EXPLORING THE EFFECTS OF SOCIAL PREFERENCE, ECONOMIC DISPARITY, AND HETEROGENEOUS ENVIRONMENTS ON SEGREGATION

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Abstract. It is believed that social preference, economic disparity, and heterogeneous environments are mechanisms for segregation. However, it is difficult to unravel the exact role of each mechanism in such a complex system. We introduce a versatile, simple and intuitive particle-interaction model that allows to easily examine the effect of each of these factors. It is amenable to numerical simulations, and allows for the derivation of the macroscopic equations. As the population size and number of groups with different economic status approach infinity, we derive various local and non-local system of PDEs for the population density. Through the analysis of the continuous limiting equations, we conclude that social preference is a necessary but not always sufficient mechanism for segregation. On the other hand, when combined with the environment and economic disparity (which on the their own also do not cause segregation), social preference does enhance segregation.

Key words.

subject classifications.

1. Introduction

The separation along the lines of age, race, religious affiliation, and social status, to name a few, is a ubiquitous phenomenon that has been observed throughout time and cultures [1, 3, 24]. It appears that social segregation is a fundamental characteristic of human nature, and yet there is still much we do not understand of the mechanisms behind it, in spite of the vast literature on this subject – see, for example, [11, 12, 19, 23, 24, 27] and references within. Much of this research was sparked by the pioneering work of Shelling in [24], which made the case that social preference alone facilitates segregation by studying simple mathematical models. One issue here is the difficulty to model social preferences. They can vary dramatically across cultures, generations, and even neighborhoods within a city, and although there is a large literature in this area, the results are often inconclusive [4]. For example, some research shows that misery likes company, – in other words, people like to be around people who are less well-off than themselves [16]. On the other hand, given the current trends in income distribution in some cities, it is clear that residential areas are, generally, segregated along income lines [10]. This motivates the need to go beyond social preferences and consider economic disparity, along with heterogeneous environments, where certain locations are more desirable than others, as reasons for social segregation. It is clear that the choice of where people live is highly influenced by location, housing density, reputation of neighborhoods, amenities, and security [9, 13, 26] but is limited by their economic power. Recently, [11] provided a mathematical framework for this phenomenon. The agent-based model considered in this reference involves individuals heterogeneous in their ability to pay for a certain residence, and included interactions between potential sellers and buyers. Through the use of numerical and linear stability analysis, it was concluded that social segregation was possible only if the social preference to be near people of similar or higher income was sufficiently

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strong. This model included market matching and the complicated dynamics prevented the authors from deriving a continuum model.

The objective of the present paper is to introduce a versatile model that can take into account different social factors with relative ease, as well as provide a flexibility to model the environment, such as city amenities, educational systems, public transportation, and highways, but is sufficiently simple to allow mathematical analysis. With this in mind, we introduce an interacting-particle model that enables us to model simple and intuitive rules of interaction. Similar interacting-particle systems are widely used physics – see, for example, [17], and have recently been popularized to model aggregation [6, 20, 21, 25].

We focus on three basic mechanisms believed to play a role in segregation: social preference, economic disparity, and a heterogeneous environment. We consider individuals of variable socio-economic status who navigate through a prescribed environment. The economic status is accounted for by the ease with which each individual navigates the environment – their *mobility*. In terms of social preference, we assume that individuals favor being surrounded by others with similar mobility. Hence, the role of mobility is two-fold: (1) it measures the ease with which a certain economic class can navigate an environment, and (2) it provides a mechanism of segregation through social preference. Interacting-particle systems provide a flexible framework and the above assumptions can be easily modified.

While individual behavior is important, certain macroscopic patterns arise when we observe the bulk behavior of the system. To study them, we derive the continuum limit of the particle-interaction model. This is another benefit of starting with interacting-particle models, – they allow us to derive the macroscopic models from first principles. We first study the case when the mobility is discrete – this is a crude way to describe the interaction between various social groups. This case leads to a system of partial differential equations of Keller-Segel type with repulsion – see, for example, [8] and references therein for other problems where such equations arise. Next, we consider the case when the mobility is continuous, leading to a reaction-advection-diffusion equation in $\mathbb{R}^{d+1}$ where d-dimensions come from physical space and the additional dimension represents the mobility space. The rigorous justification of the continuum models is a technical and non-trivial issue beyond the scope of this paper, and will appear elsewhere.

A benefit afforded by the continuum model is the ability to carry out mathematical analysis and gain insight into the global phenomenon of social segregation in a systematic way. Through the analysis of the system of PDEs derived here, we observe that in many cases social preference alone does not lead to social segregation. In particular, if the initial distribution is socially diverse, it will maintain its social diversity. On the other hand, mobility and the environment enhance segregation, but only when some social preference is present. Of course, the present work only touches the tip of the iceberg and there are many generalizations that need to be made in order to move toward realistic understanding of the full picture of social segregation.

The interaction potential. We assume that interaction between individuals is governed by an interacting potential. A system with N individuals whose interactions are governed by a non-negative potential $V_N(x) \geq 0$, $x \in \mathbb{R}^d$, leads to the system of evolution equations

$$\frac{d}{dt} x_N^k(t) = \frac{1}{N} \sum_{i=1, i\neq k}^{N} \nabla V_N(x_N^k(t) - x_N^j(t)) \text{ for } k = 1, \ldots, N. \tag{1.1}$$

In (1.1), $x_N^k(t)$ represents the physical position of individual k at time t. As the number of individuals increases, the strength and range of interaction can vary, a typical assumption is that the potential $V_N(x)$ is a rescaled version of a fixed potential $V_1(x)$:

$$V_N(x) = N^{\beta} V(N^{\beta/d} x),$$

where $\beta \in [0,1]$. The case $\beta \in (0,1)$ is known as the *mean-field limit* – the range of the interactions decreases as N grows, while the strength of the interactions increases. The case $\beta = 0$ is known as the *weakly-interacting limit* – while the range of interaction is large, the strength of the interactions is weak. The case $\beta = 1$ is known as the *hydrodynamic limit* and we do not discuss it here. Throughout this work we assume that $V(x)$ is a smooth and radially decreasing potential.

Notation: In what follows, we denote by Ω a bounded subset of $\mathbb{R}^d$ or all of $\mathbb{R}^d$. Also, for any $f \in C_0^2(\mathbb{R}^d)$ and probability measure μ we denote

$$\langle \mu, f \rangle = \int_{\Omega} f \, d\mu(x).$$

The mass of a function $h(x)$ will be denoted by

$$\int_{\Omega} h(x)\, dx = M[h]. \tag{1.2}$$

We will use $*$ to denote the standard convolution:

$$K * u(x) = \int_{\Omega} K(x-y) u(y)\, dy.$$

For any function $\rho(x)$ we denote by $\rho^{\#}(x)$ to be the symmetric non-increasing rearrangement of $\rho(x)$. We keep in mind that

$$\left|\left|\rho^{\#}(x)\right|\right|_p = ||\rho(x)||_p \text{ for } p \in [1,\infty) \tag{1.3}$$

and that for $G(x)$ symmetric and non-increasing

$$\int G(x-y) h(x) g(y)\, dx \le \int G(x-y) h^{\#}(x) g^{\#}(y)\, dx dy, \tag{1.4}$$

see for example [15].

Discrete economic status. We study two cases: (1) discrete set of mobilities which describes a population made only of individuals that are either disadvantaged or affluent – this setting could be easily modified to include any finite number of groups, and (2) continuous mobilities which describes the more realistic continuous spectrum of socio-economic statuses in a population.

Let $u(x,t), v(x,t) : \Omega \times [0,\infty) \to [0,\infty)$ represent the densities of the groups with mobility $\Gamma \gg 1$ and $\epsilon \ll 1$, respectively. The most general systems of partial differential equations we obtain as limiting models of the interacting-particle model are, in the non-local case:

$$u_t = \sigma \Delta u + \eta \Delta u^2 + \nabla \cdot (u \nabla G * (v-u)) + \Gamma \nabla \cdot (u \nabla A(x)) \text{ in } \Omega, \tag{1.5a}$$

$$v_t = \sigma \Delta v + \eta \Delta v^2 + \nabla \cdot (v \nabla G * (u-v)) + \epsilon \nabla \cdot (v \nabla A(x)) \text{ in } \Omega, \tag{1.5b}$$

$$u(x,0) = u_0(x) \ge 0 \text{ and } v(x,0) = v_0(x) \ge 0 \text{ for } t = 0, \tag{1.5c}$$

$$(\sigma \nabla u + \eta \nabla u^2 + u \nabla G * (v-u) + \Gamma \nabla A(x)) \cdot n = 0 \text{ on } \partial\Omega, \tag{1.5d}$$

$$(\sigma \nabla v + \eta \nabla v^2 + v \nabla G * (u-v) + \epsilon \nabla A(x)) \cdot n = 0 \text{ on } \partial\Omega. \tag{1.5e}$$

where $\eta,\sigma \geq 0$ are the (linear and nonlinear, respectively) diffusivities and n is the outward normal at the boundary of Ω; and in the local case:

$$u_t = \sigma\Delta u + \lambda_1 \Delta u^2 + \lambda_2 \nabla\cdot(u\nabla v) + \Gamma\nabla\cdot(u\nabla A(x)) \text{ in } \Omega, \tag{1.6a}$$

$$v_t = \sigma\Delta v + \lambda_1 \Delta v^2 + \lambda_2 \nabla\cdot(v\nabla u) + \epsilon\nabla\cdot(v\nabla A(x)) \text{ in } \Omega, \tag{1.6b}$$

$$u(x,0) = u_0(x) \geq 0 \text{ and } v(x,0) = v_0(x) \geq 0 \text{ for } t=0, \tag{1.6c}$$

$$(\sigma\nabla u + \lambda_1\nabla u^2 + \lambda_2 u\nabla v + \Gamma u\nabla A(x))\cdot n = 0 \text{ on } \partial\Omega, \tag{1.6d}$$

$$(\sigma\nabla v + \lambda_1\nabla v^2 + \lambda_2 v\nabla u + \epsilon v\nabla A(x))\cdot n = 0 \text{ on } \partial\Omega. \tag{1.6e}$$

with $\lambda_1,\lambda_2 > 0$. In the application we are considering, it is suitable to consider non-negative initial data $u_0(x)$ and $v_0(x)$. The no-flux boundary conditions imply that the total number of individuals in the city is preserved in time.

There are two diffusive terms in (1.5a) and (1.5b). The linear diffusion comes from the unpredictable human behavior (modeled as random noise) and the nonlinear diffusion comes from individuals' needs to have some personal space (leading to an overcrowding effect). The convolution term models the long-range attraction to individuals of the same group and long-range repulsion from individuals from the other group, with the function $G(x)$ as the governing interacting potential. The prescribed spatially heterogeneous environment is given by the scalar field, $A(x)$, and the population densities are advected by the gradient of this scalar field with velocity proportional to Γ for $u(x,t)$ and ϵ for $v(x,t)$. The assumptions $\Gamma \gg 1$ and $\epsilon \ll 1$ reflect the disparity between the populations with densities $u(t,x)$ and $v(t,x)$.

The system (1.5) preserves the mass and non-negativity, that is

$$\int_\Omega u(x,t)\,dx = M[u_0(x)] \text{ and } \int_\Omega v(x,t)\,dx = M[v_0(x)], \tag{1.7}$$

and

$$u(x,t), v(x,t) \geq 0$$

for all $x \in \Omega$ and $t > 0$. This system is, formally, the gradient flow of the free energy

$$\mathcal{F}(t) := \mathcal{E}(t) + \mathcal{W}(t) + \mathcal{S}(t) + \mathcal{L}(t). \tag{1.8}$$

Here, the *entropy*, which comes from the two dispersal mechanisms, is

$$\mathcal{E}(t) := \int_\Omega \eta\left(u^2 + v^2\right) + \sigma(u\log u + v\log v)\,dx,$$

the *interaction energy*, which comes from the long-range attraction within groups, is

$$\mathcal{W}(t) := -\frac{1}{2}\int_\Omega\int_\Omega G(x-y)(u(x,t)u(y,t) + v(x,t)v(y,t))\,dxdy,$$

the *segregation energy*, which describes the long-range repulsion between groups, is

$$\mathcal{S}(t) := \int_\Omega\int_\Omega u(x,t)G(x-y)v(y,t)\,dxdy,$$

and the *environment energy*, which comes from the environment landscape, is

$$\mathcal{L}(t) := \int_\Omega A(x)\left(\Gamma u(x,t) + \epsilon v(x,t)\right)dx.$$

The segregation energy, $\mathcal{S}(t)$, provides an explicit measure of how segregated the system is. A state is completely segregated if

$$\int_\Omega \int_\Omega u(x,t)G(x-y)v(y,t)\,dxdy = 0, \tag{1.9}$$

that is, if the two groups do not interact at all, minimizing the segregation energy. This is the case when either $u(x,t)$ or $v(x,t)$ are exactly equal to zero or have disjoint support. If $G(x)$ has compact support then the distance between the support of $u(x,t)$ and $v(x,t)$ simply has to be larger than the range of interaction of $G(x)$. Our first observation is that system (1.5) dissipates the total energy.

