The KPP system in a periodic flow with a heat loss

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Abstract

We consider a reaction-diffusion-advection system of the KPP type in a periodic flow with heat-loss through the boundary. We show, that, as in the case of a shear flow, the propagation speed is determined by the linearization ahead of the front and is thus independent of the Lewis number. Moreover, we show that a flame may be blown-off or be extinguished by the presence of a periodic flow. We present an explicit procedure of constructing a flow which leads to the blow-off or extinction of the flame. The period cell size has to be sufficiently small in order for the flow to extinguish a flame if the channel is wider than critical.

1 Introduction

The presence of a fluid flow may have a profound influence on the combustion processes [18]. This problem had been extensively studied in the engineering and physical literature for quite a long time. The mathematical studies of the flow effect have been intensified during the last decade: see [2, 22] for excellent recent reviews and references. A number of rigorous results have revealed the mathematical aspects of the mechanism behind the speed-up of the combustion fronts by flows, quenching of the flame, existence and stability of travelling waves.

A large majority of the mathematical results, however, have been obtained for the single reaction-diffusion-advection equation. The purpose of this paper is to analyze the qualitative behavior of solutions of the following reaction-diffusion-advection system:

$$
\begin{aligned}
T_t + u \cdot \nabla T &= \Delta T + Y g(T), \\
Y_t + u \cdot \nabla Y &= \frac{1}{\mathrm{Le}} \Delta Y - Y g(T),
\end{aligned}
\tag{1.1}
$$

where $T(t, \mathbf{x})$ is a temperature and $Y(t, \mathbf{x})$ is a concentration of deficient reactant. The Lewis number Le, the ratio of the thermal diffusivity and the diffusivity of the deficient reactant, may be an arbitrary positive number. We recall that if $\mathrm{Le} = 1$ then the system (1.1) reduces to a single reaction-diffusion equation for T, as the constraint $T + Y = 1$ holds provided that initially $T_0 + Y_0 = 1$.

The reaction-diffusion-advection system (1.1) is considered in a cylinder $D = \mathbb{R} \times \Omega$, where $\Omega \subseteq \mathbb{R}^{d-1}$ is a bounded domain, with the heat-loss boundary conditions at $\partial\Omega$:

$$
\frac{\partial T}{\partial n} + qT = 0, \quad \frac{\partial Y}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{1.2}
$$

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Here $q \geq 0$ is the heat loss parameter. The flow $u(\mathbf{x})$ is assumed to be L-periodic in the x-variable: $u(x+L,y) = u(x,y)$, time-independent and incompressible, that is,

$$\nabla \cdot u = 0. \tag{1.3}$$

We also assume that the flow has mean zero over the period:

$$\int_\Omega \int_0^L u(x,y)dxdy = 0. \tag{1.4}$$

The non-linearity $g(T)$ is assumed to be of the KPP-type:

$$g(0) = 0, \quad g(T) > 0, \; g(T) \leq g'(0)T, \; g'(T) \geq 0 \text{ for } T > 0. \tag{1.5}$$

It is well known that even in the absence of the flow the reaction-diffusion system (1.1) may exhibit a very rich behavior: oscillating modes may develop, travelling waves are not unique [8] and hence unstable, etc. However, when the non-linearity $g(T)$ is of the KPP type, the behavior is known to be more regular: for instance, the travelling front speed with $\mathbf{u} = 0$ is independent of the Lewis number [7], and the burning rate in a periodic flow is uniformly bounded above by the pulsating travelling front speed at $\mathrm{Le} = 1$ [15]. Recently a number of results on the behavior of solutions of (1.1) in a shear flow have been obtained in [4]. The same problem with an Arrhenius nonlinearity has been also studied numerically in [10, 11]. The purpose of the present paper is to extend the results of [4] to general periodic flows. In particular we show that, as in the case of a single KPP-type equation, the exponential decay of the initial temperature T_0 determines the asymptotic speed of propagation. The speed itself turns out to be independent of the Lewis number. These results are described in Section 2. We also show the possibility of flame extinction or blow-off by a periodic flow: a sufficiently strong heat-loss parameter may lead to flame quenching. Furthermore, we show that while the heat-loss parameter may be too small to prevent flame propagation in the absence of a flow, a sufficiently strong periodic flow might improve mixing toward the boundary to an extent that leads to flame quenching or blow-off. These results are described in Section 3. Section 4 contains the results of some numerical simulations that illustrate the results of this paper. In particular, it has been previously shown numerically that for a cellular flow to quench an initial data of large compact support for one equation with the ignition nonlinearity the cell has to be sufficiently small [20]. We find that a similar result holds for the KPP system with a heat-loss: a sufficiently fast periodic cellular flow will extinguish a flame in an arbitrary wide channel provided that the cell size is sufficiently small. We conjecture that the critical amplitude necessary to extinguish the flame scales as $A_c \sim W^4$, where W is the channel width. Finally, Section 5 contains some conclusions.

