), \text{the reflection of } x = (x_1, \ldots, x_n) \text{ with respect to } T_\lambda. \end{aligned}$$

Note that T_λ is our "moving plane". We define

$$w_\lambda(x) = u(x) - u(x_\lambda) \text{ for } x \in \Sigma_\lambda.$$

The mean value theorem implies that w_λ satisfies $\Delta w_\lambda = f(u(x_\lambda)) - f(u(x)) = -c(x, \lambda) w_\lambda$ in Σ_λ (this is a recurring trick that we use very often: the difference of two solutions of a semi-linear equation satisfies a "linear" equation with an unknown function c). The boundary $\partial\Sigma_\lambda$ consists of a piece of $\partial\Omega$, where $w_\lambda = -u(x_\lambda) < 0$ and of T_λ, where $w_\lambda = 0$. Summarizing, we have

$$\begin{aligned} &\Delta w_\lambda + c(x, \lambda) w_\lambda = 0 \text{ in } \Sigma_\lambda \\ &w_\lambda \le 0 \text{ and } w_\lambda \not\equiv 0 \text{ on } \partial\Sigma_\lambda, \end{aligned} \tag{4.4}$$

with a bounded function $c(x, \lambda)$. We will show that $w_\lambda < 0$ inside Σ_λ for all $\lambda \in (0, a)$. This implies in particular that w_λ assumes its maximum (equal to zero) over $\bar{\Sigma}_\lambda$ along T_λ. The Hopf lemma implies that

$$\left.\frac{\partial w_\lambda}{\partial x_1}\right|_{x_1=\lambda} = 2 \left.\frac{\partial u}{\partial x_1}\right|_{x_1=\lambda} < 0.$$

Given that λ is arbitrary it remains only to show that $w_\lambda < 0$ inside Σ_λ.

In order to show that $w_\lambda < 0$ one would like to apply the maximum principle to (4.4). However, a priori the function $c(x,\lambda)$ does not have a sign so the usual maximum principle may not be used. Still, when λ is close to a, the maximum principle for narrow domains, as well as the maximum principle for a domain of small volume imply that $w_\lambda < 0$ inside Σ_λ. Let us now decrease λ and let (λ_0, a) be the largest interval of values so that $w_\lambda < 0$ inside Σ_λ for all $\lambda \in (\lambda_0, a)$. We want to show that $\lambda_0 = 0$. If $\lambda_0 > 0$ then by continuity $w_{\lambda_0} \le 0$ in Σ_{λ_0} and w_{λ_0} is not identically equal to zero on $\partial\Sigma_{\lambda_0}$. The strong maximum principle implies that

$$w_{\lambda_0} < 0 \text{ in } \Sigma_{\lambda_0}. \tag{4.5}$$

We will show that then

$$w_{\lambda_0-\varepsilon} < 0 \text{ in } \Sigma_{\lambda_0} \tag{4.6}$$

for sufficiently small $\varepsilon < \varepsilon_0$.

Here is the key idea: choose a closed subset K of Σ_{λ_0} so that $|\Sigma_{\lambda_0} \backslash K| < \delta/2$ with $\delta > 0$ to be determined. Inequality (4.5) implies that there exists $\eta > 0$ so that $w_{\lambda_0} \le -\eta < 0$ for any $x \in K$. By continuity we have $w_{\lambda_0-\varepsilon} < 0$ for any $x \in K$. Therefore $w_{\lambda_0-\varepsilon} \le 0$ both on $\partial\Sigma_{\lambda_0-\varepsilon}$ and on ∂K and thus on $\partial(\Sigma_{\lambda_0-\varepsilon}\backslash K)$. However, when ε is sufficiently small we have $|\Sigma_{\lambda_0-\varepsilon}\backslash K| < \delta$. Choose δ so small that we may apply the maximum principle for domains of small volume to $w_{\lambda_0-\varepsilon}$ in $\Sigma_{\lambda_0-\varepsilon}\backslash K$. Hence we obtain $w_{\lambda_0-\varepsilon} \le 0$ in $\Sigma_{\lambda_0-\varepsilon}\backslash K$. The strong maximum principle implies that $w_{\lambda_0-\varepsilon} < 0$ in $\Sigma_{\lambda_0-\varepsilon}\backslash K$. Putting two and two together we see that (4.6) holds. This, however, contradicts the choice of λ_0. □

5 The Sliding Method

The sliding method differs from the moving plane method in that one compares translations of a function rather than its reflections. We will illustrate it in this and the following sections on examples taken from [8] and [10]. The following result is the simplest application of the method.

Theorem 5.1 *Let Ω be an arbitrary bounded domain in $\mathbb{R}^n$ which is convex in the x_1-direction. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of*

$$\begin{aligned} &\Delta u + f(u) = 0 \textit{ in } \Omega \\ &u = \phi \textit{ on } \partial\Omega \end{aligned} \tag{5.1}$$

with a Lipschitz continuous function f. Assume that for any three points $x' = (x_1', y)$, $x = (x_1, y)$ and $x'' = (x_1'', y)$ lying on a segment parallel to the x_1-axis, $x_1' < x_1 < x_1''$ with $x', x'' \in \partial\Omega$, the following hold:

$$\phi(x') < u(x) < \phi(x'') \textit{ if } x \in \Omega \tag{5.2}$$

and

$$\phi(x') \le \phi(x) \le \phi(x'') \textit{ if } x \in \partial\Omega. \tag{5.3}$$

Then u is monotone in x_1 in Ω:

$$u(x_1+\tau) > u(x_1, y) \textit{ for } (x_1, y), (x_1+\tau, y) \in \Omega \textit{ and } \tau > 0.$$

Furthermore, if f is differentiable, then $\dfrac{\partial u}{\partial x_1} > 0$ in Ω. Finally, u is the unique solution of (5.1) in $C^2(\Omega) \cap C(\bar{\Omega})$ satisfying (5.2).

Assumption (5.2) is usually checked in applications from the maximum principle and is not as unverifiable and restrictive in practice as it might seem at a first glance. For instance, one may look at (5.1) in a rectangle $[-a, a]_x \times [0, 1]_y$ with the Dirichlet data $\phi = 0$ and $\phi = 1$ prescribed at $x = -a$ and $x = a$, respectively, while the zero Neumann data is prescribed along the lines $y = 0$ and $y = 1$. The function f is assumed to vanish at $u = 0$ and $u = 1$: $f(0) = f(1) = 0$, $f(s) \leq 0$ with $u \notin [0, 1]$. Then Theorem 5.1 still applies with these boundary conditions, and the maximum principle implies that $0 \leq u \leq 1$ so that (5.2) holds.

Proof. For $\tau \geq 0$ we let $u^\tau(x_1, y) = u(x_1 + \tau, y)$ be a shift of u. The function u^τ is defined on the set $\Omega^\tau = \Omega - \tau \mathbf{e}_1$ obtained from Ω by sliding it to the left a distance τ parallel to the x_1-axis. The monotonicity of u may be restated as

$$u^\tau > u \text{ in } D^\tau = \Omega^\tau \cap \Omega \text{ for any } \tau > 0 \tag{5.4}$$

and this is what we will prove. The strategy will be exactly the same as in the moving plane method: we first establish (5.4) using the maximum principle for domains of a small volume for τ close to a critical value τ_0 - that is, those that have been slided almost all the way to the left. Then we will start decreasing τ, sliding the domain Ω^τ to the right, and will show that you may decrease it all the way to $\tau = 0$ keeping (5.4) enforced.

Set $w^\tau(x) = u^\tau(x) - u(x) = u(x_1 + \tau, y) - u(x_1, y)$, the function w^τ is defined in D^τ. Since u^τ satisfies the same equation as u, we have from the mean value theorem

$$\begin{aligned} &\Delta w^\tau + c^\tau(x) w^\tau = 0 \text{ in } D^\tau \\ &w^\tau \geq 0 \text{ on } \partial D^\tau \end{aligned} \tag{5.5}$$

where c^τ is uniformly bounded function. The inequality on the boundary ∂D^τ in (5.5) follows from assumptions (5.2) and (5.3). Let $\tau_0 = \sup\{\tau > 0 : \ D^\tau \neq \emptyset\}$ be the largest shift of Ω to the left that we can make so that Ω and Ω^τ still have a non-zero intersection. The volume $|D^\tau|$ is small when τ is close to τ_0. Hence we may apply the maximum principle for small domains to w^τ in D^τ for τ close to τ_0 and conclude that $w^\tau > 0$ for such τ.

Then we start sliding Ω^τ back to the right, that is, we decrease τ from τ_0 to a critical position τ_1: let (τ_1, τ_0) be a maximal interval with $\tau_1 \geq 0$ so that $w^\tau \geq 0$ in D^τ for all $\tau \in (\tau_1, \tau_0]$. We want to show that $\tau_1 = 0$ and argue by contradiction assuming that $\tau_1 > 0$.

Continuity implies that $w^{\tau_1} \geq 0$ in D^{τ_1}. Furthermore, (5.2) implies that $w^{\tau_1}(x) > 0$ for all $x \in \Omega \cap \partial D^{\tau_1}$. The strong maximum principle then implies that $w^{\tau_1} > 0$ in D^{τ_1}.

Now we use the same idea as in the proof of Lemma 4.2: choose $\delta > 0$ so that the maximum principle holds for any solution of (5.5) in a domain of volume less than δ. Carve out of D^{τ_1} a closed set $K \subset D^{\tau_1}$ so that $|D^{\tau_1} \backslash K| < \delta/2$. We know that $w^{\tau_1} > 0$ on K, hence for ε small $w^{\tau_1 - \varepsilon}$ is also positive on K. Moreover, for $\varepsilon > 0$ small, $|D^{\tau_1 - \varepsilon} \backslash K| < \delta$. Furthermore, since $\partial(D^{\tau_1 - \varepsilon} \backslash K) \subset \partial D^{\tau_1 - \varepsilon} \cup K$ we see that $w^{\tau_1 - \varepsilon} \geq 0$ on $\partial(D^{\tau_1 - \varepsilon} \backslash K)$. Thus, $w^{\tau_1 - \varepsilon}$ satisfies

$$\begin{aligned} &\Delta w^{\tau_1 - \varepsilon} + c^{\tau_1 - \varepsilon}(x) w^{\tau_1 - \varepsilon} = 0 \text{ in } D^{\tau_1 - \varepsilon} \backslash K \\ &w^\tau \geq 0 \text{ on } \partial(D^{\tau_1 - \varepsilon} \backslash K). \end{aligned} \tag{5.6}$$

The maximum principle for small domains implies that $w^{\tau_1-\varepsilon} \geq 0$ on $D^{\tau_1-\varepsilon}\backslash K$. Hence $w^{\tau_1-\varepsilon} \geq 0$ in all of $D^{\tau_1-\varepsilon}$ and thus it is positive there. However, this contradicts the choice of τ_1. Therefore $\tau_1 = 0$ and the function u is monotone in the x_1-variable.

Moreover, if f is differentiable, the derivative $u_1 = \dfrac{\partial u}{\partial x_1}$ satisfies an equation

$$\Delta u_1 + f'(u)u_1 = 0 \text{ in } \Omega.$$

As we already know $u_1 \geq 0$, and $u_1 \not\equiv 0$ identically, we conclude that $\dfrac{\partial u}{\partial x_1} > 0$ in Ω.

Finally, to show that such solution u is unique, we suppose that v is another solution. We argue exactly as before but with $w^\tau = u^\tau - v$. The same proof shows that $u^\tau \geq v$ for all $\tau \geq 0$. In particular, $u \geq v$. Interchanging the role of u and v we conclude that $u = v$. $\square$

Another beautiful application of the sliding method allows to extend lower bounds obtained in one part of a domain to a different part by moving a sub-solution around the domain and observing that it may never touch a solution.

Lemma 5.2 *Let u be a positive function in an open connected set D satisfying*

$$\Delta u + f(u) \leq 0 \textit{ in } D$$

with a Lipschitz function f. Let B be a ball with its closure $\bar{B} \subset D$, and suppose z is a function in $\bar{B}$ satisfying

$$\begin{aligned} &z \leq u \textit{ in } B \\ &\Delta z + f(z) \geq 0, \textit{ wherever } z > 0 \textit{ in } B \\ &z \leq 0 \textit{ on } \partial B. \end{aligned}$$

Then for any continuous one-parameter family of Euclidean motions (rotations and translations) $A(t)$, $0 \leq t \leq T$, so that $A(0) = \mathrm{Id}$ and $A(t)\bar{B} \subset D$ for all t, we have

$$z_t(x) = z(A(t)^{-1}x) < u(x) \textit{ in } B_t = A(t)B. \tag{5.7}$$

Proof. The rotational invariance of the Laplace operator implies that the function z_t satisfies

$$\begin{aligned} &\Delta z_t + f(z_t) \geq 0, \text{ wherever } z_t > 0 \text{ in } B_t \\ &z_t \leq 0 \text{ on } \partial B_t. \end{aligned}$$

Thus the difference $w_t = z_t - u$ satisfies $\Delta w_t + c_t(x)w_t \geq 0$ wherever $z_t > 0$ in B_t with c_t bounded in B_t, where, as always,

$$c_t(x) = \begin{cases} \dfrac{f(z_t(x)) - f(u(x))}{z_t(x) - u(x)}, \text{ if } z_t(x) \neq u(x) \\ \qquad 0, \text{ otherwise.} \end{cases}$$

In addition, $w_t < 0$ on ∂B_t.

We now argue by contradiction. Suppose that there is a first t so that the graph of z_t touches the graph of u at a point x_0. Then, for that t, still $w_t \leq 0$ in B_t, $w_t(x_0) = 0$. The point x_0 has to be inside B_t and thus $w_t \equiv 0$ in the whole component G of the set of points in B_t where $z_t > 0$ that contains x_0. Consequently, this is still true for all $\tilde{x} \in \partial G$ and thus ∂G lies inside B_t. But then $z_t(\tilde{x}) = u(\tilde{x}) > 0$ on ∂G, which contradicts the fact that $z_t = 0$ on ∂G. Hence the graph of z_t may not touch that of u and (5.7) follows. □

Lemma 5.2 is often used to "slide around" a sub-solution that is positive somewhere to show that solution itself is uniformly positive.

6 Monotonicity in Unbounded Domains

We now consider the monotonicity properties of bounded solutions of

$$\Delta u + f(u) = 0 \text{ in } \Omega \tag{6.1}$$

when the domain Ω is not bounded so that monotonicity may not be "forced" on the solution as in (5.2)-(5.3). We will consider two examples, the first one deals with the whole space and the version we present is fairly simple – the main result is that solution depends only on one variable, the second is more difficult – it addresses domains bounded by a graph of a function and shows monotonicity in any direction that does not touch the graph.

6.1 Monotonicity in $\mathbb{R}^n$

Our first example taken from the paper [6] by Berstycki, Hamel and Monneau deals with the whole space. We consider solutions of

$$\Delta u + f(u) = 0 \text{ in } \mathbb{R}^n \tag{6.2}$$

which satisfy $|u| \leq 1$ together with the asymptotic conditions

$$u(x', x_n) \to \pm 1 \text{ as } x_n \to \pm\infty \text{ uniformly in } x' = (x_1, \ldots, x_{n-1}). \tag{6.3}$$

The given function f is Lipschitz-continuous on $[-1, 1]$. We assume that there exists $\delta > 0$ so that

$$f \text{ is non-increasing on } [-1, -1+\delta] \text{ and on } [1-\delta, 1]; \text{ and } f(\pm 1) = 0. \tag{6.4}$$

The prototypical example is $f(u) = u - u^3$. We will show that any solution of (6.2) with the asymptotic conditions (6.3) is actually one-dimensional.

Theorem 6.1 *Let u be any solution of (6.2)-(6.3) such that $|u| \leq 1$. Then $u(x', x_n) = u_0(x_n)$ where u_0 is a solution of*

$$u_0'' + f(u_0) = 0 \text{ in } \mathbb{R},\ u_0(\pm\infty) = \pm 1.$$

Moreover, u is increasing with respect to x_n. Finally, such solution is unique up to a translation.

Without the uniformity assumption in (6.3) this is known as De Giorgi conjecture that is still open in $n > 3$. It has been recently established by Ambrosio and Cabré in $n = 3$. Another recent result is the proof of the De Giorgi conjecture in $n = 4, 5$ for odd solutions [16]. We will consider the proof of Cabré and Ambrosio in Section 7. The additional assumption of uniform convergence at infinity made in this section makes this much easier.

First we state a version of the maximum principle for unbounded domains.

Lemma 6.2 *Let D be an open connected set in $\mathbb{R}^n$, possibly unbounded. Assume that $\bar{D}$ is disjoint from the closure of an infinite open cone Σ. Suppose there is a function $z \in C(\bar{D})$ that is bounded above and satisfies for some continuous function $c(x)$*

$$\begin{aligned} &\Delta z + c(x)z \geq 0 \text{ in } D \text{ with } c(x) \leq 0 \\ &z \leq 0 \text{ on } \partial D. \end{aligned} \tag{6.5}$$

Then $z \leq 0$.

Proof. The idea is to decrease the cone Σ to a cone $\tilde{\Sigma}$ and to construct the principle eigenfunction ψ (with the corresponding eigenvalue λ) of the spherical Laplace-Beltrami operator in the region $G = S^{n-1}\backslash\tilde{\Sigma}$ with $\psi = 0$ on ∂G:

$$\begin{aligned} &\Delta_S \psi + \mu\psi = 0, \quad \psi > 0 \text{ in } G,\ \mu > 0, \\ &\psi = 0 \text{ on } \partial G. \end{aligned}$$

Then, going to the polar coordinates $x = r\xi$, $r > 0$, $\xi \in S^{n-1}$, one considers the function $g(x) = r^\alpha \psi(\xi)$, $\xi \in G$ defined on D, with $\alpha(n+\alpha-2) = \lambda$. With this choice of α the function g is harmonic:

$$\Delta g = \frac{\partial^2 g}{\partial r^2} + \frac{n-1}{4}\frac{\partial g}{\partial r} + \frac{1}{r^2}\Delta_S g = [\alpha(\alpha-1) + \alpha(n-1) - \mu]r^{\alpha-2}\Psi = 0.$$

Note that $g > 0$ in D, as D lies outisde the conce Σ that strictly contains $\tilde{\Sigma}$. Since the function g is harmonic, the function $\sigma = z/g$ satisfies the following equation in D:

$$\Delta\sigma + \frac{2}{g}\nabla g \cdot \nabla\sigma + c\sigma \geq 0.$$

This similar to (6.5) but with the asymptotic condition

$$\limsup_{|x|\to\infty} \sigma(x) \leq 0.$$

Moreover, $\sigma \leq 0$ on ∂D. Hence one may apply the usual maximum principle to $\sigma(x)$ and finish the proof. □

This lemma will be most important in the proof of the Berestycki-Caffarelli-Nirenberg result. For now we will need the following corollary that we will use for half-spaces.

Corollary 6.3 *Let f be a Lipschitz continuous function, non-increasing on $[-1, -1+\delta]$ and on $[1-\delta, 1]$ for some $\delta > 0$. Assume that u_1 and u_2 satisfy*

$$\Delta u_i + f(u_i) = 0 \text{ in } \Omega$$

and are such that $|u_i| \leq 1$. Assume furthermore that $u_2 \geq u_1$ on $\partial\Omega$ and that either $u_2 \geq 1-\delta$ or $u_1 \leq -1+\delta$ in Ω. If $\Omega \subset \mathbb{R}^n$ is an open connected set so that $\mathbb{R}^n\backslash\bar{\Omega}$ contains an open infinite cone then $u_2 \geq u_1$ in Ω.

Proof. Assume, for instance, that $u_2 \geq 1-\delta$ and set $w = u_1 - u_2$. Then

$$\Delta w + c(x,z)w = 0 \text{ in } \Omega$$

with

$$c(x) = \frac{f(u_1)-f(u_2)}{u_1-u_2} \leq 0 \text{ where } w \geq 0.$$

Hence we may apply the maximum principle to w and conclude that $w \leq 0$. □

Proof of Theorem 6.1. We are going to prove that u is increasing in any direction $\nu = (\nu_1, \ldots, \nu_n)$ with $\nu_n > 0$. This is sufficient as this means that

$$\frac{1}{\nu_n}\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial x_n} + \sum_{j=1}^{n-1} \alpha_j \frac{\partial u}{\partial x_j} > 0$$

for all $\alpha_j = \nu_j/\nu_n$. That means that all $\partial u/\partial x_j = 0$, $j = 1, \ldots, n-1$, so that u depends only on x_n and, moreover, $\partial u/\partial x_n > 0$.

Define $u^t(x) = u(x+t\nu)$, the goal is to show that $u^t(x) \geq u(x)$ for all $t \geq 0$. We start the sliding method with a very large t. Observe that there exists a real $a > 0$ so that $u(x', x_n) \geq 1-\delta$ for all $x_n \geq a$ and $u(x', x_n) \leq -1+\delta$ for all $x \leq -a$. Take $t \geq 2a/\nu_n$, then the functions u and u^t are such that

$$\begin{aligned} u^t(x', x_n) &\geq 1-\delta && \text{for all } x' \in \mathbb{R}^{n-1} \text{ and for all } x_n \geq -a \\ u(x', x_n) &\leq -1+\delta && \text{for all } x' \in \mathbb{R}^{n-1} \text{ and for all } x_n \leq -a \\ u^t(x', -a) &\geq u(x', -a) && \text{for all } x' \in \mathbb{R}^{n-1}. \end{aligned} \tag{6.6}$$

Hence we may apply Corollary 6.3 separately in $\Omega_1 = \mathbb{R}^{n-1} \times (-\infty, -a)$ and $\Omega_2 = \mathbb{R}^{n-1} \times (-a, +\infty)$. In both cases we conclude that $u^t \geq u$ and thus $u^t \geq u$ in all of $\mathbb{R}^n$ for $t \geq 2a/nu_n$.

Following the sliding method we start to decrease t, and let $\tau = \inf\{t > 0,\ u^t \geq u \text{ in } \mathbb{R}^n\}$ – by continuity we still have $u^\tau \geq u$ in $\mathbb{R}^n$. We need to show that $\tau = 0$ and argue by contradiction. Assume that $\tau > 0$ and consider two cases.

Case 1. Suppose that

$$\inf_{D_a}(u^\tau - u) > 0, \quad D_a = \mathbb{R}^{n-1} \times [-a, a]. \tag{6.7}$$

The function u is globally Lipschitz continuous – this follows from the standard elliptic estimates [15] that we do not repeat. This implies that there exists $\eta_0 > 0$ so that for all $\tau - \eta_0 < t < \tau$ we still have

$$u^t(x', x_n) > u(x', x_n) \text{ for all } x' \in \mathbb{R}^{n-1} \text{ and for all } -a \leq x_n \leq a. \tag{6.8}$$

As $u(x', x_n) \geq 1-\delta$ for all $x_n \geq a$, it follows that

$$u^t(x', x_n) \geq 1-\delta \text{ for all } x_n \geq a \text{ and } t > 0. \tag{6.9}$$

We may now apply Lemma 6.3 in the half-spaces $\{x_n > a\}$ and $\{x_n < -a\}$ to conclude that $u^{\tau-\eta}(x) > u(x)$ everywhere in $\mathbb{R}^n$ for all $\eta \in [0, \eta_0]$. This contradicts the choice of τ. Hence the case (6.7) is impossible.

Case 2. Suppose that

$$\inf_{D_a}(u^\tau - u) = 0, \quad D_a = \mathbb{R}^{n-1} \times [-a, a]. \tag{6.10}$$

We will use the usual trick of moving "the interesting part" of the domain to the origin and passing to the limit. Then there exists a sequence $\xi_k \in D_a$ so that $u^\tau(\xi_k) - u(\xi_k) \to 0$ as $k \to \infty$. Let us translate; $u_k(x) = u(x + \xi_k)$, then the standard elliptic estimates imply that $u_k(x)$ converge along a subsequence to a function $u_\infty(x)$, uniformly on compact sets. We have $u^\tau_\infty(0) = u_\infty(0)$ and $u^\tau_\infty \geq u_\infty$ because $u^\tau_k \geq u_k$ for all k. The strong maximum principle implies that $u^\tau_\infty = u_\infty$, that is, $u_\infty(x + \tau\mathbf{e}_n) = u_\infty(x)$ – the function u_∞ is periodic in x_n. However, as all $\xi_k \in D_a$ and hence their n-th components are uniformly bounded $|(\xi_k)_n| \leq a$, the function u_∞ must satisfy the boundary conditions (6.3). This is a contradiction. Hence this case is also impossible and thus $\tau = 0$. □

6.2 Monotonicity in general unbounded domains

We now consider the monotonicity properties of bounded solutions of

$$\begin{aligned} &\Delta u + f(u) = 0 \text{ in } \Omega \\ &u = 0 \text{ on } \partial\Omega \end{aligned} \tag{6.11}$$

when the domain Ω is not bounded so that monotonicity may not be "forced" on the solution as in (5.2)-(5.3). We assume that u is bounded; $0 < u \leq M < \infty$ in Ω and the domain Ω is defined by

$$\Omega = \{x \in \mathbb{R}^n : \ x_n > \phi(x_1, \ldots, x_{n-1}), \tag{6.12}$$

where $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a globally Lipschitz continuous function. A typical example would be a half-space Ω – the main result we are going to prove says that u has to be monotonic function of the variable x_n only in this case.

Thinking of some reaction-diffusion applications we will assume that f is Lipschitz continuous on $\mathbb{R}^+$, $f(s) > 0$ on $(0, 1)$ and $f(s) \leq 0$ for $s \geq 1$. Furthermore, we assume, roughly speaking, that f is of the KPP type (think of $f(s) = s(1 - s)$):

$$f(s) \geq \delta_0 s \text{ on } [0, s_0] \text{ for some } s_0 > 0, \tag{6.13}$$

and

$$\text{there exists } s_1 \text{ so that } f \text{ is non-increasing on } (s_1, 1). \tag{6.14}$$

The main result says that such u is unique, monotonic in x_n and tends to one as distance to the boundary tends to infinity.

Theorem 6.4 *The function u has the following properties:*
(i) it is monotonic with respect to x_n:

$$\frac{\partial u}{\partial x_n} > 0 \ \textit{in } \Omega,$$

(ii) $0 < u < 1$ *in* Ω
(iii) $u(x) \to 1$ *as* $\operatorname{dist}(x, \partial\Omega) \to \infty$*, uniformly in* Ω.

(iv) u is the unique bounded solution of (6.11) that is positive inside Ω.
(v) Let κ be the Lipschitz constant of the graph function ϕ, then given any collection of constants a_α, $\alpha = 1, \dots, n-1$ so that $\sum_{\alpha=1}^{n-1} a_\alpha^2 < \frac{1}{\kappa^2}$, we have

$$\frac{\partial u}{\partial x_n} + \sum_{\alpha=1}^{n-1} a_\alpha \frac{\partial u}{\partial x_\alpha} > 0 \text{ in } \Omega. \tag{6.15}$$

Part (v) implies that u is increasing in any direction ξ such that there exists an orthonormal change of variables $x \to z$ with the z_n-axis in the direction of ξ and $\partial\Omega = \{z_n = \tilde{\phi}(z')\}$ with a Lipschitz function $\tilde{\phi}$. Note that when $\phi = 0$, that is, when Ω is a half-space, the constants α_α in (6.15) may be arbitrary which immediately implies that all $\frac{\partial u}{\partial x_\alpha} = 0$ for $\alpha = 1, \dots, n-1$ so that u has to be a function of x_n only in this case.

The proof of Theorem 6.4 is fairly long and we split it into proofs of each part separately. The general flow is as follows. First one uses the maximum principle of Lemma 6.2 to show that $0 < u < 1$ and hence (ii) holds. Second, we show that $f(u) \to 0$ as $\operatorname{dist}(x, \partial\Omega)) \to \infty$ – roughly speaking, because otherwise u would be strictly convex which is impossible as $0 < u < 1$. It is easy to conclude from $f(u) \to 0$ that $u \to 1$. In the third step uniqueness is proved by the sliding method. Finally, monotonicity is established by constructing a solution that is positive and monotonic. Uniqueness implies that the original solutioon coincides with that one and hence is itself monotonic. Such solution is constructed first on bounded domains and then passing to the limit. The tricky part is to make sure that the limit is positive – this is done by ensuring that solution we construct stays above u. This in turn requires some Hölder estimates on u – we will depart from our general rule and obtain some regularity estimates for this sake.

Proof of (ii) in Theorem 6.4. Let us assume that $u > 1$ somewhere and let D be a connected component of the set where $u > 1$. The set D lies outside an open cone since the graph function ϕ is Lipschitz. Consider the function $z = u - 1$ in D. It satisfes

$$\Delta z = -f(u) \geq 0 \text{ in } D,$$

as $f(u) \leq 0$ in D. Furthermore, z vanishes on ∂D and thus Lemma 6.2 implies that $z \leq 0$ in D. Thus $u \leq 1$. Suppose $u(x_0) = 1$ for some $x_0 \in \Omega$, then $z = u - 1$ satisfies $z \leq 0$ in Ω, $z(x_0) = 0$ and

$$\Delta z + c(x)z = 0 \text{ in } \Omega$$

where

$$c(x) = \begin{cases} \dfrac{f(u(x))}{u(x)-1}, & \text{if } u(x) < 1 \\ 0, & \text{if } u(x) = 1. \end{cases}$$

The function $c(x)$ is bounded and hence the strong maximum principle implies that $z \equiv 0$ in Ω which contradicts the fact that $z = -1$ on $\partial\Omega$. $\square$

Proof of (iii) in Theorem 6.4. The proof that $u(x) \to 1$ as $\operatorname{dist}(x, \partial\Omega) \to \infty$ is in two steps. First, we show that u is bounded away from zero at a fixed distance away from the boundary: $u(x) \geq \varepsilon_1$ if $\operatorname{dist}(x, \partial\Omega) > R_0$. Second, we show that $f(u(y)) \to 0$ as

$\mathrm{dist}(x, \partial\Omega) \to \infty$. This implies that $u \to 1$, as u is bounded away from zero in this region, and $u = 0$ and $u = 1$ are the only zeros of $f(u)$.

Lemma 6.5 *There exist $\varepsilon_1 > 0$ and $R_0 > 0$ that depend only on the function $f(s)$ so that*

$$u(x) > \varepsilon_1 \text{ if } \mathrm{dist}(x, \partial\Omega) > R_0. \tag{6.16}$$

Proof. Let R_0 be so large that the principle eigenvalue λ_1 of the Dirichlet Laplacian in a ball $B(0; R_0)$ of radius R_0 is smaller than the constant δ_0 in (6.13). Let ϕ_1 be the corresponding positive eigenfunction with $\max \phi_1 = 1$. Then the function $z = \varepsilon\phi_1$ is a sub-solution of our equation for $0 < \varepsilon \leq s_0$, that is,

$$\begin{aligned} &\Delta z + f(z) \geq 0 \text{ in } B(0; R_0) \\ &z = 0 \text{ on } \partial B(0; R_0). \end{aligned} \tag{6.17}$$

Let us choose a large enough so that $\overline{B(a; R_0)} \subset \Omega$ and set

$$\varepsilon_0 = \min_{\bar{B}(a;R_0)} u,$$

clearly $\varepsilon_0 > 0$. We set $\varepsilon_1 = \min(\varepsilon_0, s_0)$, then, as $\phi_1 \leq 1$ we have

$$\varepsilon_1\phi_1(x - a) \leq u(x) \text{ in } B(a; R_0).$$

Lemma 5.2 (we slide the ball $B(a; R_0)$ around the domain) implies now that

$$\varepsilon_1\phi_1(x - y) \leq u(x) \text{ in } B(y; R_0)$$

for all $y \in \Omega$ with $\mathrm{dist}(y, \partial\Omega) > R_0$. In particular, $u(y) > \varepsilon_1$ for such y. $\Box$

As we have explained above, in order to complete the proof of part (iii) of Theorem 6.4 we show that $f(u(x)) \to 0$ as $\mathrm{dist}(x, \partial\Omega) \to \infty$. Let us define $C_n = \max_{B(0;1)} v = v(0) = 1/(2n)$, where

$$v(x) = \frac{1 - |x|^2}{2n}$$

is the exit time from the unit ball, solution of

$$\begin{aligned} &-\Delta v = 1 \text{ in } B(0; 1) \\ &v = 0 \text{ on } \partial B(0; 1). \end{aligned}$$

Let also $|y| \geq R_0$ with R_0 as in the previous lemma, and set

$$\delta(y) = \min\{f(s) : \ s \in [\varepsilon_1, u(y)]\}.$$

Here ε_1 is also as in (6.16). We claim that

$$\delta(y) \leq \frac{1}{C_n[\mathrm{dist}(y, \partial\Omega) - R_0]^2} = \frac{2n}{[\mathrm{dist}(y, \partial\Omega) - R_0]^2}. \tag{6.18}$$

We argue by contradiction: suppose that (6.18) fails, that is,

$$\delta(y_0) > \frac{2n}{[\mathrm{dist}(y_0, \partial\Omega) - R_0]^2}$$

for some y_0 with $\mathrm{dist}(y_0, \partial\Omega) > R_0$. Fix $R < \mathrm{dist}(y_0, \partial\Omega) - R_0$ so that

$$\frac{\delta(y_0)}{2n} > 1/R^2. \tag{6.19}$$

The function u cannot have a local minimum at y_0 since $\Delta u(y_0) = -f(y_0) < 0$, thus we may find y_1 close to y_0 so that $u(y_1) < u(y_0)$ and $\mathrm{dist}(y_1, \partial\Omega) > R_0 + R$. Lemma 6.5 implies that $u \geq \varepsilon_1$ in $B := B(y_1; R)$. Let

$$z(y) = \delta(y_0) R^2 v\left(\frac{y - y_1}{R}\right),$$

then $\max_B z = \delta(y_0)R^2/(2n)$ and z satisfies

$$\begin{aligned} -\Delta z &= \delta(y_0) \text{ in } B \\ z &= 0 \text{ on } \partial B. \end{aligned} \tag{6.20}$$

Now we do "ballooning" (as opposed to "sliding") of z: let $z^\tau(x) = \tau z(x)$, then for $\tau > 0$ small we have $z^\tau(x) < \varepsilon_1 \leq u(x)$ in B. As we increase τ there is a value τ_0 so that the graph of z^τ touches the graph of $u(y)$ at some point x_0. Since $z = 0$ on ∂B, x_0 has to be inside B. Also, as $z^\tau(y_1) \leq u(y_1)$, we have

$$u(x_0) = \tau z(x_0) \leq \tau z(y_1) = \frac{\tau\delta(y_0)R^2}{2n} \leq u(y_1) < u(y_0) < 1 \tag{6.21}$$

and hence $\tau < 2n/(\delta(y_0)R^2) < 1$ by the choice of R – see (6.19). Therefore

$$w := \tau z - u \leq 0 \text{ in } B,\ w(x_0) = \tau z(x_0) - u(x_0) = 0. \tag{6.22}$$

Note that according to (6.21), $u(x_0) < u(y_0)$ and so in an neighborhood N of x_0 we still have $u(x) < u(y_0)$. The definition of $\delta(y_0)$ implies that $\Delta u(x) \leq -\delta(y_0)$ for $x \in N$ and thus $\Delta w(x) \geq -\tau\delta + \delta > 0$ for $x \in N$ since $\tau < 1$. This contradicts the fact that w has a local maximum at x_0. □

Therefore $\delta(y)$ satisfies $\delta(y) \to 0$ as $\mathrm{dist}(y, \partial\Omega) \to \infty$ and thus $f(y) \to 0$ in this limit, which implies $u(y) \to 1$ as $u(y) \geq \varepsilon_1$ in this region.