Proposition 1.1 (Formal energy dissipation). *Let $u(x,t)$ and $v(x,t)$ be solutions to* (1.5) *then*

$$\mathcal{F}(t) + \int_0^t \mathcal{D}(s)\,ds \le \mathcal{F}(0), \forall\, t>0, \tag{1.10}$$

where the dissipation is

$$\begin{aligned}\mathcal{D}(s) = &\int_\Omega u\left|\nabla(2\eta u + \sigma \log u + G*(v-u) + \Gamma A(x))\right|^2 dx \\ &+ \int_\Omega v\left|\nabla(2\eta v + \sigma \log v + G*(u-v) + \epsilon A(x))\right|^2 dx. \end{aligned}\tag{1.11}$$

To analyze the different roles of social preference, the environment, and economic disparity we consider the case when dispersal is due only to Brownian motion, *i.e.* $\eta = 0$. We first note that if economic disparity is not included in the model then a population which is initially socially diverse will remain diverse for all time.

Proposition 1.2 (Lack of segregation). *Let $\sigma > 0$, $\eta = 0$, and $\Gamma = \epsilon = 0$. If $u(x,t)$ and $v(x,t)$ are solutions to* (1.5) *with initial conditions $u_0(x) = v_0(x)$ then $u(x,t) = v(x,t)$ for all $(x,t) \in \Omega \times (0,\infty)$.*

More can be said in the case when diffusion dominates in the system, linearizing (1.5), with $\sigma > 0$, $\eta = 0$, and $\Gamma = \epsilon = 0$, around the states $u \equiv v \equiv 1$. This gives the linear system

$$u_t = \sigma \Delta u + \Delta G * (v-u), \tag{1.12a}$$
$$v_t = \sigma \Delta v + \Delta G * (u-v). \tag{1.12b}$$

System (1.12) is linearly stable when

$$\sigma > 2\hat{G}(\xi), \tag{1.13}$$

for all $\xi > 0$ and so we expect that, when (1.13) is satisfied, the "perfectly mixed" state $u \equiv v \equiv 1$ is nonlinearly stable, so that there will be social diversity in the long run, rather than segregation.

Theorem 1.3 (Preservation of social diversity). *Let $u(x,t), v(x,t)$ be solutions to system* (1.5) *with $\sigma, \eta \ge 0$, $\Gamma = \epsilon = 0$. Furthermore, assume that the initial conditions $u_0(x)$ and $v_0(x)$ satisfy*

$$M[u_0(x)] = M[v_0(x)] = M. \tag{1.14}$$

Then, for $\sigma > C(|\Omega|, G)$ sufficiently large the following bound holds

$$||u-v||_{L^2}^2 + ||u+v-\bar{s}||_{L^2}^2 \leq \left(||u_0(x)-v_0(x)||_{L^2}^2 + ||u_0(x)+v_0(x)-\bar{s}||_{L^2}^2 \right) e^{-Ct}, \tag{1.15}$$

where $C(\sigma, ||\Delta G||_{L^\infty(\Omega)}, M)$ and

$$\bar{s} := \frac{1}{|\Omega|} \int_\Omega (u+v)\, dx = \frac{2M}{|\Omega|}. \tag{1.16}$$

The estimate (1.15) implies that in the long time limit we expect that the population density of affluent individuals is the same as that of the disadvantaged individuals throughout the city. On the other hand, when (1.13) is violated we expect (at least, partially) segregated states in the long time limit.

Condition (1.14) is not necessary and is solely made to simplify the statement and proof of the theorem. In general, if

$$M[u_0(x)] = M_u \text{ and } M[v_0(x)] = M_v,$$

then the following estimate holds,

$$||u-v-\bar{s}_1||_{L^2}^2 + ||u+v-\bar{s}||_{L^2}^2 \leq \left(||u_0(x)-v_0(x)-\bar{s}_1||_{L^2}^2 + ||u_0(x)+v_0(x)-\bar{s}||_{L^2}^2 \right) e^{-Ct},$$

with

$$\bar{s}_1 = \frac{1}{|\Omega|} \int_\Omega (u-v)\, dx.$$

The next observation is that in an environment where the resources or amenities in a city are non-uniform, such as in mono-centric cities, disparity in mobility leads to segregation: if $\Gamma \gg 1$ and $\epsilon \ll 1$, then $u(x,t)$ will be concentrated in areas where $A(x)$ is large and $v(x,t)$ will be concentrated in areas where $u(x,t)$ is low. We first state a more general result for the local system (1.6).

Theorem 1.4 (Segregated steady state for the local system). *Let $A(x) \in L^\infty(\Omega)$, $\lambda_1 = 0$, and $u_0(x), v_0(x) \in L^1(\Omega)$. There exist two positive constants, c_1, c_2, and continuous functions, $\bar{u}(x)$ and $\bar{v}(x)$, which are positive and satisfy*

$$\bar{u}(x) = c_1 \exp\left\{ -\frac{1}{\sigma} (\Gamma A(x) + \lambda_2 \bar{v}(x)) \right\}, \tag{1.17a}$$

$$\bar{v}(x) = c_2 \exp\left\{ -\frac{1}{\sigma} (\epsilon A(x) + \lambda_2 \bar{u}(x)) \right\}, \tag{1.17b}$$

that are steady-state solutions of system (1.6)*. Additionally, the following holds*

$$\int_\Omega \bar{u}(x)\, dx = M[u_0(x)] \quad \textit{and} \quad \int_\Omega \bar{v}(x)\, dx = M[v_0(x)]. \tag{1.17c}$$

Next, we state the corresponding result for the non-local system for potentials that are close to the delta kernel. For a given potential $G(x)$ we define

$$G_\delta(x) := \frac{1}{\delta^d} G\left(\frac{1}{\delta} x \right).$$

Theorem 1.5 (Segregated steady state for non-local system). *Let $A(x)\in C^1(\Omega)$, $G(x)\in L^1(\mathbb{R}^d)$, $\eta=0$, and $u_0(x),v_0(x)\in L^1(\Omega)$. There exists a $\delta_0>0$ such that for $\delta<\delta_0$ there exist constants, c_1,c_2, and continuous functions, $u_\delta(x)$ and $v_\delta(x)$, which are positive and satisfy*

$$u(x)=c_1\exp\left\{-\frac{1}{\sigma}\left(\Gamma A(x)+G_\delta*(v(x)-u(x))\right)\right\}, \tag{1.18a}$$

$$v(x)=c_2\exp\left\{-\frac{1}{\sigma}\left(\epsilon A(x)+G_\delta*(u(x)-v(x))\right)\right\}, \tag{1.18b}$$

and are steady-state solutions of system (1.5)*. Furthermore, the following hold*

$$\int_\Omega u_\delta(x)\,dx=M[u_0(x)] \quad \text{and} \quad \int_\Omega v_\delta(x)\,dx=M[v_0(x)]. \tag{1.18c}$$

From (1.17) we observe that social preference is necessary for segregation but not sufficient. Indeed, without the social preference, when $G=0$, the ground states would have the form

$$u(x)=c_1\exp\left\{-\frac{\Gamma}{\sigma}A(x)\right\}, \tag{1.19a}$$

$$v(x)=c_2\exp\left\{-\frac{\epsilon}{\sigma}A(x)\right\}, \tag{1.19b}$$

and would have qualitatively similar profiles. The proofs of Theorem 1.4 and Theorem 1.5 rely on the fact that the dispersal is due only to random noise. However, we can explore the existence of steady-state solutions in $\mathbb{R}^d$ when the dispersal is due to the over-crowing effect, which leads to the degenerate diffusion, through the use of the energy functional (1.8). In this direction we obtain some non-existence results. Since (1.5) is formally a gradient flow of (1.8) we expect that minimizers of the free energy functional are steady state solutions of (1.5). As the parabolic system (1.5) conserves mass it is reasonable to look for minimizers in the set

$$\mathcal{Y}_{M_u,M_v}:=\left\{(u,v)\in(L^1_+(\mathbb{R}^d)\times L^1_+(\mathbb{R}^d))\cap(L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)):||u||_1=M_u,||v||_1=M_v\right\}, \tag{1.20}$$

for $M_u,M_v>0$.

Proposition 1.6. *(Stationary solutions via energy minimization) Let $(u,v)\in\mathcal{Y}_{M_u,M_v}$ be a minimizer of $\mathcal{F}[u,v]$ then (u,v) satisfies*

$$\nabla(2\eta u+G*(v-u))=0,$$
$$\nabla(2\eta v+G*(u-v))=0,$$

a.e. in $x\in\mathbb{R}^d$.

Proposition 1.7. *(Regularity of steady-states) Let $(u_0(x),v_0(x))\in\mathcal{Y}_{M_u,M_v}$ then the solutions $(u(x,t),v(x,t))$ to* (1.5) *with $\sigma=\Gamma=\epsilon=0$ satisfy*

$$\int u|\nabla u|^2+v|\nabla v|^2\,dx<\infty,\ a.e\ t>0. \tag{1.21}$$

In particular, any steady-state solutions to (1.5) *with $\sigma=\Gamma=\epsilon=0$, (u,v), satisfies* (1.21) *and both u and v are C^2 on their support.*

A formal analysis of (1.8) gives us hints at the behavior of solutions to system (1.5) and also to the existence of minimizers in $\mathcal{Y}_{M_u,M_v}$. The choice of interaction potential affects the system in two key ways: the strength of its pull at the origin and its decay as $|x|\to\infty$. The former provides a measure of the risk of finite-time blow up of the solutions. The latter provides a pull of mass when it is trying to escape to $\pm\infty$. In other words, if the pull is too weak (and we can additionally rule out finite-time blow up) we expect that no minimizers will exist.

Our choice of interacting potentials turns (1.5) into a diffusion dominated system. Indeed, consider what happens to the energy as we scale $u(x)$ and $v(x)$ function into a delta function:

$$u_\lambda = \lambda^d u(\lambda x) \,\text{and}\, v_\lambda = \lambda^d v(\lambda x).$$

We obtain the rescaled energy,

$$\mathcal{F}[u_\lambda, v_\lambda] = \lambda^d\Big(||u||_2^2 + ||v||_2^2\Big) - \frac{1}{2}\Big(\int G\left(\frac{x-y}{\lambda}\right)(u(x)u(y) + v(x)v(y) - 2u(x)v(y))dxdy\Big)$$
$$\geq \lambda^d\left(||u||_2^2 + ||v||_2^2\right) - ||G||_\infty (M_u^2 + M_v^2).$$

Taking the limit as $\lambda\to\infty$ gives $\mathcal{F}[u_\lambda, v_\lambda]\to\infty$. From this heuristic, we expect the solutions to exists globally and the natural question to ask is if the solutions weak* converge to zero or if there exists a nontrivial ground state. Next, let us analyze what happens to the energy as the mass of a sequence in $\mathcal{Y}_{M_u,M_v}$ spreads out and the center of mass of u and v shift further away from each other. Take $(u(x), v(x)) \in \mathcal{Y}_{M_u,M_v}$ and consider the mass invariant scaling with a shift

$$u_\lambda(x) = \lambda^d u(\lambda(x - x_\lambda)) \,\text{and}\, v_\lambda(x) = \lambda^d v(\lambda(x + x_\lambda)),$$

where $\{x_\lambda\}_{\lambda>0}$ is a sequence in $\mathbb{R}^d$ with $\lim_{\lambda\to 0}|x_\lambda| = \infty$. This time we are interested in what happens when $\lambda\to 0$,

$$\mathcal{F}[u_\lambda, v_\lambda] = \lambda^d(||u||_2^2 + ||v||_2^2) - \frac{\lambda^d}{2}\int\int \frac{1}{\lambda^d} G\left(\frac{x-y}{\lambda}\right)(u(x)u(y) + v(x)v(y))\,dxdy$$
$$+ \lambda^d \int\int \frac{1}{\lambda^d} G\left(\frac{x-y}{\lambda} + 2x_\lambda\right) u(x)v(y)\,dxdy.$$

Note that as $\lambda\to 0$ the second term scales as $\frac{\lambda^d}{2}||G_1||_1(||u||_2^2 + ||v||_2^2)$. Thus, if $||G_1||_1 \leq 2$ the energy is always non-negative and we can find a suitable subsequence of $\mathcal{Y}_{M_u,M_v}$ that weak* converges to zero; thus, we conclude

$$\inf_{u,v\in\mathcal{Y}_{M_u,M_v}} \mathcal{F}[u,v] := I_{M_u,M_v} \leq 0.$$

Lemma 1.8 (Complete segregation). *Any non-trivial minimizer of* $\mathcal{F}[u,v]$, $(u^*, v^*) \in \mathcal{Y}_{M_u,M_v}$, *must have disjoint supports. In particular, if supp*$(G)\subset B_R(0)$ *then dist(supp*(u^*)*, supp*$(v^*)) \geq 2R$ *and*

$$S[u^*, v^*] = 0. \tag{1.22}$$

Theorem 1.9 (Nonexistence of minimizers). *Let* $G\in L^1(\mathbb{R}^d)$, $M_u, M_v > 0$. *If* $||G||_1 \leq 2\eta$ *or* G *has unbounded support there are no minimizers of* $\mathcal{F}[u,v]$ *in* $\mathcal{Y}_{M_u,M_v}$.