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2 Front Propagation in the KPP system

We consider in this section the spreading rate of the solution with the initial data that is decaying at one end of the domain with a prescribed decay rate $O\left(e^{-\lambda x}\right)$.

2.1 Pulsating KPP travelling fronts

We first recall the classical results for a single KPP reaction-diffusion equation in a uniform medium. Travelling front solutions of the Fisher-KPP equation

$$\frac{\partial T}{\partial t} = T_{xx} + g(T)(1-T) \tag{2.1}$$

of the form $U_c(x-ct)$ with

$$U_c(x) \to 1 \text{ as } x \to -\infty,\, U_c(x) \to 0 \text{ as } x \to +\infty$$

exist for all $c \geq c_* = 2\sqrt{g'(0)}$. They are distinguished by their decay rate as $x \to +\infty$:

$$U_c(x) \sim O\left(e^{-\lambda_c x}\right), \text{ as } x \to +\infty.$$

The front speed and the corresponding decay rate are related by the quadratic equation

$$c\lambda = \lambda^2 + g'(0). \tag{2.2}$$

This relation may be formally obtained by looking for an exponential solution $\tilde{T} = \exp\{-\lambda(x-ct)\}$ of the linearized equation

$$\frac{\partial \tilde{T}}{\partial t} = \tilde{T}_{xx} + g'(0)\tilde{T}. \tag{2.3}$$

More generally, solutions of (2.1) with the initial data with a prescribed exponential decay

$$T_0(x) \sim O\left(e^{-\lambda x}\right), \text{ as } x \to +\infty \tag{2.4}$$

propagate to the right with the speed $c(\lambda)$ determined by (2.2) provided that $\lambda < \lambda_* = \sqrt{g'(0)}$. More precisely, solution that satisfies (2.4) converges as $t \to +\infty$ to a travelling front solution $U_c(x - x_0 - ct)$ that propagates with the speed $c(\lambda)$.

The simple idea of the linearization of the KPP equation at infinity in order to obtain the relation between the decay rate and the front speed has been generalized by Berestycki, Nadirashvili and Hamel to the pulsating travelling fronts in a periodic medium in [3, 5]. In particular, they considered a reaction-diffusion-advection equation

$$\frac{\partial T}{\partial t} + u(\mathbf{x}) \cdot \nabla T = \Delta T + g(T)(1-T) \tag{2.5}$$

in a cylinder $D = \mathbb{R} \times \Omega$, $\Omega \subset \mathbb{R}^{d-1}$ with the Neumann boundary conditions

$$\frac{\partial T}{\partial n} = 0 \text{ on } \partial\Omega \tag{2.6}$$

on the boundary of Ω, and the front-like boundary conditions as $x_1 \to \pm\infty$:

$$T(x_1, y) \to 0 \text{ as } x_1 \to -\infty,\, T(x_1, y) \to 0 \text{ as } x_1 \to +\infty, \text{ uniformly in } y \in \Omega.$$

The domain Ω is bounded. The full system (1.1),(1.2) reduces to (2.5),(2.6) when the Lewis number $\mathrm{Le} = 1$ and the heat-loss parameter $q = 0$. A pulsating travelling front is a solution of (2.5) of the form $U_c(x_1 - ct, \mathbf{x})$ that is periodic in the second variable and satisfies

$$U_c(s, \mathbf{x}) \to 1 \text{ as } s \to -\infty,\, U_c(s, \mathbf{x}) \to 0 \text{ as } s \to +\infty$$

It has been shown in [3] that there exists $c_* > 0$ so that such fronts exist for $c \geq c_*$. Their decay rate λ is related to the front speed c as follows. Let $\eta(\lambda)$ and $\Psi_\lambda > 0$ be the principal eigenvalue and the positive eigenfunction of

$$-\Delta\Psi_\lambda + u(\mathbf{x}) \cdot \nabla\Psi_\lambda - \lambda u_1 \Psi_\lambda + 2\lambda \frac{\partial \Psi_\lambda}{\partial x_1} = \eta(\lambda)\Psi_\lambda \text{ in } [0, L] \times \Omega \tag{2.7}$$

$$\frac{\partial \Psi_\lambda}{\partial n} = 0 \text{ on } \partial\Omega$$

Ψ_λ is L-periodic in x.