Proof of (iv) in Theorem 6.4. We now show uniqueness of a positive bounded solution of (6.11). First, we show the following.

Lemma 6.6 *For any $h > 0$ the solution is bounded away from 1 in $\Omega_h = \{x \in \Omega : \phi(x') < x_n < \phi(x') + h\}$.*

Proof. The argument is by contradiction. Assume that there exists a sequence $\xi_j \in \Omega_h$ so that $u(\xi_j) \to 1$. We may shift all ξ_j to the origin, with Ω shifted to a domain $\Omega_j = \{x_n > \phi_j(x')$. The functions $\phi_j(x')$ are all translations of $\phi(x)$ and thus all have the same Lipschitz constant. Thus along a subsequence they converge to a function $\hat\phi(x)$. The shifted domains converge to $\hat\Omega = \{x_n > \hat\phi(x')\}$, while the shifted solutions converge along a subsequence (one needs the Hölder estimates on u that we will prove later to see this) to a solution of

$$\begin{aligned} &\Delta\hat u + f(\hat u) = 0 \\ &0 \leq \hat u \leq 1 \end{aligned}$$

in $\hat{\Omega}$ such that $\hat{u}(0) = 1$. This is impossible according to part (ii) of the present theorem that we have already proved. □

Let now u and w be a pair of positive bounded solutions of (6.11). Note that $d(x) := x_n - \phi(x') \to \infty$ implies $\text{dist}(x, \nabla\Omega) \to \infty$, as the function ϕ is Lipschitz. Hence part (iii) of Theorem 6.4 that we have already proved implies that both $u(x)$ and $w(x)$ tend to one uniformly as $d(x) \to \infty$. Hence there exists $A > 0$ so that

$$u(x), w(x) \geq s_1 \text{ if } d(x) \geq A, \tag{6.23}$$

with s_1 as in (6.14). We set $\Omega^A = \{x \in \Omega : \ d(x) > A\}$ and $\Omega_A = \{x \in \Omega : \ d(x) < A\}$. A key point is that once we show $u \geq w$ in Ω_A then this inequality propagates to the whole Ω. More generally, we have the following lemma.

Lemma 6.7 *Suppose that for some $\tau \geq 0$ the inequality*

$$u^\tau(x) = u(x + \tau\mathbf{e}_n) \geq w(x) \tag{6.24}$$

holds in $\bar{\Omega}_A$. Then (6.24) holds in all of Ω.

Proof. The proof is very similar to that of Corollary 6.3. Assume that (6.24) holds. The function $z = w - u^\tau$ satisfies an equation of the form $\Delta z + c(x)z = 0$ in Ω^A. Both $w, u^\tau \geq s_1$ in Ω^A, thus

$$c(x) = \frac{f(w(x)) - f(u^\tau(x))}{w(x) - u^\tau(x)} \leq 0.$$

Moreover, by assumption $z \leq 0$ on $\partial\Omega^A$ – this is all we need from (6.24). We may now apply the maximum principle for unbounded domains of Lemma 6.2 to Ω^A and conclude that $z \leq 0$ in Ω^A, that is (6.24) holds in all of Ω. □

Thus it suffices to show that $u(x) \geq w(x)$ in Ω_A. We show that by the sliding method. Note that $u^\tau(x) = u(x + \tau\mathbf{e}_n) > w(x)$ in Ω_A for large $\tau > 0$ because $u^\tau(x) \to 1$ as $\tau \to +\infty$ (according to the already proved part (iii) of the present theorem) while Lemma 6.6 implies that $w(x)$ is bounded away from 1 in Ω_A. As has been our common practice, we let

$$T = \inf\{\tau > 0 : \ u^\tau(x) \geq w(x) \text{ in } \Omega_A\}.$$

By continuity,

$$u^T(x) \geq w(x) \text{ in } \Omega_A. \tag{6.25}$$

We have only to prove that $T = 0$. Suppose that $T > 0$, then there is a sequence of points $x_j \in \Omega_A$ and a sequence $\tau^j < T$, $\tau_j \to T$, so that

$$u(x_j + \tau_j\mathbf{e}_n) < w(x_j). \tag{6.26}$$

Once again we shift the points x_j to the origin. The domain Ω is moved to Ω_j, and, as before, along a subsequence, Ω_j converge to a domain $\hat{\Omega} = \{x : \ x_n > \hat{\phi}(x')\}$ with a Lipschitz function $\hat{\phi}$. The shifts of u and w converge to positive solutions $\hat{u}$ and $\hat{w}$ of (6.11) in $\hat{\Omega}$ – this follows from the Hölder estimates that we prove below. As follows from (6.25), $\hat{u}(x + T\mathbf{e}_n) \geq \hat{w}(x)$

in $\hat{\Omega}_A$. Lemma 6.7 implies that this holds in the whole domain $\hat{\Omega}$. But passing to the limit in (6.26) we obtain that at the origin

$$0 < \hat{u}(T\mathbf{e}_n) \leq \hat{w}(0).$$

This implies that $\hat{u}^T(0) = \hat{w}(0)$, and in particular $0 \in \hat{\Omega}$, as on the boundary of $\hat{\Omega}$ we have $\hat{w} = 0$ while $\hat{u}^T > 0$ since $T > 0$.

As before the function $\hat{z} = \hat{w} - \hat{u}^T$ satisfies an elliptic equation

$$\Delta\hat{z} + \hat{c}(x)\hat{z} = 0.$$

As $\hat{z} \leq 0$ and $\hat{z}(0) = 0$ and $0 \in \hat{\Omega}$, it follows that $\hat{z} \equiv 0$. However, at the boundary $\partial\hat{\Omega}$, $\hat{w} = 0$ while $\hat{u}^T > 0$ which is a contradiction. Thus $T = 0$ and uniqueness is shown. □

Proof of (i) in Theorem 6.4. One would like to use Theorem 5.1 to show monotonicity. The problem is that the domain Ω is unbounded. This should be remedied by the fact that $u \to 1$ as $\operatorname{dist}(x, \partial\Omega) \to \infty$ – hence one may think of infinity as another part of the boundary where the value $u = 1$ is prescribed that guarantees that condition (5.2) still "holds" with $\phi(x') = 0$ and "$\phi(x'') = 1$". In order to make this precise we will consider a family of approximating domains

$$\Omega_h = \{\phi(x') < x_n < \phi(x') + h\} \tag{6.27}$$

and consider a sequence $h_n \to \infty$. We will construct a monotonic solution w_h on Ω_h and let $h \to \infty$. The sequence w_{h_n} will converge to a limit function w along a subsequence $h_n \to \infty$. The function w will be a monotonic solution of (6.11) and uniqueness (the already proved part (iv) of the Theorem 6.4) will finish the proof. However, in order to make sure that $w \neq 0$ identically we will construct w_h so that $w_h \geq u$ in Ω_h. Somewhat destructingly for the pure beauty of the proof in order to establish the latter bound we need certain Hölder estimates on u. It is with a heavy heart that we embark on the road of obtaining those Hölder estimates first.

The Hölder regularity diversion. The first lemma establishes the Hölder regularity near the boundary.

Lemma 6.8 *There exist positive constants R, C and α, $\alpha < 1$ such that*

$$u(x) \leq C|x - x_0|^\alpha \text{ if } x_0 \in \partial\Omega \text{ and } |x - x_0| \leq R. \tag{6.28}$$

One can choose C to depend only on the Lipschitz constants κ and K of the functions $\phi(x')$ and $f(s)$, respectively, with α depending only on κ.

Proof. We may take $x_0 = 0$ without loss of generality. Again we consider a cone Σ with the vertex at 0 but lying outside Ω:

$$\Sigma = \{x : \; x_n < -\kappa|x'|\} \text{ with } \kappa' > \kappa$$

so that $\bar{\Sigma} \cap \bar{\Omega} = \{0\}$. Let g be the positive harmonic function in the cone Σ^c of the form $g = r^\alpha\psi(x/|x|)$, constructed as in the proof of Lemma 6.2 – ψ is the eigenfunction of the spherical Dirichlet Laplacian in a slightly larger cone. The constant α depends on κ', of course, and by enlarging κ' we may take $\alpha < 1$, since $\kappa' = 0$ corresponds to $\alpha = 1$. We normalize ψ so that $g \geq |x|^\alpha$ in $\bar{\Omega}$. Then for some constant C_0

$$|x|^\alpha \leq g(x) \leq C_0|x|^\alpha \text{ in } \Omega. \tag{6.29}$$

Consider the barrier function

$$z = 2R^{-\alpha}g - g^2 \text{ in } G = \Omega \cap \{|x| < R\} \tag{6.30}$$

with the radius R chosen so small that

$$\Delta z + f(z) = -2|\nabla g|^2 + f\left(\frac{2}{R^\alpha}g - g^2\right) \leq 0 \text{ in } G. \tag{6.31}$$

This is possible as $|\nabla g|$ blows up at $x = 0$, as we may take $\alpha < 1$. Furthermore, if R is small we may ensure

$$z \geq 1 \text{ on } \Omega \cap \{|x| = R\} \tag{6.32}$$

because of (6.29). Now we observe that the function $w = u - z$ satisfies

$$\Delta w + c(x)w \geq 0 \text{ in } G$$

with $w \leq 0$ on ∂G. Finally, for R small the set G has a small volume and thus we may apply the maximum principle to conclude that $w \leq 0$ in G and thus

$$u \leq z \leq \frac{2}{R^\alpha}g \leq 2C_0R^{-\alpha}|x|^\alpha \text{ in } G.$$

Thus (6.28) holds. □

The interior Hölder regularity follows from Lemma 6.8:

Lemma 6.9 *The function u is uniformly Hölder continuous in $\bar{\Omega}$.*

Proof. Let α, C and R be as in Lemma 6.8. We will show that there exists a constant $M > 0$ so that

$$|u(x) - u(y)| \leq M|x - y|^\alpha \text{ for } x, y \in \Omega. \tag{6.33}$$

The function f is bounded, hence the standard interior elliptic regularity [15] implies that if $B = B(y;\rho)$ and $|x - y| \leq \rho/2$ then

$$|u(x) - u(y)| \leq \hat{C} \max_B u \cdot \left(\frac{|x-y|}{\rho}\right)^\gamma + \hat{C}\rho^{2-\gamma}|x-y|^\gamma, \text{ for } 0 < \gamma \leq 1, \tag{6.34}$$

with a constant $\hat{C}$ that depends only on n.

Note that, as $0 < u < 1$, in order to prove (6.33) we need to consider only the case $|x - y| < R/4$. To be sufficiently pedantic, we will consider four cases, depending on whether x and y are close to the boundary $\partial\Omega$.

Case 1. $\text{dist}(x, \partial\Omega), \text{dist}(x, \partial\Omega) \geq R/2$. Using (6.34) with $\gamma = 1$ and $\rho = R/2$ we see that $|\nabla u(z)| \leq C$ for $\text{dist}(z, \partial\Omega) \geq R/2$. Moreover, the Lipschitz regularity of the boundary implies that we may join x and y by a path $\mathbf{l}$ consisting of two line segments that stays at distance $R/2$ from the boundary with $|\mathbf{l}| \leq C|x-y|$. Using the above two facts we get

$$|u(x) - u(y)| \leq C|\mathbf{l}| \leq C|x-y| \leq CR^{1-\alpha}|x-y|^{\alpha}.$$

Case 2. $\operatorname{dist}(x, \partial\Omega) \leq R/2 \leq \operatorname{dist}(y, \partial\Omega)$. We may use (6.34) with $\rho = R/2$ and $\gamma = \alpha$ to get

$$|u(x) - u(y)| \leq C\left(\frac{2|x-y|}{R}\right)^{\alpha} + CR^{2-\alpha}|x-y|^{\alpha}.$$

Case 3. $\operatorname{dist}(x, \partial\Omega) \leq \operatorname{dist}(y, \partial\Omega) \leq R/2$ and $|x-y| \geq \operatorname{dist}(y, \partial\Omega)/4$. Then we may use Lemma 6.8:

$$|u(x) - u(y)| \leq \max\{u(x), u(y)\} \leq C[\operatorname{dist}(y, \partial\Omega)]^{\alpha} \leq C|x-y|^{\alpha}.$$

Case 4. $\operatorname{dist}(x, \partial\Omega) \leq \operatorname{dist}(y, \partial\Omega) \leq R/2$ and $|x-y| \leq \operatorname{dist}(y, \partial\Omega)/4$. We apply (6.34) with $\rho = \operatorname{dist}(y, \partial\Omega)$ and $\gamma = \alpha$, observing in addition that Lemma 6.8 implies that $u \leq C(\rho)^{\alpha}$ in B. This yields

$$|u(x) - u(y)| \leq C\rho^{\alpha}\left(\frac{|x-y|}{\rho}\right)^{\alpha} + \hat{C}\left(\frac{R}{2}\right)^{2-\alpha}|x-y|^{\alpha}.$$

Lemma 6.9 is proved. $\square$

Back to the proof of (i) in Theorem 6.4. We are now ready to construct the monotonic solution w. This is done in two steps. First, we consider the domain

$$\Omega_{h,R} = \{x \in \Omega_h : \ |x'| < R\}$$

with Ω_h as in (6.27). Using the constants α and M as in (6.33) we define

$$\sigma(t) = \begin{cases} Mt^{\alpha}, \text{ for } 0 \leq t \leq M^{-1/\alpha}, \\ \quad 1, \text{ for } t \geq M^{-1/\alpha}. \end{cases}$$

We consider $h > h_0 = 1 + M^{-1/\alpha}$ and define a continuous function σ_R on $\partial\Omega_{h,R}$:

$$\sigma(t) = \begin{cases} 0, & \text{for } x \in \partial\Omega, \\ 1, & \text{for } x \text{ s.t. } x_n = \phi(x') + h, \\ \sigma(x_n - \phi(x')), & \text{otherwise on } \partial\Omega_{h,R}. \end{cases}$$

Note that $\sigma_R \geq u$ on $\partial\Omega_h$ by (6.33). Let w the solution of

$$\begin{aligned} &\Delta w_{h,R} + f(w_{h,R}) = 0 \text{ in } \Omega_{h,R} \\ &w = \sigma_R \text{ on } \partial\Omega_{h,R}. \end{aligned} \tag{6.35}$$

Existence of a solution to (6.35) follows from the fact that it has a sub-solution $\underline{w} = u$ and a super-solution $\overline{w} = 1$. Indeed, start with $w_0 = \underline{w}$ and solve

$$\begin{aligned} &\Delta w_{j+1} - kw_{j+1} = -f(w_j) - kw_j \text{ in } \Omega_{h,R} \\ &w_{j+1} = \sigma_R \text{ on } \partial\Omega_{h,R}. \end{aligned}$$

Here k is the Lipschitz constant of f. First, we have

$$\Delta w_1 - kw_1 = -f(w_0) - kw_0 = -f(u) - ku = \Delta u - ku$$

and hence

$$\Delta(w_1 - u) - k(w_1 - u) = 0 \text{ in } \Omega_{h,R},$$

while $w_1 \geq u$ on $\partial\Omega_{h,R}$. Hence $w_1 \geq u \geq w_0$. The induction argument shows that

$$w_0 \leq w_1 \leq \cdots \leq \overline{w}, \tag{6.36}$$

because

$$\Delta(w_{j+1} - w_j) - k(w_{j+1} - w_j) = -k(w_j - w_{j-1}) - [f(w_j) - f(w_{j-1})] \leq 0$$

by the induction hypothesis. The last inequality in (6.36) also follows from induction applied to

$$\Delta(w_{j+1} - \bar{w}) - k(w_{j+1} - \bar{w}) = -k(w_j - \bar{w}) - (f(w_j) - f(\bar{w})) \geq 0.$$

Hence w_j converge to limit $w_{h,R}$ as $j \to +\infty$ – elliptic regularity implies that $w_{h,R}$ is a solution of (6.35).

Theorem 5.1 implies that $w_{h,R}$ is monotonic in x_n. The maximum principle implies that $w_{h,R} \geq u$.

We now pass to the limit $R \to \infty$. The same Hölder estimates as before imply that $w_{h,R}$ converges along a subsequence $R_n \to \infty$ to a function $w_h \geq u$ that satisfies

$$\begin{aligned} &\Delta w_h + f(w_h) = 0 \text{ in } \Omega_h \\ &w_h = 0 \text{ on } \partial\Omega \\ &w_h = 1 \text{ on } \{x_n = \phi(x') + h\}. \end{aligned}$$

Finally we let $h \to \infty$, and by the same argument conclude that along a subsequence w_{h_n} converges to a monotonic solution of

$$\begin{aligned} &\Delta w + f(w) = 0 \text{ in } \Omega_h \\ &w = 0 \text{ on } \partial\Omega \end{aligned}$$

with $w \geq u$. Hence uniqueness of a positive bounded solution implies that u has to coincide with w and we are done. □

Proof of (v) of Theorem 6.4. This one is trivial: all it says is that u is monotonic in any direction ξ such that there exists an orthonormal basis with $\mathbf{e}_n$ along ξ so that the boundary $\partial\Omega$ may be represented as $z = \phi(z')$ in the new variables. □

7 The De Giorgi conjecture in dimension $n = 3$

We will now describe the recent proof by Amborsio and Cabré [1, 2] of the De Giorgi conjecture.

Conjecture. *Let $u \in C^2(\mathbb{R}^n)$ be solution of*

$$\Delta u = u^3 - u \tag{7.1}$$

such that $|u| \leq 1$ *and* $\dfrac{\partial u}{\partial x_n} > 0$ *in the whole of* $\mathbb{R}^n$. *Is it true that the level sets* $\{u = \lambda\}$ *are hyper-planes, at least when* $n \leq 8$? Let us explain briefly the motivation behind the De Giorgi conjecture for a general equation

$$\Delta u = F'(u) \tag{7.2}$$

with $F(-1) = F(1) = 0$. The values $u = \pm 1$ are assumed to be the stable equilibria of $\dot{u} = F(u)$. Rescaling (7.2) ($x \to x/\varepsilon$), we obtain

$$\varepsilon^2 \Delta u = F'(u). \tag{7.3}$$

Solutions of (7.3) are minimizers of the functional

$$\mathcal{E}_\varepsilon = \int \left\{ \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} F(u) \right\} dx.$$

Intuitively, the minimizers of $\mathcal{E}_\varepsilon$ are very close to $u = \pm 1$ away from an interface of width ε that separates the two states. The second term in $\mathcal{E}_\varepsilon$ is, roughly speaking, proportional to the area of the interface between the two phases, where u is different from ± 1. The De Giorgi conjecture states that the interface (understood as a fixed level set $u = u_0$ with $0 < u_0 < 1$) must be planar. This relates the De Giorgi conjecture to the Bernstein problem: show that the level sets of all solutions of the minimal surface equation

$$\mathrm{div} \left(\frac{\nabla \psi}{\sqrt{1 + \nabla \psi|^2}} \right) = 0$$

that is, of the minimizers of the area functional,

$$\mathcal{E} = \int \sqrt{1 + |\nabla \psi|^2} dx, \tag{7.4}$$

are hyper-planes. The Bernstein problem has been solved by Bombieri, De Giorgi and Guisti who showed that this holds in dimension $n \leq 7$ but not higher. As we are talking about an interface for (7.2), the whole space has dimension $n \leq 8$ in the De Giorgi conjecture.

We will consider a slightly simpler version of the De Giorgi conjecture with an additional assumption

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1, \text{ for all } x' \in \mathbb{R}^{n-1}. \tag{7.5}$$

This imposes an assumption that we are indeed looking at an interface between $u = \pm 1$. Here the limits are not assumed to be uniform in $x' \in \mathbb{R}^{n-1}$, unlike in the work of Berestycki, Hamel and Monneau that we considered in Theorem 6.1 in Section 6.1. Non-uniformity of the limits make the problem much more difficult and the proof is known only when $n \leq 3$. We mention also that recently Ghoussoub and Gui have proved the De Giorgi conjecture in the case $n = 4, 5$ in [16] but with an assumption that the solution u is odd in some variables.

Theorem 7.1 *Let* u *be a bounded solution of*

$$\Delta u - F'(u) = 0 \text{ in } \mathbb{R}^3 \tag{7.6}$$

satisfying

$$\partial_3 u > 0 \text{ in } \mathbb{R}^3 \text{ and } \lim_{x_n\to\pm\infty} u(x', x_n) = \pm 1, \text{ for all } x' \in \mathbb{R}^{n-1}. \tag{7.7}$$

Assume that $F \in C^2(\mathbb{R})$ *and that*

$$F \geq \min\{F(-1), F(1)\} \text{ in } (-1, 1). \tag{7.8}$$

Then the level sets of u *are planes, that is, there exists a vector* $a \in \mathbb{R}^3$ *and a function* $g \in C^2(\mathbb{R})$ *so that* $u(x) = g(a \cdot x)$ *for all* $x \in \mathbb{R}^3$.

Note that the direction a is not known a priori unless we require that the limits in (7.5) are uniform. In that case $u = u(x_n)$ as we have already seen in Theorem 6.1. Otherwise, if u is a one-dimensional solution we may slightly rotate the vector a and get another such solution. Ambrosio and Cabré actually prove also the full Di Giorgi conjecture in $n = 3$ without the additional assumption (7.5) but we will not discuss this part of their result here.

Proof of Theorem 7.1. The proof of Theorem 7.1 is based on two observations. First, one proves an energy estimate.

Lemma 7.2 *Let* u *be a bounded solution of*

$$\Delta u - F'(u) = 0 \text{ in } \mathbb{R}^n,$$

where F *is an arbitrary* $C^2(\mathbb{R})$ *function. Assume that*

$$\frac{\partial u}{\partial x_n} > 0 \text{ in } \mathbb{R}^n \text{ and } \lim_{x_n\to\pm\infty} u(x', x_n) = \pm \text{ for all } x' \in \mathbb{R}^{n-1}.$$

Then there exists a constant $C > 0$ *that is independent of* R *so that*

$$\int_{B(0;R)} \left\{\frac{1}{2}|\nabla u|^2 + F(u) - F(1)\right\} dx \leq CR^{n-1}. \tag{7.9}$$

The proof of this lemma is postponed for the moment. There is no restriction on the space dimension in Lemma 7.2.

In the second step one considers the functions

$$\phi = \frac{\partial u}{\partial x_3} \text{ and } \sigma_i = \frac{\partial_i u}{\partial_3 u}.$$

We compute that

$$\nabla \cdot (\phi^2 \nabla \sigma_i) = \nabla \cdot (\partial_3 u \nabla(\partial_i u) - \partial_i u \nabla(\partial_3 u)) = \partial_3 u \Delta(\partial_i u) - \partial_i u \Delta(\partial_3 u)) = 0.$$

We used in the last step that all $\partial_i u$ satisfy

$$\Delta(\partial_i u) - F''(u)\partial_i u = 0. \tag{7.10}$$

Hence all functions σ_i satisfy

$$\nabla \cdot (\phi^2 \nabla \sigma_i) = 0. \tag{7.11}$$

Recall that by assumption (7.8), $F(u) \geq \min(F(-1), F(1))$. Suppose first that

$$\min(F(-1), F(1)) = F(1).$$

Lemma 7.2 implies that in dimension $n = 3$ (this is the only place in the proof where the assumption $n = 3$ is used) we have

$$\begin{aligned}\frac{1}{2}\int_{B(0;R)} |\phi\sigma_i|^2 dx = \frac{1}{2}\int_{B(0;R)} |\partial_i u|^2 dx \leq \frac{1}{2}\int_{B(0;R)} |\nabla u|^2 dx \\ \leq \int_{B(0;R)} \left\{\frac{1}{2}|\nabla u|^2 + F(u) - F(1)\right\} dx \leq CR^2.\end{aligned} \tag{7.12}$$

The case $\min(F(-1), F(1)) = F(-1)$ is identical after reversing the direction of x_n. The final ingredient in the proof is the next Liouville-type lemma.

Lemma 7.3 *Let $\phi \in L^\infty_{loc}(\mathbb{R}^n)$ be a positive function. Suppose that $\sigma \in H^1_{loc}(\mathbb{R}^n)$ satisfies*

$$\sigma \nabla \cdot \left(\phi^2 \nabla \sigma\right) \geq 0 \text{ in } \mathbb{R}^n \tag{7.13}$$

in the weak sense. Assume that there exists a constant C independent of $R > 0$

$$\int_{B(0;R)} (\phi\sigma)^2 \leq CR^2 \tag{7.14}$$

for all $R > 0$. Then σ is a constant.

Now, in order to show that all functions σ_i are constants (and hence finish the proof of Theorem 7.1) all we need to verify is that $\phi = \partial_3 u \in L^\infty(\mathbb{R}^n)$ to be able to apply Lemma 7.3 to equation (7.11) for σ_i and that $u \in C^{2,\alpha}(\mathbb{R}^n)$ to justify equation (7.10). However, applying the interior $W^{2,p}$ estimates with $p > n$ to the equation $\Delta u = F'(u) \in L^\infty$, in any ball $B(y; 2)$ we get

$$\|u\|_{W^{2,p}(B(y;1))} \leq C\{\|u\|_{L^\infty(B(y;2))} + \|F'(u)\|_{L^p(B(y;2))} \leq C$$

with C independent of y. The Sobolev imbedding $W^{2,p}(B(y;1)) \subset C^1(\bar{B}(y;1))$ with $p > n$ we conclude that

$$|\nabla u| \in L^\infty(\mathbb{R}^n) \tag{7.15}$$

and that $u \in C^1$. Furthermore, this implies that each derivative $w_j = \partial_j u$ satisfies weakly

$$\Delta w_j - F''(u) w_j = 0. \tag{7.16}$$

Since $F''(u)w_j \in L^\infty(\mathbb{R}^n)$, we obtain $w_j \in W^{2,p}_{loc}(\mathbb{R}^n)$ and hence $u \in C^{2,\alpha}(\mathbb{R}^n)$, as in the previous argument. This completes the proof of Theorem 7.1. $\square$

We now turn to the proof of Lemmas 7.2 and 7.3. We start with Lemma 7.3.

Proof of Lemma 7.3. Let $\zeta_R(x) = \zeta(|x|/R)$, where $\zeta(s)$ is a $C^\infty(\mathbb{R})$-cut-off function so that $0 \leq \zeta \leq 1$, and

$$\zeta(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq 1, \\ 0, & \text{if } 2 \leq s. \end{cases}$$

Multiplying (7.13) by ζ_R^2 and integrating by parts we obtain

$$\int \zeta_R^2\phi^2|\nabla\sigma|^2 \le -2\int \zeta_R\phi^2\sigma\nabla\zeta_R\cdot\nabla\sigma \le 2\left[\int_{R<|x|<2R}\zeta_R^2\phi^2|\nabla\sigma|^2\right]^{1/2}\left[\int \phi^2|\sigma|^2|\nabla\zeta_R|^2\right]^{1/2}$$
$$\le \left[\int_{R<|x|<2R}\zeta_R^2\phi^2|\nabla\sigma|^2\right]^{1/2}\left[\frac{C}{R^2}\int_{B(0;2R)}\phi^2|\sigma|^2\right]^{1/2}$$

with a constant C independent of R. Using assumption (7.14) we deduce that

$$\int \zeta_R^2\phi^2|\nabla\sigma|^2 \le C\left[\int_{R<|x|<2R}\zeta_R^2\phi^2|\nabla\sigma|^2\right]^{1/2}. \tag{7.17}$$

Therefore

$$\int \zeta_R^2\phi^2|\nabla\sigma|^2 \le C$$

and, letting $R\to\infty$ we obtain

$$\int \phi^2|\nabla\sigma|^2 \le C.$$

It follows, however, that the right side of (7.17) tends to zero as $R\to+\infty$ so that actually

$$\int \phi^2|\nabla\sigma|^2 = 0$$

and thus $\sigma = \text{const.}$ □

Proof of Lemma 7.2. The proof is by a version of the sliding technique. Consider the functions

$$u^t(x) = u(x', x_n + t).$$

We have for each $t\in\mathbb{R}$

$$\Delta u^t - F'(u^t) = 0 \tag{7.18}$$

and

$$|u^t| + |\nabla u^t| \le C$$

according to (7.15). The functions u^t satisfy

$$\lim_{t\to+\infty} u^t(x) = 1 \text{ for all } x\in\mathbb{R}^n. \tag{7.19}$$

The time derivative

$$\partial_t u^t(x) = \partial_n u(x', x_n + t) > 0 \tag{7.20}$$

is positive for all $x\in\mathbb{R}^n$.

Consider the energy of u^t in the ball $B(0;R)$ defined by

$$E_R(u^t) = \int_{B(0;R)}\left\{\frac{1}{2}|\nabla u^t|^2 + F(u^t) - F(1)\right\}dx.$$

Let us show that

$$\lim_{t\to+\infty} E_R(u^t) = 0. \tag{7.21}$$

Indeed, the term

$$\int_{B(0;R)} \left\{F(u^t) - F(1)\right\} dx \to 0 \text{ as } t \to +\infty$$

by the Lebesgue dominated convergence theorem and (7.19). In order to show that

$$\int_{B(0;R)} |\nabla u^t|^2 dx \to 0 \text{as } t \to +\infty$$

we multiply (7.19) by $u^t - 1$ and integrate by parts in $B(0;R)$. We obtain

$$\int_{B(0;R)} |\nabla u^t|^2 dx = \int_{\partial B(0;R)} (u^t - 1)\frac{\partial u^t}{\partial n} - \int_{B(0;R)} F'(u^t)(u^t - 1).$$

The integrals on the right side converge to zero by the Lebesgue dominated convergence theorem and the uniform gradient bound (7.15).

Next we compute dE_R/dt:

$$\frac{dE_R}{dt} = \int_{B(0;R)} \left[\nabla u^t \cdot \nabla(\partial_t u^t) + F'(u^t)\partial_t u^t\right] = \int_{\partial B(0;R)} \frac{\partial u^t}{\partial t}\frac{\partial u^t}{\partial n} \geq -C\int_{\partial B(0;R)} \frac{\partial u^t}{\partial t}$$

because of (7.20). Hence we have

$$\begin{aligned}
&E_R(u) = E_R(u^T) - \int_0^T \left(\frac{dE_R(u^t)}{dt}\right) dt \leq E_R(u^T) + C\int_0^T \left(\int_{\partial B(0;R)} \frac{\partial u^t}{\partial t} d\sigma(x)\right) dt \\
&= E_R(u^T) + C\int_{\partial B(0;R)} \left(\int_0^T \frac{\partial u^t}{\partial t} dt\right) d\sigma(x) = E_R(u^T) + C\int_{\partial B(0;R)} [u^T(x) - u(x)] d\sigma(x) \\
&\leq E_R(u^T) + C|\partial B(0;R)| = E_R(u^T) + CR^{n-1}.
\end{aligned}$$

We now let $T \to +\infty$, use (7.21) and obtain the estimate (7.9). □

Chapter 2

Qualitative Properties of Reaction-Diffusion Fronts

1 The one-dimensional reaction-diffusion equation

We will for a while consider solutions of reaction-diffusion equations of the form

$$\frac{\partial T}{\partial t} + u(x) \cdot \nabla T = \frac{\partial}{\partial x_n}\left(a_{nm}(x)\frac{\partial T}{\partial x_m}\right) + f(T).$$

The function $f(T)$, called the reaction rate, will always be Lipschitz, and satisfy

$$f(T) \geq 0 \text{ for } 0 \leq T \leq 1,\ f(0) = f(1) = 0. \tag{1.1}$$

We will distinguish two types of the reaction rate: we say that f is of the KPP-type if

$$f(T) > 0 \text{ for } 0 < T < 1 \tag{1.2}$$

$$f(T) \leq f'(0)T \tag{1.3}$$

while f is of the ignition type if

$$\exists \theta_0 \in (0,1) \text{ so that } f(T) = 0 \text{ for } 0 \leq T \leq \theta_0 \text{ and } f(T) > 0 \text{ for } T \in (\theta_0, 1). \tag{1.4}$$

We will consider in this section the simplest reaction-diffusion problem in one dimension without any flow $u(x)$ and with a constant diffusivity function $a(x)$:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{1}{4}f(T). \tag{1.5}$$

The factor $1/4$ is introduced for convenience.

1.1 Existence of one dimensional travelling waves

Following the historical order of development we begin with showing that travelling wave solutions of (1.5) exist when the nonlinearity $f(T)$ is of the KPP type. These are solutions of the form $T(t,x) = U(x - ct)$ with the function $U(x)$ satisfying the boundary conditions

$$U(x) \to 1 \text{ as } x \to -\infty,\ U(x) \to 0 \text{ as } x \to +\infty. \tag{1.6}$$

We will assume that f is normalized so that $f'(0) = 1$, then (1.3) implies that $f(T) \le T$.

The function $U(x)$ satisfies an ODE

$$-cU' = U'' + \frac{1}{4}f(U), \quad U(-\infty) = 1, \quad U(+\infty) = 0. \tag{1.7}$$

Introduce $V = -U'$ so that (1.7) becomes

$$\begin{aligned} \frac{dU}{dx} &= -V \\ \frac{dV}{dx} &= -cV + \frac{1}{4}f(U). \end{aligned} \tag{1.8}$$

This system has two equilibria: $(U, V) = (0, 0)$ and $(U, V) = (1, 0)$. A travelling wave corresponds to a heteroclinic orbit of (1.8) that connects the second equilibrium $(1, 0)$ to the first, $(0, 0)$. Linearization around $(0, 0)$ gives

$$\frac{d}{dx}\begin{pmatrix} U \\ V \end{pmatrix} = A_0 \begin{pmatrix} U \\ V \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & -1 \\ \frac{1}{4}f'(0) & -c \end{pmatrix}.$$

The eigenvalues of A_0 satisfy

$$\lambda^2 + c\lambda + \frac{1}{4}f'(0) = 0$$

and are both real and negative if $c^2 \ge f'(0) = 1$. Therefore for a positive travelling wave $U(x - ct)$ to exist we need $c \ge 1$ so that $(0, 0)$ is a stable point. The linearization around $(1, 0)$ gives

$$\frac{d}{dx}\begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix} = A_1 \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & -1 \\ \frac{1}{4}f'(1) & -c \end{pmatrix}.$$

The eigenvalues of A_1 satisfy

$$\lambda^2 + c\lambda + \frac{1}{4}f'(1) = 0$$

so that they have a different sign: $\lambda_1 > 0$, $\lambda_2 < 0$, and $(1, 0)$ is a saddle. Note that the unstable direction $(1, -\lambda_1)$ corresponding to $\lambda_1 > 0$ lies in the second and fourth quadrants.