On a final note, making a rigorous connection between the interacting-particle model and the continuous system (1.5) requires the global well-posedness of (1.5). The global well-posedness result for weak-solutions, which is the best we can hope for when $\sigma=0$ due to the degenerate diffusion, is stated below. However, we leave the proof for a more technical paper in preparation.

THEOREM 1.10 (Global well-posedness). *Let $\sigma=0$ and $G(x)$ be admissible. Let $u_0(x),v_0(x)\in L^\infty(\Omega)$ be non-negative initial conditions. System* (1.5) *has weak solutions $u(x,t),v(x,t)\in L^\infty(\Omega\times(0,T))$ for any $T>0$.* Note that when $\sigma>0$ the solutions are classical.

Continuous income spectrum. So far, we have assumed that the mobility is discrete. The model can be easily extended to a continuous mobility spectrum. Starting from the interacting-particle model we not only take the limit as the number of individuals approach infinity, but also as the number of groups, each with a different mobility, approaches infinity. We will assume that the mobility y is an independent variable, and it is convenient to normalize it so that $y\in[0,1]$. It is helpful to define a *mobility threshold,* $\kappa\in[0,1]$, as a parameter that measures the line between attraction and repulsion based on mobility. In other words, an individual with mobility y_1 is attracted to an individual with mobility y_2 if $|y_1-y_2|\le\kappa$. If $|y_1-y_2|>\kappa$ then the individuals repulse each other. The interactions can be governed, for example, by the potential $H(y)G(x)$, where,

$$H(y)=\kappa|y|-1, \tag{1.23}$$

determines the attractive or repulsive nature of the interaction, and $G(x)$ determines the strength of the interaction due to the spatial distance between the individuals. The population density $u(x,y,t)$ at location $x\in\mathbb{R}^d$, with mobility $y\in[0,1]$ at time t, satisfies the following partial differential equation

$$\begin{aligned}\partial_t u(x,y,t)=&\eta\nabla_x\cdot(u(x,y,t)\nabla_x u(x,t))\\ &-\nabla_x\cdot(u(x,y,t)[\nabla_x([HG]*u(x,y,t))+y\nabla_x A(x)]),\end{aligned} \tag{1.24}$$

where ∇_x denotes the spatial gradient, $*$ represents convolution in both the x and y variables, and

$$u(x,t)=\int_0^1 u(x,y,t)\,dy. \tag{1.25}$$

Notice that the first term on the right-hand side of (1.24) is reminiscent of porous-media diffusion. Indeed, the equation is advecting $u(x,y,t)$ down gradients of $u(x,t)$, which provides a measure for the total population density. Equation (1.24), with no-flux boundary conditions, conserves mass and non-negativity and also rearranges the mobility.

Heterogeneous environments: crime, safety, and economic disparity. As we have mentioned, one benefit of the models we introduce in this work is the freedom to explore the effects of heterogeneous environments. As an example, we consider a scalar field $A(x,t)$ which measures the probability that a criminal act will occur at location x and time t. Through this environment we can explore various hypothesis related to the interconnection between *crime hotspots*, defined to be spatio-temporal areas of high density of criminal activity, and the local economic status of the population. In particular, our objective here is to find mechanisms that yield the formation of crime hotspots in low-income neighborhoods, a trend which has been observed recently [7,14]. While the role the government plays with regards to crime is not exactly clear, in our model we assume that its objective is to minimize the "fear of crime" [22]. Indeed, research done in this area points to the inability of individuals to rationally calculate the objective danger and in many circumstances the perceived danger is not well-founded. This can be observed, for example, in the fact that certain subpopulations tend to be more fearful of crime – see [22] and references within.

Given a distribution of the population at time t we consider an *influence field*, $I(x,t)$, which measures the economic power of a certain location at that time, and let $p(x,t)$ be the distribution of police resources at time t. The insecurity functional at time t, $U(x,t)$, depends on $I(x,t)$ and measures how unsafe the population feels. For example, under the assumption that people get used to crime, (people who live in a high-crime neighborhood are less phased by a single criminal activity whereas the insecurity of the people who live in a safe neighborhood will significantly increase with a single criminal activity) we obtain the following examples

$$U(p(x,t)) = \sqrt{\delta + \exp(-\alpha p(x,t)) I(x,t)} \text{ or } U(p(x,t)) = \sqrt{\delta + \exp(-\alpha p(x,t)) \exp(I(x,t))}, \tag{1.26}$$

where $\delta > 0$ is a parameter that measures the minimum amount of insecurity and $\alpha > 0$ measures the effectiveness of the police recourses. The police aims to minimize the functional

$$F(p) = \int_\Omega [\varepsilon |\nabla p(x,t)|^2 + U(p(x,t))]\, dx, \tag{1.27}$$

where the first term is a regularization to provide some smoothness. Taking into account the normalization due to a fixed total amount of police resources, the variational formulation for the distribution of the police resources, given the population income distribution $I(x,t)$ is:

$$\min_{p \geq 0} F(p), \tag{1.28a}$$

$$\text{subject to} \int_\Omega p(x,t) = 1. \tag{1.28b}$$

Given the optimal use of resources, $p^*(x,t)$, which satisfies (1.28), we assume an inverse relationship between $A(x,t)$ and $p^*(x,t)$. For example,

$$A(x,t) = \exp\{-\alpha p^*(x,t)\}.$$

From a prescribed distribution of the population we generate the influence field and given this field we then use a gradient decent scheme to solve (1.28). Numerically, we observe the development of "safe-havens," regions with very low values of $A(x,t)$, in ares where there is high economic power, refer to Figure 5.5b for an illustration.

Outline: In section 2 we introduce the interacting-particle system for two groups with distinct mobility and formally derive various systems of PDEs. Section 3 is devoted to the proofs of Theorem 1.3, Theorem 1.4, and Theorem 1.5. In section 4 we introduce the particle-interaction model for groups with continuous mobility and derive the formal PDE for this system. In section 5 we discuss and illustrate some numerical experiments, including a study of the distribution of the police resources based on the minimization procedure (1.28).

2. Discrete mobility model In this section we consider the case of discrete mobility, corresponding to the case when there is a discrete number of social classes.

2.1. Interacting particle model

We consider two distinct groups in the population that interact within groups and between groups in different ways. Group one has N_1 individuals, each with influence $\Gamma \gg 1$, and group two has N_2 individuals, each with influence $\epsilon \ll 1$. The dynamics of individuals are governed by three rules:

(A1) *overcrowding effect*: although individuals from the same group are attracted in the long-range, the desire for some personal space leads to local (short-range) repulsion.

(A2) *social preference segregation*: individuals are attracted to those in their own group and repulsed from those in the other group, leading to long-range attraction within-groups and repulsion between-groups.

(A3) *environment field* $A(x)$: individuals are attracted to either low or high values of the scalar field, $A(x)$, depending on what $A(x)$ represents. For example, if $A(x)$ represents the quality of the educational system then individuals have the preference to be in areas where $A(x)$ is high. On the other hand, if $A(x)$ represents the crime density then individuals have the preference to be in areas with small values of $A(x)$. Furthermore, we assume that the velocity of each individual is proportional to its mobility.

We assume that within group, short-range repulsion is governed by the interaction potential $V_N(x)$ and the long-range mobility segregation is governed by the interaction potential $G_N(x)$. Both $V_N(x)$ and $G_N(x)$ are admissible potentials of the form

$$V_N(x) = N^{\gamma_v} V_1(N^{\gamma_v/d} x) \text{ and } G_N(x) = N^{\gamma_g} G_1(N^{\gamma_g/d} x), \tag{2.1}$$

with $\gamma_v, \gamma_g \in [0,1)$. Recall that the scaling parameters, γ_v and γ_g determine the strength and range of the interactions as the number of individuals in the system increases. Combining these effects, we obtain the following system of ODEs for the positions $x_1^k(t)$

$$\begin{aligned} \frac{d}{dt} x_1^k(t) = &-\frac{1}{N_1} \sum_{j=1, j\neq k}^{N_1} \Big(\nabla V_{N_1}(x_1^k(t) - x_1^j(t)) - \nabla G_{N_1}(x_1^k(t) - x_1^j(t)) \Big) \\ &- \frac{1}{N_2} \sum_{j=1}^{N_2} \nabla G_{N_2}(x_1^k(t) - x_2^j(t)) - \Gamma \nabla A(x_1^k(t)) \text{ for } k = 1, \ldots, N_1, \end{aligned} \tag{2.2}$$

and $x_2^k(t)$:

$$\begin{aligned} \frac{d}{dt} x_2^k(t) = &-\frac{1}{N_2} \sum_{j=1, j\neq k}^{N_2} \Big(\nabla V_{N_2}(x_2^k(t) - x_2^j(t)) - \nabla G_{N_2}(x_2^k(t) - x_2^j(t)) \Big) \\ &- \frac{1}{N_2} \sum_{j=1}^{N_2} \nabla G_{N_1}(x_2^k(t) - x_2^j(t)) - \epsilon \nabla A(x_2^k(t)) \text{ for } k = 1, \ldots, N_2. \end{aligned} \tag{2.3}$$

The dynamics of both groups is identical except for the difference in mobility. This model can be easily extended to a general number of groups, say n. This would lead to n equations similar to (2.2) and (2.3), with different mobility parameters.

2.2. Continuum limit for the two-population model

The interacting-particle system (2.2)-(2.3) can be studied numerically, but insight can be gained from its formal continuum limit. We define the empirical measures for the two groups:

$$X_1(t)=\frac{1}{N_1}\sum_{i=1}^{N_1}\delta_{x_1^i(t)} \quad \text{and} \quad X_2(t)=\frac{1}{N_2}\sum_{i=1}^{N_2}\delta_{x_2^i(t)}. \tag{2.4}$$

Our goal now is to study the behavior of the empirical measures (2.4) and obtain the limiting equations for (2.2) and (2.3) as $N_1,N_2\to\infty$.

From a macroscopic perspective, we assume that there are functions $u_0(x),v_0(x)\in\mathbb{R}^d$ which represent the initial distribution of the two groups and $u(x,t),v(x,t)\in\mathbb{R}^d\times(0,\infty)$ which represent the density distribution of the groups at time t, that is:

$$\lim_{N_1\to\infty}\langle X_1(0),f\rangle=\langle u_0(\cdot),f\rangle \text{ and } \lim_{N_2\to\infty}\langle X_2(0),f\rangle=\langle v_0(\cdot),f\rangle, \tag{2.5}$$
$$\lim_{N_1\to\infty}\langle X_1(t),f\rangle=\langle u(\cdot,t),f\rangle \text{ and } \lim_{N_2\to\infty}\langle X_2(t),f\rangle=\langle v(\cdot,t),f\rangle,\ \forall t>0.$$

Taking the existence of these functions for granted, our goal is to find the equation for the macroscopic densities $u(x,t)$ and $v(x,t)$. It is convenient to set, with $x\in\mathbb{R}^d$, $y_1\in(\mathbb{R}^d)^{N_1}$, and $y_2\in(\mathbb{R}^d)^{N_2}$:

$$g(x,y_1,y_2)=\frac{1}{N_1}\sum_{j=1}^{N_1}\Big(V_{N_1}(x-y_1^j)-G_{N_1}(x-y_1^j)\Big)+\frac{1}{N_2}\sum_{j=1}^{N_2}G_{N_2}(x-y_2^j)+\Gamma A(x), \tag{2.6}$$

and

$$h(x,y_1,y_2)=\frac{1}{N_2}\sum_{j=1}^{N_2}\Big(V_{N_2}(x-y_2^j)-G_{N_2}(x-y_2^j)\Big)+\frac{1}{N_1}\sum_{j=1}^{N_1}G_{N_1}(x-y_1^j)+\epsilon A(x), \tag{2.7}$$

so that equations (2.2) and (2.3) become

$$\frac{d}{dt}x_1^k(t)=-\nabla g(x_1^k(t),x_1(t),x_2(t)) \quad k=1,\dots,N_1, \tag{2.8a}$$
$$\frac{d}{dt}x_2^k(t)=-\nabla h(x_2^k(t),x_1(t),x_2(t)) \quad k=1,\dots,N_2. \tag{2.8b}$$