Then λ and c are related by

$$c\lambda = \lambda^2 + g'(0) - \eta(\lambda). \tag{2.8}$$

It has been also shown in [3] that the function $h_0(\lambda) = \eta(\lambda) - \lambda^2$ is concave and satisfies

$$h_0(0) = 0, \quad h'(0) = 0. \tag{2.9}$$

This implies the following result [3].

Theorem 2.1 *There exists c_* so that equation (2.8) has no positive solution λ for $c < c_*$, one solution $\lambda_* > 0$ for $c = c_*$ and two solutions $0 < \lambda < \lambda'$ for $c > c_*$. The pulsating travelling fronts $U_c(s, x, y)$ exist for all $c \geq c_*$ and has the decay rate λ, where λ is the smaller solution of (2.8).*

Hence (2.8) serves as a generalization of the quadratic equation (2.2) to the periodic case. The eigenvalue problem (2.7) may be formally obtained in a way similar to the uniform case: one looks for solutions of the linearized problem

$$\frac{\partial \tilde{T}}{\partial t} + u(\mathbf{x}) \cdot \nabla \tilde{T} = \Delta \tilde{T} + g'(0)\tilde{T}$$

of the form

$$\tilde{T}(t, x, y) = \exp\{-\lambda(x - ct)\}\Psi_\lambda(x, y) \tag{2.10}$$

with a positive, L-periodic in x function $\Psi_\lambda(x, y)$. A direct calculation shows that Ψ_λ has to satisfy the eigenvalue problem (2.7)-(2.8).

2.2 Fronts in a KPP system

We show in this section that, as in the case of a single equation (2.1) in a uniform medium, the linearized version of the reaction-diffusion-advection system (1.1) predicts the correct speed of propagation in terms of the rate of the exponential decay of the initial data. Linearizing (1.1) at infinity where $Y = 1$ and looking for solutions of the form (2.10) we obtain the following eigenvalue problem

$$\begin{aligned}
&-\Delta\Psi_\lambda + u(\mathbf{x}) \cdot \nabla\Psi_\lambda - \lambda u_1 \Psi_\lambda + 2\lambda \frac{\partial \Psi_\lambda}{\partial x_1} = \mu(\lambda)\Psi_\lambda \text{ in } [0, L] \times \Omega \\
&\frac{\partial \Psi_\lambda}{\partial n} + q\Psi_\lambda = 0 \text{ on } \partial\Omega, \\
&\Psi_\lambda \text{ is periodic in } x, \ \Psi_\lambda \geq 0 \text{ in } \Omega.
\end{aligned} \tag{2.11}$$

The function $\tilde{T}(t, x, y) = \exp\{-\lambda(x - ct)\}\Psi_\lambda(x, y)$ then satisfies the linearized problem

$$\begin{aligned}
&\frac{\partial \tilde{T}}{\partial t} + u(\mathbf{x}) \cdot \nabla \tilde{T} = \Delta \tilde{T} + g'(0)\tilde{T} \\
&\frac{\partial \tilde{T}}{\partial n} + q\tilde{T} = 0 \text{ on } \partial\Omega,
\end{aligned}$$

provided that c and λ are related by

$$c\lambda = \lambda^2 + g'(0) - \mu(\lambda). \tag{2.12}$$

However, a direct analog of Theorem 2.1 no longer holds in the heat-loss case in general, that is, when $q > 0$. The reason lies in the simple fact that when the domain Ω is of a sufficiently small volume, $\mu(0) > g'(0)$. This brings about the main difference between the two eigenvalue problems

(2.7) and (2.11) that arise in the adiabatic and the heat-loss cases, respectively. In particular, the minimal speed c_* may not be defined as in Theorem 2.1. It is also possible that, while $\mu(0) < 1$, the minimal speed $c_* < 0$. These two possibilities, that we refer to as flame extinction and blow-off, respectively, are considered in Section 3.