Let us look at the the triangle D formed by the lines $l_1 = \{V = \gamma U\}$, $l_2 = \{V = \alpha(1 - U)\}$ and the interval $l_3 = \{[0, 1]\}$ on the U-axis. We check that with an appropriate choice of γ and α all trajectories of (1.8) point into D on the boundary ∂D if $c \ge 1$. That means that the unstable manifold of $(1, 0)$ has to end at $(0, 0)$ since it may not cross the boundary of the triangle. That is, U and V stay positive along a heteroclinic orbit that starts at $(1, 0)$ and ends at $(0, 0)$ – this is a monotonic positive travelling wave we want to exist. In particular that will show that travelling waves exist for all $c \ge c_* = 1$.

Along l_3 we have $V = 0$ and

$$\frac{dU}{dx} = 0, \quad \frac{dV}{dx} = \frac{1}{4}f(U) > 0$$

so that trajectories point upward, that is, into D. Along l_1 we have $\dfrac{dU}{dx} = -V < 0$ and

$$\frac{dV}{dU} = c - \frac{f(U)}{4V} = c - \frac{f(U)}{4\gamma U}.$$

That means that the trajectory points into D if the slope $\dfrac{dV}{dU} \geq \gamma$ along l_1. This is true if

$$c - \frac{f(U)}{4\gamma U} \geq \gamma$$

for all $U \in [0,1]$. This is equivalent to

$$c\gamma - \gamma^2 \geq \frac{f(U)}{4U}. \tag{1.9}$$

We have $\dfrac{f(U)}{U} \leq 1$ and hence (1.9) holds provided that

$$c\gamma - \gamma^2 \geq \frac{1}{4}. \tag{1.10}$$

Such $\gamma > 0$ exists if $c \geq 1$. Let us check that with this choice of c and γ all trajectories point into D also along the segment l_2. Indeed we have along l_2: $\dfrac{dU}{dx} = -V < 0$ and

$$\frac{dV}{dU} = c - \frac{f(U)}{4V} = c - \frac{f(U)}{4\alpha(1-U)}.$$

That means that the trajectory points into D if the slope $\dfrac{dV}{dU} \geq -\alpha$ along l_1. This is true if

$$c - \frac{f(U)}{4\alpha(1-U)} \geq -\alpha$$

for all $U \in [0,1]$, or

$$c\alpha + \alpha^2 \geq \frac{f(U)}{4(1-U)}.$$

This is true for instance if $\alpha \geq \inf\limits_{0 \leq U \leq 1} \dfrac{f(U)}{4(1-U)}$.

Therefore a travelling front exists provided that $c \geq 1$. However, we have also shown that no travelling front exists for $c < 1$. Thus we have proved the following theorem.

Theorem 1.1 *A travelling front solution of (1.7) with a KPP-type nonlinearity exists for all* $c \geq 1$.

Remark 1.2 We note that the travelling waves that propagate with the speeds $c > c_* = 1$ are in a sense superfluous. More precisely, their existence is unrelated to diffusion: let $U_0(x)$ be solution of

$$\frac{dU_0}{dx} = -\frac{1}{4} f(U_0), \quad U_0(0) = 1/2. \tag{1.11}$$

Such solution exists and satisfies the boundary conditions

$$U_0(x) \to 1 \text{ as } x \to -\infty \text{ and } U_0(x) \to 0 \text{ as } x \to +\infty. \tag{1.12}$$

Then given any $c > 0$ the function $T(t,x) = U_0\left(\frac{x}{c} - t\right)$ is a traveling wave solution of

$$\frac{\partial T}{\partial t} = 0 \cdot \frac{\partial^2 T}{\partial x^2} + \frac{1}{4} f(T).$$

Thus these travelling waves exist even at zero diffusion coefficient and are therefore not quite "reaction-diffusion" waves.

The shortcoming of having non-physical waves is absent in the case of ignition nonlinearity.

Theorem 1.3 *Let $f(T)$ be of ignition type. Then there exists a unique $c = c_*$ so that a travelling wave solution of (1.5) of the form $U(x - ct)$ exists.*

The proof is based on the ODE methods similar to those in the KPP case, and is left as an exercise for the reader. We note, however, that in the ignition case spurious waves at zero diffusivity do not exist: solutions of (1.11) do not satisfy the boundary conditions (1.12). This explains qualitatively uniqueness of the travelling front speed. We also remark that if we fix the travelling wave so that $U(0) = \theta_0$ then the travelling wave is given by $U(x) = \theta_0 \exp(-cx)$ for $x \geq 0$. Hence we have to find c so that in the variables $(U, V = -U')$ the stable manifold of the point $(1,0)$ in (1.8) would pass through the point $(\theta_0, c\theta_0)$. Not surprisingly such c is unique.

1.2 The front-like initial data

We look now at the behavior of solutions of (1.5) with a general initial data $T_0(x) = T(0,x)$ such that $T_0(x) = 1$ for $x \leq x_0$, $T_0(x) = 0$ for $x \geq x_1$ and $0 \leq T_0(x) \leq 1$. The main result is that such initial data propagates with the speed $c_* = 1$ of the slowest travelling front in the KPP case and with the speed of the unique travelling wave in the ignition case. More precisely, we have the following.

Theorem 1.4 *Let $T(t,x)$ be solution of (1.5) with the initial data $T_0(x)$ as above. Then given any $x \in \mathbb{R}$ we have*

$$\lim_{t\to\infty} T(t, x + ct) = \begin{cases} 0, & \text{if } c > c_*, \\ 1, & \text{if } c < c_*. \end{cases} \tag{1.13}$$

Here $c_* = 1$ is the minimal speed in the KPP case and the unique travelling front speed in the ignition case.

This means qualitatively that T moves with the speed c_*. More precise statements on the convergence to a travelling front in the ignition case may be obtained by spectral methods, or in the KPP case as in the original paper [21], but we will not go into details – see [30] for detailed references.

We will prove (1.13) only in the ignition case. The idea is to use the travelling wave solution $U(x - ct)$ to construct a super-solution and a sub-solution. Let $U(x)$ be the travelling wave, solution of

$$-c_* U' = U'' + f(U), \quad U(0) = \theta_0.$$

We dropped the factor 1/4 in front of $f(U)$ as it is useful only in the KPP case. We look for a sub-solution for T of the form

$$\psi_l(t,x) = U(x - c_* t + x_1 + \xi_1(t)) - q_1(t,x).$$

The functions $\xi_1(t)$ and $q_1(t,x,z)$ are to be chosen so as to make ψ_l be a sub-solution. The shift x_1 will be then used to make sure that initially we have $\psi_l(0,x) \le T_0(x)$. In order for ψ_l to be a sub-solution we need

$$G[\psi_l] = \frac{\partial \psi_l}{\partial t} - \frac{\partial^2 \psi_l}{\partial x^2} - f(\psi_l) \le 0.$$

We have

$$G[\psi_l] = \dot{\xi}_1 U' - \frac{\partial q_1}{\partial t} + \frac{\partial^2 q_1}{\partial x^2} + f(U) - f(U - q_1).$$

With an appropriate choice of x_1, that is, by shifting U sufficiently to the left we may ensure that $T_0(x) \ge U(x) - q_{10}(x)$ with $0 \le q_{10}(x) \le (1-\theta_0)/2$ and $q_{10}(x) \in L^1(\mathbb{R})$. Then we choose $q_1(t,x)$ to be the solution of

$$\frac{\partial q_1}{\partial t} = \frac{\partial^2 q_1}{\partial x^2}, \quad q_1(0,x) = q_{10}(x) \tag{1.14}$$

so that we have

$$\|q_1(t)\|_\infty \le \frac{C}{\sqrt{t}} \|q_{10}\|_{L^1(D)} \tag{1.15}$$

for $t \ge 1$.

We may find $\delta > 0$ so that if $U \in (1-\delta, 1)$ and $q_1 \in (0, (1-\theta_0)/2)$ then $f(U) \le f(U - q_1)$. Hence we have in this range of U:

$$G[\psi_l] \le \dot{\xi}_1 U' \le 0 \tag{1.16}$$

provided that $\dot{\xi}_1 \ge 0$. Furthermore, if δ is sufficiently small and $U \in (0,\delta)$ then $f(U) = f(U-\delta) = 0$ and hence in this range of U we have (1.16) with the equality sign on the left. Finally, if $U \in (\delta, 1-\delta)$ then $|f(U) - f(U - q_1)| \le K|q|$ and $U' \le -\beta$ with positive constants K and β that depend on $\delta > 0$. Hence $G[\psi_l] \le 0$ everywhere provided that

$$\dot{\xi}_1(t) \ge \frac{K\|q_1(t)\|_\infty}{\beta}. \tag{1.17}$$

Thus we may choose

$$\xi_1(t) = C\sqrt{t}. \tag{1.18}$$

Therefore we obtain a lower bound for T:

$$T(t,x) \ge U(x - c_* t + C\sqrt{t}) - q_1(t,x,z). \tag{1.19}$$

In order to obtain an upper bound we set $\psi_u = U(x - c_* t - x_2 - \xi_2(t)) + q_2(t,x)$ and look for $\xi_2(t)$ and $q_2(t,x)$ so that $G[\psi_u] \ge 0$. The constant x_2 is chosen so that

$$T_0(x) \le U(x - x_2) + q_2(0,x)$$

with $q_2(0,x) \in L^1(\mathbb{R})$ and $0 \le q_2(0,x) \le \theta_0/2$, as with $q_1(0,x)$. The function $q_2(t,x)$ is then chosen to satisfy the same heat equation (1.14) as q_1. Hence it obeys the same time decay bounds as q_1. With the above choice of q_2 we have

$$G(\psi_u) = -\dot{\xi}_2 U' + f(U) - f(U+q_2).$$

Once again, we consider three regions of values for U. First, if $1-\delta \le U \le 1$ with a sufficiently small $\delta > 0$ then $f(U) - f(U+q_2) \ge 0$, as $q_2 \ge 0$. Hence $G[\psi_u] \ge 0$ in this region provided that $\dot{\xi}_2 \ge 0$. Second, as $q_2 \le \theta_0/2$ we have $f(U) = f(U+q_2) = 0$ if $0 \le U \le \delta$ with a sufficiently small $\delta > 0$. Hence $G[\psi_u] \ge 0$ in that region under the same condition $\dot{\xi}_2 \ge 0$. Finally, if $U \in (\delta, 1-\delta)$ then $U' \le -\beta$ with $\beta > 0$ and $|f(U) - f(U+q_2)| \le K\|q_2\|_\infty$. That means that $G[\psi_u] \ge 0$ if we choose ξ_2 so that

$$\dot{\xi}_2 \ge \frac{K\|q_2\|_\infty}{\beta}.$$

Therefore we may choose

$$\xi_2(t) = C\sqrt{t},$$

as with $\xi_1(t)$. Thus we obtain upper and lower bounds

$$U(x - c_* t + \xi_1(t) + x_1) - q_1(t,x) \le T(t,x) \le U(x - c_* t - \xi_2(t) - x_2) + q_2(t,x) \tag{1.20}$$

that imply in particular that

$$U(x - c_* t + C_0[1+\sqrt{t}]) - \frac{C_0}{\sqrt{t}} \le T(t,x) \le U(x - c_* t - C_0[1+\sqrt{t}]) + \frac{C_0}{\sqrt{t}} \tag{1.21}$$

with a constant C_0 determined by the initial conditions. Now, if we take $x = x_0 + ct$ with $c < c_*$ and use the lower bound in (1.21) we get $T(t, x_0 + ct) \to 1$ as $t \to \infty$. On the other hand, if we take $x = x_0 + ct$ with $c > c_*$ and use the upper bound we obtain $T(t, x_0 + ct) \to 0$ as $t \to \infty$.

One may obtain the KPP result from the ignition case by cutting of a KPP-type $f(T)$ at a point $\theta_0 > 0$ and then letting $\theta_0 \to 0$. We omit the details. □

We confess that a better effort using the spectral methods and functional analysis shows that in the ignition case solution actually converges to a travelling wave exponentially fast in time. However, this requires a different technique that we do not go into here.

1.3 Compactly supported initial data

When the initial data T_0 has compact support, two scenarios are possible in the ignition case. First, if the support is too small, solution may become extinct:

$$T(t,x) \to 0 \text{ as } t \to \infty, \text{ uniformly on } \mathbb{R}. \tag{1.22}$$

On the other hand, if the support is sufficiently large, two fronts will form: one on the right, another on the left that will propagate to the left and to the right so that eventually

$$T(t,x) \to 1 \text{ as } t \to \infty, \text{ uniformly on compact sets.} \tag{1.23}$$

Extinction is impossible in the KPP case: solution with an arbitrarily small initial support satisfies (1.23). The difference between the KPP and ignition case is best seen from the linearization at small T: in the ignition case we get the heat equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

that has solutions decaying in time. In the KPP case the linearization is

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + f'(0)T$$

that has solutions growing in time.

We consider solutions of

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + f(T) \tag{1.24}$$

with an initial data of the form

$$T_0(x) = \begin{cases} 1, & |x| \leq x_0, \\ 0, & |x| \geq x_1, \end{cases} \tag{1.25}$$

and with $0 \leq T_0 \leq 1$. The first result concerns extinction.

Theorem 1.5 *Let f be of the ignition type. There exists $l > 0$ so that if $x_1 \leq l$ then solution becomes extinct, that is, (1.22) holds.*

Proof. The proof is simple: the comparison principle implies that $T(t,x) \leq \Psi(t,x)$, solution of

$$\frac{\partial \Psi}{\partial t} = \frac{\partial^2 \Psi}{\partial x^2} + M\Psi, \quad \Psi(0,x) = T_0(x)$$

with the constant M chosen so that $f(T) \leq MT$. However, Ψ is given explicitly by

$$\Psi(t,x) = \frac{e^t}{\sqrt{2\pi t}} \int e^{-(x-y)^2/4t} T_0(y)dy \leq \frac{e^t}{\sqrt{2\pi t}} \|T_0\|_{L^1} \leq \frac{e^t}{\sqrt{2\pi t}} |x_1|.$$

Hence $T(t=1,x) \leq \Psi(t=1,x) \leq \theta_0$ provided that $|x_1| \leq C\theta_0$. However, if $T(t=1,x) \leq \theta_0$ then uniqueness of the solution of the Cauchy problem for (1.24) implies that $T(t,x)$ satisfies the heat equation for $t > 1$:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}.$$

Then the limit behavior (1.22) follows from the fact that $\|T(t=1)\|_{L^1} < \infty$. $\square$

On the other hand, if $T_0 = 1$ on a sufficiently large set, then the flame propagates: (1.23) holds.

Theorem 1.6 *If f is of the ignition type then there exists $L > 0$ so that if $x_0 \geq L$ then (1.23) holds. If f is of the KPP type then (1.23) holds provided that $x_1 > 0$.*

Proof. We will consider the ignition case but the KPP case is similar. The proof is in two steps. First we will find $d > 0$ and a sub-solution $\phi(x)$ on the interval $[-d, d]$ so that

$$\phi''(x) + f(\phi) \geq 0, \quad \phi(-d) = \phi(d) = 0 \tag{1.26}$$

and $0 \leq \phi \leq 1$. We will then consider a special solution $\Phi(t, x)$ of (1.24) with the initial data $\Phi(t, x) = \phi(x)$. It will turn out to be monotonically increasing in time and thus converge to a limit as $t \to +\infty$. It will only remain to identify the limit as $\Phi_\infty \equiv 1$.

Choose θ_1, θ_2 so that $\theta_2 > \theta_1 > \theta_0$ and define $f_1(T)$ by

$$f_1(T) = \begin{cases} 0, & T \leq \theta_1 \\ \dfrac{f(\theta_2)(T - \theta_1)}{\theta_2 - \theta_1}, & \theta_1 \leq T \leq \theta_2 \\ f(T), & \theta_2 \leq T \leq 1 \end{cases}$$

The function $f(T)$ is Lipschitz continuous, and hence we may choose θ_1 and θ_2 so that $f_1(T) \leq f(T)$. Therefore if ϕ satisfies

$$\phi'' + f_1(\phi) = 0 \tag{1.27}$$

then ϕ satisfies (1.26). We are going to exhibit an explicit solution $\phi(x)$ of (1.27) with the "initial" conditions

$$\phi(0) = \theta_2, \quad \frac{d\phi}{dx}(0) = 0.$$

Indeed, $\phi(x)$ is given explicitly by

$$\phi(r) = \theta_1 + (\theta_2 - \theta_1)\cos\left(x\sqrt{\alpha}\right), \quad \alpha = \frac{f(\theta_2)}{\theta_2 - \theta_1} \text{ for } x \leq R_1 = \frac{\pi}{2\sqrt{\alpha}}. \tag{1.28}$$

Furthermore, we have

$$\phi(x) = B(d - |x|), \quad \text{for } R_1 \leq |x| \tag{1.29}$$

with B and d determined by matching (1.28) and (1.29) at $x = R_1$: $B = (\theta_2 - \theta_1)\sqrt{\alpha}$ and $d = R_1 + \dfrac{\theta_1}{B}$. The function $\phi(x)$ satisfies (1.26). We choose x_0 sufficiently large so that $T_0(x) \geq \phi(x)$. We now let $\Phi(t, x)$ be solution of (1.24) with the initial data $\Phi(0, x) = \phi(x)$, as we discussed above. Then

$$T(t, x) \geq \Phi(t, x) \geq \phi(x), \tag{1.30}$$

as the first inequality follows from the comparison principle, while the second from the fact that $\phi(x)$ is a sub-solution. This implies that $v(t, x) = \frac{\partial \Phi}{\partial t} > 00$ for $t > 0$. Indeed, given any $h > 0$, let $v_h(t, x) = \Phi(t + h, x) - \Phi(t, x)$. Then $v_h(0, x) > 0$ is positive as implied by the second inequality in (1.30). The function v_h satisfies an equation of the form

$$\frac{\partial v_h}{\partial t} = \frac{\partial^2 v_h}{\partial x^2} + \frac{f(\Phi(t + h, x)) - f(\Phi(t, x))}{\Phi(t + h, x) - \Phi(t, x)} v_h.$$

Hence the maximum principle implies that $v_h(t,x) > 0$ for all $t > 0$, and all $h > 0$. Thus the function $\Phi(t,x)$ is monotonically increasing in time and has a limit $\bar{\Phi}(x)$ as $t \to +\infty$. The limit function has to be a non-negative solution of

$$)\bar{\Phi}_{xx} + f(\bar{\Phi}) = 0. \tag{1.31}$$

As $\bar{\Phi}(x) \geq \phi(x)$, for $x \in [-d,d]$ we may apply Lemma 5.2 and conclude that $\bar{\Phi}(x) \geq \theta_2$ everywhere. This implies that $\bar{\Phi}$ may not attain a local minimum in $\mathbb{R}$, hence either it is equal to a constant T_0, or it is monotonic on two sides of the point x_0 where it attains its maximum, and thus has limits as $x \to \pm\infty$. In the former case the constant $T_0 \geq \theta - 2$ has to be such that $f(T_0) = 0$ and hence $\bar{\Phi} = 1$. In the latter case there exist two sequences $x_n \to -\infty$ and $y_n \to +\infty$ so that $\bar{\Phi}_x(x_n) \to 0$ and $\bar{\Phi}_x(y_n) \to 0$ as $n \to +\infty$. Integrating (1.31) between x_n and y_n we get

$$\bar{\Phi}_x(y_n) - \bar{\Phi}_x(x_n) + \int_{x_n}^{y_n} f(\bar{\Phi})dx = 0.$$

Passing to the limit $n \to +\infty$ we obtain $f(\bar{\Phi}) \equiv 0$, hence $\bar{\Phi} \equiv 1$. □

It is not difficult to construct a sub-solution that propagates to the left and right with the speed c_*. That means that not only (1.23) holds for $x_1 > l$ but also the front expands at the speed of the travelling front.

2 Reaction-Diffusion in Flows

We will now describe some of the recent results on the behavior of solutions of reaction-diffusion equations in flows.

2.1 The spreading rate in a shear flow: the KPP case

A shear flow in $\mathbb{R}^n$ is a unidirectional flow of the form $u = (u(x), 0)$. Here we have decomposed $x = (x,y)$, $x \in \mathbb{R}^{n-1}$, and $u(y)$ is a scalar function. A reaction-diffusion-advection equation in such a flow has the form

$$\frac{\partial T}{\partial t} + u(y)\frac{\partial T}{\partial x} = \Delta T + f(T). \tag{2.1}$$

We will consider this problem in a cylinder $D = \mathbb{R}_x \times \Omega_y$, where Ω is a nice bounded domain. We will assume that

$$\int_\Omega u(y)dy = 0, \tag{2.2}$$

this is simply a normalization condition. The boundary conditions along $\partial\Omega$ are homogeneous Neumann ("adiabatic" in the physical language):

$$\frac{\partial T}{\partial n} = 0 \text{ on } \mathbb{R} \times \partial\Omega. \tag{2.3}$$

The nonlinearity $f(T)$ will be usually taken of KPP type, though many results are applicable to the ignition nonlinearity as well. Hence we will assume that

$$f(T) > 0 \text{ for } 0 < T < 1,\ f(0) = f(1) = 0,\ f(T) \leq f'(0)T.$$

Before we have set $f'(0)=1$ for convenience and had a factor of $1/4$ in front of $f(T)$ in (2.1) so as to make the minimal front speed $c_*=1$. We now drop both of these, as c_* will not be overly explicit. We first recall the no-flow case when solutions propagate with the minimal speed $c_*=2\sqrt{f'(0)}$. A quick formal way to see why this happens is through an observation that the behavior of the KPP solutions is determined by the linearization at small T. More precisely, if we start with

$$\frac{\partial T}{\partial t}=\Delta T+f(T) \tag{2.4}$$

and linearize around $T=0$ we get

$$\frac{\partial q}{\partial t}=\Delta q+f'(0)q.$$

Let us look for the solutions of this equation of the form

$$q(t,x)=e^{-\lambda(x-ct)} \tag{2.5}$$

with $\lambda>0$. A direct calculation shows that c and λ are related by

$$\lambda^2-\lambda c+f'(0)=0. \tag{2.6}$$

A middle-school algebra leads to the conclusion that such a $\lambda>0$ exists if and only if $c\geq c_*=2\sqrt{f'(0)}$ (the factor of two here is related to the lack of $1/4$ in front of $f(T)$). This is exactly the correct range of speeds for the one-dimensional KPP equation.

The above naive idea of linearization actually applies in the presence of a shear and other flows. However, the ansatz (2.5) should be modified (depending on the flow type), in the shear case the correct ansatz is

$$q(t,x)=e^{-\lambda(x-ct)}\psi(y) \tag{2.7}$$

with $\lambda>0$ and $\psi>0$. We insert this ansatz into the linearization

$$\frac{\partial q}{\partial t}+u(y)\frac{\partial q}{\partial x}=\Delta q+f'(0)q \tag{2.8}$$

and obtain an eigenvalue problem for ψ in Ω:

$$\begin{aligned}-\Delta\psi-\lambda u(y)\psi&=(\lambda^2-c\lambda+f'(0))\psi \text{ in } \Omega \\ \frac{\partial\psi}{\partial n}&=0 \text{ on } \partial\Omega.\end{aligned} \tag{2.9}$$

As the quadratic equation (2.6) in the homogeneous case, the eigenvalue problem (2.9) is the key point to understanding the propagation phenomena. In order to understand for which c an eigenvalue $\lambda>0$ exists we consider an auxiliary problem

$$\begin{aligned}-\Delta\psi-tu(y)\psi&=\mu(t)\psi \text{ in } \Omega \\ \frac{\partial\psi}{\partial n}&=0 \text{ on } \partial\Omega, \\ \|\psi\|_{L^\infty(\Omega)}&=1.\end{aligned} \tag{2.10}$$

Here $\mu(t)$ is the leading eigenvalue that corresponds to a positive eigenfunction ψ. Its existence is guaranteed by the general Sturm-Liouville theory. The direct analog of (2.6) is

$$\lambda^2-c\lambda+f'(0)=\mu(\lambda). \tag{2.11}$$

Proposition 2.1 *There exists a unique* c_* *so that (2.11) has two real solutions* $\lambda_{1,2}$ *for* $c > c_*$, *no real solutions for* $c < c_*$ *and one solution for* $c = c_*$.

Not surprisingly, c_* will turn out to be the minimal speed of a travelling front solution of (2.1): a travelling front is a solution of (2.1) of the form $U(x - ct, y)$ with the Neumann boundary conditions at $\mathbb{R}_x \times \partial\Omega_y$ that satisfies

$$U(x,y) \to 1 \text{ as } x \to -\infty,\ U(x,y) \to 0 \text{ as } x \to +\infty, \text{ both uniformly in } \Omega. \tag{2.12}$$

It turns that such fronts exist for all $c \geq c_*$ so that c_* is the minimal front speed. We postpone the proof of this result till later. Instead, we will show that c_* is the asymptotic speed of spreading in the positive x-direction.

We may also look for left-going solutions of the form

$$q(t,x) = e^{\lambda(x+ct)}\psi(y), \tag{2.13}$$

which leads to an eigenvalue problem

$$\begin{aligned} -\Delta\psi + \lambda u(y)\psi &= (\lambda^2 - c\lambda + f'(0))\psi \text{ in } \Omega \\ \frac{\partial\psi}{\partial n} &= 0 \text{ on } \partial\Omega \end{aligned} \tag{2.14}$$

that coincides with (2.9) but with u replaced by $-u$. Proposition 2.1 implies that there exists a minimal speed c_*^l so that (2.14) has no positive eigenvalues for $c < c_*^l$, two positive eigenvalues $\lambda_{1,2} > 0$ for $c > c_*^l$ and one positive eigenvalue λ_* for $c = c_*^l$.

Proof of Proposition 2.1. First, we observe that $\mu(t)$ has a variational characterization:

$$\mu(t) = \inf \frac{\int |\nabla\phi|^2 - t\int u(y)\phi^2(y)}{\int \phi^2}.$$

Thus $\mu(t)$ is concave as a minimum of a family of affine functions of t. Furthermore, $\mu(0) = 0$ (and $\psi|_{t=0} = \text{const}$), and we may compute formally

$$-\Delta\psi' - tu(y)\psi' - u(y)\psi = \mu'(t)\psi + \mu(t)\psi'. \tag{2.15}$$

Here prime denotes derivative with respect to t and ψ is normalized so that $\int_\Omega \psi^2 = 1$. Multiplying (2.15) by ψ and integrating, using the normalization of ψ, we obtain

$$\mu'(t) = -\int u(y)\psi^2(y)dy. \tag{2.16}$$

In particular $\psi = 1$ at $t = 0$ so that (2.2) implies that $\mu'(0) = 0$. Now the conclusion of Proposition 2.1 follows from (2.11) and the facts that $\mu(t)$ is concave, $\mu(0) = 0$ and $\mu'(0) = 0$.

Let us now justify our formal differentiation with respect to t. Let $\psi_t(x)$ be the principal eigenfunction of (2.10). Note that $\psi_t(x) \to 1$ as $t \to 0$. Integrating (2.10) over Ω we obtain

$$-t\int u(y)\psi_t(y)dy = \mu(t)\int \psi_t(y)dy.$$

Therefore, as $\mu(0) = 0$, we have

$$\frac{\mu(t) - \mu(0)}{t} = -\frac{\int u(y)\psi_t(y)dy}{\int \psi_t(y)dy} \to \int u(y)dy = 0 \text{ as } t \to 0.$$

This finishes the proof of Proposition 2.1. □

We now show that the speeds c_* and c_*^l defined above define the asymptotic speed of spreading.

Theorem 2.2 *Let the initial data $T_0(x)$ for (2.1) be continuous and satisfy $T_0(x, y) = 0$ for $|x| > x_1$ and $T_0(x, y) > 0$ on an open set. Then we have $T(t, x + ct, y) \to 0$ as $t \to +\infty$, uniformly on compact sets, provided that $c > c_*$ or $c < -c_*^l$. On the other hand, if $-c_*^l \le c < c_*$ then $T(t, x + ct, y) \to 1$ as $t \to +\infty$, uniformly on compact sets.*

Proof. First we consider the case $c' > c_*$ (or $c' < -c_*^l$ that is essentially identical) which is much easier. We simply introduce the function

$$q(t, x, y) = Ae^{-\lambda(x-ct)}\psi(y)$$

with λ, c and ψ as in (2.9), and with $c_* < c < c'$. Then q satisfies

$$\frac{\partial q}{\partial t} + u(y)\frac{\partial q}{\partial x} = \Delta q + f'(0)q \ge \Delta q + f(q).$$

Moreover, $q(0, x, y) \ge T_0(x, y)$ provided that the factor A is large enough. Hence the comparison principle implies that

$$T(t, x, y) \le q(t, x, y)$$

and thus $T(t, x + c't, y) \to 0$ as $t \to +\infty$.

The second part of the theorem that deals with the case $-c_*^l < c < c_*$ is more difficult. The argument we present is from a paper by Mallordy and Roquejoffre [22]. The idea is to construct a function described in the following lemma.

Lemma 2.3 *Given $c < c_*$ but sufficiently close to c_*, and a real number $\theta > 0$ there exists a non-negative function $\phi(x, y) \le \theta$ that has compact support, is everywhere less than θ, satisfies the Neumann boundary conditions at $\partial\Omega$, and satisfies*

$$-c\frac{\partial \phi}{\partial x} + u(y)\frac{\partial \phi}{\partial x} = \Delta\phi + (f'(0) - \delta)\phi \le \Delta\phi + f(\phi) \text{ on supp } \phi.$$

Similarly, given $c < c_*^l$ but sufficiently close to c_*^l, and a real number $\theta > 0$ there exists a non-negative function $\phi^l(x, y) \le \theta$ that has compact support, is everywhere less than θ, satisfies the Neumann boundary conditions at $\partial\Omega$, and is a sub-solution for

$$c\frac{\partial \phi^l}{\partial x} + u(y)\frac{\partial \phi^l}{\partial x} = \Delta\phi^l + (f'(0) - \delta)\phi^l \le \Delta\phi^l + f(\phi^l) \text{ on supp } \phi^l.$$

The constant $\delta > 0$ above is chosen so that $f(T) \ge (f'(0) - \delta)T$ for $0 \le T \le \theta$.

We postpone the proof of this lemma and finish the proof of Theorem 2.2 first.

Given a function ϕ as in Lemma 2.3 with $c < c_*$ but close to c_*, we consider the function $\Phi(t,x,y)$ that solves (2.1) with the initial data $\Phi(t,x,y) = \phi(x,y)$. Then, as in the proof of Theorem 1.6, the function $\Psi(t,x,y) = \Phi(t,x+ct,y)$ is monotonic in t and thus has a limit Ψ_∞. It has to satisfy

$$-c\frac{\partial \Psi_\infty}{\partial x} + u(y)\frac{\partial \Psi_\infty}{\partial x} = \Delta\Psi_\infty + f(\Psi_\infty)$$

with the Neumann boundary conditions. One then argues that Ψ_∞ has to converge to a constant as $x \to \pm\infty$, that is, either to 0 or 1. However, as $0 < c < c_*$, the left and right limits have to coincide. Thus $\Psi_\infty = 0$ or 1 everywhere. However, $\Psi_\infty > 0$ where $\phi > 0$ so that $\Psi_\infty = 1$.

Similarly, given a function ϕ^l as in Lemma 2.3 with $c < c_*^l$ but close to c_*^l, we consider the function $\Phi^l(t,x,y)$ that solves (2.1) with the initial data $\Phi^l(t,x,y) = \phi^l(x,y)$. Then, as before, the function $\Psi^l(t,x,y) = \Phi^l(t,x-ct,y)$ is monotonic in t and thus has a limit Ψ^l_∞ which satisfies

$$c\frac{\partial \Psi^l_\infty}{\partial x} + u(y)\frac{\partial \Psi^l_\infty}{\partial x} = \Delta\Psi^l_\infty + f(\Psi^l_\infty)$$

with the Neumann boundary conditions. We conclude as in the previous paragraph that $\Psi^l_\infty = 1$.

To finish the proof of Theorem 2.2 we observe that given the initial data T_0 that is positive somewhere, $T(t=1) > 0$ everywhere. Hence we may choose $\theta > 0$ small and a corresponding $\phi(x,y)$ so that $T(t=1) > \phi$. Then $T(t,x,y) \geq \Phi(t,x,y)$ and thus $T(t,x,y) \to 1$ uniformly on compact set in the reference frame moving with the speed c to the right, that is, $T(t,x+ct,y) \to 1$. Thus we are done for the case when $c < c_*$ is close to c_*. Similarly we show that $T(t,x-ct,y) \to 1$ for $c < c_*^l$ but close to c_*^l, using comparison to ϕ^l with a sufficiently small θ.

The above facts imply that $T(t,x+ct,y) \to 1$ for all $-c_*^l < c < c_*$. Indeed, let $-c_*^l < c < c_*$ and choose $X_1(t) = -c_1 t$, $X_2(t) = c_2 t$ with c_1 close to c_*^l and c_2 close to c_*. Choose a time t_0 so that $T(t,X_1(t),z) > 1-\varepsilon$, $T(t,X_1(t),z) > 1-\varepsilon$ for all $t \geq t_0$. Define $D(t) = [X_1(t), X_2(t)] \times \Omega$ and let $h_0 = \inf_{D(t_0)} T(t_0,x,y)$. Consider the function $h(t)$ that solves $\dot{h} = f_\varepsilon(h)$ with with $f_\varepsilon(h) = mh(1-2\varepsilon-h)$. Here the constant m is chosen so that $f(h) > f_\varepsilon(h)$. The initial data for $h(t)$ is given by $h(t_0) = h_0$ so that $T(t_0,x,y) \geq h_0$ in $D(t_0)$. The funcion $h(t)$ is then a sub-solution for $T(t,x,y)$ in $D(t)$ for $t \geq t_0$, that is, $T(t,x,y) \geq h(t)$ for all $(x,y) \in D(t)$. Thus, $T(t,x,y) \geq 1-3\varepsilon$ for a sufficiently large t in all of $D(t)$. □

We now prove Lemma 2.3.