System (2.8) is equivalent to (2.2) and (2.3) provided $\nabla V(0)=\nabla G(0)=0$. Now, for $f\in C_0^1(\mathbb{R}^d)$ we have

$$\langle X_i(t),f\rangle=\frac{1}{N_i}\sum_{k=1}^{N_i}f(x_i^k(t)) \quad \text{for} \quad i\in\{1,2\},$$

and taking the time derivative yields

$$\frac{d}{dt}\langle X_1(t),f\rangle=\frac{1}{N_1}\sum_{k=1}^{N_1}\nabla f\left(x_1^k(t)\right)\cdot\frac{d}{dt}x_1^k(t)=-\frac{1}{N_1}\sum_{k=1}^{N_1}\nabla f\left(x_1^k(t)\right)\cdot\nabla g\left(x_1^k(t),x_1(t),x_2(t)\right),$$
$$\frac{d}{dt}\langle X_2(t),f\rangle=-\frac{1}{N_2}\sum_{k=1}^{N_2}\nabla f\left(x_2^k(t)\right)\cdot\nabla h\left(x_2^k(t),x_1(t),x_2(t)\right).$$

This gives the integral formulation

$$\langle X_1(t),f\rangle = \langle X_1(0),f\rangle + \int_0^t \langle X_1(s), -\nabla g(\cdot,x_1(s),x_2(s))\cdot\nabla f(\cdot)\rangle\,ds, \tag{2.9a}$$

$$\langle X_2(t),f\rangle = \langle X_2(0),f\rangle + \int_0^t \langle X_2(s), -\nabla h(\cdot,x_1(s),x_2(s))\cdot\nabla f(\cdot)\rangle\,ds. \tag{2.9b}$$

For simplicity, we consider the case $d=1$, noting that the general dimension case follows the same procedure. In this case, we can write

$$\begin{aligned}
&\langle X_1(t), -g'(\cdot,x_1(t)x_2(t))f'(\cdot)\rangle = \\
&-\frac{1}{N_1^2}\sum_{i,j=1}^{N_1}\Big(V'_{N_1}(x_1^i(t)-x_1^j(t)) - G'_{N_1}(x_1^i(t)-x_1^j(t))\Big) f'(x_1^i(t)) \\
&\quad -\frac{1}{N_1}\sum_{i=1}^{N_1}\left(\frac{1}{N_2}\sum_{j=1}^{N_2} G'_{N_2}(x_1^i(t)-x_2^j(t)) + \Gamma A'(x_1^i(t))\right) f'(x_1^i(t)).
\end{aligned}$$

The type of limiting equations that we will obtain depends on the assumptions on the scaling parameters. Accordingly, we derive below two types of continuous models.

A non-local model. First, consider the case $\gamma_v\in(0,1)$ and $\gamma_g=0$, so that

$$V_N(x) = N^{\gamma_v}V_1(N^{\gamma_v/d}x)\,\text{and}\,G_N(x)=G_1(x),$$

"Physically", this means that individuals within the same group are repulsed locally and attracted long-range, describing the balance of individuals' social nature and the need for personal space. On the other hand, individuals are repulsed at a long-range from individuals belonging to another group. We make use of the *local equilibrium hypothesis*, which states that locally the individuals are distributed in a uniform manner – see [20,21]. In this context, this means that the average distance between individuals in group one close to a location x is $(N_1u(x,t))^{-1}$, and similarly for group two. As $\gamma_g=0$, the attractive term can be re-written as (we write $G(x)=G_1(x)$ for simplicity)

$$\frac{1}{N_1^2}\sum_{i,j=1}^{N_1} G'(x_1^i(t)-x_1^j(t))f'(x_1^i(t)) = \left\langle X_1(t), \frac{1}{N_1}\sum_{j=1}^{N_1} G'(\cdot - x_1^j(t))f'(\cdot)\right\rangle.$$

If the potential $G(x)$ decays sufficiently fast as $|x|\to\pm\infty$, after taking into account the local equilibrium hypothesis, the sum in the term above can be approximated by an integral:

$$\frac{1}{N_1}\sum_{j=1}^{N_1} G'(x-x_1^j(t)) \approx \int_{\mathbb{R}} G'(x-y)u(y,t)\,dy. \tag{2.10}$$

Similarly, the segregation term is

$$-\frac{1}{N_1}\sum_{i=1}^{N_1}\frac{1}{N_2}\sum_{j=1}^{N_2} G'(x_1^i(t)-x_2^j(t))f'(x_1^i(t) = \left\langle X_1(t), -\frac{1}{N_2}\sum_{j=1}^{N_2} G'(\cdot - x_2^j(t))f'(\cdot)\right\rangle,$$

and

$$-\frac{1}{N_2}\sum_{j=1}^{N_2}G'(x-x_2^j(t))\approx-\int_{\mathbb{R}}G'(x-y)v(y,t)\,dy. \tag{2.11}$$

As $\gamma_v\in(0,1)$, the repulsion within groups has to be treated differently from above. We use the fact that $V_N'(x)$ is odd, and that $V_N(x)$ is small for $|x|\ll N^{-\gamma_v}$, to perform a Taylor expansion on $f'(x_1^i(t))$:

$$\begin{aligned}\frac{1}{N_1^2}\sum_{i,j=1}^{N_1}V_{N_1}'(x_1^i(t)-x_1^j(t))f'(x_1^i(t))&=\frac{1}{2N_1^2}\sum_{i,j=1}^{N_1}(f'(x_1^i(t))-f'(x_1^j(t)))V_{N_1}'(x_1^i(t)-x_1^j(t))\\&\approx\frac{1}{2N_1^2}\sum_{i,j=1}^{N_1}f''(x_1^i(t))(x_1^i(t)-x_1^j(t)))V'(x_1^i(t)-x_1^j(t))\\&=\left\langle X_1(t),f''(\cdot)\frac{1}{2N_1}\sum_{j=1}^{N_1}V_{N_1}'(\cdot-x_1^j(t))(\cdot-x_1^j(t))\right\rangle.\end{aligned}$$

Let us denote $V_1(x)=V(x)$:

$$\frac{1}{2N_1}\sum_{j=1}^{N_1}V_{N_1}'(x-x_1^j(t))(x-x_1^j(t))\approx\frac{1}{2N_1}\sum_{j=1}^{N_1}V'\left(N_1^{\gamma_v}(x-x_1^j(t))\right)N_1^{\gamma_v}(x-x_1^j(t)).$$

Now, as $\gamma_v>0$, and $V(x)$ is decaying at infinity, we may look only near the point x in the above sum, where the density of the points x_1^j is $1/(Nu(t,x))$. This leads to

$$\begin{aligned}\frac{1}{2N_1}\sum_{j=-\infty}^{\infty}V_{N_1}'(x-x_1^j(t))(x-x_1^j(t)))\approx\frac{u(x,t)}{2}\int_{\mathbb{R}}V'(y)y\,dy=-\frac{u(x,t)}{2}\int V(y)\,dy\\=-M[V]\frac{u(x,t)}{2}.\end{aligned}$$

Thus, the contribution of the short-range repulsion is

$$\begin{aligned}\langle X_1(s),-V'(\cdot,x_1(s),x_2(s))f'(\cdot)\rangle&=\left\langle X_1(s),\frac{M[V]}{2}f''(\cdot)\right\rangle\\&=\left\langle X_1(s),-\frac{M[V]}{2}u'(x,t)f'(\cdot)\right\rangle.\end{aligned} \tag{2.12}$$

Finally, the advective term coming from the heterogeneous environment leads to

$$-\frac{1}{N_1}\sum_{i=1}^{N_1}\Gamma A'(x_1^i(t))f'(x_1^i(t))=-\langle X_1(s),f'(\cdot)\Gamma A'(\cdot)\rangle. \tag{2.13}$$

Combining (2.10)-(2.13), we obtain

$$\begin{aligned}&\langle X_1(t),-g'(\cdot,x_1(t),x_2(t))f'(\cdot)\rangle=\left\langle X_1(t),-\frac{M[V]}{2}f'(\cdot)u'(\cdot,t)\right\rangle\\&+\langle X_1(s),f'(\cdot)G'*u(\cdot,t)\rangle-\langle X_1(t),f'(\cdot)G'*v(\cdot,t)\rangle-\langle X_1(t),f'(\cdot)\Gamma A'(\cdot)\rangle\\&=\left\langle X_1(t),f'(\cdot)\left(-\frac{M[V]}{2}u'(\cdot,t)+G'*(u(\cdot,t)-v(\cdot,t))-\Gamma A'(\cdot)\right)\right\rangle.\end{aligned}$$

Substituting this into (2.9), and passing to the limit $N_1, N_2 \to +\infty$, with the help of (2.5), we obtain

$$\langle u(t)-u(0), f(\cdot)\rangle = \int_0^t \left\langle u(s), f'(\cdot)\left(-\frac{M[V]}{2}u'(\cdot,s) + G' * (u(\cdot,s)-v(\cdot,s)) - \Gamma A'(\cdot)\right)\right\rangle,$$

with a similar equation for the function $v(x,t)$. These equations are the weak form of the following system

$$u_t = \eta\Delta u^2 + \nabla\cdot(u\nabla G*(v-u)) + \Gamma\nabla\cdot(u\nabla A(x)), \tag{2.14a}$$

$$v_t = \eta\Delta v^2 + \nabla\cdot(u\nabla G*(u-v)) + \epsilon\nabla\cdot(v\nabla A(x)). \tag{2.14b}$$

The system (2.14) is equivalent to system (1.5) with $\sigma = 0$ and $\eta = \frac{M[V]}{2}$.

A local model. If both $\gamma_v \in (0,1)$ and $\gamma_g \in (0,1)$, then, following the procedures as in the non-local case, we obtain the *local* system of equations (1.6) with $\sigma = 0$ and

$$\lambda_1 = \frac{1}{2}(M[V] - M[G]) \text{ and } \lambda_2 = \frac{1}{2}M[G]. \tag{2.15}$$

We refer to λ_2 as the social preference parameter. The mass of the attractive potential $G(x)$ (within groups) relative to that of the repulsive potential $V(x)$ determines the sign of λ_1. For well-posedness of the system (1.6), one has to assume that diffusion overpowers aggregation so that $\lambda_1 > 0$.

The interaction between individuals can, of course, be modeled in numerous ways. In fact, another interesting model to consider would be to have separate kernels for within group interactions and between group interactions. For example, a Morse-type potential could be used for the within group interactions.

2.3. Individual preference through random noise

Thus far, we have assumed that all individuals follow the exact same rules of interaction. In reality, however, individuals have personal preferences that do not necessarily follow the deterministic dynamics. We may account for personalized preferences through a random noise, in which case the position of the individuals

$$\left\{x_i^k(t)\right\}_{t>0}, k = 1,\ldots N_i, \text{ and } i = \{1,2\},$$

is described as a stochastic process described by a stochastic differential equation

$$dx_i^k(t) = \left[T(x_1(t),x_2(t)) + y_i A(x_i^k(t))\right] dt + \sigma(x_1(t),x_2(t)dW_i^k(t),$$

for $k = 1,\ldots N_i$ and $i = \{1,2\}$. Here W_i^k is a family of independent standard Wiener processes. The function T includes all of the within group and between group interactions we have considered above. Following the formal derivations of [18], which is easily adapted to systems, we obtain the general models (1.5) and (1.6). We refer an interested reader to [18] for more details.

3. Social diversity vs. segregation

In this section, we explore the question of whether a population preserves social diversity or moves toward a segregated state. We begin with the proof Proposition 1.1. We only show the formal derivation that can be made rigorous by regularizing the entropy energy and then proving appropriate bounds. This last step allows us to pass to the limit (see for example the proof of Proposition 1 in [2]).

Proof. (Proposition 1.1) Let $u(x,t)$ and $v(x,t)$ be as in the hypothesis of Proposition 1.1. Taking the time derivatives of each of the energies, we get

$$\frac{d}{dt}\mathcal{E}(t) := \int_\Omega 2\eta(uu_t + vv_t) + \sigma(u_t \log u + v_t \log v)\, dx. \tag{3.1}$$

$$\frac{d}{dt}\mathcal{W}(t) := -\int_\Omega G*uu_t + G*vv_t\, dx. \tag{3.2}$$

$$\frac{d}{dt}\mathcal{S}(t) := \int_\Omega (G*vu_t + G*uv_t)\, dxdy. \tag{3.3}$$

$$\frac{d}{dt}\mathcal{L}(t) := \int_\Omega A(x)(\Gamma u_t(x,t) + \epsilon v_t(x,t))\, dx. \tag{3.4}$$

From (3.1)-(3.4), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(t) = &\int_\Omega (2\eta u + \sigma \log u + G*(v-u) + \Gamma_1 A(x))u_t\, dx \\ &+ \int_\Omega (2\eta v + \sigma\log v + G*(u-v) + \epsilon A(x))v_t\, dx \\ = &-\int_\Omega u\left|2\eta\nabla u + \sigma\frac{1}{u}\nabla u + \nabla G*(v-u) + \Gamma_1\nabla A(x)\right|^2 dx \\ &-\int_\Omega v\left|2\eta\nabla v + \sigma\frac{1}{v}\nabla v + \nabla G*(u-v) + \epsilon\nabla A(x)\right|^2 dx,\end{aligned}$$

where we used the fact that u and v are solutions to system (1.5) and integrated by parts once.
□

Social diversity. Here, we prove Theorem 1.3.