Nevertheless, in the presence of the heat-loss parameter we still have the following result.

Proposition 2.2 *Assume that $\mu(0) < g'(0)$. There exists c_* so that equation (2.12) has no positive solution λ for $c < c_*$, one solution $\lambda_* > 0$ for $c = c_*$ and two solutions $0 < \lambda < \lambda'$ for $c > c_*$.*

The proof of Proposition 2.2 is very close to that of Theorem 2.1 of [3]. We present it for the convenience of the reader but postpone until the end of this section.

One may expect that, as in the uniform and scalar case (as well as in the presence of a shear (unidirectional) flow) solutions of the KPP system (1.1)-(1.2) with a heat-loss that have initial data

$$T_0(x,y) \sim O\left(e^{-\lambda x}\right)$$

that decay at an exponential rate λ, propagate with the asymptotic speed $c(\lambda)$ determined by (2.12). The main result of this section is that this is indeed the case. This is a generalization of the corresponding result in [4] for a shear flow.

Consider the reaction-diffusion-advection system (1.1) with the boundary conditions (1.2). The periodic flow u is assumed to satisfy (1.3), (1.4). The initial conditions $T_0(x,y)$ for temperature and $Y_0(x,y)$ for concentration satisfy

$$0 \le T_0(x,y), M := \sup T_0(x,y) > 0, \quad 0 \le Y_0 \le 1, \quad \forall (x,y) \in D \tag{2.13}$$
$$C^{-1}e^{-\lambda x} \le T_0 \le Ce^{-\lambda x}, \quad x \ge 0,\ y \in \bar{\Omega} \tag{2.14}$$
$$1 - Y_0(x,y) \le Ce^{-\lambda' x}, \quad x \ge 0,\ y \in \bar{\Omega},\ \lambda' > 0 \tag{2.15}$$

with a positive constant $C > 0$.

Theorem 2.3 *Assume that $g(T) = T$, $\mu(0) < 1$ and let λ_* be as in Proposition 2.2. Any solution of the problem (1.1),(1.2), with the initial data satisfying (2.13)-(2.15) with $\lambda < \lambda_*$ propagates with the speed $c(\lambda)$ determined by (2.12), in the following sense: for any $\tilde{c} > c$ and any $x \in \mathbb{R}$, $y \in \Omega$ we have $T(t, x + \tilde{c}t, y) \to 0$ as $t \to \infty$, while for each $x \in \mathbb{R}$, $y \in \Omega$ there exists a constant $a > 0$ so that $T(t, x + ct, y) > a$ for all $t > 0$.*

In particular, Theorem 2.3 generalizes the result of [21] where it has been shown that solutions of a scalar KPP equation equation in a periodic medium with compactly supported initial data propagate with the minimal speed of a pulsating travelling front. However, our main result here concerns only the propagation speed and does not provide any information about convergence of the solution of the Cauchy problem to a travelling wave. The problem of existence of pulsating travelling fronts for the KPP system remains open. We note that the assumption that $g(T) = T$ is made only to simplify the proof and present the basic idea of the alternating construction of sub- and super-solutions in a clear fashion.

The fact that the speed of propagation in a periodic flow does not depend on the Lewis number generalizes this observation for the adiabatic case in a uniform medium made in [7]. We also note that an upper bound for the spreading rate of a solution of the adiabatic KPP system with a compactly supported initial data by the minimal pulsating travelling front speed for a single equation has been obtained in [15].

Proof of Theorem 2.3. The main tool in obtaining asymptotic speed of propagation and convergence to a travelling front solution in the case of a single reaction-diffusion equation is the maximum principle and comparison to appropriate perturbations of travelling wave solutions [19]. Unfortunately, neither the maximum principle holds nor pulsating travelling fronts are known to exist in the case of systems which creates the main technical difficulties. Nevertheless the proof of Theorem 2.3 is based on the technique of constructing sub- and super-solutions that propagate with the same speed. These sub- and super-solutions are constructed by iteration. Using the fact that $\bar{Y} = 1$ is a super-solution for concentration $Y(t,x,y)$ we construct a super-solution for temperature $T(t,x,y)$ which is unbounded but still propagates with the speed determined by (2.12). Using this super-solution for temperature we construct a sub-solution for the concentration which in turn allows to construct a sub-solution for the temperature. The main point is that the sub- and super-solutions for temperature and concentration propagate with the same speed $c(\lambda)$ given by (2.12). Therefore the propagation speed of the solution with an initial data as in (2.13)-(2.15) is also given by (2.12).