Proof of Lemma 2.3. Let λ^* be the unique solution of (2.11) with $c = c^*$ and $\mu(\lambda^*)$ be the corresponding eigenvalue. The function $\mu(t)$ may be analytically continued into the complex plane in a neighborhood of $t = \lambda^*$ as a simple eigenvalue $\mu(\lambda)$ of the operator $-\Delta_y - \lambda u(y)$. There exists a family of eigenfunctions ψ_λ in $L^2(\Omega)$, analytic in λ.

Define the function $F_c(\lambda) = \mu(\lambda) + c\lambda - \lambda^2 - f'(0)$ in a neighborhood of λ^*. It is analytic in this neighborhood and converges locally uniformly to F_{c_*} as $c \to c_*$. Moreover, $F_{c_*}(\lambda^*) = 0$, thus the Rouché theorem yields the existence of a family λ_c so that $F_c(\lambda_c) = 0$ and $\lambda_c \to \lambda^*$

as $c \to c_*$. Hence $\psi_c = \psi_{\lambda_c}$ satisfies

$$-\Delta\psi_c - \lambda_c u(y)\psi_c \lambda_c^2 - c\lambda_c + f'(0))\psi_c \text{ in } \Omega \qquad (2.17)$$
$$\frac{\partial \psi_c}{\partial n} = 0 \text{ on } \partial\Omega.$$

Note that $\psi_c \to \psi_{c_*}$ in $L^2(\Omega)$ as $c \to c_*$. The classical elliptic estimates allow to bootstrap this convergence to $L^\infty(\Omega)$. Thus $\mathrm{Re}\psi_c \to \psi_{c_*} > 0$ for c near c_*.

We observe that if $c < c_*$ is real, then λ_c may not be real by definition of c_* – hence $\mathrm{Im}\lambda \neq 0$. Let us repeat the above construction with $f'(0)$ replaced by $f'(0) - \delta$ – as we are building a sub-solution. Then we get $c_*^\delta < c_*$ so that $c_*^\delta \to c_*$ as $\delta \to 0$. Then, given $c < c_*^\delta$ near c_*^δ we construct λ_c^δ and ϕ_c^δ that satisfy (2.17) with $f'(0)$ replaced by $f'(0) - \delta$. The function $\tilde\phi(x,y) = \mathrm{Re}(e^{-\lambda_c^\delta x}\phi_c^\delta(y))$ is a real-valued solution of

$$-c\frac{\partial\tilde\phi}{\partial x} + u(y)\frac{\partial\tilde\phi}{\partial x} = \Delta\tilde\phi + [f'(0) - \delta]\tilde\phi \text{ on supp } \tilde\phi,\ \ \frac{\partial\tilde\phi}{\partial n} = 0 \text{ on } \partial\Omega.$$

We may multiply $\tilde\phi$ by a sufficiently small factor to arrange that $[f'(0) - \delta]\tilde\phi \leq f(\tilde\phi)$. Then $\tilde\phi$ satisfies

$$-c\frac{\partial\tilde\phi}{\partial x} + u(y)\frac{\partial\tilde\phi}{\partial x} \leq \Delta\tilde\phi + f(\tilde\phi) \text{ on supp } \tilde\phi,\ \ \frac{\partial\tilde\phi}{\partial n} = 0 \text{ on } \partial\Omega.$$

Now we recall that λ_c^δ has to be complex as $c < c_*^\delta$ and $\mathrm{Re}\psi_c^\delta > 0$. Thus $\tilde\phi$ vanishes along some curves $x = x_1(y)$, $x = x_2(y)$ so that $\tilde\phi > 0$ on $D = \{x_1(y) < x < x_2(y)\}$. Therefore, as a function ϕ required by lemma we may take $\phi(x,y) = \tilde\phi(x,y)\chi_D(x,y)$. □

2.2 The fast shear flow asymptotics: the KPP case

Let us consider a chemical reaction occurring in a fast shear flow, that is, we consider

$$\frac{\partial T}{\partial t} + Au(y)\frac{\partial T}{\partial x} = \Delta T + f(T) \qquad (2.18)$$

with a large flow amplitude $A \gg 1$. We are interested in the behavior of the spreading rate $c_*(A)$ as $A \to \infty$. The mean-zero condition (2.2) is still kept so that any increase in $c_*(A)$ would not bee due to the simple translation of the front by a large mean flow. Rather, the speed up should be due to front wrinkling: as A increase the front gets more and more curved (wrinkled) that results in the increase of the area of interaction of the hot and cold material. As a result the cold material is converted into hot at a faster rate and the spreading rate increases. The purpose of the next proposition [5] is to quantify this phenomenon.

Theorem 2.4 *Assume $u(y) \not\equiv 0$ satisfies (2.2). Then the spreading rate $c_*(A)$ satisfies the following properties: (i) $c_*(A)$ is increasing A, (ii) $c_*(A)/A$ is decreasing in A, and (iii) $c_*(A)/A$ converges as $A \to \infty$ to a positive limit $\rho > 0$.*

Proof. The eigenvalue problem (2.10) with $u(y)$ replaced by $Au(y)$ takes the form

$$-\Delta\psi - Atu(y)\psi = \mu_A(At)\psi \text{ in } \Omega \qquad (2.19)$$
$$\frac{\partial\psi}{\partial n} = 0 \text{ on } \partial\Omega.$$

Hence $\mu_A(t) = \mu(At)$, and (compare to (2.11)) the spreading rate $c_*(A)$ is the smallest number c so that the equation

$$\lambda^2 - c\lambda + f'(0) = \mu(A\lambda) \tag{2.20}$$

has a real non-negative solution λ. Geometrically, $c = c_*(A)$ is such that the graph of the convex function $g(\lambda) = \lambda^2 - c\lambda + f'(0)$ is tangent to the graph of the function $s_A(\lambda) = \mu(A\lambda)$. Recall (see the proof of Proposition 2.1) that the function $\mu(t)$ is concave, $\mu(0) = 0$ and $\mu'(0) = 0$. This implies that $c_*(A)$ is an increasing function of A.

On the other hand, introducing $t = A\lambda$ we see that $\alpha_*(A) = c_*(A)/A$ may be characterized as the number α so that equation

$$\mu(t) = \frac{t^2}{A^2} + f'(0) - \alpha t$$

has a unique solution. That means that the graphs of $\mu(t)$ and $\tilde{g}(t) = \frac{t^2}{A^2} + f'(0) - \alpha(A)t$ are tangent to each other. It is geometrically clear that $\alpha(A)$ is a decreasing function of A and that $\alpha(A)$ tends to the slope ρ of the tangent line to the graph of $\mu(t)$ that passes through the point $(0, f'(0))$. $\square$

2.3 Travelling fronts and the spreading rate: the ignition case

We will now examine the behavior of solutions of the reaction-diffusion equation

$$\frac{\partial T}{\partial t} + u(y)\frac{\partial T}{\partial x} = \Delta T + f(T), \tag{2.21}$$

when the non-linearity $f(T)$ is of the ignition type:

$$f(T) = 0 \text{ for } 0 \le T \le \theta_0,\ f(T) > 0 \text{ for } \theta_0 < T < 1,\ f(1) = 0. \tag{2.22}$$

This equation, as before, is considered in a cylinder $D = \mathbb{R}_x \times \Omega_y$ with the Neumann boundary conditions at $\partial\Omega$:

$$\frac{\partial T}{\partial n} = 0 \text{ on } \mathbb{R} \times \partial\Omega. \tag{2.23}$$

A travelling front is a solution of (2.21) with the boundary conditions (2.23) of the form $U(x - ct, y)$ that tends to the values 1 and 0 at the left and right ends of the domain, respectively:

$$\begin{aligned} &-c\frac{\partial U}{\partial x} + u(y)\frac{\partial U}{\partial x} = \Delta U + f(U), \\ &\frac{\partial U}{\partial n} = 0 \text{ on } \mathbb{R} \times \partial\Omega \\ &\lim_{x\to-\infty} U(x,y) = 1, \quad \lim_{x\to+\infty} U(x,y) = 0. \end{aligned} \tag{2.24}$$

The limits above are assumed to be uniform.

We will show that a travelling front solution $U(x - ct, y)$ of (2.21) exists for a unique $c^* > 0$. Moreover, given a general front-like initial condition $T_0(x)$ so that

$$T_0(x,y) \to 0 \text{ as } x \to +\infty \text{ sufficiently fast} \tag{2.25}$$

and

$$\text{there exists } \eta > 0 \text{ so that } T_0(x,y) > \theta_0 + \eta \text{ for } x < x_1, \tag{2.26}$$

solution propagates with the speed c^*:

$$\lim_{t\to\infty} T(t, x+ct, y) = \begin{cases} 1, & c < c^* \\ 0, & c > c^*. \end{cases} \tag{2.27}$$

One may actually prove global stability of the travelling front: there exists a shift $x_0 \in \mathbb{R}$ so that

$$|T(t, x+ct, y) - U(x+x_0, y)| \le Ce^{-\delta t}. \tag{2.28}$$

We will not prove (2.28), however, as the proof is highly technical: see [26] for the proof.

Theorem 2.5 *(Existence of a travelling front) Let f be a Lipschitz function of the ignition type (2.22) and let the flow $u(y) \in C(\Omega)$ have mean zero:*

$$\int_\Omega u(y)dy = 0.$$

Then there exists a unique $c = c^* > 0$ so that a travelling front, that is, solution of (2.24) exists. Moreover, this solution satisfies

$$0 < U < 1 \tag{2.29}$$

and

$$\frac{\partial U}{\partial x} < 0. \tag{2.30}$$

Moreover, $U(x,y)$ approaches exponentially fast the values 0 and 1 at the two ends of the domain.

This theorem has been proved in [7, 9]. The proof is quite technical. Before embarking on it we show that the asymptotic speed of propagation of a general front-like data is also c^*. This, incidentally, provides an independent proof of the uniqueness of the travelling front speed.

Theorem 2.6 *Let $T_0(x)$ be front-like initial data so that (2.25) and (2.26) hold. Then solution propagates with the speed c^* of the unique travelling front in the sense that (2.27) holds.*

Proof. The proof is very similar to that of Theorem 1.4 with an additional twist due to the presence of the flow. Let $T(t,x,z)$ be the solution of the Cauchy problem

$$T_t + u(y)T_x = \Delta T + f(T) \tag{2.31}$$

with the initial data $T_0(x,z)$ that satisfies (2.25) and (2.26). We will construct sub- and super-solutions for $T(t,x,z)$ that both propagate with the speed c^* – then (2.27) follows easily. This construction follows [29]. We look for a sub-solution for T of the form

$$\psi_l(t,x,z) = U(x - c^*t + x_1 + \xi_1(t)) - q_1(t,x,z).$$

Here $U(x, y)$ is the travelling wave solution of (2.24) normalized so that $\max_y U(0, y) = \theta_0$. The functions $\xi_1(t)$ and $q_1(t, x, z)$ are to be chosen. In order for ψ_l to be a sub-solution we need

$$G[\psi_l] = \frac{\partial \psi_l}{\partial t} + u(y)\frac{\partial \psi_l}{\partial x} - \Delta\psi_l - f(\psi_l) \leq 0.$$

We have

$$G[\psi_l] = \dot{\xi}_1\frac{\partial U}{\partial x} - \frac{\partial q_1}{\partial t} - u(y)\frac{\partial q_1}{\partial x} + \Delta q_1 + f(U) - f(U - q_1).$$

With an appropriate choice of x_1, that is, by shifting Φ_0 sufficiently to the left we may ensure that $T_0(x, y) \geq \Phi_0(x) - q_{10}(x)$ with $0 \leq q_{10}(x) \leq (1 - \theta_0)/2$ and $q_{10}(x) \in L^1(\mathbb{R})$. Then we choose $q_1(t, x, y)$ to be the solution of

$$\frac{\partial q_1}{\partial t} + u(y)\frac{\partial q_1}{\partial x} = \Delta q_1, \quad q_1(0, x, y) = q_{10}(x), \frac{\partial q_1}{\partial n} = 0 \text{ on } \partial\Omega. \tag{2.32}$$

The following lemma provides a uniform $L^1 - L^\infty$ decay estimate for q_1 that is independent of the advection term. For simplicity we consider the case that dimension $n = 2$ and $\Omega = [0, 1]$.

Lemma 2.7 *There exists a constant $C > 0$ that is independent of the flow $u(y)$ so that*

$$\|q_1(t)\|_\infty \leq \frac{C}{\sqrt{t}}\|q_{10}\|_{L^1(D)} \tag{2.33}$$

for $t \geq 1$.

An interesting point is that the constant in (2.33) is independent from the flow u. We will actually prove its more general version that works for an arbitrary incompressible flow u, not necessarily shear. We also note that this $L^1 - L^\infty$ estimate behaves in a one-dimensional way for large times, as one would expect for a strip.

We continue with the proof of Theorem 2.6 and postpone the proof of Lemma 2.7. We may find $\delta > 0$ so that if $U \in (1 - \delta, 1)$ and $q_1 \in (0, (1 - \theta_0)/2)$ then $f(U) \leq f(U - \delta)$. Hence we have in this range of U:

$$G[\psi_l] \leq \dot{\xi}_1\frac{\partial U}{\partial x} \leq 0 \tag{2.34}$$

provided that $\dot{\xi}_1 \geq 0$. Furthermore, if $U \in (0, \delta)$ then $f(U) = f(U - \delta) = 0$ and hence in this range of U we also have (2.34). Finally, if $U \in (\delta, 1 - \delta)$ then $|f(U) - f(U - q)| \leq K|q|$ and $\dfrac{\partial U}{\partial x} \leq -\beta$. Hence $G[\psi_l] \leq 0$ everywhere provided that

$$\dot{\xi}_1(t) \geq \frac{K\|q(t)\|_\infty}{\beta}. \tag{2.35}$$

Thus we may choose

$$\xi_1(t) = C\sqrt{t}. \tag{2.36}$$

Therefore we obtain a lower bound for T:

$$T(t, x, y) \geq U(x - c^*t + C\sqrt{t}) - q_1(t, x, y). \tag{2.37}$$

In order to obtain an upper bound we set $\psi_u = U(x - c^*t - x_2 - \xi_2(t)) + q_2(t, x, y)$ and look for $\xi_2(t)$ and $q_2(t, x, y)$ so that $G[\psi_u] \geq 0$. The constant x_2 is chosen so that

$$T_0(x, y) \leq U(x - x_2) + q_2(0, x, y)$$

with $q_2(0, x, y) \in L^1(D)$ and $0 \leq q_2(0, x, y) \leq \theta_0/2$, as with $q_1(0, x, y)$. The function $q_2(t, x, y)$ is then chosen to satisfy the same advection-diffusion equation (2.32) similarly to q_1. Hence it obeys the same time decay bounds as q_1. With the above choice of q_2 we have

$$G(\psi_u) = -\dot{\xi}_2 \frac{\partial U}{\partial x} + f(U) - f(U + q_2).$$

Once again, we consider three regions of values for U. First, if $1-\delta \leq U \leq 1$ with a sufficiently small $\delta > 0$ then $f(U) - f(U + q_2) \geq 0$, as $q_2 \geq 0$. Hence $G[\psi_u] \geq 0$ in this region provided that $\dot{\xi}_2 \geq 0$. Second, as $q_2 \leq \theta_0/2$ we have $f(U) = f(U + q_2) = 0$ if $0 \leq U \leq \delta$ with a sufficiently small $\delta > 0$. Hence $G[\psi_u] \geq 0$ in that region under the same condition $\dot{\xi}_2 \geq 0$. Finally, if $U \in (\delta, 1 - \delta)$ then $\dfrac{\partial U}{\partial x} \leq -\beta$ with $\beta > 0$ and $|f(U) - f(U + q_2)| \leq K\|q_2\|_\infty$. That means that $G[\psi_u] \geq 0$ if we choose ξ_2 so that

$$\dot{\xi}_2 \geq \frac{K\|q_2\|_\infty}{\beta}.$$

Therefore we may choose

$$\xi_2(t) = C\sqrt{t},$$

as with $\xi_1(t)$. Therefore we obtain upper and lower bounds

$$U(x - c^*t + \xi_1(t) + x_1) - q_1(t, x, y) \leq T(t, x, y) \leq U(x - c^*t - \xi_2(t) - x_2) + q_2(t, x, y)$$

that imply in particular that

$$U(x - c^*t + C_0[1 + \sqrt{t}]) - \frac{C_0}{\sqrt{t}} \leq T(t, x, y) \leq U(x - c^*t - C_0[1 + \sqrt{t}]) + \frac{C_0}{\sqrt{t}} \tag{2.38}$$

with a constant C_0 determined by the initial conditions. Now (2.27) follows immediately from the asymptotic conditions for $U(x, y)$ in (2.24). $\square$

Proof of Lemma 2.7. We will show that there exists a universal constant $C > 0$ so that the solution of

$$\frac{\partial \psi}{\partial t} + \mathbf{u} \cdot \nabla \psi = \sigma \Delta \psi \tag{2.39}$$
$$\psi(0, x, y) = \psi_0(x, y) \geq 0,$$

with the Neumann boundary conditions at $y = 0$ and $y = 1$, and $\mathbf{u}$ sufficiently regular, satisfies

$$\|\psi(t)\|_{L^\infty(D)} \leq Cn^2(t)\|\psi_0\|_{L^1(D)}. \tag{2.40}$$

Here $n(t)$ is the unique solution of

$$\frac{n^4(t)}{1 + n^3(t)} = \frac{C}{\sigma t}. \tag{2.41}$$

We multiply (2.39) by ψ and integrate over the domain D to obtain

$$\frac{1}{2}\frac{d}{dt}\|\psi\|_2^2 = -\sigma\|\nabla\psi\|_2^2. \tag{2.42}$$

We now prove the following version of the Nash inequality [23] for the strip $D = \mathbb{R}_x \times [0,1]$ in two dimensions:

$$\|\nabla\psi\|_2^2 \geq C\frac{\|\psi\|_2^6}{\|\psi\|_1^4 + \|\psi\|_1\|\psi\|_2^3}. \tag{2.43}$$

The proof of (2.43) is similar to that of the usual Nash inequality. We represent ψ in terms of its Fourier series-integral:

$$\psi(x,y) = \sum_{n\geq 0}\int_{\mathbb{R}} e^{ikx}\cos(\pi n y)\,\hat{\psi}_n(k)\frac{dk}{2\pi},$$

where

$$\hat{\psi}_n(k) = 2\int_D e^{-ikx}\cos(\pi n y)\,\psi(x,y)dxdy.$$

Therefore we have

$$|\hat{\psi}_n(k)| \leq 2\|\psi\|_{L^1}. \tag{2.44}$$

The Plancherel formula becomes

$$\begin{aligned}\int_D |\psi(x,y)|^2 dxdy &= \sum_{n,m\geq 0}\int_{\mathbb{R}^2} e^{ikx-ipx}\cos(\pi n y)\cos(\pi m y)\,\hat{\psi}_n(k)\hat{\psi}_m^*(p)\frac{dkdpdxdy}{(2\pi)^2}\\ &= \frac{1}{2}\sum_{n\geq 0}\int|\hat{\psi}_n(k)|^2\frac{dk}{2\pi}\end{aligned}$$

and similarly

$$\int_D |\nabla\psi(x,y)|^2 dxdy = \frac{1}{2}\sum_{n\geq 0}\int_{\mathbb{R}}\left(k^2+\pi^2 n^2\right)|\hat{\psi}_n(k)|^2\frac{dk}{2\pi}.$$

Let $\rho > 0$ be a positive number to be chosen later. Then using the above Plancherel formula we may write

$$\|\psi\|_2^2 = I + II,$$

with the first term that may be bounded using (2.44)

$$I = \frac{1}{2}\sum_{0\leq n\leq\rho}\int_{|k|\leq\rho}|\hat{\psi}_n(k)|^2\frac{dk}{2\pi} \leq C\rho([\rho]+1)\|\psi\|_1^2 \leq C\rho(\rho+1)\|\psi\|_1^2.$$

The rest may be bounded by

$$II \leq \frac{C}{\rho^2}\sum_{n\geq 0}\int_{k\in\mathbb{R}}(k^2+4\pi^2n^2)|\hat{\psi}_n(k)|^2 dk \leq \frac{C}{\rho^2}\|\nabla\psi\|_2^2.$$

Therefore we have for all $\rho > 0$:

$$\|\psi\|_2^2 \le C\rho(\rho+1)\|\psi\|_1^2 + \frac{C}{\rho^2}\|\nabla\psi\|_2^2.$$

We choose ρ so that

$$\rho^3 = \frac{\|\nabla\psi\|_2^2}{\|\psi\|_1^2}$$

and obtain

$$\begin{aligned}\|\psi\|_2^2 &\le \frac{C\|\nabla\psi\|_2^{2/3}}{\|\psi\|_1^{2/3}}\left(\frac{\|\nabla\psi\|_2^{2/3}}{\|\psi\|_1^{2/3}}+1\right)\|\psi\|_1^2 + \frac{C\|\nabla\psi\|_2^2\|\psi\|_1^{4/3}}{\|\nabla\psi\|_2^{4/3}}\\ &= C\|\psi\|_1^{4/3}\|\nabla\psi\|_2^{2/3} + C\|\nabla\psi\|_2^{4/3}\|\psi\|_1^{2/3}.\end{aligned}$$

This is a quadratic inequality $ax^2+bx-c\ge 0$ with $x=\|\nabla\psi\|_2^{2/3}$, $a=C\|\psi\|_1^{2/3}$, $b=C\|\psi\|_1^{4/3}$, and $c=\|\psi\|_2^2$ and hence

$$x \ge \frac{-b+\sqrt{b^2+4ac}}{2a} = \frac{2c}{b+\sqrt{b^2+4ac}} \ge \frac{c}{\sqrt{b^2+4ac}}.$$

This implies that

$$\|\nabla\psi\|_2^{2/3} \ge C\|\psi\|_2^2\left(\|\psi\|_1^{8/3}+\|\psi\|_1^{2/3}\|\psi\|_2^2\right)^{-1/2}$$

and therefore

$$\|\nabla\psi\|_2^2 \ge C\|\psi\|_2^6\left(4\|\psi\|_1^{8/3}+4\|\psi\|_1^{2/3}\|\psi\|_2^2\right)^{-3/2} \ge \frac{C\|\psi\|_2^6}{\|\psi\|_1^4+\|\psi\|_1\|\psi\|_2^3}.$$

Hence (2.43) indeed holds.

We insert (2.43) into (2.42) and use the conservation of the L^1-norm of ψ (recall that the initial data is non-negative) obtain

$$\frac{d\|\psi\|_2}{dt} \le -\frac{C\sigma\|\psi\|_2^5}{\|\psi_0\|_1^4+\|\psi_0\|_1\|\psi\|_2^3}. \tag{2.45}$$

Integrating (2.45) in time we have

$$C\sigma t \le \frac{\|\psi_0\|_1^4}{\|\psi\|_2^4} + \frac{\|\psi_0\|_1}{\|\psi\|_2} \le \frac{1}{z(t)}\left[1+\frac{1}{z^3(t)}\right],$$

where $z(t)=\|\psi(t)\|_2/\|\psi_0\|_1$, and thus

$$\frac{z^4(t)}{1+z^3(t)} \le \frac{1}{C\sigma t}. \tag{2.46}$$

The function on the left side of (2.46) is monotonically increasing and hence we have

$$\|\psi(t)\|_2 \le n(t)\|\psi_0\|_1, \tag{2.47}$$

where $n(t)$ is the solution of (2.41).

Let us denote by $\mathcal{P}_t$ the solution operator for (2.39): $\psi(t) = \mathcal{P}_t\psi_0$. Then (2.47) implies that $\|\mathcal{P}_t\|_{L^1\to L^2} \le n(t)$. The adjoint operator $\mathcal{P}_t^*$ is the solution operator for

$$\begin{aligned}&\frac{\partial\tilde{\psi}}{\partial t} - \mathbf{u}\cdot\nabla\tilde{\psi} = \sigma\Delta\tilde{\psi} \\ &\tilde{\psi}(0,x) = \tilde{\psi}_0(x), \quad x \in D\end{aligned} \tag{2.48}$$

with the Neumann boundary conditions at $z = 0, \lambda$. Note that the preceding estimates rely only on the anti-symmetry of the convection operator $\mathbf{u}\cdot\nabla$. Therefore we have the bound $\|\mathcal{P}_t^*\|_{L^1\to L^2} \le n(t)$ and hence $\|\mathcal{P}_t\|_{L^2\to L^\infty} \le n(t)$ so that

$$\|\psi(t)\|_{L^\infty} \le n(t/2)\|\psi(t/2)\|_{L^2} \le n^2(t/2)\|\psi_0\|_{L^1}$$

and thus (2.40) indeed holds.

The estimate (1.15) follows from the observation that for large $t \gg 1$, when $n(t)$ is small, we have the bound

$$n^2(t) \le \frac{C}{(\sigma t)^{1/2}}.$$

This finishes the proof of Lemma 2.7. □

2.4 Fast shear flow asymptotics: the general case

We now obtain the estimates for the travelling front speed in a fast shear flow.

Theorem 2.8 *Let (c_A, U^A) be the travelling wave solution of*

$$\begin{aligned}&-c^A\frac{\partial U^A}{\partial x} + Au(y)\frac{\partial U^A}{\partial x} = \Delta U^A + f(U^A), \\ &\frac{\partial U^A}{\partial n} = 0, \ \textit{on } \partial\Omega, \\ &U^A(-\infty,y) = 1, \quad U^A(+\infty,y) = 0\end{aligned} \tag{2.49}$$

in a cylinder $D = \{x \in \mathbb{R}, y \in \times\Omega\}$. The nonlinearity $f(s)$ is either of the KPP or of the iginition type. Then there exist two constants $C_{1,2}$ that depend on the flow profile $u(y)$ and the nonlinearity f so that

$$C_1A \le c^A \le C_2A. \tag{2.50}$$

Proof. We present the proof due to S. Heinze [18]. The upper bound in (2.50) follows from Theorem 2.6 on the spreading rate of general solutions to

$$\frac{\partial T}{\partial t} + Au(y)\frac{\partial T}{\partial x} = \Delta T + f(T) \tag{2.51}$$

with rapidly decaying initial data. We observe that the function $\psi(t,x) = e^{-\lambda(x-ct)}$ is a super-solution of (2.51) provided that

$$c = \lambda + \frac{M}{\lambda} + A\|u\|_{L^\infty}.$$

The constant M is chosen so that $f(T) \leq MT$. Hence solutions may not progatae faster than the above speed and the upper bound in (2.50) follows.

The lower bound is slightly more intricate. Let

$$F(s) = \int_0^s f(\xi)d\xi$$

be the antiderivative of f. Define the corrector $\chi(y)$ as

$$\Delta\chi = u(y), \quad \frac{\partial\chi}{\partial n} = 0 \text{ on } \partial\Omega, \quad \int_\Omega \chi(y)dy = 0. \tag{2.52}$$

First, we integrate (2.49) over the cylinder $D = \mathbb{R} \times \Omega$ to obtain

$$c^A = \int f(U^A)dxdy. \tag{2.53}$$

Next, we multiply (2.49) by U^A and integrate to obtain

$$\frac{c^A}{2} + \int |\nabla U^A|^2 = \int U^A f(U^A) \leq \int f(U^A) = c^A.$$

Hence we have

$$\int |\nabla U^A|^2 \leq \frac{c^A}{2}. \tag{2.54}$$

Multiply (2.49) by $f(U^A)\chi(y)$ and integrate over the whole cylinder to obtain

$$\begin{aligned} &-c^A \int f(U^A)\chi(y)U_x^A + A\int u(y)\chi(y)f(U^A)U_x^A \\ &= -\int f'(U^A)\chi(y)|\nabla U^A|^2 - \int f(U^A)\frac{\partial\chi}{\partial y}U_y^A + \int \chi(y)f^2(U^A). \end{aligned}$$

The first integral on the left side is equal to (after evaluating the integral in x) $F(1)\int \chi(y)dy$ and hence vanishes. Combining the other terms we obtain

$$-F(1)A\int u(y)\chi(y)dy = -\int f'(U^A)\chi(y)|\nabla U^A|^2 - \int f(U^A)\frac{\partial\chi}{\partial y}\frac{\partial U^A}{\partial y} + \chi\int f^2(U^A).$$

Now, using the definition (2.52) of the function χ, and the fact that the function f is bounded, we obtain

$$F(1)A\int |\nabla\chi|^2 dy \leq C\int |\nabla U^A|^2 + C\sqrt{\int f^2(U^A)}\sqrt{\int |\nabla U^A|^2} + C\int f^2(U^A).$$

This inequality together with the bounds (2.53) and (2.54) imply that

$$CA \leq c^A.$$

Hence the lower bound in (2.50) holds. $\square$

2.5 Existence of a travelling front in a shear flow

We now turn to the proof of existence of travelling fronts in a shear flow.

Proof of Theorem 2.5: I. Existence. The proof of existence of a travelling wave proceeds as follows – this is a typical road in the proof of existence of travelling fronts.

First, we consider the problem in a finite region $D_a = [-a, a]_x \times \Omega$:

$$
\begin{aligned}
&-c_a \frac{\partial U_a}{\partial x} + u(y)\frac{\partial U_a}{\partial x} = \Delta U_a + f(U_a), \\
&\frac{\partial U_a}{\partial n} = 0 \text{ on } \mathbb{R} \times \partial\Omega \\
&U(-a, y) = 1, \quad U(a, y) = 0.
\end{aligned} \tag{2.55}
$$

As we anticipate passing to the limit $a \to +\infty$ we impose an additional constraint:

$$
\max_{x \geq 0} U_a(x, y) = \theta_0. \tag{2.56}
$$

The maximum principle implies that then the maximum equal to θ_0 is achieved at $x = 0$ – reaction vanishes for $x \geq 0$ and hence (2.55) is simply an advection-diffusion equation in this region. This nails the front down at $x = 0$ and prevents the limit of U_a as $a \to +\infty$ from being equal identically to 0 or 1.

We will show that a solution (c_a, U_a) of (2.55)-(2.56) exists. This is done using a priori estimates and the degree arguments.

The second step in the proof is passage to the limit $a \to +\infty$. The same a priori estimates independent of the domain size a show that the sequence (c_a, U_a) converges along a subsequence to a travelling front solution (c, U) of (2.24).

Step 1. Existence of a solution in D_a. First we get a priori estimates on c_a and U_a assuming that solution of (2.55)-(2.56) exists.

Lemma 2.9 *There exists a constant $K > 0$ that is independent of $a \geq 1$ so that any solution (c_a, U_a) of (2.55)-(2.56) satisfies*

$$
|c_a| \leq K \tag{2.57}
$$

and

$$
\|u\|_{C^1(\bar{D}_a)} \leq K. \tag{2.58}
$$

Proof. First, we observe that the function $\psi_A(x) = Ae^{-\alpha(x+a)}$ is a super-solution for (2.55) if $A > 1$ and

$$
c_a \geq \alpha + \frac{M}{\alpha} + \|u\|_\infty, \tag{2.59}
$$

that is,

$$
-c_a \frac{\partial \psi_A}{\partial x} + u(y)\frac{\partial \psi_A}{\partial x} \geq \Delta \psi_A + f(\psi_A), \tag{2.60}
$$

provided that (2.59) holds with

$$
M = \sup_{0 \leq T \leq 1} \frac{f(T)}{T}.
$$

Furthermore, we have

$$T(-a,y) = 1 < A = \psi_A(-a), \quad T(a,y) = 0 \le \psi_A(a) \tag{2.61}$$

at the two ends of the domain D_a. This together with (2.60) implies that

$$T(x,y) \le \psi_A(x) \tag{2.62}$$

for all $(x,y) \in D_a$ and $A > 1$. Indeed, consider the family of functions $\psi_A(x) = A\psi(x)$. Then all ψ_A are super-solutions in the sense that the inequality (2.60) holds. Moreover, for $A > 2e^{2a}$ sufficiently large we have $\psi_A(x) > 2 > T(x,y)$ for all $(x,y) \in D_a$. We start decreasing A and let A_0 be the largest value of A so that the equation $\psi_A(x) = T(x,y)$ has a solution. We claim that $A_0 = 1$. Indeed, otherwise the point (x_0,y_0) that solves $\psi_{A_0}(x_0) = T(x_0,y_0)$ may not be at the boundary of D_a and hence has to be in in its interior. The continuity of $\psi_A(x)$ with respect to A implies that such point is unique for $A = A_0$ and hence the graphs of $\psi_{A_0}(x)$ and $T(x,y)$ are tangent at (x_0,y_0). Let $w = \psi_{A_0}(x) - T(x,y)$, then the function w satisfies

$$-c_a\frac{\partial w}{\partial x} + u(y)\frac{\partial w}{\partial x} \ge \Delta w + \gamma(x,y)w, \quad \gamma = \frac{f(\psi_A) - f(T)}{\psi_A - T}.$$

The strong maximum principle implies that w may not attain an interior minimum at $w = 0$ so that $\psi_{A_0}(x) \equiv T(x,y)$ which is a contradiction, as they differ on the boundary. Hence we conclude that $A_0 = 1$ and thus (2.62) holds for all $A > 1$. Passing to the limit $A \to 1$ we obtain that (2.59) implies that

$$T(x,z) \le e^{-\alpha(x+a)}. \tag{2.63}$$

However, existence of such a super-solution contradicts (2.56) if $\alpha \ge \ln(\theta_0^{-1})/a$ because (2.56) implies that there exists y_0 so that $T(0,y_0) = \theta_0$. Hence we must have

$$\begin{aligned} c_a &\le \inf_{\alpha \ge \ln(\theta_0^{-1})/a}\left(\alpha + \frac{1}{\alpha}\right) + \|u\|_\infty \\ &\le 2\chi\left(a \ge \ln(\theta_0^{-1})\right) + \left(\frac{\ln(\theta_0^{-1})}{a} + \frac{a}{\ln(\theta_0^{-1})}\right)\chi\left(a \le \ln(\theta_0^{-1})\right) + \|u\|_\infty \\ &\le 2 + \frac{\ln(\theta_0^{-1})}{a} + \|u\|_\infty. \end{aligned} \tag{2.64}$$

This provides an upper bound on c_a independent of a.

In order to prove the lower bound we proceed similarly. Observe that the function $\phi = 1 - e^{\alpha(x-a)}$ is a sub-solution for T_a if

$$c_a \le -\alpha - \|u\|_\infty. \tag{2.65}$$

That is, if (2.65) holds then $T(x,y) \le 1 - e^{\alpha(x-a)}$. This is shown in a way similar to the proof of (2.63) under assumption (2.59) above – start with $\phi_A = 1 - Ae^{\alpha(x-a)}$ and start decreasing A. However, $\phi(0) = 1 - e^{-\alpha a} > \theta_0$ for

$$a > \frac{\ln((1-\theta_0)^{-1})}{\alpha}. \tag{2.66}$$

This implies that $\max_{x\geq 0} T(x,y) \geq \phi(0) > \theta_0$ provided that both (2.65) and (2.66) hold. This is a contradiction to (2.56). Hence we must have

$$c_a \geq \sup_{\alpha > \frac{\ln((1-\theta_0)^{-1})}{a}} [-\alpha - \|u\|_\infty] = -\frac{\ln((1-\theta_0)^{-1})}{a} - \|u\|_\infty. \tag{2.67}$$

This is the lower bound on c_a and the proof of (2.57) is complete.