Proof. (Theorem 1.3) Let $u(x,t)$ and $v(x,t)$ be solutions to system (1.5) with $\Gamma = \epsilon = \eta = 0$, and define

$$w = u - v \quad \text{and} \quad s = u + v - \bar{s},$$

where $\bar{s}$ is defined in (1.16). First, assume that $\eta = 0$:

$$w_t = \sigma\Delta w - \nabla\cdot(s\nabla G*w) - \bar{s}\Delta G*w, \tag{3.5a}$$

$$s_t = \sigma\Delta s - \nabla\cdot(w\nabla G*w), \tag{3.5b}$$

with initial conditions

$$w(x,0) = u_0(x) - v_0(x) \text{ and } s(x,0) = u_0(x) + v_0(x) - \bar{s},$$

and no-flux boundary conditions. Multiplying (3.5a) by $w(x,t)$ and (3.5b) by $s(x,t)$ we obtain the estimates

$$\begin{aligned}\frac{d}{dt}\left(||w||^2_{L^2(\Omega)} + ||s||^2_{L^2(\Omega)}\right) = &-\sigma\left(||\nabla w||^2_{L^2} + ||\nabla s||^2_{L^2}\right) + \int s(\nabla G*w)\nabla w\, dx \\ &+ \int w(\nabla G*w)\nabla s\, dx - \int \bar{s}(\Delta G*w)w\, dx := -I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Note that both $w(x,t)$ and $s(x,t)$ have mean zero mass, the latter due to (1.14) and the former due to its definition. Therefore, we can apply Poincaré's Inequality to I_1 and obtain

$$-I_1 < -C_p\sigma\left(||w||^2_{L^2(\Omega)} + ||s||^2_{L^2(\Omega)}\right). \tag{3.6}$$

Next, we integrate by parts the term I_2 and add it to I_3 to obtain

$$I_2 + I_3 = -\int sw\,(\Delta G * w)\,dx \leq \frac{1}{2}||\Delta G||_{L^\infty(\Omega)}||w||_{L^1(\Omega)}\left(||w||^2_{L^2(\Omega)} + ||s||^2_{L^2(\Omega)}\right). \tag{3.7}$$

The final bound seen in (3.7) is obtained by the use of Young's inequality for convolutions and the Cauchy-Schwarz inequality. For the last term we obtain

$$I_4 \leq \bar{s}||\Delta G||_{L^1(\Omega)}||w||^2_{L^2(\Omega)}. \tag{3.8}$$

Thus, combining (3.6), (3.7), and (3.8) we obtain

$$\begin{aligned}\frac{d}{dt}\left(||w||^2_{L^2(\Omega)} + ||s||^2_{L^2(\Omega)}\right) \leq &\left(-C_p\sigma + \frac{1}{2}||\Delta G||_{L^\infty(\Omega)}||w||_{L^1(\Omega)} + \bar{s}||\Delta G||_{L_1(\Omega)}\right)||w||^2_{L^2} \\ &+ \left(-C_p\sigma + \frac{1}{2}||\Delta G||_{L^\infty(\Omega)}||w||_{L^1(\Omega)}\right)||s||^2_{L^2(\Omega)}.\end{aligned}$$

Choosing $\sigma > \frac{1}{C_p}\left(\frac{1}{2}||\Delta G||_{L^\infty(\Omega)}||w||_{L^1(\Omega)} + \bar{s}||\Delta G||_{L_1(\Omega)}\right)$ we obtain the differential inequality

$$\frac{d}{dt}y \leq -Cy, \quad y(0) = ||u_0(x) - v_0(x)||^2_{L^2} + ||u_0(x) + v_0(x) - \bar{s}||^2_{L^2}, \tag{3.9}$$

where $y = ||w||^2_{L^2} + ||s||^2_{L^2}$ and $C = C(\sigma, \bar{s}, G, M)$. Integrating (3.9) we obtain estimate (1.15).
□

Segregation due to mobility disparity and environment. This section is devoted to the proofs of Theorem 1.4 and Theorem 1.5. To prove the former we need the following lemma.
Lemma 3.1. *Let $A(x) \in L^\infty(\Omega)$. There are solutions, $\bar{u}(x), \bar{v}(x) \in L^1(\Omega)$, to*

$$u(x) = c_1 \exp\left\{-\frac{1}{\sigma}(\Gamma A(x) + \lambda_2 v(x))\right\}, \tag{3.10a}$$

$$v(x) = c_2 \exp\left\{-\frac{1}{\sigma}(\epsilon A(x) + \lambda_2 u(x))\right\}. \tag{3.10b}$$

In addition, for any $M_1, M_2 > 0$ we may choose c_1 and c_2 so that

$$c_1\int_\Omega \bar{u}(x)\,dx = M_1 \quad \textit{and} \quad c_2\int_\Omega \bar{v}(x)\,dx = M_2. \tag{3.11}$$

Proof. (Lemma 3.1) The proof is in two steps: in the first step we prove existence of a fixed point of system (3.10) when c_1 and c_2 are arbitrary positive constants and in the second step we prove that two such constants exist so that (3.11) holds.

Step 1: We set $\sigma = 1$ and $\lambda_2 = 1$ without loss of generality. For a fixed $\bar{x} \in \Omega$, we set the constants

$$\tilde{c}_1 = c_1 e^{-\Gamma A(\bar{x})} \quad \text{and} \quad \tilde{c}_2 = c_2 e^{-\epsilon A(\bar{x})},$$

so that

$$u(\bar{x}) = \tilde{c}_1 \exp\{-\tilde{c}_2 \exp\{-u(\bar{x})\}\}. \tag{3.12}$$

Note that given $u(\bar{x})$ we can find $v(\bar{x})$ using (3.10b). First, observe that for $u(\bar{x}) = 0$ the left-hand-side of (3.12) is smaller than the right-hand-side. However, for $u(\bar{x}) > \tilde{c}_1$ the right-hand-side is smaller than the left-hand-side. Thus, by continuity and the intermediate value theorem, there exists a positive fixed point $\bar{u}(\bar{x})$, and, correspondingly, $\bar{v}(\bar{x}) > 0$, that satisfy (3.10) for $\bar{x}$. Following this procedure for all $x \in \Omega$ we, obtain the existence of functions $\bar{u}(x)$ and $\bar{v}(x)$ which satisfy (3.10) in all of the domain. Moreover, the fixed points can be chosen so that $\bar{u}(x)$ and $\bar{v}(x)$ are continuous.

Step 2: Next, we prove that there exists c_1 and c_2 such that (3.11) holds. Note that $\bar{u}(x)$ and $\bar{v}(x)$ which satisfy (3.10) with finite c_1, c_2 are in $L^\infty(\Omega)$ and consequently in $L^1(\Omega)$. For simplicity, we define

$$f(x) = e^{-\Gamma A(x)} \,\text{and}\, g(x) = e^{-\epsilon A(x)}.$$

As we want $M[\bar{u}(x)] = M_1$ and $M[\bar{v}(x)] = M_2$ we set

$$\int_\Omega \bar{v}(x)\,dx = c_2 \int_\Omega g(x) e^{-\bar{u}(x)}\,dx = M_2,$$

which solving for c_2 gives

$$c_2 = \frac{M_2}{\int_\Omega g(x) e^{-\bar{u}(x)}\,dx}. \tag{3.13}$$

Therefore, given $M_2 > 0$ we may set c_2 as above, and insert expression (1.18b) into (1.18a). This will give a single equation for the function $\bar{u}(x)$, parametrized by the constant c_1, and our task is to show that there exists $c_1 > 0$ so that $M[\bar{u}(x)] = M_1$. Consider the mass of $\bar{u}(x)$:

$$\begin{aligned}\int_\Omega \bar{u}(x)\,dx &= c_1 \int_\Omega f(x) \exp\{-c_2 g(x) \exp\{-\bar{u}(x)\}\}\,dx \\ &= c_1 \int_\Omega f(x) \exp\left\{-\frac{M_2}{\int_\Omega g(y) e^{-\bar{u}(y)}\,dy} g(x) \exp\{-\bar{u}(x)\}\right\} dx,\end{aligned} \tag{3.14}$$

where we have used (3.13). Note that if $c_1 = 0$, then $M[\bar{u}(x)] = 0$. In addition, we have the following lower bound for the mass of $\bar{u}(x)$

$$\begin{aligned}\int_\Omega \bar{u}(x)\,dx &= c_1 \int_\Omega f(x) \exp\left\{-\frac{M_2}{\int_\Omega g(y) e^{-\bar{u}(y)}\,dy} g(x) \exp\{-\bar{u}(x)\}\right\} dx \\ &\geq c_1 e^{-\Gamma A_{max}} \exp\left\{-\frac{M_2 \kappa}{e^{-\epsilon A_{max}}}\right\} \mu\left\{x \in \Omega : e^{-\bar{u}(x)} \leq \kappa \int_\Omega e^{-\bar{u}(y)}\,dy\right\},\end{aligned} \tag{3.15}$$

for any $\kappa \geq 1$ and A_{max} is the maximum value of A in Ω. Now, we have the bound

$$\mu\left\{x \in \Omega : e^{-\bar{u}(x)} \geq \kappa \int_\Omega e^{-\bar{u}(y)}\,dy\right\} \leq \gamma |\Omega|, \tag{3.16}$$

for $\kappa \geq 1$ and $\gamma < 1$ such that

$$\kappa\gamma|\Omega| > 1.$$

Thus, we have the bound

$$\int_\Omega \bar{u}(x)\,dx \geq c_1 e^{-\Gamma A_{max}} \exp\left\{-\frac{M_2\kappa}{e^{-\epsilon A_{max}}}\right\}|\Omega|(1-\gamma). \tag{3.17}$$

Note that γ and κ only depend on the size of Ω. From (3.17) we conclude that for c_1 sufficiently large (and depending on M_2) we have $M[\bar{u}(x)] > M_1$. Thus, by continuity there exists a c_1 such that $M[\bar{u}] = M_1$ and c_2 is then determined from (3.13). With this we conclude the proof.
□

We are now ready to prove Theorem 1.4

Proof. (Theorem 1.4) Inspired by the Fokker-Plank type nature of the system (1.6), we rewrite it

$$u_t = \sigma\nabla\cdot\left(e^{-\frac{1}{\sigma}(\Gamma A(x)+\lambda_2 v(x))}\nabla\cdot\left(e^{\frac{1}{\sigma}(\Gamma A(x)+\lambda_2 v(x))}u\right)\right), \tag{3.18a}$$

$$v_t = \sigma\nabla\cdot\left(e^{-\frac{1}{\sigma}(\epsilon A(x)+\lambda_2 u(x))}\nabla\cdot\left(e^{\frac{1}{\sigma}(\epsilon A(x)+\lambda_2 u(x))}v\right)\right), \tag{3.18b}$$

with the initial conditions $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$ and no-flux boundary conditions. From (3.18), we observe that $\bar{u}(x)$ and $\bar{v}(x)$, satisfying (3.10), are steady-state solutions to (1.6) for any $c_1, c_2 > 0$ (setting $\sigma = \lambda_2 = 1$). Therefore, we are left to verify that they satisfy the no-flux boundary conditions (1.6d) and (1.6e) and the mass property (1.17c). Taking the gradient of $\bar{u}(x)$ and $\bar{v}(x)$ from (3.10) we observe that

$$\sigma\nabla\bar{u}(x) = \bar{u}(x)\nabla(-\Gamma A(x) + \lambda_2\bar{v}(x)),$$
$$\sigma\nabla\bar{v}(x) = \bar{v}(x)\nabla(-\epsilon A(x) + \lambda_2\bar{u}(x)),$$

guaranteeing that the no-flux boundary conditions are satisfied for $\bar{u}(x)$ and $\bar{v}(x)$. Finally, from Lemma 3.1 we can set $M_1 = M[u_0(x)]$ and $M_2 = M[v_0(x)]$ and with this we conclude. □

We now prove Theorem 1.5 in three steps. The first step consists of proving that (1.18) with $\delta = 0$ has a non-trivial solution $(u_0, v_0) \in C^1(\Omega) \times C^1(\Omega)$ for any $c_1, c_2 > 0$. Second, we invoke the implicit function theorem to show that the same holds in a small neighborhood of $\delta = 0$. Finally, we prove that there are constants $c_1, c_2 > 0$ such that (1.18c) holds.