Step 1. A super-solution for temperature. The maximum principle implies that that the concentration $Y(t,x,y) \leq 1$. Hence a function $\bar{T}$ that satisfies (here and below $x \in \mathbb{R}$, $y \in \Omega$)

$$\begin{aligned} &\bar{T}_t + u \cdot \nabla \bar{T} \geq \Delta \bar{T} + \bar{T}, \\ &\frac{\partial \bar{T}}{\partial n} + q\bar{T} = 0, \text{ on } \partial\Omega, \\ &\bar{T}(0,x,y) \geq T_0(x,y), \end{aligned} \tag{2.16}$$

is a super-solution:

$$T(t,x,y) \leq \bar{T}(t,x,y). \tag{2.17}$$

We are looking for such a super-solution in the form

$$\bar{T}(t,x,y) = Me^{-\lambda(x-ct)}\Psi_\lambda(x,y), \tag{2.18}$$

where $\Psi_\lambda(x,y)$ is the principal eigenfunction of the eigenvalue problem (2.11) that we re-write as

$$\begin{aligned} &\mathcal{L}_\lambda \Psi_\lambda = \mu(\lambda)\Psi_\lambda \\ &\frac{\partial \Psi_\lambda}{\partial n} + q\Psi_\lambda = 0, \quad \text{on } \partial\Omega, \\ &\Psi_\lambda \text{ is periodic in } x \text{ and } \Psi_\lambda \geq 0, \end{aligned} \tag{2.19}$$

with

$$\mathcal{L}_\lambda = -\Delta + u \cdot \nabla - \lambda u_1 + 2\lambda\partial_x \tag{2.20}$$

The function Ψ_λ is normalized so that

$$\int_\Omega \int_0^L \Psi_\lambda^2(x,y)dxdy = 1. \tag{2.21}$$

A direct calculation shows that the function $\bar{T}$ defined by (2.18) is a super-solution provided that $c(\lambda)$ is given by (2.12). Therefore, (2.17) holds provided that the constant M is chosen large enough to ensure that $T_0(x,y) \leq \bar{T}(0,x,y)$.

Step 2. A sub-solution for concentration. Since the function $\bar{T}$ given by (2.18) is a super-solution for T, the function $\underline{Y}$ that satisfies

$$\begin{aligned} &\frac{\partial \underline{Y}}{\partial t} + u \cdot \nabla \underline{Y} \leq \frac{1}{\mathrm{Le}} \Delta \underline{Y} - \bar{T}\underline{Y} \\ &\frac{\partial \underline{Y}}{\partial n} = 0, \text{ on } \partial\Omega, \\ &\underline{Y}(0,x,y) \leq Y_0(x,y), \end{aligned} \tag{2.22}$$

is a sub-solution for concentration $Y(t,x,y)$:

$$\underline{Y}(t,x,y) \leq Y(t,x,y). \tag{2.23}$$

We are looking for a solution of (2.22) in the following form

$$\underline{Y}(t,x,y) = 1 - \beta e^{-\gamma(x-ct)}\Phi_\gamma(x,y). \tag{2.24}$$

Here β and γ are some constants to be determined later. The function $\Phi_\gamma > 0$ is the principal normalized eigenfunction of the following eigenvalue problem:

$$\left(-\frac{1}{\mathrm{Le}}\Delta + 2\gamma\partial_x + u\cdot\nabla - \gamma u_1\right)\Phi_\gamma = \mu_\nu(\gamma)\Phi_\gamma, \quad \int_\Omega\int_0^L \Phi_\gamma^2(x,y)dxdy = 1 \tag{2.25}$$

$$\frac{\partial\Phi_\gamma}{\partial n} = 0 \text{ on } \partial\Omega,\ \Phi_\gamma \text{ is periodic in } x.$$

Substituting expression (2.24) for $\underline{Y}$ into (2.22), we see that for the function $\underline{Y}$ defined by (2.24) to be a sub-solution for the concentration Y we need

$$\left(-c\gamma - \frac{1}{\mathrm{Le}}\gamma^2 - \mu_\nu(\gamma)\right)\Phi_\gamma \leq -\frac{1}{\beta}Me^{-(\lambda-\gamma)\xi}(1-e^{-\gamma\xi}\Phi_\gamma), \quad \xi = x - ct.$$