In order to prove (2.57) we observe that the usual elliptic estimates up to the boundary and the uniform bound (2.56) on c_a imply that the $W^{2,p}$-norm of u is bounded in any rectangle $[x, x+1] \times \Omega$. This and the Sobolev imbedding theorem with a sufficiently large p imply the uniform C^1-bound (2.57). □

Lemma 2.10 *There exists a solution (c_a, T_a) of (2.55)-(2.56).*

Proof. Consider the space $\mathcal{E} = C^1(D_a) \times \mathbb{R}$. Given $(c,v) \in \mathcal{E}$ let $q = \phi_\tau(c,v)$ be the unique solution of the linear inhomogeneous equation

$$\begin{aligned}
&-c\frac{\partial q}{\partial x} + \tau u(y)\frac{\partial q}{\partial x} = \Delta q + \tau f(v), \\
&\frac{\partial q}{\partial n} = 0 \text{ on } \mathbb{R} \times \partial\Omega \\
&q(-a,y) = 1, \quad q(a,y) = 0.
\end{aligned} \tag{2.68}$$

The problem (2.55) is translated into $q = \phi_1(c,q)$. In order to include the constraint (2.56) we let

$$h_\tau(c,v) = \max_{x\geq 0} \phi_\tau(c,v). \tag{2.69}$$

Thus the problem (2.55)-(2.56) is now written in the space $\mathcal{E}$ as

$$\begin{aligned}
T_a &= \phi_1(c_a, T_a) \\
h_1(c_a, T_a) &= \theta_0.
\end{aligned}$$

This may be re-written as a fixed point problem

$$(c_a, T_a) = \mathcal{F}_1(c_a, T_a),$$

where

$$\mathcal{F}_\tau(c,v) = (c - h_\tau(c,v) + \theta_0, \phi_\tau(c,v)), \quad \tau \in [0,1].$$

The mapping $\mathcal{F}_\tau(c,v) : \mathcal{E} \to \mathcal{E}$ is continuous and compact. Lemma 2.9 implies that

$$(c,v) \neq \mathcal{F}_\tau(c,v) \text{ for all } (c,v) \in \partial\mathcal{O} \text{ and all } \tau \in [0,1],$$

where

$$\mathcal{O} = \{(c,v) \in \mathcal{E} : \ \|u\|_{C^1(D_a)} \leq K, \ |c| \leq K\}.$$

hence the Leray-Schauder degree $d(I - \mathcal{F}_1, \mathcal{O}, 0)$ is well-defined and

$$d(I - \mathcal{F}_1, \mathcal{O}, 0) = d(I - \mathcal{F}_0, \mathcal{O}, 0))$$

by the homotopy invariance. Now, $\mathcal{F}_0(c,v)=(c-\hat{h}(c)+\theta_0,\psi_c)$, with $\psi_c(x)$ the solution of

$$-\psi_c''+c\psi_c'=0 \text{ in } (-a,a)$$
$$\psi_c(-a)=1,\quad \psi_c(a)=0,$$

and $\hat{h}(c)=\psi_c(0)$. This may be computed explicitly: $\hat{h}(c)=\dfrac{1-e^{-ca}}{e^{ca}-e^{-ca}}$, hence the equation $(c,v)=\mathcal{F}_0(c,v)$ uniquely determines $c=c_*^0$ and consequently $\psi_{c_*^0}$. Let us do another homotopy: define

$$\mathcal{F}_0^\tau(c,v)=(c-\hat{h}(c)+\theta_0,(1-\tau)\psi_c+\tau\psi_{c_*^0}),$$

then $d(I-\mathcal{F}_0,\mathcal{O},0))=d(I-\mathcal{F}_0^\tau,\mathcal{O},0))$ for all $\tau\in[0,1]$ as $\mathcal{F}_0^\tau$ may not have a fixed point on $\partial\mathcal{O}$, as $|c_0^*|<K$. Therefore we conclude that

$$d(I-\mathcal{F}_1,\mathcal{O},0)=d(I-\mathcal{F}_0^1,\mathcal{O},0)$$

with $\mathcal{F}_0^1(c,v)=(c-\hat{h}(c)+\theta_0,\psi_{c_*^0})$. The latter degree may be computed as a product $d(I-\mathcal{F}_0^1,\mathcal{O},0))=d(\hat{h}(c)-\theta,(-K,K),0))d(I-\psi_{c_*^0},B_K,0)$, where B_K is te ball of radius K in $C^1(D_a)$. Since $\psi_*\in B_K$ and $\hat{h}(c)$ has one simple zero on $(-K,K)$ we conclude that $(I-\mathcal{F}_1,\mathcal{O},0)\neq 0$ and hence solution of (2.55)-(2.56) exists which completes the proof of Lemma 2.10. □

Step II. Properties of U_a. First we show that U_a is monotonic in x.

Lemma 2.11 *Any solution $U_a(x,y)$ of ((2.55)-(2.56) is monotonic in x:*

$$\frac{\partial U_a}{\partial x}<0.$$

Proof. The proof is identical to that of Theorem 5.1 of Chapter I – we may apply the sliding technique in x since the flow $u(y)$ is independent of x. Hence given $\lambda>0$, the difference $w_\lambda(x,y)=U_a(x,y)-U_a(x+\lambda,y)$ satisfies

$$-c_a\frac{\partial w_\lambda}{\partial x}+u(y)\frac{\partial w_\lambda}{\partial x}=\Delta w_\lambda+c(x,\lambda)w_\lambda.$$

The rest of the proof is essentially unchanged. □

Next we show that $c_a\geq 0$. For that we need

Lemma 2.12 *There exists a constant $\delta>0$ that is independent of a so that*

$$\int_{D_a} f(U_a)dxdy\geq\delta>0. \tag{2.70}$$

Proof. Let $\lambda\in(\theta_0,1)$ be a number so that $f(\lambda)>0$. There exists a point $(x_0,y_0)\in D_a$ so that $f(x_0,y_0)=\lambda$. The uniform upper bound on $|\nabla U_a|$ in (2.58) implies that there exists $\eta>0$ so that $f(x,y)>\theta_0+\frac{\lambda-\theta_0}{2}$ in a ball of radius η around (x_0,y_0). Then (2.70) follows immediately from the fact that the function f is non-negative everywhere and strictly positive in this ball. □

Step III. Passage to the limit $a \to +\infty$. The uniform estimates in Lemma 2.9 imply that after extracting a subsequence $a_n \to +\infty$ we obtain a sequence (c_{a_n}, U_{a_n}) that converges uniformly in $\mathbb{R} \times C^1(\mathbb{R} \times \Omega)$ on compact sets to (c, U). The function U satisfies

$$\begin{aligned} & -c\frac{\partial U}{\partial x} + u(y)\frac{\partial U}{\partial x} = \Delta U + f(U), \\ & \frac{\partial U}{\partial n} = 0 \text{ on } \mathbb{R} \times \partial\Omega \\ & \frac{\partial U}{\partial x} < 0, \quad 0 < U < 1, \quad \sup_{x \ge 0} U(x,y) = \theta_0. \end{aligned} \tag{2.71}$$

It remains to establish the correct limits of U as $x \to \pm\infty$.

Lemma 2.13 *The following integrals are bounded:*

$$\int_D f(U)dx < \infty, \quad \int_D |\nabla U|^2 < \infty. \tag{2.72}$$

Proof. First we note that $f(U) = 0$ for $x \ge 0$. Integrating (2.71) over $Q_z = [-z, 0] \times \Omega$ we get

$$\int_{Q_z} f(U)dxdz = A(0) - A(-z) \tag{2.73}$$

with

$$A(s) = c\int U(s,y)dy - \int u(y)U(s,y)dy + \Phi'(s), \quad \Phi(s) = \int U(s,y)dy.$$

Therefore if $\int_{Q_z} f(U)dxdz \to +\infty$ as $z \to +\infty$ then $\Phi'(-z) \to -\infty$ as $z \to +\infty$, because all other terms in the right side of (2.73) remain bounded. However, $\Phi'(-z)$ may not tend to $-\infty$ as $z \to +\infty$ because $\Phi(s)$ remains bounded. The second estimate in (2.72) is proved identically after multiplying (2.71) by U and integrating over Q_z. $\square$

The monotonicity of $U(x,y)$ in x implies that the limits

$$\beta_\pm(y) = \lim_{x \to \pm\infty} U(x,y)$$

exist. The L^2-bound on the gradient of U in (2.72) implies that $\beta_\pm(y) = \beta_\pm$ are two constants. The estimate on the total reaction rate in (2.72) implies that $f(\beta_\pm) = 0$ so that $\beta_\pm \in [0, \theta_0] \cup \{1\}$. We need to show that $\beta_- = 1$ and $\beta_+ = 0$. Note that monotonicity of U implies that $\beta_+ \le \beta_-$. The normalization condition (2.56) implies that $\beta_+ \le \theta_0$ and that $\beta_- \ge \theta_0$ – hence either $\beta_- = 1$ or $\beta_- = \theta_0$.

First we show that if $\beta_- = \theta_0$ then $U \equiv \theta_0$.

Lemma 2.14 *If $\beta_- = \theta_0$ then $U \equiv \theta_0$.*

Proof. Indeed, if $\beta_- = \theta_0$, then monotonicity of $U(x,z)$ implies that $U(x,z) \le \theta_0$ everywhere in D. Thus $U(x,z)$ solves in this case the linear equation

$$-c\frac{\partial U}{\partial x} + u(y)\frac{\partial U}{\partial x} = \Delta U \text{ in } D. \tag{2.74}$$

The L^2-estimate on the gradient in (2.72) implies that

$$S_n = \int \frac{\partial U(s_n, y)}{\partial x} dy \to 0 \text{ along a subsequence } s_n \to +\infty$$

and

$$R_n = \int \frac{\partial U(r_n, y)}{\partial x} dy \to 0 \text{ along a subsequence } r_n \to -\infty.$$

Integrating (2.74) between $x = r_n$ and $x = s_n$ and passing to the limit $n \to \infty$, using the mean-zero assumption (2.2) we obtain

$$c(\beta_- - \beta_+) = 0. \tag{2.75}$$

Furthermore, multiplying (2.74) by U and integrating we obtain in the same manner

$$\frac{c(\beta_-^2 - \beta_+^2)}{2}|\Omega| = -\int |\nabla U|^2.$$

This together with (2.75) imply that

$$\int |\nabla U|^2 dx dy = 0$$

and hence $U \equiv$ const, thus $U \equiv \theta_0$. $\Box$

Lemma 2.14 implies that if $\beta_- = \theta_0$ then $U_{a_n}(x, z) \to \theta_0$ and $\nabla U_{a_n}(x, y) \to 0$, uniformly on compact sets.

Lemma 2.15 *Let $\delta > 0$ be the constant defined in Lemma 2.12. Then the speed c satisfies*

$$c > \frac{\delta}{|\Omega|}. \tag{2.76}$$

Proof. We will consider two possibilities separately: that $\beta_- = 1$ and that $\beta_- = \theta_0$. *Case 1.* $\beta_- = \theta_0$. Integrate (2.55) over the domain $Q_{a_n} = [-a_n, 0] \times \Omega$ to obtain

$$c_{a_n}|\Omega| - c_{a_n}\int_\Omega U_{a_n}(0, y)dy + \int_\Omega u(y)U_{a_n}(0, y)dy = \int_\Omega \frac{\partial U_{a_n}}{\partial x}(0, y)dy - \int_\Omega \frac{\partial U_{a_n}}{\partial x}(-a_n, y)dy$$
$$+ \int_{Q_{a_n}} f(U_{a_n})dxdy.$$

However, as $U_{aq_n} \to \theta_0$ and $|\nabla U_{a_n}| \to 0$ uniformly on compact sets, the third integral on the left side (recall that u has mean zero) and the first integral on the right side converge to zero as $a_n \to \infty$. Furthermore, $\dfrac{\partial U_a(-a, y)}{\partial x} < 0$ and thus we get using (2.70)

$$c|\Omega|(1 - \theta_0) \geq \delta.$$

Hence (2.76) holds in this case.

Case 2. $\beta_- = 1$. Then, as in the proof of Lemma 2.12 we argue that

$$\int f(U) \geq \delta,$$

as $U(x,y)$ satisfies the same uniform gradient bound (2.58) as U_a. Integrating (2.71) over the whole domain we obtain

$$|\Omega|(\beta_- - \beta_+) = \int g(U)dxdy \geq \delta$$

and hence (2.76) holds, as $\beta_- - \beta_+ = 1 - \beta_+ \geq 1 - \theta_0$. □

We now show that $\beta_+ = 0$. This implies that $\beta_- = 1$ and we are done.

Lemma 2.16 *The right limit* $\beta_+ = 0$.

Proof. Choose $\gamma > 0$ so that $\gamma < c_{a_n}$ for n large enough. Let μ be the principal eigenvalue of

$$-\Delta\Psi - tu(y)\Psi = \mu(t)\Psi \text{ in } \Omega$$
$$\frac{\partial\Psi}{\partial n} = 0 \text{ on } \partial\Omega.$$

Recall that the function $\mu(t)$ is concave, $\mu(0) = 0$ and $\mu'(0) = 0$. Hence there exists a unique $\lambda > 0$ so that

$$\mu(\lambda) = \lambda^2 - \gamma\lambda.$$

The corresponding eigenfucntion $\Psi > 0$ satisfies

$$-\Delta\Psi - \lambda[u(y) - \gamma]\Psi = \lambda^2\Psi \text{ in } \Omega$$
$$\frac{\partial\Psi}{\partial n} = 0 \text{ on } \partial\Omega.$$

Consider the function $\Phi(x,y) = e^{-\lambda x}\Psi(y)$, then we have for $x \geq 0$:

$$-c_a\frac{\partial U_a}{\partial x} + u(y)\frac{\partial U_a}{\partial x} = \Delta U_a$$

so that

$$-\gamma\frac{\partial U_a}{\partial x} + u(y)\frac{\partial U_a}{\partial x} = \Delta U_a + (c_a - \gamma)\frac{\partial U_a}{\partial x} \leq \Delta U_a$$

and

$$-\gamma\frac{\partial\Phi}{\partial x} + u(y)\frac{\partial\Phi}{\partial x} = \Delta\Phi.$$

Moreover, if we normalize Ψ so that $\Psi(y) \geq \theta_0$, we have $\Phi(0,y) \geq U_a(0,y)$. The maximum principle implies then that

$$U_a(x,y) \leq \Psi(x,y) \text{ for all } x \geq 0.$$

Passing to the limit we obtain $U(x,y) \leq \Psi(x,y)$ and thus $\beta_+ = 0$. □

This finishes the proof of Theorem 2.5.

Chapter 3

Asymptotic Bounds in PDE's

Asymptic problems when a physical parameter is large arise in infinitely many physical problems. We will be concerned in this chapter with rigorous asymptotic bounds on physically interesting quantites when a relevant parameter is large (or small) but finite. We will consider two problems related to issues of convection-diffusion. The first is the issue of bounds in turbulent transport in the Boussinesq model. The second is the structure of solutions of an advection-diffusion problem in a bounded domain with a large Péclet number.

1 Bounds for turbulent transport

The first example is due to Constantin and Doering [12]. They consider convection convection in a three-dimensional box. The botom ($z = 0$) is held at a constant temperature $T_b = 1$, while the top at $z = h$ is held at temperature $T_{top} = 0$. Gravity acts along the z-direction. The non-dimanesional Boussinesq equations at the infinite Prandtl number for the incompressible flow $\mathbf{u}$ and temperature T have the form

$$\begin{aligned} &T_t + \mathbf{u}\cdot\nabla T = \Delta T \\ &-\Delta \mathbf{u} + \nabla p = RT\hat{\mathbf{z}} \\ &\nabla\cdot\mathbf{u} = 0. \end{aligned} \tag{1.1}$$

The boundary conditions at the top and bottom are

$$\mathbf{u}\,|_{z=0} = \mathbf{u}|_{z=1} = 0,\quad T\,|_{z=0} = 1,\quad T|_{z=1} = 0, \tag{1.2}$$

while solution is periodic in x and y with periods L_x and L_y, respectively.

Let us introduce the long-time average defined by

$$\langle f\rangle = \lim_{t\to\infty}\frac{1}{t}\int_0^t f(s)ds$$

and the L^2-norm

$$\|f\| = \left(\int |f|^2\right)^{1/2}.$$

The Nusselt number is defined as the average dissipation rate

$$\mathrm{Nu} = \frac{1}{\mathcal{A}}\langle\|\nabla T\|^2, \quad \mathcal{A} = L_x L_y.$$

Theorem 1.1 *There exists a constant C so that $Nu \leq CR^{2/5}$.*

One may actually prove a better upper bound of the form $CR^{1/3}(1+\log_+ R)^{2/3}$ [11] but the proof is somewhat more technical.

Proof. We decompose the temperature field into a steady "background field" and a fluctuation:

$$T(t,x,y,z) = \tau(z) + \theta(t,x,y,z).$$

The background field is taken to satisfy the boundary conditions, that is, $\tau(0) = 1$, $\tau(1) = 0$ but is otherwise arbitrary. The idea is to choose it later in an appropriate fashion. The fluctuation θ satisfies

$$\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \Delta\theta - \tau' w + \tau'' \tag{1.3}$$

with the boundary conditions

$$\theta\,|_{z=0} = \theta|_{z=1} = 0.$$

Multiplying (1.3) by θ and integrating we obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 = -\|\nabla\theta\|^2 - \int \tau'(z) w\theta dV - \int \tau'(z)\frac{\partial\theta}{\partial z}dV. \tag{1.4}$$

The total dissipation rate is

$$\|\nabla T\|^2 = \mathcal{A}\int_0^1 [\tau'(z)]^2 dz + 2\int \tau'(z)\frac{\partial\theta}{\partial z}dV + \|\nabla\theta\|^2.$$

Taking the long time average in (1.4) we obtain then

$$\mathrm{Nu} = \frac{1}{\mathcal{A}}\langle\|\nabla T\|^2\rangle = \int_0^1 [\tau'(z)]^2 dz - \frac{1}{\mathcal{A}}\left\langle\int (|\nabla\theta|^2 + 2\tau'(z)w\theta)dV\right\rangle.$$

The idea is to constract a background $\tau(z)$ so that the quadratic form

$$Q(\theta) = \int (|\nabla\theta|^2 + 2\tau'(z)w\theta)dV$$

is non-negative for all functions $\theta(x,y,z)$ satisfying the boundary conditions with

$$-\Delta^2 w = R\Delta_H\theta, \quad \Delta_H\theta := \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}. \tag{1.5}$$

In order to check that (1.5) hodls we first observe that incompressibility of $\mathbf{u}$ implies that $\Delta p = RT_z$. Then, applying Laplacian to

$$-\Delta w + p_z = RT,$$

we obtain (1.5), keeping in mind that $\Delta_H \tau = 0$.

We will take

$$\tau(z) = \begin{cases} 1 - \dfrac{z}{2\delta} & \text{for } 0 \le z \le \delta, \\ \dfrac{1}{2} & \text{for } \delta \le z \le 1-\delta,. \\ \dfrac{1-z}{2\delta} & \text{for } 1-\delta \le z \le 1. \end{cases} \tag{1.6}$$

Here $\delta \ll 1$ is a small parameter measuring the width of the boundary layer. We will show that when $\delta > 0$ is sufficiently small the form $Q(\theta)$ is non-negative. It suffices to show that for each horizontal wave number $\mathbf{k}$ we have

$$Q_{\mathbf{k}}(\theta_{\mathbf{k}}) = \int_0^1 \left[\left| \frac{d\theta_{\mathbf{k}}}{dz} \right|^2 + k^2 |\theta_{\mathbf{k}}|^2 + \tau'(z)(w_{\mathbf{k}}^* \theta_{\mathbf{k}} + w_{\mathbf{k}} \theta_{\mathbf{k}}^*) \right] dz \ge 0.$$

The Fourier coefficients $\theta_{\mathbf{k}}$ satisfy $\theta_{-\mathbf{k}}(z) = \theta_{\mathbf{k}}^*(z)$ and $\theta_{\mathbf{k}}(0) = \theta_{\mathbf{k}}(1) = 0$. The function $w_{\mathbf{k}}$ satisfies

$$\left(-\frac{d^2}{dz^2} + k^2 \right)^2 w_{\mathbf{k}} = Rk^2 \theta_{\mathbf{k}} \tag{1.7}$$

with the boundary conditions

$$w_{\mathbf{k}}(0) = w_{\mathbf{k}}(1) = 0, \quad \frac{dw_{\mathbf{k}}}{dz}(0) = \frac{dw_{\mathbf{k}}}{dz}(1) = 0. \tag{1.8}$$

The Neumann boundary conditions in (1.8) follow from the incompressibility and no-slip boundary conditions for $\mathbf{u}$. We will prove the following lemma.

Lemma 1.2 *There exists a constant $C > 0$ so that*

$$|w_{\mathbf{k}}(z)| \le CRz^2 k \|\theta_{\mathbf{k}}\|, \quad 0 \le z \le 1/2 \tag{1.9}$$

and

$$|w_{\mathbf{k}}(z)| \le CR(1-z)^2 k \|\theta_{\mathbf{k}}\|, \quad 1/2 \le z \le 1. \tag{1.10}$$

We postpone the proof of this lemma and finish the proof of Theorem 1.1 first. We look at the indefinite terms in $Q_{\mathbf{k}}(\theta_{\mathbf{k}})$:

$$\begin{aligned} &\int_0^1 \tau'(z)[w_{\mathbf{k}}^* \theta_{\mathbf{k}} + w_{\mathbf{k}} \theta_{\mathbf{k}}^*] dz \le \frac{1}{\delta} \int_0^\delta |w_{\mathbf{k}}(z)||\theta_{\mathbf{k}}(z)| dz + \frac{1}{\delta} \int_{1-\delta}^1 |w_{\mathbf{k}}(z)||\theta_{\mathbf{k}}(z)| dz \\ &\le \frac{CR}{\delta} \int_0^\delta z^2 k \|\theta_{\mathbf{k}}\| \sqrt{z} \left(\int_0^{1/2} \left| \frac{d\theta_{\mathbf{k}}(z')}{dz'} \right|^2 dz' \right)^{1/2} dz \\ &+ \frac{CR}{\delta} \int_{1-\delta}^1 (1-z)^2 k \|\theta_{\mathbf{k}}\| \sqrt{1-z} \left(\int_{1/2}^1 \left| \frac{d\theta_{\mathbf{k}}(z')}{dz'} \right|^2 dz' \right)^{1/2} dz \\ &\le CR\delta^{5/2} k \|\theta_{\mathbf{k}}\| \left\| \frac{d\theta_{\mathbf{k}}}{dz} \right\| \le CR\delta^{5/2} \left(k^2 \|\theta_{\mathbf{k}}\|^2 + \left\| \frac{d\theta_{\mathbf{k}}}{dz} \right\|^2 \right). \end{aligned}$$

Hence $Q_{\mathbf{k}}(\theta_{\mathbf{k}}) \geq 0$ provided that $R\delta^{5/2} = C$. With this choice of δ we get

$$\text{Nu} \leq \int_0^1 |\tau'(z)|^2 dz \leq \frac{1}{\delta} \leq CR^{2/5}.$$

Hence it remains only to prove Lemma 1.2 to finish the proof of Theorem 1.1.

Proof of Lemma 1.2. We first observe that the homogeneous Dirichlet and Neumann boundary conditions for $w_{\mathbf{k}}$ imply that

$$|w_{\mathbf{k}}(z)| = \left| \int_0^z \int_0^{z'} \frac{d^2 w_{\mathbf{k}}(s)}{ds^2} ds dz' \right| \leq Cz^2 \left\| \frac{d^2 w_{\mathbf{k}}}{dz^2} \right\|_\infty . \tag{1.11}$$

The right hand side above will be estimated using the following lemma.

Lemma 1.3 *Let $f(z)$ be a smooth function satisfying both homogeneous Dirichlet and Neumann boundary conditions on $[0,1]$. Then*

$$\left\| \frac{d^2 f}{dz^2} \right\|_\infty \leq \left(2 \left\| \frac{d^4 f}{dz^4} \right\| \left\| \frac{d^2 f}{dz^2} \right\| \right)^{1/2} . \tag{1.12}$$

We postpone the proof of this lemma for the moment.

Let us define $D = \dfrac{d}{dz}$ and re-write (1.7) as

$$\left(-D^2 + k^2\right)^2 w_{\mathbf{k}} = Rk^2 \theta_{\mathbf{k}}.$$

This implies that

$$\begin{aligned} & R^2 k^4 \|\theta_{\mathbf{k}}\|^2 = \|D^4 w_{\mathbf{k}}\|^2 - 2k^2 \int_0^1 \left[D^4 w_{\mathbf{k}}^* D^2 w_{\mathbf{k}} + D^4 w_{\mathbf{k}} D^2 w_{\mathbf{k}}^* \right] dz \\ & + 6k^4 \|D^2 w_{\mathbf{k}}\|^2 + 4k^6 \|D w_{\mathbf{k}}\|^2 + k^8 \|w_{\mathbf{k}}\|^2 \\ & \geq \|D^4 w_{\mathbf{k}}\|^2 + 6k^4 \|D^2 w_{\mathbf{k}}\|^2 - a \|D^4 w_{\mathbf{k}}\|^2 - \frac{4k^4}{a} \|D^2 w_{\mathbf{k}}\|^2. \end{aligned}$$

We choose $a \in (2/3, 1)$ to get

$$C \left[\|D^4 w_{\mathbf{k}}\|^2 + k^4 \|D^2 w_{\mathbf{k}}\|^2 \right] \leq R^2 k^4 \|\theta_{\mathbf{k}}\|^2.$$

Lemma 1.3, applied to the real and imaginary parts of $w_{\mathbf{k}} = a_{\mathbf{k}} + i b_{\mathbf{k}}$, and (1.11) imply then that

$$\begin{aligned} & |w_{\mathbf{k}}(z)|^2 \leq Cz^4 \left\| \frac{d^2 w_{\mathbf{k}}}{dz^2} \right\|_\infty^2 \leq Cz^4 \left(\|D^2 a_{\mathbf{k}}\|_\infty^2 + \|D^2 b_{\mathbf{k}}\|_\infty^2 \right) \\ & \leq Cz^4 \left(\left\|D^4 a_{\mathbf{k}}\right\| \left\|D^2 a_{\mathbf{k}}\right\| + \left\|D^4 b_{\mathbf{k}}\right\| \left\|D^2 b_{\mathbf{k}}\right\| \right) \leq Cz^4 \left(\frac{1}{k^2} \|D^4 w_{\mathbf{k}}\|^2 + k^2 \|D^2 w_{\mathbf{k}}\|^2 \right) \\ & \leq Cz^4 R^2 k^2 \|\theta_{\mathbf{k}}\|^2. \end{aligned}$$

This proves Lemma 1.2. $\square$

Proof of Lemma 1.3. The boundary codnitions for the function $f(z)$ imply that it has at least two distinct inflection points z_1 and z_2 on the interval $[0,1]$ so that $D^2 f(z_1) = D^2 f(z_2) = 0$. Then we write

$$(D^2 f(z))^2 = 2\int_{z_1}^{z} D^2 f(z')D^3 f(z')dz'$$

and for any $s \in (0,1)$:

$$D^3 f(z') = D^3 f(s) + \int_s^{z'} D^4 f(z'')dz''.$$

Hence we have

$$(D^2 f(z))^2 = 2D^3 f(s)\int_{z_1}^{z} D^2 f(z')dz' + 2\int_{z_1}^{z} D^2 f(z')\int_s^{z'} D^4 f(z'')dz''dz'.$$

We integrate this equation in $s \in [z_1, z_2]$ and observe that the first term on the right vanishes:

$$(z_2 - z_1)(D^2 f(z))^2 = 2\int_{z_1}^{z_2}\int_{z_1}^{z} D^2 f(z')\int_s^{z'} D^4 f(z'')dz''dz'ds.$$

Finally, we apply the Scvhwartz inequality:

$$(z_2 - z_1)(D^2 f(z))^2 \le 2\|D^4 f\|\int_{z_1}^{z_2}\int_{z_1}^{z} D^2 f(z')\sqrt{z'-s}dz'ds$$
$$\le 2\|D^4 f\|\int_{z_1}^{z_2}\int_{z_1}^{z} D^2 f(z')dz'ds \le 2(z_2 - z_1)\|D^4 f\|\|D^2 f\|$$

and hence the conclusion of Lemma 1.3 follows. □

2 Cellular flows in a bounded domain

We now turn to a problem that has spatially inhomogeneous coefficients but neverthteless solutions have a certain structure, at least in an asymptotic limit. We consider the steady advection-diffusion problem

$$\varepsilon\Delta\phi^\varepsilon - u\cdot\nabla\phi^\varepsilon = 0 \tag{2.1}$$

in a simply connected bounded domain $\Omega \subset \mathbb{R}^2$. The flow u is incompressible: $\nabla\cdot u = 0$, and non-penetrating through the boundary of Ω: $u \cdot n = 0$ at $\partial\Omega$ (see Figure 2.1). The small parameter $\varepsilon = \mathrm{Pe}^{-1} \ll 1$ is the inverse of the Péclet number. Equation (2.1) is supplemented by the Dirichlet boundary data:

$$\phi^\varepsilon(\mathbf{x}) = T_0(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \tag{2.2}$$

As the flow u is assumed to be incompressible, a stream function $H(x,y)$ exists so that $u = \nabla^\perp H = (H_y, -H_x)$. Furthermore, since we assume that the normal component of u at the boundary $\partial\Omega$ vanishes, $\partial\Omega$ has to be contained in a level set of H: $\partial\Omega \subseteq \{H = H_0\}$. Hence either Ω is bounded by a closed streamline of the flow u or by a collection of separatrices of u that connect a finite number of singular points of H lying on the level set $\{H = H_0\}$. The latter case is of the most interest to us. We will assume without loss of generality that the critical value $H_0 = 0$. All the critical points of H are assumed to be non-degenerate. Then the set Ω is a union of finite number of flow cells $\mathcal{C}_j$ bounded by separatrices of u, as in Figure 2.1. The streamlines of the flow (level sets of the stream function) are assumed to be sufficiently regular inside each flow cell away from the saddle points of $H(\mathbf{x})$.

We will also assume that the boundary data $T_0(\mathbf{x}) \neq \text{const}$, $\mathbf{x} \in \partial\Omega$ is sufficiently smooth but is not uniform to avoid the trivial case of the constant solution.

The problem of the qualitative behavior of solutions of (2.1)-(2.2) has been studied in various areas where passive scalar advection arises, such as oceanography, meteorology, etc. One of the main interesting effects is the non-trivial coupling of the effects of diffusion and strong advection at a high Péclet number. If we neglect the coupling and approximate (2.1) by

$$u \cdot \nabla\phi = 0 \tag{2.3}$$

we see that ϕ has to be approximately a constant along the flow lines of u. This is, however, impossible near the boundary where the data is prescribed. Another region where (2.3) need not hold is near the stationary points of u. Hence a boundary layer will form along the boundary as well as along the streamlines that pass near the stationary points of the flow, that is, near the separatrices. The width l of the boundary layer should be such that the diffusion term in (2.1) balances the advection term, an easy calculation shows that $l = O(\varepsilon^{1/2})$. The total dissipation rate scales in the corresponding fashion:

$$\int_\Omega |\nabla\phi^\varepsilon(\mathbf{x})|^2 d\mathbf{x} \sim O(\varepsilon^{-1/2}). \tag{2.4}$$

Numerical and physical evidence supports this qualitative picture: see Figure 2.2 for a numerical illustration.

We describe in this section some results that confirm that picture. We prove a dissipation bound as in (2.4).

Theorem 2.1 *Let us assume that $\partial\Omega$ is a piecewise smooth curve and the boundary data T_0 in (2.2) is sufficiently smooth. Then there exists a constant $C = C(u, T_0, \Omega) > 0$ that is independent of $\varepsilon > 0$ so that*

$$\frac{1}{C\sqrt{\varepsilon}} \leq \int_\Omega |\nabla\phi(\mathbf{x})|^2 d\mathbf{x} \leq \frac{C}{\sqrt{\varepsilon}}. \tag{2.5}$$

Convergence of solution to a constant inside a cell is quantified as follows. Let $\mathcal{D}(h) = \{\mathbf{x} : \ |H(\mathbf{x})| \geq h\}$, $h > 0$ be a domain strictly inside the flow cells, at distance $O(h)$ away from the separatrices.

Theorem 2.2 *There exist constants K_j^ε so that we have inside each cell $\mathcal{C}_j$*

$$\sup_{\mathbf{x}\in\mathcal{D}(N\sqrt{\varepsilon})} \left|\phi^\varepsilon(\mathbf{x}) - K_j^\varepsilon\right| \leq \frac{C}{N^{3/2}}. \tag{2.6}$$

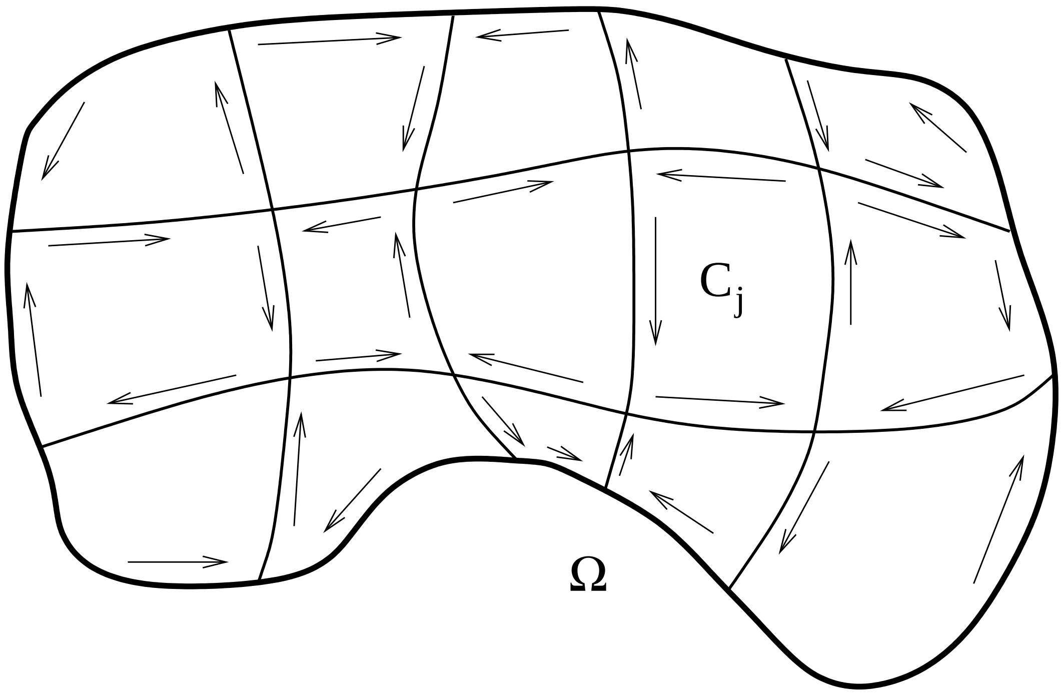

Figure 2.1: The domain Ω is partitioned by flow separatrices into cells $\mathcal{C}_j$.