Proof. Without loss of generality assume that $\sigma = 1$.

Step 1: Define $f_1(x) = \exp\{-\Gamma A(x)\}$, $f_2(x) = \exp\{-\epsilon A(x)\}$, and $g(x) = f_1(x)f_2(x)$. Note that

$$u(x)v(x) = c_1c_2 g(x),$$

and since $c_1c_2g(x) > 0$ we can express v in terms of u

$$v(x) = \frac{c_1c_2g(x)}{u(x)}.$$

This reduces the problem to solving the following fixed point

$$u(x) = c_1 f_1(x) e^{u(x) - \frac{c_1c_2g(x)}{u(x)}}.$$

For $\bar{x}\in\Omega$ fixed this is equivalent to solving $u=ae^{u-\frac{b}{u}}$ where a,b are positive constants, which always has a non-trivial solution as

$$\lim_{u\to 0} f''(u)=0,$$

where we define $f(u):=ae^{u-\frac{b}{u}}$. Repeating this process for all $x\in\Omega$ gives a solution $(u_0(x),v_0(x))$ to (1.18). Moreover, u_0 and v_0 inherit the regularity of $A(x)$ and there exist $K,\epsilon_1>0$ such that $\epsilon_1<u_0(x),v_0(x)<K$.

Step 2: Let $F(\delta,u):[0,\infty)\times C^1(\Omega)\to C^1(\Omega)$ defined by

$$F(\delta,u):=u(x)-c_1 f_1(x)\exp\left\{G_\delta*\left(u(x)-\frac{c_1c_2g(x)}{u(x)}\right)\right\}.$$

Note that F is a C^1 map, $F(0,u_0)=0$ and $D_uF(0,u_0)v:C^1(\Omega)\to C^1(\Omega)$ defined by

$$D_uF(0,u_0)v=h(x)v,$$

where $h(x)<\infty$ and strictly bounded from 0, is an isomorphism. Then the implicit function theorem defines a unique mapping (δ,u_δ) for δ near 0 and u_δ near u_0 such that $F(\delta,u_\delta)=0$.

Step 3: In the previous step we proved the existence of $u_\delta(x)$ and $v_\delta(x)$ for any positive and finite constants c_1 and c_2. Next we prove the claim that for any given $0<M_1,M_2<\infty$ there are corresponding c_1 and c_2 that allow $u_\delta(x)$ and $v_\delta(x)$ to satisfy (1.18c). Recall that we want

$$\int v_\delta(x)\,dx=c_1c_2\int\frac{g(x)}{u_\delta(x)}\,dx=M_2,$$

which gives that

$$c_2=\frac{M_2}{c_1\int\frac{g(x)}{u_\delta(x)}\,dx}.$$

Now, consider the mass of $u_\delta(x)$

$$\int_\Omega u_\delta(x)\,dx=c_1\int_\Omega f_1(x)\exp\left\{G*\left(u_\delta(x)-\frac{M_2}{\int\frac{g(y)}{u_\delta(y)}\,dy}\frac{g(x)}{u_\delta(x)}\right)\right\}dx.$$

Again, when $c_1=0$ we see that $M[u_\delta(x)]=0$. In addition, we have the following lower bound for the mass of $u_\delta(x)$

$$\begin{aligned}\int_\Omega u_\delta(x)\,dx&\geq c_1\int_\Omega f_1(x)\exp\left\{-\frac{M_2}{\int\frac{g(y)}{u_\delta(y)}\,dy}\int G(x-y)\frac{g(y)}{u_\delta(y)}\,dy\right\}dx\\&\geq c_1e^{-\Gamma A_{max}}\exp\{-M_2\,||G||_{L^\infty}\}.\end{aligned}\tag{3.19}$$

As before, this implies that we can choose c_1 sufficiently large so that $M[u_\delta(x)]>M_1$. By continuity, we conclude the proof.
∎

Energy minimizers. This subsection is devoted to the proofs of results related to the energy minimizers of (1.8) and its connection to the steady-state solutions to system (1.5). We begin with the proof of Lemma 1.8.

Proof. (Lemma 1.8) Suppose there exists $(u^*,v^*)\in\mathcal{Y}_{M_u,M_v}$ with

$$\operatorname{supp}(u^*)\cap\operatorname{supp}(v^*)\neq\emptyset, \tag{3.20}$$

such that $F[u^*,v^*]=I_{M_u,M_v}$. By (3.20) it must be the case that

$$\mathcal{S}(u^*,v^*)>0.$$

Let $u^{\#}(x)$ be the symmetric decreasing rearrangement of $u(x)$ (similarly for $v(x)$) and consider the sequence

$$u_n(x):=u^{\#}(x+x_n)\,\text{and}\,v_n(x):=v^{\#}(x-x_n),$$

for $\{x_n\}$ a sequence in $\mathbb{R}^d$ satisfying $\lim_{n\to\infty}|x_n|=\infty$. Recall that $||u||_2=||u_n||_2$ for all n and furthermore by the Riesz rearrangement inequality (1.4), we know that

$$-\mathcal{W}(u_n,v_n)\leq-\mathcal{W}(u,v).$$

Finally, we have that

$$\lim_{n\to\infty}\mathcal{S}(u_n,v_n)=0.$$

Thus, there exists an N sufficiently large so that $\mathcal{F}(u_N,v_N)<\mathcal{F}(u^*,v^*)$. which is a contradiction. □

Remark 3.1. *Note that for* (1.22) *to hold it is necessary that $G(x)$ have bounded support.*

It is useful to rewrite the energy (1.8) with $\sigma=0$ as

$$\mathcal{F}[u,v](x)=\mathcal{F}_1(u)+\mathcal{F}_1(v)+\mathcal{S}(u,v), \tag{3.21}$$

where

$$\mathcal{F}_1(w)=\eta\int w^2\,dx-\frac{1}{2}\int\int G(x-y)w(x)w(y)\,dxdy. \tag{3.22}$$

Let us define

$$\liminf_{w\in\mathcal{Y}_M}\mathcal{F}_1[w]:=I_M,$$

where

$$\mathcal{Y}_M=\left\{u\in L^1\cap L^2:||u||_1=M\right\}.$$

We now state a result which reduces the problem of finding minimizers to (1.8) in $\mathcal{Y}_{M_u,M_v}$ for $M_u,M_v>0$ to finding minimizers of (3.22) in $\mathcal{Y}_M$ for $M>0$.

Lemma 3.2 (Reduction of minimizer problem). *Let G_1 have compact support, i.e. supp$(G_1)\subset B_R(0)$ for some $R>0$.*

- *(i) If the infimum of $\mathcal{F}_1$ in $\mathcal{Y}_M$ for $M>0$ is achieved by a function with compact support, then there exist a minimizer $(u,v)\in\mathcal{Y}_{M_u,M_v}\cap C_0^2(\Omega)\times C_0^2(\Omega)$ of $\mathcal{F}$. In particular, if (u^*,v^*) such that $u^*\in\mathcal{Y}_{M_u}$ is a minimizer of $\mathcal{F}_1(u)$ and $v^*\in\mathcal{Y}_{M_v}$ is a minimizer of $\mathcal{F}_1(v)$ with bounded support then there exists $x_1\in\mathbb{R}^d$ such that $(u^*(\cdot+x_1),v^*(\cdot-x_1))$ is a minimizer of $\mathcal{F}$ in $\mathcal{Y}_{M_u,M_v}$.*

(ii) If all minimizers of $\mathcal{F}_1$ in $\mathcal{Y}_M$ for $M>0$ have unbounded support then $\mathcal{F}$ does not achieve its minimizer in $\mathcal{Y}_{M_u,M_v}$ for any $M_u,M_v>0$.

(i) Any $(u^*,v^*)\in\mathcal{Y}_{M_u,M_v}$ which minimize $\mathcal{F}[u,v]$ must minimize $\mathcal{F}_1$ individually.

Proof. The proof of (i) is clear. To prove (ii), recall that any minimizer of $\mathcal{F}$ must have compact support, thus we assume for contradiction that there exits a minimizer $(u,v)\in\mathcal{Y}_{M_1,M_2}$. Since u cannot be a minimizer for $\mathcal{F}_1$ then

$$\mathcal{F}_1(u)+\mathcal{F}_1(v)>\mathcal{F}_1(u^*)+\mathcal{F}_1(v^*)$$

where $(u^*,v^*)\in\mathcal{Y}_{M_u,M_v}$ are both minimizer of $\mathcal{F}_1$. Now, define

$$u_n=(u^*)^{\#}(\cdot-x_n) \quad \text{and} \quad v_n=(v^*)^{\#}(\cdot+x_n)$$

for a sequence $\{x_n\}\subset\mathbb{R}^d$. Let N be such that $|x_N|$ be sufficiently large with

$$S(u_N,v_N)<\mathcal{F}_1(u)+\mathcal{F}_1(v)-\mathcal{F}_1(u^*)-\mathcal{F}_1(v^*).$$

From this we conclude that $\mathcal{F}(u_N,v_N)<\mathcal{F}(u,v)$ which is a contradiction. The proof of (iii) is the same as that of (ii). □

We are now ready to prove Proposition 1.6.

Proof. By Proposition 3.1 in [5] we know that any minimizer, u, of $\mathcal{F}_1$ in $\mathcal{Y}_M$ satisfies

$$u\nabla\cdot(2\eta u-G*u)=0 \quad \text{a.e.} \quad \text{in} \quad \mathbb{R}^d.$$

This in conjunction with the fact that any minimizers, (u,v), of $\mathcal{F}[u,v]$ in $\mathcal{Y}_{M_u,M_v}$ must have disjoint support, specifically the support of u and the support of v must be separated by $2R$ if $supp(G)\subset B_R(0)$. This gives the result. □

Proof. (Proposition (1.7)) From Proposition 1.1 with $\sigma=\Gamma=\epsilon=0$ we obtain that

$$\mathcal{F}(t)+\int_0^t\int_\Omega u\left|\nabla(2\eta u+G*(v-u)\right|^2+v\left|\nabla(2\eta v+G*(u-v)\right|^2\,dx\,ds\le\mathcal{F}(0), \tag{3.23}$$

for all $t>0$. Thus, we have

$$\begin{aligned}&4\eta^2\int_\Omega u|\nabla u|^2+v|\nabla v|^2\,dx+4\eta\int_\Omega u\nabla u\nabla G*(u-v)\\&+v\nabla v\nabla G*(u-v)\,dx+\int(u+v)\left|\nabla G*(u-v)\right|^2dx<\infty,\end{aligned}$$

for almost every $t>0$. An application of Cauchy-Schwarz inequality gives (1.21). Additionally, if (u,v) are steady-state solutions then they satisfy

$$u\left|\nabla(2\eta u+G*(v-u)\right|^2+v\left|\nabla(2\eta v+G*(u-v)\right|^2=0,$$

for almost all $x\in\mathbb{R}^d$. This implies that

$$2\eta u+G*(v-u)=C_1 \,\text{and}\, 2\eta v+G*(u-v)=C_2,$$

for some constants C_1,C_2 almost everywhere on every connected component of the supports. By properties of the convolution we have that u and v inherit the regularity of G.

□

Proof. (Theorem 1.9) For an interaction potential $G(x)$ with unbounded support we follow a similar argument of that given in the proof of Lemma 1.8. Taking the distance between the center of mass of a sequence u_n and sequence v_n to infinity will decrease the segregation energy to zero and a minimizer will never be achieved. Thus, without loss of generality, let us take $G(x)$ to have compact support with $||G||_1 < 2$. Assume that there exists a minimizer $(u,v) \in \mathcal{Y}_{M_u,M_v}$, we have the lower bound

$$\mathcal{F}(u,v) \geq \left(||u||_2^2 + ||v||_2^2\right)\left(1 - \frac{||G||_1}{2}\right).$$

if $||G|| < 2$ then $\mathcal{F}(u,v) > 0$ which is a contradiction as $I_{M_1,M_2} \leq 0$. □

4. Continuous income spectrum In this section, we consider the case where the mobility of individuals is continuous, which models a continuum spectrum of incomes in the population. The dynamics of individuals is governed similarly to the case of discrete mobility with some modifications. First, since the mobility is continuous, an individual will be attracted to another individual if the difference in their mobility (representing the difference in economic status) is within the *mobility threshold*, κ, (see section 1) and they are repulsed otherwise. This brings about a second change which is that everyone has a short-range repulsion from everyone else regardless of their mobility.