This condition is satisfied provided that

$$-c\gamma - \frac{1}{\mathrm{Le}}\gamma^2 - \mu_\nu(\gamma) \leq -\frac{1}{\beta}Me^{-(\lambda-\gamma)\xi}\left(\frac{1}{m_\gamma} - e^{-\gamma\xi}\right) \tag{2.26}$$

with

$$m_\gamma = \inf \Phi_\gamma(x,y) > 0$$

We now choose the constants γ and β as follows. We let $\gamma < \lambda$ be sufficiently small. It suffices to ensure that

$$-c\gamma - \frac{1}{\mathrm{Le}}\gamma^2 - \mu_\nu(\gamma) \leq -\frac{1}{\beta}M\left(\frac{1}{m_\gamma} - e^{-\gamma\xi}\right) \tag{2.27}$$

in order for (2.26) to hold for $\xi > 0$. Let us first show that the function

$$s(\gamma) = -c\gamma - \frac{1}{\mathrm{Le}}\gamma^2 - \mu_\nu(\gamma)$$

is negative for small γ. First, we observe that $\mu_\nu(0) = 0$ since $\Phi_0 = |\Omega^{-1/2}|$ is the corresponding eigenfunction. Thus we have $s(0) = 0$. Next, differentiating (2.25) with respect to γ at $\gamma = 0$ we obtain

$$-\frac{1}{\mathrm{Le}}\Delta\Phi_0' + u\cdot\nabla\Phi_0' + 2\frac{\partial\Phi_0}{\partial x} - u_1\Phi_0 = \mu_\nu(0)\Phi_0' + \mu_\nu'(0)\Phi_0. \tag{2.28}$$

Here prime denotes derivative with respect to γ. Multiplying (2.28) by $\Phi_0 = |\Omega|^{-1/2}$, integrating over the period cell, and using the fact that the flow u has mean zero we obtain

$$\mu_\nu'(0) = 0$$

Thus $ds(0)/d\gamma = -c < 0$ and therefore $s(\gamma) < 0$ for small γ. The formal differentiation may be justified as in [6]. Therefore, in order for (2.27) (and hence (2.26)) to hold for $\xi > 0$ we may choose β to be

$$\beta = \frac{M}{m_\gamma(c\gamma + \gamma^2/Le + \mu_\nu(\gamma))} \tag{2.29}$$

with a sufficiently small $\gamma > 0$.

Furthermore, in order for (2.26) to hold for $\xi < 0$ we need

$$-c\gamma - \frac{1}{\mathrm{Le}}\gamma^2 - \mu_\nu(\gamma) \le -\frac{M}{\beta}e^{(\lambda-\gamma)|\xi|}\left(\frac{1}{m_\gamma} - e^{\gamma|\xi|}\right) \tag{2.30}$$

However, with β as in (2.29) condition (2.30) becomes

$$1 \ge (1 - m_\gamma e^{\gamma|\xi|})e^{(\gamma-\lambda)|\xi|}$$

which is true since $0 < \gamma < \lambda$. Therefore, the function $\underline{Y}$ given by (2.24) is a sub-solution for Y, that is, (2.23) holds, provided that $\gamma \in (0, \lambda)$ and β is given by (2.29). We then set

$$\underline{V}(t, x, y) = \max(0, 1 - \alpha e^{-\gamma(x-ct)}) \tag{2.31}$$

with

$$\alpha = \max\{1, \beta \max \Phi_\gamma\}.$$

We have then $\underline{V} \le Y$ after possibly increasing α so as to guarantee that $\underline{V}(0, x) \le Y_0(x, y)$.