The flow of the proof is as follows: the upper bound in (2.5) is obtained first. Next we establish convergence of the solution to a constant K_j^ε inside each cell $\mathcal{C}_j$ as in (2.6). The proof of this fact employs the idea of integration and averaging along streamlines. The fact that solution is nearly constant at a distance $O(\sqrt{\varepsilon})$ away from the boundary, where the prescribed data is non-constant, implies the lower bound on the dissipation rate. The results are directly applicable if homogeneous Neumann boundary conditions are prescribed on a part of the boundary, while non-uniform Dirichlet boundary conditions are prescribed on the rest of $\partial\Omega$. The generalization to that case is straightforward.

2.1 The effective diffusivity at a high Péclet number

We would like first to recall the connection of the high Péclet number limit of (2.1)-(2.2) to the effective diffusivity of a particle in a periodic flow. Let $u(y)$ be a mean-zero periodic flow with a period cell $\mathcal{C}$. Consider the advection-diffusion problem

$$\frac{\partial T}{\partial \tau} + u(y) \cdot \nabla_y T = \kappa \Delta_y T. \tag{2.7}$$

The long-time large-scale behavior of T is studied using the diffusive rescaling $y = x/\delta$, $\tau = t/\delta^2$, where $\delta \ll 1$ is a small parameter. We will look at the rescaled variables $x, t = O(1)$ which means that the original variables $y = O(1/\delta)$ and $\tau = O(1/\delta^2)$ are large. Equation (2.7) becomes in the new variables

$$\frac{\partial T^\delta}{\partial t} + \frac{1}{\delta} u\left(\frac{x}{\delta}\right) \cdot \nabla T^\delta = \kappa \Delta T^\delta. \tag{2.8}$$

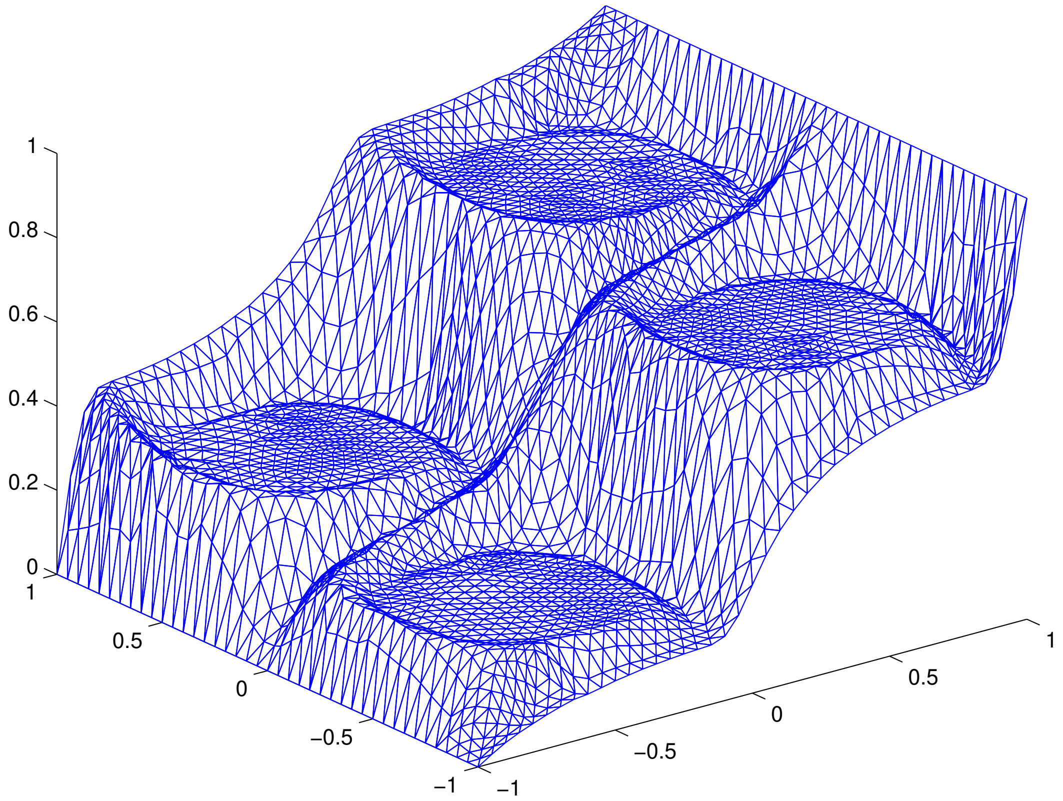

Figure 2.2: The temperature distribution for periodic cellular flows computed in MATLAB. $u = \nabla^{\perp} H$, $H = \sin(\pi x)\sin(\pi y)$; four cells, $\mathrm{Pe} = 20$.

We look for solution as a formal asymptotic series

$$T^{\delta}(t,x) = T_0(t,x) + \delta T_1(t,x,z) + \delta^2 T_2(t,x,z) + \dots, \quad z = \frac{x}{\delta}.$$

Inserting this expansion into (2.8) we obtain in the leading order

$$\kappa \Delta_z T_1 - u(z)\cdot\nabla_z T_1 = u(z)\cdot\nabla_x T_0.$$

Hence T_1 has the form

$$T_1(t,x,z) = \sum_j \chi_j(z)\frac{\partial T_0(t,x)}{\partial x_j} + T_{10}(x).$$

The functions χ_j satisfy an inhomogeneous version of (2.1): they are periodic in z and mean-zero solutions of

$$\kappa \Delta_z \chi_j - u\cdot\nabla_z \chi_j = u_j \tag{2.9}$$

in a period cell $\mathcal{C}$ of the flow u. They are called the first order correctors. Note that solutions of (2.9) exist because u has mean zero. The functions $T_{10}(x)$ are to be determined later (we will not need them to compute the effective diffusivity). In the next order we obtain

$$\frac{\partial T_0}{\partial t} + u(z)\cdot\nabla_z T_2 + u(z)\cdot\nabla_x T_1 = \kappa\Delta_x T_0 + \kappa\Delta_z T_2 + 2\nabla_x\cdot\nabla_z T_1.$$

Integrating this equation in z over the cell $\mathcal{C}$ we obtain

$$\frac{\partial T_0}{\partial t} = \frac{\partial}{\partial x_n}\left(D^h_{nm}\frac{\partial T_0}{\partial x_m}\right). \tag{2.10}$$

The symmeric part of the homogenized diffusion matrix is given by

$$D^h_{nm} = \kappa\delta_{nm} - \frac{1}{2|\mathcal{C}|}\int_{\mathcal{C}}[u_n(z)\chi_m(z) + u_n(z)\chi_m(z)]dz = \kappa\left[\delta_{nm} + \frac{1}{|\mathcal{C}|}\int_{\mathcal{C}}\nabla\chi_n(z)\cdot\nabla\chi_m(z)dz\right]. \tag{2.11}$$

We may introduce $\rho_j(x) = \chi_j(x) + x_j$ to make (2.9) have the form (2.1):

$$\kappa\Delta\rho_j - u\cdot\nabla\rho_j = 0.$$

Thus the total dissipation rate (2.4) is closely related to the small $\kappa \ll 1$ asymptotics of the effective diffusion matrix D^h. In particular, (2.4) with $\varepsilon = \kappa$ predicts that in the presence of a cellular flow, $D^h \sim \sqrt{\kappa} \gg \kappa$ – this reflects the effective diffusivity enhancement by a periodic flow. A rigorous drivation of tthe homogenzation limit may be found in [4].

2.2 A uniform upper bound

We prove in this section the uniform upper bound on the total dissipation rate in the inequality (2.5) in Theorem 2.1.

Theorem 2.3 *Let us assume that $\partial\Omega$ is a piecewise smooth curve and T_0 is sufficiently smooth. There exists a constant $C = C(u, T_0, \Omega)$ so that*

$$\int_\Omega |\nabla\phi(\mathbf{x})|^2 d\mathbf{x} \le \frac{C}{\sqrt{\varepsilon}}. \tag{2.12}$$

Proof. Let ψ^ε be a test function to be specified later. We multiply (2.1) by the function $q^\varepsilon = \phi^\varepsilon - \psi^\varepsilon$ and obtain after integration by parts:

$$\varepsilon\int_{\partial\Omega} q^\varepsilon\frac{\partial\phi^\varepsilon}{\partial n}dS - \varepsilon\int_\Omega(\nabla\phi^\varepsilon - \nabla\psi^\varepsilon)\cdot\nabla\phi^\varepsilon d\mathbf{x} - \int_\Omega(\phi^\varepsilon - \psi^\varepsilon)u\cdot\nabla\phi^\varepsilon d\mathbf{x} = 0.$$

Using incompressibility of the flow u we get

$$\begin{aligned}&\varepsilon\int_\Omega|\nabla\phi^\varepsilon|^2 d\mathbf{x} \le \varepsilon\int_{\partial\Omega} q^\varepsilon\frac{\partial\phi^\varepsilon}{\partial n}dS + \varepsilon\int_\Omega|\nabla\psi^\varepsilon|^2 d\mathbf{x} + \frac{\varepsilon}{4}\int_\Omega|\nabla\phi^\varepsilon|^2 d\mathbf{x} \\ &+\alpha\int_\Omega|\psi^\varepsilon|^2 d\mathbf{x} + \frac{1}{\alpha}\int_\Omega|u\cdot\nabla\phi^\varepsilon|^2 d\mathbf{x}\end{aligned} \tag{2.13}$$

with the constant α to be chosen. We now multiply (2.1) by $u\cdot\nabla\phi^\varepsilon$ and integrate to get

$$\int_\Omega|u\cdot\nabla\phi^\varepsilon|^2 d\mathbf{x} = \varepsilon\int_\Omega(u\cdot\nabla\phi^\varepsilon)\Delta\phi^\varepsilon d\mathbf{x} = \varepsilon\int_{\partial\Omega}(u\cdot\nabla\phi^\varepsilon)\frac{\partial\phi^\varepsilon}{\partial n}dS - \varepsilon\int_\Omega\nabla(u\cdot\nabla\phi^\varepsilon)\cdot\nabla\phi^\varepsilon d\mathbf{x}.$$

Once again using incompressibility of u and the definition of the constant M we obtain from the above

$$\begin{aligned}\int_\Omega |u\cdot\nabla\phi^\varepsilon|^2 d\mathbf{x} &= \varepsilon\int_{\partial\Omega}(u\cdot\nabla\phi^\varepsilon)\frac{\partial\phi^\varepsilon}{\partial n}dS - \frac{1}{2}\varepsilon\int_\Omega (u\cdot\nabla\left(|\nabla\phi^\varepsilon|^2\right))d\mathbf{x} - \varepsilon\int_\Omega \frac{\partial u_n}{\partial x_m}\frac{\partial\phi^\varepsilon}{\partial x_m}\frac{\partial\phi^\varepsilon}{\partial x_n}d\mathbf{x}\\ &\le \varepsilon\int_{\partial\Omega}(u\cdot\nabla\phi^\varepsilon)\frac{\partial\phi^\varepsilon}{\partial n}dS + \varepsilon M\int_\Omega|\nabla\phi^\varepsilon|^2 d\mathbf{x}. \end{aligned}\tag{2.14}$$

We insert (2.14) into (2.13) to get

$$\begin{aligned}\varepsilon\int_\Omega|\nabla\phi^\varepsilon|^2d\mathbf{x} &\le \varepsilon\int_{\partial\Omega} q^\varepsilon\frac{\partial\phi^\varepsilon}{\partial n}dS + \varepsilon\int_\Omega|\nabla\psi^\varepsilon|^2d\mathbf{x} + \frac{\varepsilon}{4}\int_\Omega|\nabla\phi^\varepsilon|^2d\mathbf{x} + \alpha\int_\Omega|\psi^\varepsilon|^2d\mathbf{x}\\ &+\frac{\varepsilon}{\alpha}\left(\int_{\partial\Omega}(u\cdot\nabla\phi^\varepsilon)\frac{\partial\phi^\varepsilon}{\partial n}dS + M\int_\Omega|\nabla\phi^\varepsilon|^2d\mathbf{x}\right).\end{aligned}$$

With the choice $\alpha = 4M$ the above becomes

$$\frac{\varepsilon}{2}\int_\Omega|\nabla\phi^\varepsilon|^2d\mathbf{x} \le \varepsilon\int_{\partial\Omega}\left[q^\varepsilon + \frac{1}{4M}(u\cdot\nabla\phi^\varepsilon)\right]\frac{\partial\phi^\varepsilon}{\partial n}dS + \varepsilon\int_\Omega|\nabla\psi^\varepsilon|^2d\mathbf{x} + 4M\int_\Omega|\psi^\varepsilon|^2d\mathbf{x}.$$

It remains to require that $q^\varepsilon + \frac{1}{4M}(u\cdot\nabla\phi^\varepsilon) = 0$ on the boundary $\partial\Omega$. However, $\partial\Omega$ is a streamline of u so that $u\cdot\nabla\phi^\varepsilon = u\cdot\nabla T_0$ is a given function. That imposes a boundary condition on the function ψ^ε:

$$\psi^\varepsilon\,|_{\partial\Omega}(\mathbf{x}) = T_0(\mathbf{x}) + \frac{1}{4M}(u\cdot\nabla T_0(\mathbf{x})).\tag{2.15}$$

Then, provided that (2.15) holds we obtain

$$\frac{\varepsilon}{2}\int_\Omega|\nabla\phi^\varepsilon|^2d\mathbf{x} \le \varepsilon\int_\Omega|\nabla\psi^\varepsilon|^2d\mathbf{x} + 4M\int_\Omega|\psi^\varepsilon|^2d\mathbf{x}.\tag{2.16}$$

We may choose a function ψ^ε so that it satisfies the boundary conditions (2.15), vanishes identically at distances larger that $\sqrt{\varepsilon}$ away from $\partial\Omega$ and satisfies the uniform bounds $\|\psi^\varepsilon\|_{L^\infty(\Omega)} \le C$, $\|\nabla\psi^\varepsilon\|_{L^\infty(\Omega)} \le C/\sqrt{\varepsilon}$. Using such a test function in (2.16) we obtain the upper bound (2.12). □

2.3 Convergence to constants

In this section we obtain the lower bound of the inequality (2.5) in Theorem 2.1 and prove Theorem 2.2: we show that solution of (2.1)-(2.2) is close to a constant inside each cell of the flow when ε is small. We denote by $\mathcal{L}_j(\gamma) = \{(x,y:\ H(x,y)=\gamma\}$ the level set of $H(x,y)$ inside a cell $\mathcal{C}_j$. We will usually omit the subscript j to simplify the notation as long as we consider one cell and this does not cause any confusion. We denote by $\mathcal{D}_j(\gamma)$ the region bounded by $\mathcal{L}_j(\gamma)$ inside each cell and by $\mathcal{D}_j(\alpha,\beta) = \mathcal{D}_j(\beta)\backslash\mathcal{D}_j(\alpha)$ the annulus between two level sets. We have the following proposition.

Proposition 2.4 *Let $\phi^\varepsilon(\mathbf{x})$ be solution of (2.1)-(2.2) and let $M_j^\varepsilon(\alpha) = \sup_{\mathbf{x}\in\mathcal{L}_j(\alpha)} \phi^\varepsilon(\mathbf{x})$, and $m_j^\varepsilon(\alpha) = \inf_{\mathbf{x}\in\mathcal{L}_j(\alpha)} \phi^\varepsilon(\mathbf{x})$. Then there exists a constant $C > 0$ so that*

$$M_j^\varepsilon(\alpha) - m_j^\varepsilon(\alpha) \le C\left(\frac{\varepsilon}{\alpha^2}\right)^{3/4}. \tag{2.17}$$

This proposition states that the width of the boundary layer is not larger than $O(\sqrt{\varepsilon})$, as the oscillation on the level set $H = N\sqrt{\varepsilon}$ is bounded by $C/N^{3/2}$. The proof is based on the following key lemma.

Lemma 2.5 *(The level-set oscillation inequality) Let $\mathcal{L}_j(\alpha)$ and $\mathcal{L}_j(\beta)$ be two level sets of the stream function $H(x)$ in a cell $\mathcal{C}_j$ with $\mathcal{D}_j(\alpha) \subset \mathcal{D}_j(\beta)$. Then we have*

$$(M_\varepsilon(\alpha) - m_\varepsilon(\alpha))|F(\alpha,\beta)| \le \varepsilon \int_{\mathcal{L}_j(\alpha)} \left|\frac{\partial\phi^\varepsilon}{\partial n}\right| ds + \varepsilon \int_{\mathcal{L}_j(\beta)} \left|\frac{\partial\phi^\varepsilon}{\partial n}\right| ds, \tag{2.18}$$

where $F(\alpha,\beta)$ is the flux between two level sets

$$F(\alpha,\beta) = \int_\gamma (u\cdot n)ds, \quad \gamma = \gamma(t), \quad t\in[0,1], \quad \gamma(0)\in\mathcal{L}_j(\alpha), \quad \gamma(1)\in\mathcal{L}_j(\beta). \tag{2.19}$$

Here γ is any smooth curve that connects the level sets $\mathcal{L}_j(\alpha)$ and $\mathcal{L}_j(\beta)$ and does not intersect itself.

We will assume without loss of generality that $F(\alpha,\beta) \ge 0$. Note that the flux between two level sets is independent of the choice of the curve γ because of the incompressibility of the flow u.

We now prove the level-set oscillation inequality (Lemma 2.5). As we restrict our analysis to one cell we drop the subscript j in all the involved quantities. The idea of the proof is to construct a set $\mathcal{R}$ bounded by a pair of gradient curves of ϕ^ε and parts of the streamlines $\mathcal{L}(\alpha)$ and $\mathcal{L}(\beta)$ if possible. The gradient curves would be chosen so that the difference in the values of the function ϕ^ε between these curves is at least as large as the oscillation of ϕ^ε along $\mathcal{L}(\alpha)$. Integrating equation (2.1) over $\mathcal{R}$ we get then (2.18). The main technicality is the construction of the set $\mathcal{R}$: see Figures 2.3 and 2.4 below for a geometric depiction of $\mathcal{R}$.

We turn now to the construction of $\mathcal{R}$. Let us define the oscillation function $d(\gamma) = M^\varepsilon(\gamma) - m^\varepsilon(\gamma)$. The maximum principle implies that if the level set $\mathcal{L}(\gamma)$ is contained inside the region $\mathcal{D}(\gamma')$ bounded by the level set $\mathcal{C}(\gamma')$, then $d(\gamma) < d(\gamma')$. We denote by $\mathbf{x}_m(\alpha)$ and $\mathbf{x}_M(\alpha)$ the points where ϕ^ε attains its minimum and maximum on the level set $\mathcal{L}(\alpha)$: $M^\varepsilon(\alpha) = \phi^\varepsilon(\mathbf{x}_M(\alpha))$ and $m^\varepsilon(\alpha) = \phi^\varepsilon(\mathbf{x}_m(\alpha))$.

Consider the gradient curves

$$\frac{d\gamma_m}{dt} = -\nabla\phi^\varepsilon(\gamma_m(t)), \quad \gamma_m(0) = \mathbf{x}_m(\alpha), \tag{2.20}$$

and

$$\frac{d\gamma_M}{dt} = \nabla\phi^\varepsilon(\gamma_M(t)), \quad \gamma_M(0) = \mathbf{x}_M(\alpha). \tag{2.21}$$

The function ϕ^ε may have critical points in $\mathcal{D}(\alpha,\beta)$ and the gradient curves γ_M and γ_m potentially may tend to those points as $t \to +\infty$. However, all critical points of ϕ^ε are isolated saddle points as it may have neither internal maxima nor minima according to the maximum principle. Moreover, as ϕ^ε satisfies an elliptic problem in Ω it may have only finitely many critical points in the interior away from the boundary. Thus there are only finitely many critical points of ϕ^ε inside $\mathcal{D}(\alpha,\beta)$ that we denote by $\xi_1,\dots,\xi_{N_\varepsilon}$. Note that both $\mathbf{x}_M(\alpha),\mathbf{x}_m(\alpha) \neq \xi_k$ for all k because of the strong maximum principle [15]. Let us consider the disks $U_j^r = \{|\mathbf{x}-\xi_j| \le r\}$, $j=1,\dots,N_\varepsilon$ centered at the singular points, and let $U^r = \cup_{j=1}^{N_\varepsilon} U_j^r$. Note also that $|\nabla\phi^\varepsilon(\mathbf{x})| > C(\varepsilon,r)$ for $\mathbf{x} \in \mathcal{D}^r(\alpha,\beta) = \mathcal{D}(\alpha,\beta)\backslash U^r$. Therefore $\phi^\varepsilon(\gamma_M(t)) > M^\varepsilon(\alpha) + C(\varepsilon)t$ if $\gamma_M(s) \in \mathcal{D}^r(\alpha,\beta)$ for $0 \le s \le t$ and hence the curve $\gamma_M(t)$ must leave the set $\mathcal{D}^r(\alpha,\beta)$ at a finite time since the function ϕ^ε is uniformly bounded. However, the curve $\gamma_M(t)$, $t>0$ may not intersect the level set $\mathcal{L}(\alpha)$ because $\phi^\varepsilon(\gamma_M(t))$ is strictly increasing for $t<t_0$ provided that it stays inside $\mathcal{D}^r(\alpha,\beta)$ for all $t<t_0$. Hence there are two possibilities: either both γ_M and γ_m exit the set $\mathcal{D}^r(\alpha,\beta)$ at $\mathcal{L}(\beta)$ or one of them crosses $\partial\mathcal{D}^r(\alpha,\beta)$ at one of ∂U_j^r. We consider these two cases separately. First, we assume that we may choose $r>0$ so small that the curves γ_M and γ_m do not intersect the circles $U_j^r = \{|x-\xi_j| = r\}$, $j=1,\dots,N_\varepsilon$, and then we treat the other case.

Case 1: There exists $r>0$ so small that both $\gamma_m(t)$ and $\gamma_M(t)$ exit $\mathcal{D}^r(\alpha,\beta)$ at $\mathcal{L}(\beta)$. We denote the corresponding exit times by t_m and t_M, that is $\gamma_m(t_m) \in \mathcal{L}(\beta)$ and $\gamma_M(t_M) \in \mathcal{L}(\beta)$, while $\gamma_m(s) \in \mathcal{D}^r(\alpha,\beta)$ for $0 \le s \le t_m$ and $\gamma_M(s) \in \mathcal{D}^r(\alpha,\beta)$ for $0 \le s \le t_M$. With a slight abuse of notation we denote $\gamma_m = \{\gamma_m(s),\ 0 \le s \le t_m\}$ and $\gamma_M = \{\gamma_M(s),\ 0 \le s \le t_M\}$. The curves γ_m and γ_M both have a finite length since $|\nabla\phi^\varepsilon|$ is uniformly bounded above and below in $D^r(\alpha,\beta)$ (by constants that may depend on ε and r). These curves may not intersect since $\phi^\varepsilon(\mathbf{x}) > M^\varepsilon > m^\varepsilon > \phi^\varepsilon(\mathbf{y})$ for all $\mathbf{x} \in \gamma_M$ and $\mathbf{y} \in \gamma_m$. Let $\mathcal{R}$ be a domain bounded by γ_m, γ_M and parts of the streamlines $\gamma_\alpha \in \mathcal{L}(\alpha)$ and $\gamma_\beta \in \mathcal{L}(\beta)$ (see Figure 2.3). There are two such domains, $\mathcal{R}$ and $\mathcal{D}(\alpha,\beta)\backslash\mathcal{R}$. We fix $\mathcal{R}$ so that $u \cdot n > 0$ on $\gamma_M(t)$ for t sufficiently small – this guarantees that "each streamline of u goes out of $\mathcal{R}$ when it intersects γ_M for the last time." Furthermore, we have $u \cdot n < 0$ on γ_m for t sufficiently small so that "each streamline of u goes into $\mathcal{R}$ when it intersects γ_m for the first time." Here n is the outward normal to $\partial\mathcal{R}$. Integrating (2.1) over $\mathcal{R}$ we obtain

$$0 = \int_{\mathcal{R}} (\varepsilon\Delta\phi^\varepsilon - u\cdot\nabla\phi^\varepsilon)d\mathbf{x} = \varepsilon\int_{\gamma_\alpha}\frac{\partial\phi^\varepsilon}{\partial n}ds + \varepsilon\int_{\gamma_\beta}\frac{\partial\phi^\varepsilon}{\partial n}ds - \int_{\gamma_m}(u\cdot n)\phi^\varepsilon ds - \int_{\gamma_M}(u\cdot n)\phi^\varepsilon ds, \quad (2.22)$$

because $u \cdot n \equiv 0$ on γ_α, γ_β and $\dfrac{\partial\phi^\varepsilon}{\partial n} = 0$ on γ_m, γ_M since the latter are gradient curves of ϕ^ε.

We will use the following fact.

Lemma 2.6 *Let $\gamma:\ [0,1] \to \mathcal{D}(\alpha,\beta)$ be any non-self intersecting smooth curve that connects $\mathcal{L}(\alpha)$ and $\mathcal{L}(\beta)$: $\gamma(0) \in \mathcal{L}(\alpha)$, $\gamma(1) \in \mathcal{L}(\beta)$, has a finite length and is not tangent to $\mathcal{L}(\alpha)$ at $t=0$. Fix the unit normal n to γ so that $n(t)$ is continuous and $u \cdot n$ is non-negative when a streamline of u intersects γ for the last time, that is, $u \cdot n(\tau(\xi)) \ge 0$ for all ξ between α and β, with $\tau(\xi) = \sup\{t:\ \gamma(t) \in \mathcal{L}(\xi)\}$. Let $f(\mathbf{x}) \ge 0$ be a continuous function monotonically increasing along γ. Then we have*

$$F(\alpha,\beta)\inf_{\mathbf{x}\in\gamma} f \le \int_\gamma (u\cdot n) f ds \le F(\alpha,\beta)\sup_{\mathbf{x}\in\gamma} f, \quad (2.23)$$

where $F(\alpha,\beta)$ is the flux (2.19).

Proof. First, we observe that $u\cdot n(\tau(\xi))\geq 0$ for all $\xi\in[\alpha,\beta]$ provided that $u\cdot n(t)>0$ for t sufficiently small. The inequality (2.23) is shown as follows. For any $N\in\mathbb{N}$ we may approximate f along γ by two piecewise constant (along γ) monotonically increasing functions $\bar f_N$ and $\tilde f_N$ so that

$$\int_\gamma (u\cdot n)f ds-\frac{1}{N}\leq\int_\gamma (u\cdot n)\tilde f_N ds\leq\int_\gamma (u\cdot n)f ds\leq\int_\gamma (u\cdot n)\bar f_N ds\leq\int_\gamma (u\cdot n)f ds+\frac{1}{N}\quad (2.24)$$

and $|f-\tilde f_N|\leq 1/N$, $|f-\bar f|\leq 1/N$. Therefore it suffices to prove (2.23) for a step function f that has finitely many discontinuities, the general case follows after passing to the limit $N\to\infty$ in (2.24). We assume below that f is a step function. Let $\alpha_1,\dots,\alpha_p$ be values of the stream function H such that f has jumps only on the level sets $\mathcal{L}(\alpha_k)$, $k=1,\dots,p$. We order them so that $\mathcal{L}(\alpha_k)\subset\mathcal{D}(\alpha_{k+1})$. Then we may represent γ as the union

$$\gamma=\cup_{k=1}^p\gamma_k,\quad \gamma_k\subset\mathcal{D}(\alpha_k,\alpha_{k+1}).$$

Here γ_k is the part of γ contained in the annulus $\mathcal{D}(\alpha_k,\alpha_{k+1})$. We may further split the subset γ_k as a union $\gamma_k=\gamma_k'\cup\gamma_k''$. Here the set $\gamma_k'=\cup_{l=1}^{s_k}\gamma_{kl}'$ is a union of finitely many curves γ_{kl}' that connect the level sets $\mathcal{L}(\alpha_k)$ and $\mathcal{L}(\alpha_{k+1})$. There can be only finitely many of such curves since γ has a finite length and the distance between $\mathcal{L}(\alpha_k)$ and $\mathcal{L}(\alpha_{k+1})$ is positive. The set $\gamma_k''=\cup_l\gamma_{kl}''$ consists of curves that start and end on the same level set $\mathcal{L}(\alpha_k)$ or $\mathcal{L}(\alpha_{k+1})$. We note that the function f is constant on each curve γ_{kl}' and γ_{kl}''. Therefore we have using incompressibility of u

$$\int_{\gamma_k''}f(u\cdot n)ds=\sum_l\int_{\gamma_{kl}''}f(u\cdot n)ds=0.$$

We also have

$$\int_{\gamma_{kl}'}f(u\cdot n)ds=(-1)^{l+1}f_{kl}F(\alpha_k,\alpha_{k+1}),$$

where f_{kl} is the constant value of f on the curve γ_{kl}'. This implies that

$$\int_{\gamma_k}f(u\cdot n)ds=\int_{\gamma_k'}f(u\cdot n)ds=F(\alpha_k,\alpha_{k+1})\sum_{l=1}^{s_k}(-1)^{l+1}f_{kl}.$$

However, f_{kl} is an increasing function of l and the total number of times s_k that γ crosses from $\mathcal{L}(\alpha_k)$ to $\mathcal{L}(\alpha_{k+1})$ must be odd. Thus the above may be bounded below by

$$\int_{\gamma_k}f(u\cdot n)ds\geq f_{k1}F(\alpha_k,\alpha_{k+1})\geq F(\alpha_k,\alpha_{k+1})\inf_\gamma f.$$

Summing the above over k we obtain the first inequality in (2.23). The second inequality is proved in the same way. □

We now apply (2.23) to the curves γ_m and γ_M with $f=\phi^\varepsilon$. Since $\max_{\gamma_m}\phi^\varepsilon=m^\varepsilon(\alpha)$ and $\min_{\gamma_M}\phi^\varepsilon=M^\varepsilon(\alpha)$, we have

$$\int_{\gamma_m}(u\cdot n)\phi^\varepsilon ds\geq -m^\varepsilon(\alpha)F(\alpha,\beta),\quad \int_{\gamma_M}(u\cdot n)\phi^\varepsilon ds\geq M^\varepsilon(\alpha)F(\alpha,\beta),\qquad (2.25)$$

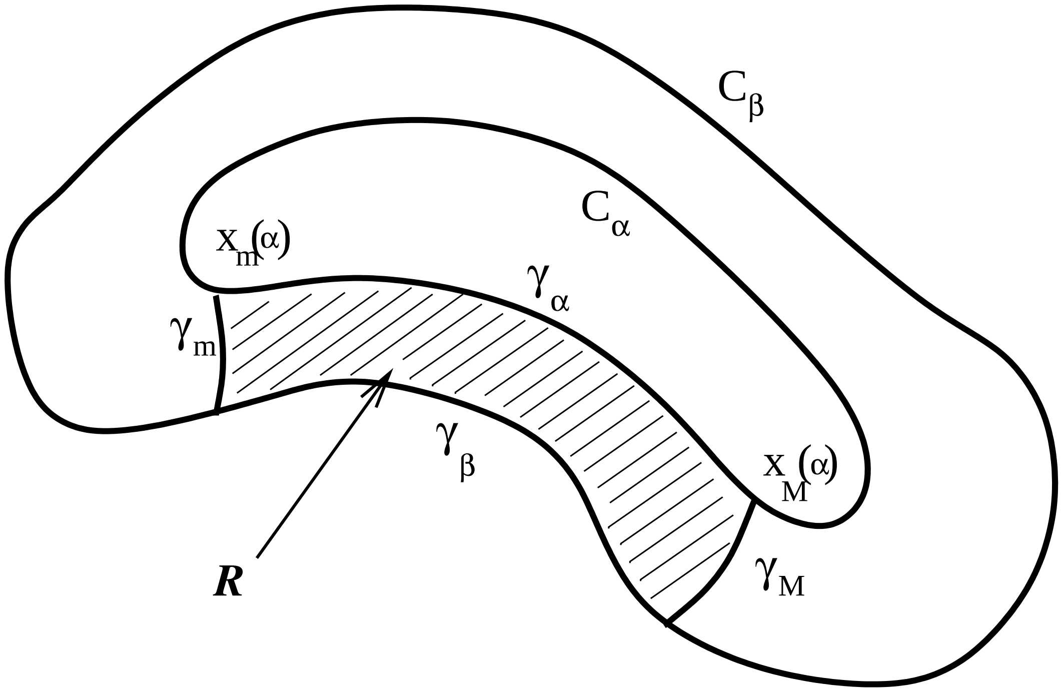

Figure 2.3: The non-critical case

so that

$$\int_{\gamma_m}(u\cdot n)\phi^\varepsilon ds+\int_{\gamma_M}(u\cdot n)\phi^\varepsilon ds\geq (M^\varepsilon(\alpha)-m^\varepsilon(\alpha))F(\alpha,\beta).$$

Clearly we also have

$$\left|\int_{\partial\gamma_\beta}\frac{\partial\phi^\varepsilon}{\partial n}ds\right|\leq\int_{\mathcal{L}(\beta)}\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|ds,$$

and

$$\left|\int_{\partial\gamma_\alpha}\frac{\partial\phi^\varepsilon}{\partial n}ds\right|\leq\int_{\mathcal{L}(\alpha)}\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|ds.$$

The claim of Lemma 2.5 now follows from the last three inequalities and (2.22) in the case when ϕ^ε has no critical points in $\mathcal{D}(\alpha,\beta)$ or when γ_m and γ_M exit $D^r(\alpha,\beta)$ along $\mathcal{L}(\beta)$.