4.1. Particle-interaction model We begin with n different groups each with N_i agents, for $i = 1...n$, which are interacting within and between groups. Every member of group i has mobility $y_i \in [0,1]$, where we have normalized the maximum mobility for ease of notation. Assume also that $y_i = \frac{i}{n}$ so that the mobility is uniformly distributed. This assumption is not necessary and is only made to simplify the derivation of the continuum model. As in the two-population case, we denote by $x_{N_i}^k(t) \in \mathbb{R}^d$ the spatial position of the k^{th} individual in group i which has N_i members. As before, the short-range repulsion dynamics is governed by the potential $V_N(x)$ and the mobility segregation is governed by the potential $H(y)G_N(x)$. Recall that $H(y_i - y_j)$ measures the difference is mobility between an individual from group i and an individual from group j, and the sign of H determines whether the individuals are attracted or repulsed from each other (one possible form for $H(y)$ is given in (1.23)). The interacting potential $G_N(x)$ takes the physical positioning of the two individuals into account and determines the strength and direction of the interactions. Of course, we assume that $V_N(x)$ and $G_N(x)$ are both admissible potentials and satisfy the scaling (2.1). Finally, individuals are advected by the velocity field $\nabla_x A(x)$ with speed proportional to the mobility.

Combining the ideas discussed above yields the system of evolution equations

$$\begin{aligned}\frac{dx_{N_i}^k(t)}{dt} = &-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{N}\sum_{l=1}^{N}\nabla_x\Big[V_N(x_{N_i}^k(t) - x_{N_j}^l(t)) + H(y_i - y_j)G_N(x_{N_i}^k(t) - x_{N_j}^l(t))\Big] \\ &- y_i\nabla_x A(x_{N_i}^k(t)),\end{aligned} \tag{4.1}$$

for $k = 1,...N_i$, and $i = 1...n$. Note that, unlike the the case of discrete mobility, to derive the macroscopic equation we need to keep track of the mobility of each individual along with their spatial location. For this purpose let we let the number of particles in each group be the same $N_i = N$ for all $i = 1,...,n$ (although we continue to use N_i to differentiate between groups). Let $Y_{N_i}^k(t) = (x_{N_i}^k(t), y_i)$ and define the empirical measure

$$Y_{N,n}(t) = \frac{1}{Nn}\sum_{i=1}^{n}\sum_{j=1}^{N}\delta_{Y^k_{N_i}(t)}. \tag{4.2}$$

Our objective now is the study of the limit of (4.2) as the number of individuals, N, and the number of groups, n, both approach infinity. Assume that there are functions $u_0(x,y)\in\mathbb{R}^d\times[0,1]$ and $u(x,y,t)\in\mathbb{R}^d\times[0,1]\times[0,\infty)$ such that for all $f\in C^2_0(\mathbb{R}^{d+1})$

$$\langle Y_{nN}(0),f\rangle \to \langle u_0(x,y),f\rangle \quad \text{and} \quad \langle Y_{nN}(t),f\rangle \to \langle u(x,y,t),f\rangle \text{ as } n,N\to\infty.$$

Taking for granted the existence of $u(x,y,t)$ our aim is to determine what equation the funciton $u(x,y,t)$ satisfies. As in the case for the two-population model, we define

$$g_{N,n}(x,y,t) = \frac{1}{nN}\sum_{j=1}^{n}\sum_{l=1}^{N}\nabla_x\Big(V_N(x-x^l_{N_j}(t)) + H(y-y_j)G_N(x-x^l_{N_j}(t)) - yA(x)\Big),$$

where ∇_x is the gradient with respect to the spatial variable. Using the definition of $g(x,y,t)$, the system of evolution equations (4.1) can be written more compactly as

$$\frac{dx^k_{N_i}(t)}{dt} = -\nabla_x g_{N,n}\left(x,\frac{i}{n},t\right), \tag{4.3}$$

for $k=1,...,N$ and $i=1...n$. Let $f\in C^2_0(\mathbb{R}^{d+1})$ then for N and n finite we have

$$\langle Y_{N,n}(t),f(x,y)\rangle = \frac{1}{nN}\sum_{i=1}^{n}\sum_{k=1}^{N} f\left(x^k_{N_i}(t),\frac{i}{n}\right). \tag{4.4}$$

Taking the time derivative of (4.4) and substituting in the evolution equation (4.3) we obtain

$$\begin{aligned}\frac{d}{dt}\langle Y_{N,n}(t),f(x,y)\rangle &= \frac{1}{nN}\sum_{i=1}^{n}\sum_{k=1}^{N}\nabla_x f\left(x^k_{N_i}(t),\frac{i}{n}\right)\\ &= -\frac{1}{nN}\sum_{j=1}^{n}\sum_{k=1}^{N}\nabla_x f\left(x^k_{N_i}(t),\frac{i}{n}\right)\cdot\nabla_x g_{N,n}\left(x^k_{N_i}(t),\frac{i}{n},t\right).\end{aligned}$$

This gives the integral equation

$$\langle Y_{N,n}(t),f(x,y)\rangle = \langle Y_{N,n}(0),f(x,y)\rangle + \int_0^t \langle Y_{N,n}(s), -\nabla_x g_{N,n}\cdot\nabla_x f\rangle\, ds. \tag{4.5}$$

Once again we restrict our work to $d=1$ since it lends itself to a cleaner and clearer derivation. It is also convenient to give a short hand notation to the spatial derivative: for any function $h(x,y)$ we denote $h'=\partial_x h$. Using this notation we obtain the following simplification of the spatial gradient of the function $g_{N,n}(x,y,t)$

$$\partial_x g_{N,n}(x,y,t) = \frac{1}{nN}\sum_{j=1}^{n}\sum_{l=1}^{N}\left[V'_N(x-x^l_{N_j}(t)) - H\left(y-\frac{j}{n}\right)G'_N(x-x^l_{N_j}(t))\right] + yA'(x).$$

Once more the only problematic term in (4.5) is the term which is integrated over time, which in one dimension is

$$
\begin{aligned}
<Y_{N,n}(t),-\nabla_x g_{N,n}\cdot\nabla_x f> = &-\frac{1}{(nN)^2}\sum_{i,j=1}^{n}\sum_{k,l=1}^{N} f'(x_{N_i}^k(t))V_N'(x_{N_i}^k(t)-x_{N_j}^l(t))\\
&-\frac{1}{(nN)^2}\sum_{i,j=1}^{n}\sum_{k,l=1}^{N} f'(x_{N_i}^k(t))H\left(\frac{i}{n}-\frac{j}{n}\right)G_N'(x_{N_i}^k(t)-x_{N_j}^l(t))\\
&+\frac{1}{nN}\sum_{i=1}^{n}\sum_{k=1}^{N} y_i A'(x_{N_i}^k(t))f'(x_{N_i}^k(t))\\
&:= I_1+I_2+I_3.
\end{aligned} \tag{4.6}
$$

As we are assuming short-range repulsion and long-range mobility segregation we take $\gamma_v\in(0,1)$ and $\gamma_g=0$. To understand what the *local equilibrium hypothesis* implies in this context, it is instructive to look at the interactions of a particular individual with each group separately. For example, consider an individual x and his/her interactions with individuals from group i (which has mobility y_i), we expect that as $N\to\infty$ the distances between individuals near x in group i at time t is approximately

$$
\frac{1}{Nu(x,y_i,t)}. \tag{4.7}
$$

We analyze the term in (4.6) separately and begin with the mobility segregation term I_2:

$$
\begin{aligned}
I_2 &= -\frac{1}{(nN)^2}\sum_{i,k=1}^{n}\sum_{j,l=1}^{N} f'(x_{N_i}^j(t),y_i)H(y_i-y_k)G_N'(x_{N_i}^j(t)-x_{N_k}^l(t))\\
&=\left\langle Y_{nN}(t), f'(\cdot)\frac{1}{nN}\sum_{k=1}^{n}\sum_{l=1}^{N} H(\cdot-y_k)G_N(\cdot-x_{N_k}^l(t))\right\rangle.
\end{aligned}
$$

Assuming that $G'(x)$ decays sufficiently fast as $|x|\to\pm\infty$ then by (4.7) we can approximate the position of the individuals of group k at $\frac{l}{Nu(x_l,\frac{k}{n},t)}$ for $l=-N/2,...,N/2$. This gives the approximation

$$
\begin{aligned}
\frac{1}{nN}\sum_{k=1}^{n}\sum_{l=-N/2}^{N/2} H(\cdot-y_k)G_N(\cdot-x_{N_k}^l(t)) &= \frac{1}{n}\sum_{k=1}^{n} H(\cdot-y_k)\left(\frac{1}{N}\sum_{l=-N/2}^{N/2} G\left(\cdot-\frac{l}{Nu(x_l,\frac{k}{n},t)}\right)\right)\\
&\approx \frac{1}{n}\sum_{k=1}^{n} H\left(\cdot-\frac{k}{n}\right)\int_{\mathbb{R}} G'(x-\tilde{x})u\left(\tilde{x},\frac{k}{n},t\right)d\tilde{x}\\
&\approx \int_0^1\int_{\mathbb{R}} H(\cdot-\tilde{y})G'(\cdot-\tilde{x})u(\tilde{x},\tilde{y},t)\,d\tilde{x}d\tilde{y} \qquad (4.8)\\
&= HG*u(x,y,t). \qquad (4.9)
\end{aligned}
$$

In (4.8) the operator $*$ represents the convolution operator in both x and y. For the repulsion term, I_1, by symmetrization and then by Taylor expansion of f' we obtain

$$\begin{aligned}
I_1 &= \frac{1}{2(nN)^2}\sum_{i,j=1}^{n}\sum_{k,l=1}^{N}[f'(x_{N_i}^k(t)) - f'(X_{N_j}^l(t))]V_N'(x_{N_i}^k(t) - x_{N_j}^l(t))\\
&\approx \frac{1}{2(nN)^2}\sum_{i,j=1}^{n}\sum_{k,l=1}^{N} f''(x_{N_i}^k(t))V_N'(x_{N_i}^k(t) - x_{N_j}^l(t))(x_{N_i}^k(t) - x_{N_j}^l(t))\\
&= \left\langle Y_{N,n}(t), f''\frac{1}{2(nN)}\sum_{j=1}^{n}\sum_{l=1}^{N}V_N'\left(x_{N_i}^k(t) - x_{N_j}^l(t)\right)(x_{N_i}^k(t) - x_{N_j}^l(t))\right\rangle.
\end{aligned}$$

Again we look at the individuals in each group near x separately and by the local equilibrium hypothesis we have that

$$x_{N_i}^k(t) - x_{N_j}^l(t) \approx \frac{l}{Nu(\cdot,\frac{k}{n},t)}.$$

Using (2.1) and denoting $V_1(x) = V(x)$ we obtain

$$\begin{aligned}
\frac{1}{nN}\sum_{k=1}^{n}\sum_{l=1}^{N}V_{nN}'(\cdot - x_{N_k}^l(t))(\cdot - x_{N_k}^l) &\approx \frac{1}{n}\sum_{k=1}^{N}\left(\frac{N^{\gamma_v}}{N}\sum_{l=-N/2}^{N/2}V'\left(\frac{N^{\gamma_v} l}{Nu(\cdot,\frac{k}{n},t)}\right)\frac{N^{\gamma_v} l}{Nu(\cdot,\frac{k}{n},t)}\right)\\
&\approx \frac{1}{n}\sum_{k=1}^{n}\left(\int_{\mathbb{R}}V'\left(\frac{z}{u(\cdot,\frac{k}{n},t)}\right)\frac{z}{u(\cdot,\frac{k}{n},t)}\,dz\right)\\
&\approx \frac{1}{n}\sum_{k=1}^{n}u\left(\cdot,\frac{k}{n},t\right)\int V'(w)w\,dw\\
&\approx -M[V]\int_0^1 u(\cdot,y,t)\,dy.
\end{aligned}$$

In the last estimate we integrated by parts and approximated the Riemann sum by an integral. Recall that $u(x,t)$ defined in (1.25) measures the population density at location x and time t over all mobilities. The term that is contributed by the environment, I_3, is derived similarly and we omit the steps. Combining the above calculations gives

$$\begin{aligned}
\left\langle Y_{nN}(s), -g_{N,n}'(\cdot,s)f'(s)\right\rangle &= \left\langle Y_{nN}(s), -\frac{1}{2}u(\cdot,\cdot,s)M[V]f''\right\rangle\\
&\quad + \left\langle Y_{nN}(s), f'\left(HS_1' * u(\cdot,s) + A'(x)y\right)\right\rangle,
\end{aligned}$$

which is the weak version of (1.24).

5. Numerical results

Numerical simulations for the interacting-particle model. We performed various numerical simulations of the interacting-particle system with multiple groups of varying economic status. Figure 5.2a illustrates the equilibrium state when only the dynamic rules (A1) and (A2) are considered. We observe some segregation but some social diversity is preserved. On the other hand, the inclusion of the effects of a monocentric environment, where all resources are concentrated in the city center, leads to an exaggerated segregation. Individuals with a higher mobility live in the center of the city while the other groups form regions surrounding the city center with increasing distance to the city center as the mobility decreases.

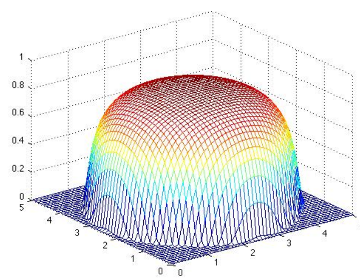
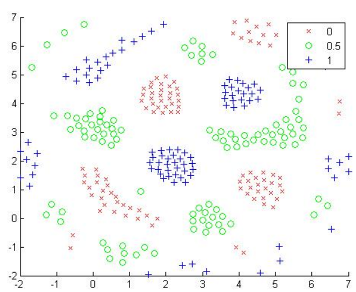
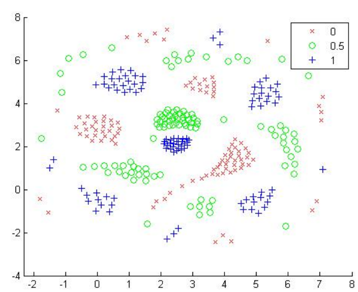


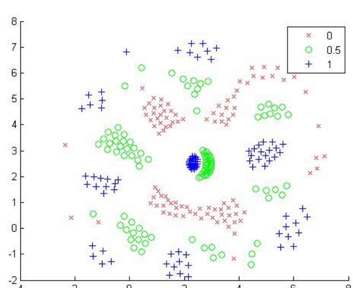
Fig. 5.1: A monocentric environment where all amenities are located in the city center.



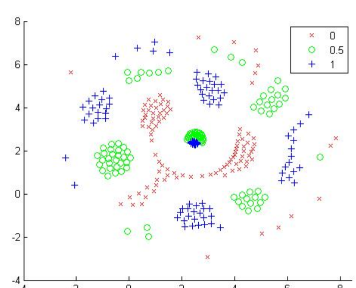
(a) $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$



(b) $\Gamma_1 = 1, \Gamma_2 = .5, \Gamma_3 = 0$

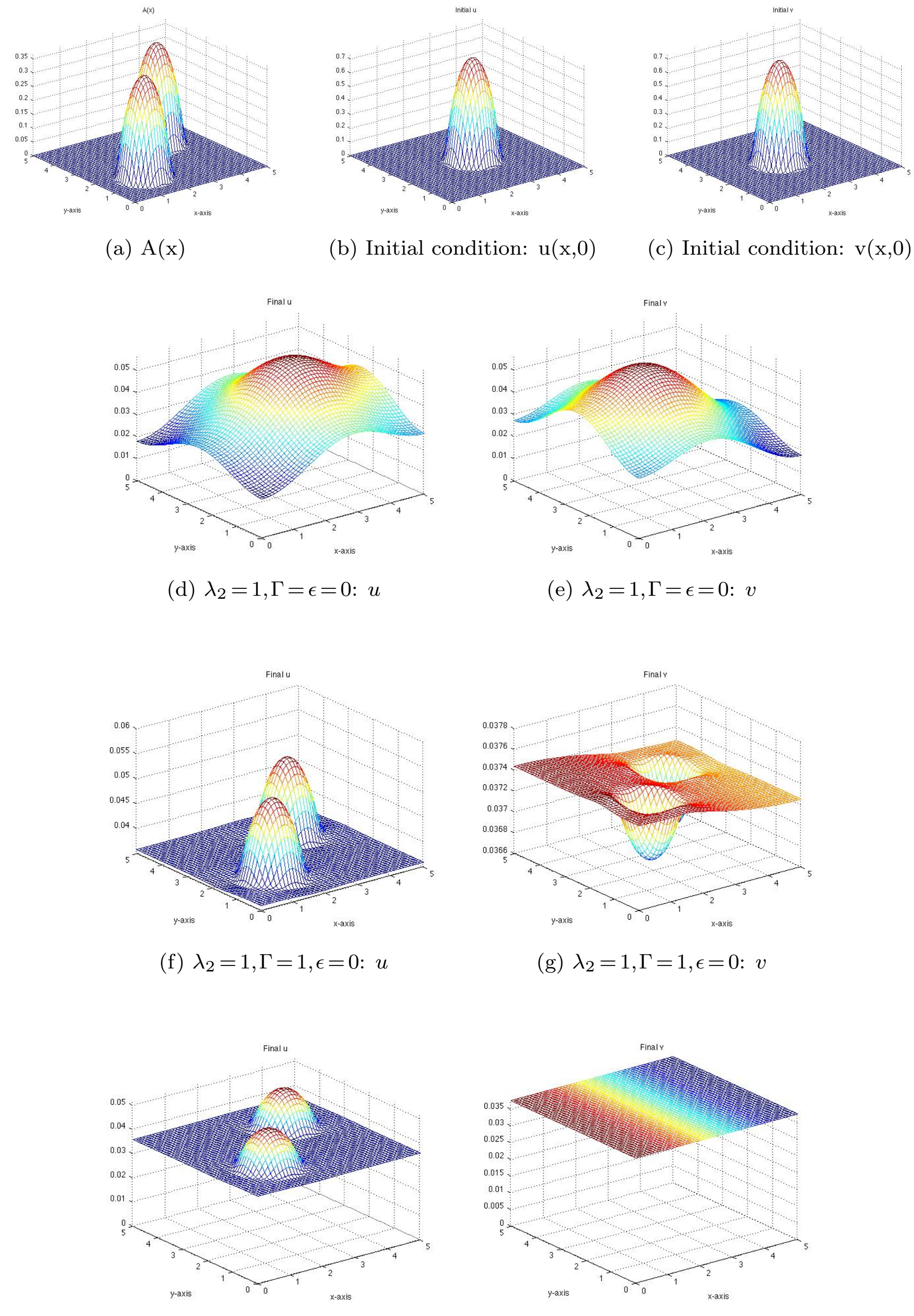


(c) $\Gamma_1 = 5, \Gamma_2 = 2.5, \Gamma_3 = 0$



(d) $\Gamma_1 = 10, \Gamma_2 = 5, \Gamma_3 = 0$

Fig. 5.2: Results of numerical simulations of the interacting particle system with three sub-populations, with mobilities Γ_1, Γ_2 and Γ_3, and an initial random distribution. Figure 5.2a illustrates the equilibrium distribution when only social preference is taken into account. Figure 5.2b, 5.2c,5.2d illustrates the equilibrium state when both the environment and economic disparity are taken into account with increasing mobility gap.



(a) A(x) (b) Initial condition: u(x,0) (c) Initial condition: v(x,0)

(d) $\lambda_2=1,\Gamma=\epsilon=0$: u (e) $\lambda_2=1,\Gamma=\epsilon=0$: v

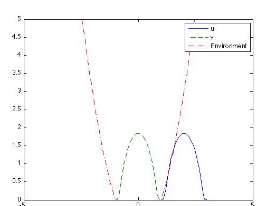
(f) $\lambda_2=1,\Gamma=1,\epsilon=0$: u (g) $\lambda_2=1,\Gamma=1,\epsilon=0$: v

(h) $\lambda_2=0,\Gamma=1,\epsilon=0$: u (i) $\lambda_2=0,\Gamma=1,\epsilon=0$: v

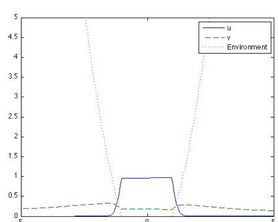
Fig. 5.3: Numerical solutions to (1.6) with the heterogeneous environment illustrated in Figure 5.3a and initial conditions for $u(x,t)$ and $v(x,t)$ illustrated in Figures 5.3b and 5.3c respectively with varying λ_2 and Γ ($\epsilon=0$).

Numerical solutions to the local continuum limit: two-dimensions. Figure 5.3 illustrates numerical solutions to (1.6) where the environment $A(x)$ represents a city with two main regions where all amenities are located, see Figure 5.3a. Initially, both populations (disadvantaged individuals and affluent individuals occupy the same space) - this is illustrated in Figure 5.3b and Figure 5.3c. In Figure 5.3d and Figure 5.3e we observe the steady-state solutions for $u(x,t)$ and $v(x,t)$ respectively when there is no mobility or environmental influence. We observe that due to the social preference $u(x,t)$ and $v(x,t)$ occupy different spaces. On the other hand when we include mobility and the environment we observe that $u(x,t)$ occupies the space where $A(x,t)$ is large and $v(x,t)$ occupies the remaining of the domain - see Figures 5.3f and 5.3g. Finally, Figures 5.3h and 5.3i illustrates the case when the social preference is zero and we see that $v(x,t)$ disperses.

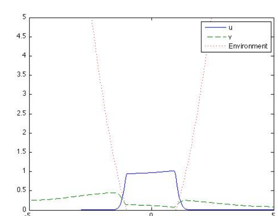
Numerical solutions to the local continuum limit: one-dimension. Figure 5.4 illustrates numerical solutions to (1.6) where $A(x)$ represents a monocentric city. Initially, the disadvantaged individuals occupy the center of the city, while the affluent individuals are outside - this is illustrated in Figure 5.4a. Figure 5.4b illustrates the distributions at a later time when the social preference is one and the economic disparity is five. We observe that the affluent individuals take over the center of the city and displace the disadvantaged individuals. Figure 5.4c illustrates the case when the social preference is a little stronger, and we observe that this forces the disadvantaged population toward one side of the city. Finally, Figure 5.4d illustrates the case when the social preference is small, and we see that even though the disadvantaged individuals disperse, the majority of the population remains in the city-center.



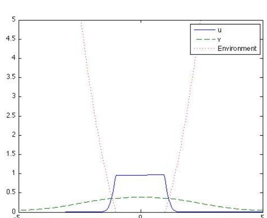
(a) Initial conditions



(b) Social preference, λ_2: 1, Mobility disparity, γ: 5



(c) Social preference, λ_2: 2, Mobility disparity, γ: 5



(d) Social preference, λ_2: .1, Mobility disparity, γ: 5

Fig. 5.4: Numerical solutions to (1.6) with different values of the parameters λ_2 and Γ ($\epsilon = 0$) with $A(x)$.

Crime, safety, and economic disparity. As mentioned in the introduction, and is well known from everyday life, poor neighborhoods have a disproportionate amount of crime [7, 14]. Here, we briefly describe how one can explore the effect of segregation along income lines on crime hotspots using a mathematical model giving results that are qualitatively similar to empirical observations. We summarize the assumptions for this model:

(i) The police enforcement aims to minimize the feeling of insecurity of the community.

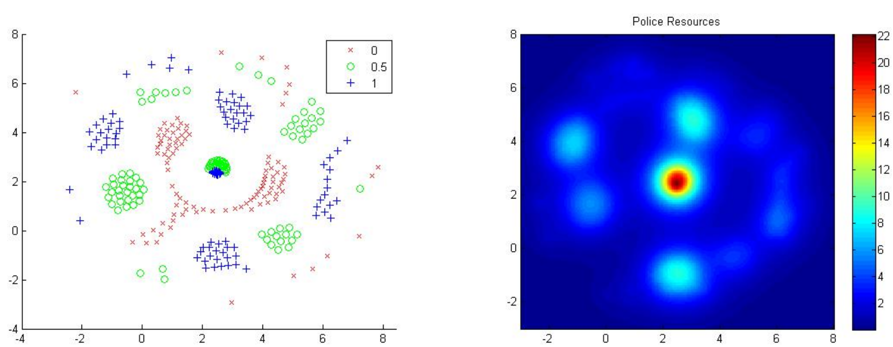
(ii) There are limited resources.

(iii) Insecurity is higher among people that have more resources.

Given a distribution of individuals with prescribed mobilities, we calculate the influence field, $I(x)$, by adding delta functions that are centered at every location where an individual lives and with a mass which depends on the mobility of the individual. Once we obtain the influence field, the objective is to find the distribution of the police resources $p(x)$ that minimizes the functional (1.27), as in (1.28). This is done numerically with the use a gradient descent method, which requires a soft version of the constrain (1.28b). Thus, we minimize

$$F(t)=\int_\Omega \epsilon\left(|\nabla p(x)|^2+U(p(x))\right)\,dx+\frac{1}{\epsilon}\left(\int_\Omega p(x)\,dx-1\right)^2, \tag{5.1}$$

for $\epsilon \ll 1$, which gives more weight to the terms that requires the constraint on the total amount of the police enforcement to be satisfied.



(a) Distribution of the population (b) Distribution of the police resources

Fig. 5.5

Figure 5.5a illustrates the distribution of our population from which we generate an influence field, $I(x)$. Figure 5.5b illustrates the optimal distribution of resources obtained based on the given $I(x)$. We observe the development of safe-havens for regions with a high influence field, where the crime fields are very low. While this does not generate the *crime hotspots* it essentially removes the possibility of hotspots in regions of high influence. The next step, which we leave for a separate publication is to couple the environment attractiveness field $A(x)$ we have considered in the PDE models, to the amount of crime (which would be described via the local police resources) produced via the above minimization procedure. This will lead to a coupled system of PDEs for the population density and the police enforcement.

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