Case 2: It remains to consider the second case when γ_m or γ_M exit the set $\mathcal{D}^r(\alpha,\beta)$ at the boundary ∂U^r for all $r>0$. Then we pick $r>0$ sufficiently small to be specified below. In particular we require that the starting points $\mathbf{x}_M(\alpha)$ and $\mathbf{x}_m(\alpha)$ are not contained in any of U_j^r, $j=1,\dots,N_\varepsilon$ – this is possible since $\mathbf{x}_m(\alpha)$ and $\mathbf{x}_M(\alpha)$ are not critical points of ϕ^ε as implied by the strong maximum principle. Then one (or both) of the curves γ_m and γ_M defined by (2.20) and (2.21) should exit $\mathcal{D}^r(\alpha,\beta)$ at the boundary $\partial U^r=\cup_{j=1}^{N_\varepsilon}\partial U_j^r$. Let us assume that this happens to γ_M and that it exits $\mathcal{D}^r_{\alpha\beta}$ at a point on $\partial U^r_{j_1}$ at a time $\tilde t^1_M$. We continue γ_M past the time $\tilde t^1_M$ as follows (see Figure 2.4). Let $\tilde\eta_M^{j_1}=\gamma_M(\tilde t^1_M)$ be the point where γ_M intersected $\partial U^r_{j_1}$ and let also $\eta_M^{j_1}$ be the point where ϕ^ε reaches its maximum over $\partial U^r_{j_1}$

$$\phi^\varepsilon(\eta_M^{j_1})=\sup_{\mathbf{x}\in\partial U^r_{j_1}}\phi^\varepsilon(\mathbf{x}).$$

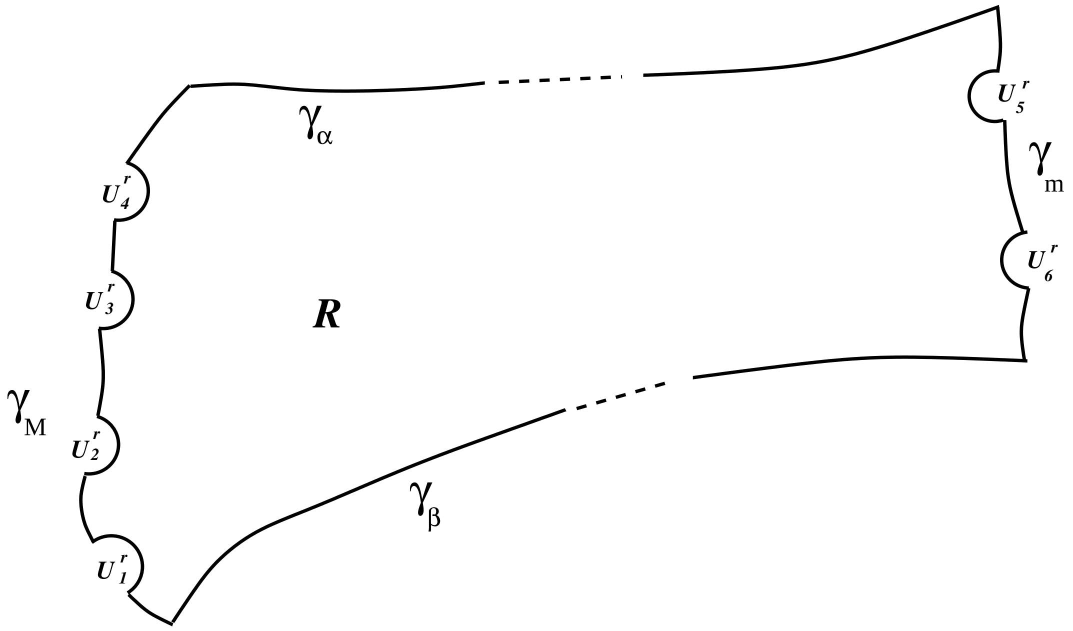

Figure 2.4: The critical case

The vector $\nabla\phi^\varepsilon(\eta_M^{j_1})$ points in the direction of the outer normal to $\partial U_{j_1}^r$ by the maximum principle. We stop γ_M at $\tilde{\eta}_M^{j_1}$ and continue it along the circle $\partial U_{j_1}^r$ to $\eta_M^{j_1}$ in either direction with the speed equal to one, so that $\gamma_M(t_M^1) = \eta_M^{j_1}$. Then γ_M follows the gradient curve going out of $\eta_M^{j_1}$ for $t \geq t_M^1$ until it hits either $\mathcal{L}(\beta)$ or another circle $\partial U_{j_2}^r$ at a point $\tilde{\eta}_M^{j_2}$ at a time $\tilde{t}_M^2$. In the former case we stop the curve γ_M, while in the latter we continue it in the same fashion as at $\partial U_{j_1}^r$, connecting γ_M to η_M^2, the maximum of ϕ^ε along $\partial U_{j_2}^r$, etc. Eventually γ_M has to cross the level set $\mathcal{L}(\beta)$ at some finite time t_M^β. Indeed, we have $\phi^\varepsilon(\tilde{\eta}_M^{j_k}) < \phi^\varepsilon(\eta_M^{j_k}) < \phi^\varepsilon(\tilde{\eta}_M^{j_{k+1}}) < \phi^\varepsilon(\eta_M^{j_{k+1}})$ which implies that the curve γ_M may not hit the same circle ∂U_j^r twice. Given that the total number of critical points N_ε is finite and that γ_M may not stay inside $\mathcal{D}^r(\alpha,\beta)$ for an infinite time we conclude that the exit time t_M^β is finite. A similar construction may be applied to the curve γ_m with $\eta_m^{j_k}$ being the point where ϕ^ε attains its minimum on $U_{j_k}^r$.

In order to guarantee that the curves γ_m and γ_M constructed in such way do not intersect, we require that r is so small that

$$0 < \sup_{\partial U_j^r}\phi^\varepsilon - \inf_{\partial U_j^r}\phi^\varepsilon < \frac{\delta}{1+N_\varepsilon}(\phi^\varepsilon(x_M(\alpha)) - \phi^\varepsilon(x_m(\alpha))), \quad j = 1,\ldots,N_\varepsilon \tag{2.26}$$

where δ is a small parameter. Observe that the sequence $\phi^\varepsilon(\eta_M^{j_k})$ is increasing in k, $\phi^\varepsilon(\eta_M^{j_1}) > \phi^\varepsilon(x_M(\alpha))$ and $\phi^\varepsilon(\gamma(s)) > \phi^\varepsilon(\eta_M^{j_k})$ for $t_M^k < s < \tilde{t}_M^{k+1}$. We also have

$$\phi^\varepsilon(\gamma_M(s)) > \phi^\varepsilon(\eta_M^{j_k}) - \frac{\delta}{1+N_\varepsilon}(\phi^\varepsilon(x_M(\alpha)) - \phi^\varepsilon(x_m(\alpha)))$$

for $\tilde{t}_M^k < s < t_M^k$. That implies a lower bound

$$\phi^\varepsilon(\gamma_M(s)) > \phi^\varepsilon(x_M(\alpha)) - \frac{\delta}{1+N_\varepsilon}(\phi^\varepsilon(x_M(\alpha)) - \phi^\varepsilon(x_m(\alpha))) \tag{2.27}$$

for all $0 < s < t_M^\beta$. Similarly we have

$$\phi^\varepsilon(\gamma_m(s)) < \phi^\varepsilon(x_m(\alpha)) + \frac{\delta}{1+N_\varepsilon}(\phi^\varepsilon(x_M(\alpha)) - \phi^\varepsilon(x_m(\alpha)))$$

for all $0 < s < t_m^\beta$. That implies an estimate

$$\phi^\varepsilon(\gamma_M(s)) - \phi^\varepsilon(\gamma_m(s')) > \left(1 - \frac{2\delta}{1+N_\varepsilon}\right)(\phi^\varepsilon(x_M(\alpha)) - \phi^\varepsilon(x_m(\alpha))) \tag{2.28}$$

for all s and s' so that γ_M and γ_m may not intersect provided that $\delta < 1/2$.

We may now proceed as in the first part of the proof. Let $\mathcal{R}$ be the domain bounded by γ_m, γ_M and parts of the level sets $\mathcal{L}(\alpha)$ and $\mathcal{L}(\beta)$, as depicted on Figure 2.4, chosen so that $u \cdot n > 0$ for t sufficiently small, that is, so that each streamline of u goes out of $\mathcal{R}$ when it crosses γ_M for the last time. Integrating (2.1) over $\mathcal{R}$ we now obtain instead of (2.22):

$$\begin{aligned} 0 \;=\; & \int_{\mathcal{R}} (\varepsilon\Delta\phi^\varepsilon + u\cdot\nabla\phi^\varepsilon)d\mathbf{x} = \varepsilon\int_{\gamma_\alpha}\frac{\partial\phi^\varepsilon}{\partial n}ds + \varepsilon\int_{\gamma_\beta}\frac{\partial\phi^\varepsilon}{\partial n}ds + \varepsilon\sum_{k=1}^{N_\varepsilon}\int_{\gamma_M\cap\,\partial U_k^r}\frac{\partial\phi^\varepsilon}{\partial n}ds \\ + \;& \varepsilon\sum_{k=1}^{N_\varepsilon}\int_{\gamma_m\cap\,\partial U_k^r}\frac{\partial\phi^\varepsilon}{\partial n}ds - \int_{\gamma_m}(u\cdot n)\phi^\varepsilon ds - \int_{\gamma_M}(u\cdot n)\phi^\varepsilon ds, \end{aligned} \tag{2.29}$$

where $\gamma_\alpha = \partial\mathcal{R}\cap\mathcal{L}(\alpha)$ and similarly for γ_β. The function $\phi^\varepsilon(\gamma_M(s))$ is no longer necessarily monotonically increasing in s, as monotonicity might be broken for $\tilde{t}_M^j < s < t_M^j$. However, we may adjust its values on these intervals, interpolating linearly between $\phi^\varepsilon(\tilde{\eta}_M^{j_k})$ and $\phi^\varepsilon(\eta_m^{j_k})$, to make the new function $\tilde{\phi}^\varepsilon(s)$ monotonic in s. The oscillation bound (2.26) implies that

$$\begin{aligned} &\left|\int_{\gamma_M}(u\cdot n)\tilde{\phi}^\varepsilon ds - \int_{\gamma_M}(u\cdot n)\phi^\varepsilon ds\right| \le \sum_{k=1}^{N_\varepsilon}\left|\int_{\gamma_M\cap\,\partial U_k^r}(u\cdot n)(\tilde{\phi}^\varepsilon - \phi^\varepsilon)ds\right| \\ &\le \|u\|_\infty N_\varepsilon\frac{\delta}{1+N_\varepsilon}(\phi^\varepsilon(x_M(\alpha)) - \phi^\varepsilon(x_m(\alpha))) \le C\delta. \end{aligned} \tag{2.30}$$

The estimate (2.23) may be applied to $\tilde{\phi}^\varepsilon$ which together with (2.27) and (2.30) implies:

$$\int_{\gamma_M}(u\cdot n)\phi^\varepsilon ds \ge \int_{\gamma_M}(u\cdot n)\tilde{\phi}^\varepsilon ds - C\delta \ge [M^\varepsilon(\alpha) - C\delta]F(\alpha,\beta) - C\delta = M^\varepsilon(\alpha)F(\alpha,\beta) - C\delta. \tag{2.31}$$

Similarly we obtain

$$\int_{\gamma_m}(u\cdot n)\phi^\varepsilon ds \ge \int_{\gamma_m}(u\cdot n)\tilde{\phi}^\varepsilon ds - C\delta \ge -[m^\varepsilon(\alpha) + C\delta]F(\alpha,\beta) - C\delta = -m^\varepsilon(\alpha)F(\alpha,\beta) - C\delta. \tag{2.32}$$

Furthermore, we may choose $r < 1$ so small that $|\nabla\phi^\varepsilon| < \delta/(1+N_\varepsilon)$ on all ∂U_j^r, $j = 1,\ldots,N_\varepsilon$ – this is possible since the centers of U_j^r are singular points of $\nabla\phi^\varepsilon$. Then we obtain

$$\left|\sum_{k=1}^{N_\varepsilon}\int_{\gamma_M\cap\,\partial U_k^r}\frac{\partial\phi^\varepsilon}{\partial n}ds\right| \le N_\varepsilon 2\pi r\frac{\delta}{1+N_\varepsilon} \le C\delta.$$

Using the above estimates in (2.29) we get

$$\begin{aligned}
&\varepsilon\int_{\mathcal{L}(\alpha)}\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|ds+\varepsilon\int_{\mathcal{L}(\beta)}\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|ds\geq\varepsilon\left|\int_{\mathcal{L}(\alpha)}\frac{\partial\phi^\varepsilon}{\partial n}ds+\int_{\mathcal{L}(\beta)}\frac{\partial\phi^\varepsilon}{\partial n}ds\right|\\
&=\left|\varepsilon\sum_{k=1}^{N_\varepsilon}\int\limits_{\gamma_M\cap\ \partial U_k^r}\frac{\partial\phi^\varepsilon}{\partial n}ds+\varepsilon\sum_{k=1}^{N_\varepsilon}\int\limits_{\gamma_m\cap\ \partial U_k^r}\frac{\partial\phi^\varepsilon}{\partial n}ds-\int_{\gamma_m}(u\cdot n)\phi^\varepsilon ds-\int_{\gamma_M}(u\cdot n)\phi^\varepsilon ds\right|\\
&\geq(M^\varepsilon(\alpha)-m^\varepsilon(\alpha))F(\alpha,\beta)-C\delta.
\end{aligned}$$

This proves Lemma 2.5 in case 2, as δ is arbitrary, and thus the proof of this lemma is complete. □

We now prove Proposition 2.4. We use the inequality (2.18) for a pair of level sets $\mathcal{L}((\alpha+\beta)/2+H)$ and $\mathcal{L}(\beta+H)$ with $0\leq H\leq\dfrac{\alpha-\beta}{2}$ to obtain

$$\begin{aligned}
&\left(M^\varepsilon\left(\frac{\alpha+\beta}{2}+H\right)-m^\varepsilon\left(\frac{\alpha+\beta}{2}+H\right)\right)F\left(\frac{\alpha+\beta}{2}+H,\beta+H\right)\\
&\leq\varepsilon\int\limits_{\mathcal{L}(\frac{\alpha+\beta}{2}+H)}\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|ds+\varepsilon\int\limits_{\mathcal{L}(\beta+H)}\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|ds.
\end{aligned}$$

However, we have

$$M^\varepsilon(\alpha)-m^\varepsilon(\alpha)\leq M^\varepsilon\left(\frac{\alpha+\beta}{2}+H\right)-m^\varepsilon\left(\frac{\alpha+\beta}{2}+H\right)$$

according to the maximum principle. Therefore we get

$$(M^\varepsilon(\alpha)-m^\varepsilon(\alpha))\,F\left(\frac{\alpha+\beta}{2}+H,\beta+H\right)\leq\varepsilon\int\limits_{\mathcal{L}(\frac{\alpha+\beta}{2}+H)}\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|ds+\varepsilon\int\limits_{\mathcal{L}(\beta+H)}\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|ds.\quad(2.33)$$

We integrate (2.33) with respect to H to obtain

$$(M^\varepsilon(\alpha)-m^\varepsilon(\alpha))\int_0^{(\alpha-\beta)/2}F\left(\frac{\alpha+\beta}{2}+H,\beta+H\right)dH\leq\varepsilon\int_\beta^\alpha\int_{\mathcal{L}(H)}\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|dsdH.\quad(2.34)$$

The integral on the right side of inequality (2.34) may be re-written in the orthogonal curvilinear coordinates (H,θ) as

$$\begin{aligned}
&\int_\beta^\alpha\int_{\mathcal{L}(h)}\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|dsdH=\int_\beta^\alpha\int\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|\frac{d\theta dH}{|\nabla\theta|}\leq\int_{\mathcal{D}(\alpha,\beta)}|\nabla\phi^\varepsilon|\frac{|J|dxdy}{|\nabla\theta|}\\
&=\int_{\mathcal{D}(\alpha,\beta)}|\nabla\phi^\varepsilon||\nabla H|dxdy\leq\left(\int_{\mathcal{D}(\alpha,\beta)}|\nabla H|^2dxdy\right)^{1/2}\left(\int_{\mathcal{C}}|\nabla\phi^\varepsilon|^2dxdy\right)^{1/2}\\
&\leq\frac{C}{\varepsilon^{1/4}}\left(\int_{\mathcal{D}(\alpha,\beta)}|\nabla H|dxdy\right)^{1/2}\leq\frac{C(\alpha-\beta)^{1/2}}{\varepsilon^{1/4}}
\end{aligned}$$

where $J = |\nabla H||\nabla \theta|$ is the Jacobian.

The left side of (2.34) satisfies

$$(M^\varepsilon(\alpha) - m^\varepsilon(\alpha)) \int_0^{(\alpha-\beta)/2} F\left(\frac{\alpha+\beta}{2} + H, \beta + H\right) dH \geq C(M^\varepsilon(\alpha) - m^\varepsilon(\alpha))(\alpha-\beta)^2.$$

The above estimates imply that

$$M^\varepsilon(\alpha) - m^\varepsilon(\alpha) \leq C\frac{\varepsilon(\alpha-\beta)^{1/2}}{(\alpha-\beta)^2\varepsilon^{1/4}} \leq C\left(\frac{\varepsilon}{\alpha^2}\right)^{3/4}$$

with the choice $\beta = \alpha/2$. This finishes the proof of Proposition 2.4. □

Proof of Theorem 2.2. Proposition 2.4 shows that the variation of $\phi^\varepsilon(x)$ on a level set $\mathcal{L}(N\sqrt{\varepsilon})$ is bounded by

$$M^\varepsilon(N\sqrt{\varepsilon}) - m^\varepsilon(N\sqrt{\varepsilon}) \leq \frac{C}{N^{3/2}}.$$

The maximum principle then implies that (2.6) holds. □.

Theorem 2.2 implies a lower bound on the L^2-norm of the gradient of solution in Theorem 2.1. Indeed, any function that is approximately equal to a constant at a distance $N\sqrt{\varepsilon}$ away from the boundary where it is not a constant must satisfy the lower bound in (2.5). This completes the proof of Theorems 2.1 and 2.2. □

Chapter 4

The Wigner Transform

1 The semiclassical limit of the Schrödinger equation

The material of this chapter is taken to a great extent from [24].

1.1 The Schrödinger equation as a paraxial wave equation

It is convenient to analyze the high frequency asymptotics of wave propagation in a simple setup, that of the Schrödinger equation where waves go in only one direction. The Schrödinger equation arises in many contexts in wave propagation, not only in quantum mechanics. It provides a convenient albeit idealized set-up to analyze high frequency asymptotics as waves go in only one direction in this model. We recall briefly the parabolic or paraxial approximation for the forward propagation of time harmonic waves.

The wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \tag{1.1}$$

where $c(x, y, z)$ is the propagation speed, has time harmonic solutions of the form $e^{i\omega t}u(x, y, z)$ with the complex wave function u satisfying the Helmholtz or reduced wave equation

$$\Delta u + \frac{\omega^2}{c^2(x, y, z)}u = 0 \tag{1.2}$$

Let c_0 be a uniform reference speed, $k = \omega/c_0$ be the wave number and $c_0/c(x, y, z) = n(x, y, z)$ be the index of refraction. The reduced wave equation has then the form

$$\Delta u + k^2 n^2(x, y, z)u = 0. \tag{1.3}$$

When waves are approximately planar and move in one direction primarily, say the z direction, we look for solutions of the form

$$u = e^{ikz}\phi(z, \mathbf{x}) \tag{1.4}$$

where $\mathbf{x} = (x, y)$ denotes the transverse variables. We insert this into the reduced wave equation and obtain

$$\phi_{zz} + 2ik\phi_z - k^2\phi + \Delta_\perp\phi + k^2 n(x, y, z)\phi = 0.$$

Assuming that the variations of the function ϕ in z are slow compared to those in x, we may drop the ϕ_{zz} term to get the Schrödinger equation

$$2ik\phi_z + \Delta_\perp \phi + k^2\mu(z, \mathbf{x})\phi = 0, \quad \mathbf{x} = (x, y). \tag{1.5}$$

Here $\Delta_\perp$ is the Laplacian in the transverse variables and $\mu = n^2 - 1$ is the fluctuation in the refractive index. Note that the direction of propagation z plays the role of time and $-k^2\mu(z, \mathbf{x})$ is the ("time" dependent) potential. Of course (1.5) is only an approximation to the full reduced wave equation. It is valid only when the variations in the index of refraction are smooth. The full justification of the passage to the parabolic approximation is well beyond the scope of these notes. We will simply accept (1.5) as an important and very useful approximation that is well suited for analysis and numerical studies since we now have an initial value problem for ϕ, assuming that $\phi(0, \mathbf{x})$ is known, rather than a boundary value problem for u.

We shall now analyze high frequency solutions to the Schrödinger equation without regard to its origin through the parabolic approximation. So we write (1.5) in standard (dimensionless) form

$$i\phi_t + \frac{1}{2}\Delta\phi - V(\mathbf{x})\phi = 0 \tag{1.6}$$

with $\mathbf{x} \in R^d$ and initial conditions

$$\phi(0, \mathbf{x}) = \phi_0(\mathbf{x}) \tag{1.7}$$

Note that for simplicity the potential is taken to be time independent but all the analysis extends to the time dependent case as well. In particular, time-dependent potentials make many results in a random medium much simpler to prove. Note that if the Schrödinger equation comes as the parabolic approximation of the wave equation, then there is no reason for the potential to be "time"-independent.

If the potential and the initial data are smooth and decaying at infinity then the solution will be smooth and decaying. The evolution is unitary and preserves the L^2 norm

$$\int_{R^d} |\phi(t, \mathbf{x})|^2 d\mathbf{x} = \int_{R^d} |\phi(\mathbf{x})|^2 d\mathbf{x}. \tag{1.8}$$

1.2 Standard high frequency asymptotics

We consider high frequency asymptotics which concerns approximate solutions of (1.6) that are good approximations to oscillatory solutions. For such solutions the propagation distance is long compared to the wavelength, the propagation time is large compared to the period and the potential $V(\mathbf{x})$ varies slowly. To make this precise we introduce slow time and space variables $t \to t/\varepsilon$, $\mathbf{x} \to \mathbf{x}/\varepsilon$ and the scaled wave function $\phi^\varepsilon(t, \mathbf{x}) = \phi(t/\varepsilon, \mathbf{x}/\varepsilon)$ which satisfies the scaled Schrödinger equation

$$i\varepsilon\phi_t^\varepsilon + \frac{\varepsilon^2}{2}\Delta\phi^\varepsilon - V(\mathbf{x})\phi^\varepsilon = 0. \tag{1.9}$$

In the standard high frequency approximation [20] we consider initial data of the form

$$\phi^\varepsilon(0,\mathbf{x}) = e^{iS_0(\mathbf{x})/\varepsilon} A_0(\mathbf{x}) \tag{1.10}$$

with a smooth, real valued initial phase function $S_0(\mathbf{x})$ and a smooth compactly supported complex valued initial amplitude $A_0(\mathbf{x})$. We then look for an asymptotic solution of (1.9) in the same form as the initial data (1.10), with an evolved phase and amplitude

$$\phi^\varepsilon(t,\mathbf{x}) \sim e^{iS(t,\mathbf{x})/\varepsilon} A(t,\mathbf{x}). \tag{1.11}$$

Inserting this form into (1.9) and equating the powers of ε we get evolution equations for the phase and amplitude

$$S_t + \frac{1}{2}|\nabla S|^2 + V(\mathbf{x}) = 0, \qquad S(0,\mathbf{x}) = S_0(\mathbf{x}) \tag{1.12}$$

and

$$(|A|^2)_t + \nabla\cdot(|A|^2\nabla S) = 0, \qquad |A(0,\mathbf{x})|^2 = |A_0(\mathbf{x})|^2. \tag{1.13}$$

The phase equation (1.12) is called the *eiconal* and the amplitude equation (1.13) the *transport* equation. The eiconal equation that evolves the phase is nonlinear and, in general, it will have a solution only up to some finite time t^* that depends on the initial phase. The solution can be constructed by characteristics, the rays, and singularities in the phase arise when the rays form envelopes which are the caustics. They are called caustics because the amplitude is singular there, which means that there is wave energy concentration. Let

$$H(\mathbf{x},\mathbf{k}) = \frac{1}{2}|\mathbf{k}|^2 + V(\mathbf{x}) \tag{1.14}$$

be the Hamiltonian so that the eiconal equation (1.12) is

$$S_t + H(\mathbf{x},\mathbf{k}) = 0.$$

$$(|A|^2)_t + \nabla\cdot(|A|^2\nabla_{\mathbf{k}}H(\mathbf{x},\mathbf{k})|_{\mathbf{k}=\nabla S}) = 0.$$

This form of the eiconal and transport equations is general and remains valid in the case of symmetric hyperbolic systems. The Lagrangian is given by

$$\mathcal{L}(\mathbf{x},\dot{\mathbf{x}}) = \sup_{\mathbf{k}}\{\mathbf{k}\cdot\dot{\mathbf{x}} - H(\mathbf{x},\mathbf{k})\} = \frac{1}{2}|\dot{\mathbf{x}}|^2 - V(\mathbf{x}) \tag{1.15}$$

and then the solution of the eiconal equation is given by the principle of least action

$$S(t,\mathbf{x}) = \inf_{\mathbf{x}(\cdot),\ \mathbf{x}(t)=\mathbf{x}} \{S_0(\mathbf{x}(0)) + \int_0^t \mathcal{L}(\mathbf{x}(s),\dot{\mathbf{x}}(s))ds\} \tag{1.16}$$

The Euler equations for the minimizing path are Lagrange's equations

$$\frac{d^2\mathbf{x}}{dt^2} + \nabla V(\mathbf{x}) = 0\ , \quad \mathbf{x}(t) = \mathbf{x}\ , \quad \nabla S_0(\mathbf{x}(0)) = \frac{d}{dt}\mathbf{x}(0) \tag{1.17}$$

and the solutions of this two point boundary value problem are the rays.

The transport equation (1.13) can be interpreted as follows. At two time points $0 \le t_1 < t_2$ consider the level surfaces $S(t_1, \mathbf{x}) = S^*$ and $S(t_2, \mathbf{x}) = S^*$, which are wavefronts at these two times. Let $d\Sigma_1$ be an element of area on the wavefront at t_1 around the point $\mathbf{x}_1$ and let $d\Sigma_2$ be the element of surface area carried by a tube of rays from $\mathbf{x}_1$ to the wavefront at t_2. Then

$$|A(t_2, \mathbf{x}_2)|^2 d\Sigma_2 = |A(t_1, \mathbf{x}_1)|^2 d\Sigma_1 \tag{1.18}$$

where $\mathbf{x}_2$ is the point around $d\Sigma_2$ on the wavefront at t_2. This relation is obtained from the transport equation (1.13) by integrating it over the tube of rays and using the divergence theorem. The relation (1.18) is the intensity law of geometrical optics: The ratio of the intensity at $(t_2, \mathbf{x}_2)$ to that at $(t_1, \mathbf{x}_1)$ is inversely proportional to the ratio of the ray tube areas, which is the Jacobian of the ray map. Thus, converging rays increase intensities as is intuitively clear.

The fact that expressions of the form $e^{iS/\varepsilon}A$ are asymptotic solutions is easy to see when everything is smooth since the L^2 norm is preserved by the Schrödinger equation. The main issue in more detailed asymptotics is to extend this elementary theory so that diffraction effects are included. This was done by J.B. Keller [19] and a lot of the work from that period is in the 1964 review of Keller and Lewis that is contained in [20]. Nowadays this subject has evolved into an important branch of the microlocal analysis of which we will touch only the surface.

2 The Wigner distribution

A useful tool in the study of high frequency (semi-classical) limits of the wave equations is the Wigner transform (also called the Wigner distribution) [28][1]. For any smooth function ϕ, rapidly decaying at infinity, the Wigner distribution is defined by

$$W(\mathbf{x}, \mathbf{k}) = \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{y}} \phi(\mathbf{x} - \frac{\mathbf{y}}{2}) \phi^*(\mathbf{x} + \frac{\mathbf{y}}{2}) \frac{d\mathbf{y}}{(2\pi)^d} \tag{2.1}$$

where ϕ^* is the complex conjugate of ϕ. The Wigner distribution is defined on phase space and has many important properties. It is real and its $\mathbf{k}$-integral is the modulus square of the function ϕ,

$$\int_{\mathbb{R}^d} W(\mathbf{x}, \mathbf{k}) d\mathbf{k} = |\phi(\mathbf{x})|^2, \tag{2.2}$$

so we may think of $W(\mathbf{x}, \mathbf{k})$ as wave number resolved energy density – this was the original Wigner motivation. This is not quite right though because $W(\mathbf{x}, \mathbf{k})$ is not always positive. However, it does become positive in the high frequency limit.

The energy flux is expressed through $W(\mathbf{x}, \mathbf{k})$ by

$$\boldsymbol{\mathcal{F}} = \frac{1}{2i}(\phi \nabla \overline{\phi} - \overline{\phi} \nabla \phi) = \int_{\mathbb{R}^d} \mathbf{k} W(\mathbf{x}, \mathbf{k}) d\mathbf{k} \tag{2.3}$$

[1] Actually in [28] Wigner gives credit to Szillard for introducing this object

and its second moment in $\mathbf{k}$ is

$$\iint |\mathbf{k}|^2 W(\mathbf{x},\mathbf{k})d\mathbf{k}d\mathbf{x} = \int |\nabla\phi(\mathbf{x})|^2 d\mathbf{x}. \tag{2.4}$$

We begin our study of the Wigner transform by calculating it for wave functions of the form (1.11) that enter as the data in high frequency calculations

$$\phi^\varepsilon(\mathbf{x}) = e^{iS(\mathbf{x})/\varepsilon}A(\mathbf{x}), \tag{2.5}$$

with S smooth and bounded and A smooth and of compact support. The Wigner distribution is

$$W(\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{y}} e^{iS(\mathbf{x}-\frac{\mathbf{y}}{2})/\varepsilon} A(\mathbf{x}-\frac{\mathbf{y}}{2}) e^{-iS(\mathbf{x}+\frac{\mathbf{y}}{2})/\varepsilon}\bar{A}(\mathbf{x}+\frac{\mathbf{y}}{2})d\mathbf{y} \tag{2.6}$$

and we see that the rescaled version

$$W^\varepsilon(\mathbf{x},\mathbf{k}) = \frac{1}{\varepsilon^d}W\left(\mathbf{x},\frac{\mathbf{k}}{\varepsilon}\right) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{y}} e^{iS(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})/\varepsilon} A(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2}) e^{-iS(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})/\varepsilon}\bar{A}(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})d\mathbf{y} \tag{2.7}$$

converges weakly (as a distribution in $\mathcal{S}'$) to

$$W(\mathbf{x},\mathbf{k}) = \delta(\mathbf{k}-\nabla S(\mathbf{x}))|A(\mathbf{x})|^2. \tag{2.8}$$

That is, if Z is a test function in $\mathcal{S}(\mathbb{R}^{2d})$

$$\int_{\mathbb{R}^{2d}} W^\varepsilon Z d\mathbf{x}d\mathbf{k} \to \int_{\mathbb{R}^d} Z(\mathbf{x},\nabla S(\mathbf{x}))|A(\mathbf{x})|^2 d\mathbf{x} \tag{2.9}$$

as $\varepsilon \to 0$. This calculation tells us two things. One is the correct scaling of the Wigner distribution for the high frequency limit

$$W^\varepsilon(\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{y}}\phi_\varepsilon(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\phi_\varepsilon^*(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})d\mathbf{y}, \tag{2.10}$$

or that for the *evolved* Wigner density

$$W^\varepsilon(t,\mathbf{x},\mathbf{k}) = \int_{R^d} e^{i\mathbf{k}\cdot\mathbf{y}}\phi^\varepsilon(t,\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\bar{\phi}^\varepsilon(t,\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\frac{d\mathbf{y}}{(2\pi)^d}$$

of the solution $\phi^\varepsilon(t,\mathbf{x})$ of the scaled Schrödinger equation (1.9). The second is that in the high frequency limit, and for the kind of inhomogeneous, near-plane waves that we are considering, the limit Wigner function is a measure, a delta function supported on the surface $\mathbf{k} = \nabla S(\mathbf{x})$ in phase space.

From (2.8) we conclude that as $\varepsilon \to 0$ the scaled Wigner distribution of the solution $\phi^\varepsilon(t,\mathbf{x})$ of (1.9) with initial data (1.10) is given by

$$W(t,\mathbf{x},\mathbf{k}) = |A(t,\mathbf{x})|^2\delta(\mathbf{k}-\nabla S(t,\mathbf{x})), \tag{2.11}$$

where $S(t,\mathbf{x})$ and $A(t,\mathbf{x})$ are solutions of the eiconal and transport equations (1.12) and (1.13), respectively.

We will now derive the high frequency approximation of the scaled Wigner distribution directly from the differential equations. Let us assume that the initial Wigner distribution $W_0^\varepsilon(\mathbf{x},\mathbf{k})$ tends to a smooth function $W_0(\mathbf{x},\mathbf{k})$ that decays at infinity. Note that this is not the case with the Wigner function corresponding to $\phi^\varepsilon(0,\mathbf{x})$ given by (1.10) but it is the case for random initial wave functions. The evolution equation for $W^\varepsilon(t,\mathbf{x},\mathbf{k})$ corresponding to the Schrödinger equation (1.9) is the Wigner equation

$$W_t^\varepsilon + \mathbf{k}\cdot\nabla_{\mathbf{x}}W^\varepsilon + \mathcal{L}^\varepsilon W^\varepsilon = 0. \tag{2.12}$$

Here the operator $\mathcal{L}^\varepsilon$ is defined by

$$\mathcal{L}^\varepsilon Z(\mathbf{x},\mathbf{k}) = \frac{i}{\varepsilon}\int_{R^d} e^{i\mathbf{p}\cdot\mathbf{x}}\hat{V}(\mathbf{p})\left[Z(\mathbf{x},\mathbf{k}+\frac{\varepsilon\mathbf{p}}{2}) - Z(\mathbf{x},\mathbf{k}-\frac{\varepsilon\mathbf{p}}{2})\right]\frac{d\mathbf{p}}{(2\pi)^d} \tag{2.13}$$

on any smooth function Z in phase space. The Fourier transform is denoted by a hat

$$\hat{V}(\mathbf{p}) = \int e^{-i\mathbf{p}\cdot\mathbf{x}}V(\mathbf{x})d\mathbf{x}. \tag{2.14}$$

From (2.13) we can find easily the limit of the operator $\mathcal{L}^\varepsilon$ as $\varepsilon\to 0$. For any smooth and decaying function $Z(\mathbf{x},\mathbf{k})$ we have

$$\mathcal{L}^\varepsilon Z(\mathbf{x},\mathbf{k}) \to -\nabla_{\mathbf{x}}V\cdot\nabla_{\mathbf{k}}Z. \tag{2.15}$$

Thus, the limit Wigner equation is the Liouville equation in phase space

$$W_t + \mathbf{k}\cdot\nabla_{\mathbf{x}}W - \nabla V\cdot\nabla_{\mathbf{k}}W = 0 \tag{2.16}$$

with the initial condition $W(0,\mathbf{x},\mathbf{k}) = W_0(\mathbf{x},\mathbf{k})$. This is a linear partial differential equation that can be solved by characteristics. When the initial Wigner distribution has the high frequency form

$$W_0(\mathbf{x},\mathbf{k}) = |A_0(\mathbf{x})|^2\delta(\mathbf{k}-\nabla S_0(\mathbf{x})) \tag{2.17}$$

then it is easy to see that the solution of (2.16) is given by

$$W(t,\mathbf{x},\mathbf{k}) = |A(t,\mathbf{x})|^2\delta(\mathbf{k}-\nabla S(t,\mathbf{x})) \tag{2.18}$$

where $S(t,\mathbf{x})$ and $A(t,\mathbf{x})$ are solutions of the eiconal and transport equations (1.12) and (1.13), respectively. We see, therefore, that from the Wigner distribution we can recover all the information in the standard high frequency approximation. In addition, it provides flexibility to deal with initial data that is not of the form (2.17).

3 General properties of the Wigner distribution

We want to consider the Wigner distributions of high frequency waves, i.e. of functions $f_\varepsilon(\mathbf{x})$ which are oscillating on a scale ε as $\varepsilon\to 0$, not necessarily of the form (1.11). Our exposition follows the ideas of L.Tartar and P.Gerard [27, 13, 14]. We consider the rescaled Wigner distribution

$$W_\varepsilon(\mathbf{x},\mathbf{k}) = \int\frac{d\mathbf{y}}{(2\pi)^d}e^{i\mathbf{k}\cdot\mathbf{y}}f_\varepsilon(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})f_\varepsilon^*(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2}). \tag{3.1}$$

Proposition 3.1 *Let the family f_ε be uniformly bound in $L^2(\mathbb{R}^d)$. Then the Wigner transform W_ε converges weakly along a susbequence $\varepsilon_k \to 0$ to a distribution $W(\mathbf{x},\mathbf{k}) \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$. Any such limit point $W(\mathbf{x},\mathbf{k})$ is a non-negative measure of bounded total mass.*

Note that the positivity of the limit distribution together with the identity

$$\int W_\varepsilon(\mathbf{x},\mathbf{k})d\mathbf{k} = |f_\varepsilon(\mathbf{x})|^2$$

justifies the physical interpretation of $W(\mathbf{x},\mathbf{k})$ as the phase space energy density.

Examples. 1. *The WKB case*: As we have shown before the Wigner measure of the family $f_\varepsilon(\mathbf{x}) = A(\mathbf{x})\exp\{iS(\mathbf{x})/\varepsilon\}$ is $W(\mathbf{x},\mathbf{k}) = |A(\mathbf{x})|^2\delta(\mathbf{k}-\nabla S(\mathbf{x}))$.
2. *The localized case*: the Wigner transform of the family $f_\varepsilon(\mathbf{x}) = \varepsilon^{-d/2}\phi(\mathbf{x}/\varepsilon)$ with a compactly supported function $\phi(\mathbf{x})$ is given by $W(\mathbf{x},\mathbf{k}) = (2\pi)^{-d}|\hat\phi(\mathbf{k})|^2\delta(\mathbf{x})$. This is verified as follows:

$$\begin{aligned}
&\langle a, W_\varepsilon\rangle = \int a(\mathbf{x},\mathbf{k})\phi\left(\frac{\mathbf{x}}{\varepsilon}-\frac{\mathbf{y}}{2}\right)\phi^*\left(\frac{\mathbf{x}}{\varepsilon}+\frac{\mathbf{y}}{2}\right)e^{i\mathbf{k}\cdot\mathbf{y}}\frac{d\mathbf{y}d\mathbf{x}d\mathbf{k}}{(2\pi\varepsilon)^d}\\
&= \int a(\varepsilon\mathbf{x},\mathbf{k})\phi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\phi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)e^{i\mathbf{k}\cdot\mathbf{y}}\frac{d\mathbf{y}d\mathbf{x}d\mathbf{k}}{(2\pi)^d} \to \int a(0,\mathbf{k})\phi(\mathbf{x})\phi^*(\mathbf{z})e^{i\mathbf{k}\cdot(\mathbf{z}-\mathbf{x})}\frac{d\mathbf{z}d\mathbf{x}d\mathbf{k}}{(2\pi)^d}\\
&= \int\int a(0,\mathbf{k})|\hat\phi(\mathbf{k})|^2\frac{d\mathbf{x}d\mathbf{k}}{(2\pi)^d}.
\end{aligned}$$

The proof of Proposition 3.1 is based on the connection between the Wigner transforms and the semiclassical pseudodifferential operators. They arise as follows. Let $a(\mathbf{x},\mathbf{k})$ be a test function in $\mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d)$. Then

$$\langle a, W_\varepsilon\rangle = (a^w(\mathbf{x},\varepsilon D)f_\varepsilon, f_\varepsilon), \tag{3.2}$$

where $\langle,\rangle$ is the usual inner product on $\mathbb{R}^d\times\mathbb{R}^d$, $(,)$ is inner product on $\mathbb{R}^d$, and the Weyl operator $a^w(\mathbf{x},\varepsilon D)$ is defined by

$$\begin{aligned}
a^w(\mathbf{x},\varepsilon D)f(\mathbf{x}) &= \int\frac{d\mathbf{y}d\mathbf{k}}{(2\pi)^d}e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}a(\frac{\mathbf{x}+\mathbf{y}}{2},\varepsilon\mathbf{k})f(\mathbf{y}) \\
&= \int\frac{d\mathbf{y}}{(2\pi\varepsilon)^d}\hat a(\frac{\mathbf{x}+\mathbf{y}}{2},\frac{\mathbf{y}-\mathbf{x}}{\varepsilon})f(\mathbf{y}).
\end{aligned}\tag{3.3}$$

Here $\hat a$ is the Fourier transform of $a(\mathbf{x},\mathbf{k})$ in the variable $\mathbf{k}$ only:

$$\hat a(\mathbf{x},\mathbf{y}) = \int e^{-i\mathbf{k}\cdot\mathbf{y}}a(\mathbf{x},\mathbf{k})d\mathbf{k}.$$

We recall now some basics of the calculus of the semiclassical operators that trivialize the proof of Proposition 3.1.

First, an alternative way to get to limits of Wigner distributions is to introduce

$$\widetilde{W}_\varepsilon(\mathbf{x},\mathbf{k}) = \int\frac{d\mathbf{y}}{(2\pi)^d}e^{i\mathbf{k}\cdot\mathbf{y}}f_\varepsilon(\mathbf{x}-\varepsilon\mathbf{y})f_\varepsilon^*(\mathbf{x}) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}/\varepsilon}}{(2\pi\varepsilon)^d}\hat f_\varepsilon(\frac{\mathbf{k}}{\varepsilon})f_\varepsilon^*(\mathbf{x}). \tag{3.4}$$

and weakly, for $a(\mathbf{x},\mathbf{k}) \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\langle a, \widetilde{W}_\varepsilon \rangle = (a(\mathbf{x},\varepsilon D) f_\varepsilon, f_\varepsilon). \tag{3.5}$$

Now the semiclassical operators $a(\mathbf{x},\varepsilon D)$ are defined by

$$a(\mathbf{x},\varepsilon D) f(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} a(\mathbf{x},\varepsilon\mathbf{k}) \hat{f}(\mathbf{k}) = \int \frac{d\mathbf{y}}{(2\pi\varepsilon)^d} \hat{a}(\mathbf{x}, \frac{\mathbf{y}-\mathbf{x}}{\varepsilon}) f(\mathbf{y}). \tag{3.6}$$

and they are frequently more convenient for calculations than (3.3). An important fact is that (3.6) and (3.3) are asymptotically equivalent as $\varepsilon \to 0$

Lemma 3.2 *The Weyl operators $a^w(\mathbf{x},\varepsilon D)$ and the semiclassical operators $a(\mathbf{x},\varepsilon D)$ are asymptotically equivalent, that is*

$$\|a(\mathbf{x},\varepsilon D) - a^w(\mathbf{x},\varepsilon D)\|_{L^2\to L^2} \to 0 \tag{3.7}$$

as $\varepsilon \to 0$.

We recall first Schur's lemma.

Lemma 3.3 *If $K = K(\mathbf{x},\mathbf{y})$ is a continuous function on $\mathbb{R}^d \times \mathbb{R}^d$ and there exists C such that*

$$\int_{\mathbb{R}^d} |K(\mathbf{x},\mathbf{y})| d\mathbf{y} \le C, \quad \int_{\mathbb{R}^d} |K(\mathbf{x},\mathbf{y})| d\mathbf{x} \le C, \tag{3.8}$$

then the operator

$$Af(\mathbf{x}) = \int_{\mathbb{R}^d} K(\mathbf{x},\mathbf{y}) f(y) d\mathbf{y} \tag{3.9}$$

is bounded on L^2 and

$$\|A\|_{L^2\to L^2} \le C. \tag{3.10}$$

Proof. The simplest way to check (3.10) is to verify that A is a bounded operator $L^1 \to L^1$ and $L^\infty \to L^\infty$ and use the Riesz-Thorin interpolation theorem. $\square$

Proof of Lemma 3.2. Then given a function $f \in L^2$ we have

$$\begin{aligned} a(\mathbf{x},\varepsilon D) f &= \int \frac{d\mathbf{y}}{\varepsilon^d} \tilde{a}(\mathbf{x}, \frac{\mathbf{x}-\mathbf{y}}{\varepsilon}) f(\mathbf{y}) \\ a^w(\mathbf{x},\varepsilon D) f &= \int \frac{d\mathbf{y}}{\varepsilon^d} \tilde{a}(\frac{\mathbf{x}+\mathbf{y}}{2}, \frac{\mathbf{x}-\mathbf{y}}{\varepsilon}) f(\mathbf{y}) \end{aligned} \tag{3.11}$$

and thus

$$(a^w(\mathbf{x},\varepsilon D) - a(\mathbf{x},\varepsilon D)) f = \int \left\{ \tilde{a}(\frac{\mathbf{x}+\mathbf{y}}{2}, \frac{\mathbf{x}-\mathbf{y}}{\varepsilon}) - \tilde{a}\left(\mathbf{x}, \frac{\mathbf{x}-\mathbf{y}}{\varepsilon}\right) \right\} f(\mathbf{y}) \frac{d\mathbf{y}}{\varepsilon^d}.$$

However, we have

$$\int \left| \tilde{a}\left(\frac{\mathbf{x}+\mathbf{y}}{2}, \frac{\mathbf{x}-\mathbf{y}}{\varepsilon}\right) - \tilde{a}(\mathbf{x}, \frac{\mathbf{x}-\mathbf{y}}{\varepsilon}) \right| \frac{d\mathbf{y}}{\varepsilon^d} = \int \left| \tilde{a}\left(\mathbf{x} - \frac{\varepsilon\mathbf{y}}{2}, \mathbf{y}\right) - \tilde{a}(\mathbf{x},\mathbf{y}) \right| d\mathbf{y} \le C(a)\varepsilon$$

and

$$\int \left| \tilde{a}\left(\frac{\mathbf{x}+\mathbf{y}}{2}, \frac{\mathbf{x}-\mathbf{y}}{\varepsilon}\right) - \tilde{a}(\mathbf{x}, \frac{\mathbf{x}-\mathbf{y}}{\varepsilon}) \right| \frac{d\mathbf{x}}{\varepsilon^d} = \int \left| \tilde{a}\left(\mathbf{y} + \frac{\varepsilon \mathbf{x}}{2}, \mathbf{x}\right) - \tilde{a}(\mathbf{y}+\varepsilon\mathbf{x}, \mathbf{x}) \right| d\mathbf{x} \le C(a)\varepsilon.$$

Then (3.7) follows by (3.10). □

The next Lemma proves the uniform L^2-bound on $a(\mathbf{x}, \varepsilon D)$ and the H^s-estimates we will use later.

Lemma 3.4 *Let $a(\mathbf{x}, \mathbf{k}) \in \mathcal{S}(R_x^d \times R_k^d)$. Then for $s > 0$*

$$\begin{aligned} \|a(\mathbf{x}, \varepsilon D)\|_{L^2 \to L^2} &\le C(a) && (3.12)\\ \varepsilon^s \|a(\mathbf{x}, \varepsilon D)\|_{H^{-s} \to L^2} &\le C_s(a) && (3.13)\\ \varepsilon^s \|a(\mathbf{x}, \varepsilon D)\|_{L^2 \to H^s} &\le C_s(a). && (3.14) \end{aligned}$$

Proof of Lemma 3.4. We have

$$a(\mathbf{x}, \varepsilon D) f = \int \frac{d\mathbf{y}}{(2\pi\varepsilon)^d} \hat{a}(\mathbf{x}, \frac{\mathbf{y}-\mathbf{x}}{\varepsilon}) f(\mathbf{y}), \tag{3.15}$$

where hat denotes the Fourier transform in $\mathbf{k}$. Since

$$\int \frac{d\mathbf{y}}{(2\pi\varepsilon)^d} \left| \hat{a}(\mathbf{x}, \frac{\mathbf{y}-\mathbf{x}}{\varepsilon}) \right| = \int \frac{d\mathbf{y}}{(2\pi)^d} \left| \hat{a}(\mathbf{x}, \mathbf{y}) \right| \tag{3.16}$$

and

$$\int \frac{d\mathbf{x}}{(2\pi\varepsilon)^d} \left| \hat{a}(\mathbf{x}, \frac{\mathbf{y}-\mathbf{x}}{\varepsilon}) \right| = \int \frac{d\mathbf{z}}{(2\pi)^d} \left| \hat{a}(\mathbf{y}+\varepsilon\mathbf{z}, \mathbf{z}) \right| \le \int \frac{d\mathbf{z}}{(2\pi)^d} \sup_{\mathbf{x}} \left| \hat{a}(\mathbf{x}, \mathbf{z}) \right|,$$

the estimate (3.12) follows by Lemma 3.3. To show that (3.13) holds, we define a function $g(\mathbf{x})$ by its Fourier transform

$$\hat{g}(\mathbf{k}) = \frac{\hat{f}(\mathbf{k})}{(1+|\mathbf{k}|^2)^{s/2}}, \tag{3.17}$$

so that $\|f\|_{H^{-s}} = \|g\|_{L^2}$. Then

$$\begin{aligned} \varepsilon^s a(\mathbf{x}, \varepsilon D) f &= \varepsilon^s \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} a(\mathbf{x}, \varepsilon\mathbf{k}) (1+|\mathbf{k}|^2)^{s/2} \hat{g}(\mathbf{k}) \\ &= \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} a(\mathbf{x}, \varepsilon\mathbf{k}) (\varepsilon^2 + \varepsilon^2|\mathbf{k}|^2)^{s/2} \hat{g}(\mathbf{k}), \end{aligned}$$

and (3.13) follows by Schur's Lemma. The estimate (3.14) for integer s is obtained by differentiating (3.6) s times with respect to $\mathbf{x}$. □

We now show that the product of two semiclassical operators corresponds to the operator which is nearly the product of their symbols.

Lemma 3.5 *The product of two operators $a(\mathbf{x}, \varepsilon D)$, $b(\mathbf{x}, \varepsilon D)$ is*

$$b(\mathbf{x}, \varepsilon D) a(\mathbf{x}, \varepsilon D) = (ba)(\mathbf{x}, \varepsilon D) + \frac{\varepsilon}{i} (\nabla_{\mathbf{k}} b \cdot \nabla_{\mathbf{x}} a)(\mathbf{x}, \varepsilon D) + \varepsilon^2 Q_\varepsilon, \tag{3.18}$$

where the operators Q_ε are uniformly bounded on L^2. The product of the Weyl quantized operators is

$$b^w(\mathbf{x},\varepsilon D)a^w(\mathbf{x},\varepsilon D) = (ba)(\mathbf{x},\varepsilon D) + \frac{\varepsilon}{2i}\left[(\nabla_{\mathbf{k}} b\cdot\nabla_{\mathbf{x}} a)^w(\mathbf{x},\varepsilon D) - (\nabla_{\mathbf{x}} b\cdot\nabla_{\mathbf{k}} a)^w(\mathbf{x},\varepsilon D)\right] + \varepsilon^2 R_\varepsilon, \tag{3.19}$$

where the operators R_ε are uniformly bounded on L^2. One of the symbols a and b above may grow polynomially in $\mathbf{k}$.

Proof. We have

$$\begin{aligned} b(\mathbf{x},\varepsilon D)a(\mathbf{x},\varepsilon D)f &= \int \frac{d\mathbf{p}d\mathbf{q}e^{i\mathbf{x}\cdot(\mathbf{p}+\mathbf{q})}}{(2\pi)^{2d}} b(\mathbf{x},\varepsilon\mathbf{p}+\varepsilon\mathbf{q})\hat a(\mathbf{p},\varepsilon\mathbf{q})\hat f(\mathbf{q}) \\ &\approx \int \frac{d\mathbf{p}d\mathbf{q}e^{i\mathbf{x}\cdot(\mathbf{p}+\mathbf{q})}}{(2\pi)^{2d}} b(\mathbf{x},\varepsilon\mathbf{q})\hat a(\mathbf{p},\varepsilon\mathbf{q})\hat f(\mathbf{q}) + \varepsilon\int \frac{d\mathbf{p}d\mathbf{q}e^{i\mathbf{x}\cdot(\mathbf{p}+\mathbf{q})}}{(2\pi)^{2d}} \mathbf{p}\cdot\nabla_{\mathbf{k}} b(\mathbf{x},\varepsilon\mathbf{q})\hat a(\mathbf{p},\varepsilon\mathbf{q})\hat f(\mathbf{q}) \\ &= (ba)(\mathbf{x},\varepsilon D)f + \frac{\varepsilon}{i}(\nabla_{\mathbf{k}} b\cdot\nabla_{\mathbf{x}} a)(\mathbf{x},\varepsilon D)f. \end{aligned}$$

Here the hat denotes the Fourier transform in $\mathbf{x}$. The Taylor expansion above may be justified in a straighforward manner. The proof of (3.19) is similar if a bit more tedious. A detailed proof is given in the Appendix to [14]. □

The adjoint operators $a(\mathbf{x},\varepsilon D)^*$ are given by the next lemma.

Lemma 3.6 *The operators $a^w(\mathbf{x},\varepsilon D)$ and $a(\mathbf{x},\varepsilon D)$ have the following adjoints:*

$$[a^w(\mathbf{x},\varepsilon D)]^* = [a^*]^w(\mathbf{x},\varepsilon D) \tag{3.20}$$

and

$$a(\mathbf{x},\varepsilon D)^* = a^*(\mathbf{x},\varepsilon D) + \varepsilon R_\varepsilon \tag{3.21}$$

with R_ε uniformly bounded on L^2.

Proof. The proof is again by the Taylor expansion of the symbol and a straighforward estimate of the remainder. Detailes are left as an exercise for the reader. □

Proof of Proposition 3.1. We note that if the functions f_ε are uniformly bounded in L^2, then the L^2-bound (3.12), together with (3.7) and (3.5) show that W_ε has a weak limit μ in $\mathcal{S}'(R^d\times R^d)$ along a subsequence $\varepsilon_k\to 0$. The family $\widetilde{W_\varepsilon}$ has the same weak limits as W_ε, as $\varepsilon\to 0$, and if μ is a limit Wigner distribution of f_ε then

$$\lim_{\varepsilon\to 0}(a(\mathbf{x},\varepsilon D)f_\varepsilon, f_\varepsilon) = \langle a,\mu\rangle = \int a(\mathbf{x},\mathbf{k})\mu(d\mathbf{x}d\mathbf{k}) \tag{3.22}$$

with the limit taken along the subsequence ε_j corresponding to the subsequence W_{ε_j} converging weakly to μ. In order to show that all the limit points μ of the family W_ε are non-negative measrues, it is sufficient to verify that

$$\langle a,\mu\rangle \geq 0$$

for all test functions $a(\mathbf{x}, \mathbf{k})$ of the form $a(\mathbf{x}, \mathbf{k}) = |b(\mathbf{x}, \mathbf{k})|^2$ because those are dense in the set of positive test functions. Then (3.20) and the product rule (3.19) implies that for such test functions

$$\langle a, \mu \rangle = \lim_{\varepsilon \to 0} (a^w(\mathbf{x}, \varepsilon D) f_\varepsilon, f_\varepsilon) = \lim_{\varepsilon \to 0} (b^w(\mathbf{x}, \varepsilon D) f_\varepsilon, b^w(\mathbf{x}, \varepsilon D) f_\varepsilon) \geq 0, \tag{3.23}$$

where the limit is along a convergent subsequence. □

Another important property is that the Wigner measure is a local notion in space.

Lemma 3.7 (Localization) *Let $f_\varepsilon(\mathbf{x})$ be a family of uniformly bounded functions in L^2 and let $\mu_f(\mathbf{x}, \mathbf{k})$ be any limit Wigner measure. Let $\phi(\mathbf{x})$ be a smooth function. Then the Wigner measure of the family $g_\varepsilon(\mathbf{x}) = \phi(\mathbf{x}) f_\varepsilon(\mathbf{x})$ is $|\phi(\mathbf{x})|^2 \mu_f(\mathbf{x}, \mathbf{k})$. Moreover, let f_ε, g_ε be two uniformly bounded families of L^2 functions which coincide in an open neighbourhood of a point $\mathbf{x}_0$. Then any limit Wigner measures μ_f and μ_g coincide in this neighborhood.*

Proof. Let $a(\mathbf{x}, \mathbf{k})$ be a test function compactly supported in $\mathbf{x}$, such that its Fourier transform in $\mathbf{k}$ is compactly supported. Then

$$\begin{aligned} &(a(\mathbf{x}, \varepsilon D) g_\varepsilon, g_\varepsilon) - (|\phi(\mathbf{x})|^2 a(\mathbf{x}, \varepsilon D) f_\varepsilon, f_\varepsilon) \\ &= \int d\mathbf{x} d\mathbf{z} f_\varepsilon^*(\mathbf{x}) \phi^*(\mathbf{x}) \hat{a}(\mathbf{x}, \mathbf{z}) (\phi(\mathbf{x} + \varepsilon \mathbf{z}) - \phi(\mathbf{x})) f_\varepsilon(\mathbf{x} + \varepsilon \mathbf{z}) \to 0 \end{aligned}$$

as $\varepsilon \to 0$. This proves the first statement which implies in turn the second statement. □

The localization property is quite useful because it allows the consideration of Wigner measures for families of functions f_ε that are uniformly bounded in L^2_{loc}.

Another useful and intuitively clear property is that the Wigner measure of waves going in different directions is the sum of the individual Wigner measures.

Lemma 3.8 (Orthogonality) *Let f_ε, g_ε be two families of functions with Wigner measures μ_f and μ_g, which are mutually singular. Then the Wigner measure of the sum $f_\varepsilon + g_\varepsilon$ is $\mu_f + \mu_g$.*

Proof. Let $a(\mathbf{x}, \mathbf{k})$ be a positive test function of the form $a(\mathbf{x}, \mathbf{k}) = |b(\mathbf{x}, \mathbf{k})|^2$, then by the product rule (3.18) and the Schwartz inequality we have

$$\begin{aligned} |(a(\mathbf{x}, \varepsilon D) f_\varepsilon, g_\varepsilon)| &= |(b(\mathbf{x}, \varepsilon D) f_\varepsilon, b(\mathbf{x}, \varepsilon D) g_\varepsilon)| + O(\varepsilon) \\ &\leq |(b(\mathbf{x}, \varepsilon D) f_\varepsilon, b(\mathbf{x}, \varepsilon D) f_\varepsilon)|^{1/2} |(b(\mathbf{x}, \varepsilon D) g_\varepsilon, b(\mathbf{x}, \varepsilon D) g_\varepsilon)|^{1/2} + O(\varepsilon) \\ &\to \; < a, \mu_f >^{1/2} < a, \mu_g >^{1/2} . \end{aligned} \tag{3.24}$$

Then, since μ_f and μ_g are mutually singular, we can split $a(\mathbf{x}, \mathbf{k})$ into a sum of three terms:

$$a = a_1 + a_2 + a_3, \tag{3.25}$$

so that a_1 is orthogonal to μ_f, a_2 is orthogonal to μ_g, and the integral of a_3 with respect to both measures can be made arbitrarily small. Then the right side of (3.24) is arbitrarily small and thus its left side goes to zero in the limit $\varepsilon \to 0$. □

The above properties: positivity, orthogonality and localization show that the Wigner measure may be indeed reasonably interpreted as the phase space energy density. We now show that it is well suited for families of functions that depend on a small parameter in an oscillatory manner, the ε-oscillatory families of [13]. The ε-oscillatory property is conveniently characterized by the following definition.

Definition 3.9 *A family of functions f_ε that is bounded in L^2_{loc} is said to be ε-oscillatory if for every smooth and compactly supported function $\phi(\mathbf{x})$*

$$\limsup_{\varepsilon\to 0}\int_{|\boldsymbol{\xi}|\geq R/\varepsilon} |\widehat{\phi f_\varepsilon}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \to 0 \quad as \quad R\to +\infty. \tag{3.26}$$

A simple and intuitive sufficient condition for (3.26) is that there exist a positive integer j and a constant C independent of ε such that

$$\varepsilon^j \left|\left| \frac{\partial^j f_\varepsilon}{\partial x^j}\right|\right|_{L^2_{loc}} \leq C. \tag{3.27}$$

This condition holds, for instance, for high frequency plane waves

$$f_\varepsilon(\mathbf{x}) = Ae^{i\boldsymbol{\xi}\cdot\mathbf{x}/\varepsilon}. \tag{3.28}$$

Another natural example of ε-oscillatory functions is

$$g_\varepsilon(\mathbf{x}) = g(\frac{\mathbf{x}}{\varepsilon}), \tag{3.29}$$

where $g(\mathbf{x})$ is a periodic function with bounded gradient.

The main reason for introducing ε-oscillatory functions is the following theorem concerning weak convergence of energy, i.e. of the integral of the square of the wave function.

Proposition 3.10 *Let f_ε be a bounded family in L^2_{loc} with limit Wigner measure $\mu(\mathbf{x},\mathbf{k})$. Then for any smooth function of compact support $\theta(\mathbf{x})$*

$$\iint |\theta(\mathbf{x})|^2\mu(d\mathbf{x},d\mathbf{k}) \leq \limsup_{\varepsilon\to 0}\int_{R^d} |\theta(\mathbf{x})f_\varepsilon(\mathbf{x})|^2 d\mathbf{x} \tag{3.30}$$

with equality holding if and only if f_ε is ε-oscillatory. In this case $\limsup$ *can be replaced by* $\lim$ *on the right side of (3.30).*

Proof. We present the proof from [14]. First we recall that given a fixed uniformly continuous funciton $\phi(\mathbf{x})$ the Wigner measure of a family $g_\varepsilon = \phi f_\varepsilon$ is given by $\mu[\phi f^\varepsilon] = |\phi|^2\mu[f^\varepsilon]$. Now choose $\psi \in C_0^\infty(\mathbb{R}^n)$, $0\leq\psi\leq 1$ with $\psi(\mathbf{k}) = 1$ in a neighborhood of $\mathbf{k}=0$. Then we have

$$\int \psi(\mathbf{k})|\phi(\mathbf{x})|^2 d\mu[f^\varepsilon] = \lim_{\varepsilon\to 0}\langle W_\varepsilon[\phi f^\varepsilon],\psi\rangle = \lim_{\varepsilon\to 0}\langle \psi(\varepsilon D)(\phi f^\varepsilon), \phi f^\varepsilon\rangle.$$

The Plancherel formula implies that the right side is bounded by

$$\lim_{\varepsilon\to 0}\|\phi f^\varepsilon\|^2 = \int |\phi|^2 d\nu.$$

Replacing ψ by $\psi(\mathbf{k}/R)$ and letting $R\to\infty$ we obtain the inequality

$$\int \mu(d\mathbf{k}) \leq \nu.$$

In order to investigate the equality we write

$$\int \psi(\mathbf{k}/R)|\phi(\mathbf{x})|^2 d\mu[f^\varepsilon] = \int |\phi|^2 d\nu - \lim_{\varepsilon\to 0}\langle(1-\psi(\varepsilon D/R))(\phi f^\varepsilon), \phi f^\varepsilon\rangle.$$

Passing to hte limit as $R\to\infty$ we conclude that $\int \mu(d\mathbf{k}) = \nu$ if and only if f^ε is an ε-oscillatory family. □

With this proposition and the positivity property we can interpret $\mu(\mathbf{x},\mathbf{k})$ as a phase space energy density, that is, energy density resolved over directions and wavenumbers.

4 The evolution of the Wigner transform

The formalism of semiclassical pseudodifferential operators provides a fast and elegant way to derive the Wigner transport equation (2.16). Let ϕ^ε satisfy the Schrödinger equation

$$i\varepsilon\frac{\partial\phi^\varepsilon}{\partial t} + \frac{\varepsilon^2}{2}\Delta\phi^\varepsilon - V(\mathbf{x})\phi^\varepsilon = 0 \tag{4.1}$$

with the initial data $\phi_0^\varepsilon(\mathbf{x})$ with the corresponding Wigner measure $W_0(\mathbf{x},\mathbf{k})$. Then for any test function $a(\mathbf{x},\mathbf{k})\in\mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d)$ we have, using (4.1) and the product formula (3.18):

$$\begin{aligned}\langle a, \frac{\partial W}{\partial t}\rangle &= \lim_{\varepsilon\to 0}\left[(a(\mathbf{x},\varepsilon D)\phi^\varepsilon, \frac{\partial\phi^\varepsilon}{\partial t}) + (a(\mathbf{x},\varepsilon D)\frac{\partial\phi^\varepsilon}{\partial t}, \phi^\varepsilon)\right]\\ &= \lim_{\varepsilon\to 0}\frac{1}{i\varepsilon}([\varepsilon^2\Delta - V(\mathbf{x}), a(\mathbf{x},\varepsilon D)]\phi^\varepsilon,\phi^\varepsilon) = \langle a, -\mathbf{k}\cdot\nabla_{\mathbf{x}}W + \nabla_{\mathbf{x}}V\cdot\nabla_{\mathbf{k}}W\rangle,\end{aligned}$$

which is the same as (2.16).

We now address more general equations than the linear Schrödinger equation. We begin with systems of equations with constant coefficients of the form

$$\begin{aligned}&\varepsilon\frac{\partial u}{\partial t} + P(\varepsilon D)u^\varepsilon = 0\\ &u^\varepsilon(t=0) = u_0^\varepsilon\end{aligned} \tag{4.2}$$

with u^ε being a $\mathbb{C}$-valued function. The operator $P(\varepsilon D)$ is associated with a multiplier $P(\mathbf{k})$. We assume that $P\in C^1(\mathbb{R}^d)$ and $P^*(\mathbf{k}) = -P(\mathbf{k})$. We impose the following conditions: thereexists a closed set $F\in\mathbb{R}^d_{\mathbf{k}}$ so that for all $\mathbf{k}\notin F$ the eigenvalues $\lambda_q(\mathbf{k})$ may be ordered as

$$\lambda_1(\mathbf{k}) < \cdots < \lambda_p(\mathbf{k})$$

with the multiplicities r_q independent of $\mathbf{k}$.

The initial data u_0^ε is uniformly bounded in $L^2(\mathbb{R}^d)$, ε-oscillatory and compact at infinity. The total energy $n_\varepsilon(t,x) = |u^\varepsilon(t,x)|^2$ is conserved:

$$\int n^\varepsilon(t,x) = \int n_0^\varepsilon(t,x).$$

We assume that $\mathrm{Tr}W_0$ vanishes on the exceptional set F defined above.

Theorem 4.1 *Under the above assumptions the energy density $n^\varepsilon(t,\mathbf{x})$ converges weakly to the measure $n^0(t,\mathbf{x})$ given by*

$$n^0(t,\mathbf{x}) = \sum_{q=1}^{p} \int_{\mathbb{R}^d\setminus F} w_q^0(\mathbf{x} - t\nabla\lambda_q(\mathbf{k}), d\mathbf{k}). \tag{4.3}$$

Here $w_q^0(\mathbf{x},\mathbf{k}) = Tr(\Pi_q W_0)(\mathbf{x},\mathbf{k})$, where $\Pi_q(\mathbf{k})$ is the orthogonal projection onto the eiegnspace corresponding to $\lambda_q(\mathbf{k})$.

Proof. The Fourier transform gives

$$\varepsilon\hat{u}_t^\varepsilon + P(\varepsilon\mathbf{k})\hat{u}^\varepsilon = 0.$$

We now set

$$\hat{u}_q^\varepsilon(t,\mathbf{k}) = \Pi_q(\varepsilon\mathbf{k})\hat{u}^\varepsilon(t,\mathbf{k})\chi(\varepsilon\mathbf{k} \notin F)$$

and obtain the initial value problem

$$\begin{aligned}&\varepsilon\frac{\partial\hat{u}_q^\varepsilon}{\partial t} + i\lambda_q(\varepsilon\mathbf{k})\hat{u}_q^\varepsilon = 0\\ &\hat{u}_q^\varepsilon(t=0,\mathbf{k}) = u_{0,q}^\varepsilon(\mathbf{k}) := \Pi_q(\varepsilon\mathbf{k})\hat{u}_0^\varepsilon(\mathbf{k})\chi(\varepsilon\mathbf{k}\notin F).\end{aligned}$$

Define the scalar Wigner transforms

$$w_q^\varepsilon(t,\mathbf{x},\mathbf{k}) = W^\varepsilon[u_q^\varepsilon(t)](\mathbf{x},\mathbf{k}).$$

The function w_q^ε satisfies an evolution equation

$$\frac{\partial\hat{w}_q^\varepsilon}{\partial t} + i\zeta\int_{-1/2}^{1/2}\nabla\lambda_q(\mathbf{k}+\varepsilon s\zeta)ds\hat{w}_q^\varepsilon = 0 \tag{4.4}$$

with

$$\hat{w}_\varepsilon(,\zeta,\mathbf{k}) = \frac{1}{(2\pi\varepsilon)^d}\bar{\hat{u}}_q^\varepsilon\left(t,\frac{\mathbf{k}}{\varepsilon}-\frac{\zeta}{2}\right)\cdot\hat{u}_q^\varepsilon\left(t,\frac{\mathbf{k}}{\varepsilon}+\frac{\zeta}{2}\right).$$

Lemma 4.2 *The family w_q^ε converges locally uniformly with respect to t to the solution of*

$$\frac{\partial w_q}{\partial t} + \nabla\lambda(\mathbf{k})\cdot\nabla_{\mathbf{x}}w_q = 0 \tag{4.5}$$

with the initial data $w_q(t=0) = w_q^0$.

Proof. The proof is immediate from (4.4). □

In order to finish the proof of Theorem 4.1 we simply observe that

$$w^\varepsilon = \sum_q \mathrm{Tr}(\Pi_q W^\varepsilon \Pi_q).$$

Now observe that the evolution (4.5) preserves the total mass and thus solution remains ε-oscilatory and hence everything converges as it should. □

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