Step 3. A sub-solution for temperature. We now construct a sub-solution $\underline{T}(t, x, y)$ for the temperature $T(t, x, y)$ that also propagates with the speed $c(\lambda)$ given by (2.12). We are looking for such a sub-solution in the following form

$$\underline{T}(t, x, y) = \Psi_\lambda(x, y)e^{-\lambda(x-ct)} - K\Psi_{\lambda+\delta}(x, y)e^{-(\lambda+\delta)(x-ct)}, \tag{2.32}$$

where Ψ_λ and $\Psi_{\lambda+\delta}$ are the principal eigenfunctions of the eigenvalue problem (2.19) corresponding to λ and $\lambda + \delta$, respectively. The positive constants K and δ are to be determined later. Since $\underline{V}$ defined by (2.31) is smaller than the concentration Y, any solution of

$$\begin{aligned} &\underline{T}_t + u \cdot \nabla \underline{T} \le \Delta \underline{T} + \underline{V}\ \underline{T} \\ &\frac{\partial \underline{T}}{\partial n} + q\underline{T} = 0, \text{ on } \partial\Omega, \\ &\underline{T}(0, x, y) \le T_0(x, y), \end{aligned} \tag{2.33}$$

is a sub-solution for T:

$$T(t, x, y) \ge \underline{T}(t, x, y). \tag{2.34}$$

Substituting (2.32) into (2.33), we observe that for $\underline{T}$ given by (2.32) to be a sub-solution we need

$$(c\lambda - \lambda^2 + \mathcal{L}_\lambda\Psi_\lambda)e^{-\lambda\xi} - K(c(\lambda+\delta) - (\lambda+\delta)^2 + \mathcal{L}_{\lambda+\delta})\Psi_{\lambda+\delta})e^{-(\lambda+\delta)\xi} \le \underline{V}(\Psi_\lambda e^{-\lambda\xi} - K\Psi_{\lambda+\delta}e^{-(\lambda+\delta)\xi}),$$

that is,

$$(c\lambda - \lambda^2 + \mu(\lambda)) - K\frac{\Psi_{\lambda+\delta}}{\Psi_\lambda}(c(\lambda+\delta) - (\lambda+\delta)^2 + \mu(\lambda+\delta))e^{-\delta\xi} \le \underline{V}(1 - K\frac{\Psi_{\lambda+\delta}}{\Psi_\lambda}e^{-\delta\xi}). \tag{2.35}$$

Here, as before, we denote $\xi = x - ct$.

Consider first the case when $\alpha e^{-\gamma\xi} < 1$. Substituting expression (2.31) for $\underline{V}$ into (2.35) and taking into account expression (2.12) for the speed c, we obtain that (2.35) is equivalent in this region to

$$-K\left[c(\lambda+\delta) - (\lambda+\delta)^2 + \mu(\lambda+\delta) - 1\right]\frac{\Psi_{\lambda+\delta}}{\Psi_\lambda}e^{-\delta\xi} \le -\alpha e^{-\gamma\xi} + \alpha K\frac{\Psi_{\lambda+\delta}}{\Psi_\lambda}e^{-(\gamma+\delta)\xi}. \tag{2.36}$$

We define the function

$$r(\delta) = c(\lambda+\delta) - (\lambda+\delta)^2 + \mu(\lambda+\delta) - 1 \tag{2.37}$$

with $c = c(\lambda)$ given, as usual, by (2.12). Let us show that $r(\delta) > 0$ for small δ. Indeed, as (2.12) implies that $r(0) = 0$, it suffices to show that $dr(0)/d\delta > 0$. We have, using (2.12),

$$\frac{dr(0)}{d\delta} = c - 2\lambda + \mu'(\lambda) = -\lambda\frac{dc(\lambda)}{d\lambda}.$$

However, Proposition 2.2 implies that

$$\frac{dc(\lambda)}{d\lambda} < 0$$

for all $\lambda < \lambda^*$. Therefore, $\dfrac{dr(0)}{d\delta} > 0$ and hence, indeed, $r(\delta) > 0$ for $\delta > 0$ sufficiently small.

Hence, in order for (2.36) to hold in the region, where $\underline{V} > 0$, it suffices to choose $\delta \in (0,\gamma)$ so small that $r(\delta) > 0$ and then set

$$K = \frac{\alpha}{r(\delta)} \sup \frac{\Psi_\lambda}{\Psi_{\lambda+\delta}}. \tag{2.38}$$

Consider next the region where $\underline{V} = 0$, that is, $\xi \le \xi_0 = \gamma^{-1}\ln\alpha$, as follows from (2.31). In this case (2.35) becomes

$$1 - K\frac{\Psi_{\lambda+\delta}}{\Psi_\lambda}(r(\delta)+1)e^{-\delta\xi} \le 0$$

It is sufficient to verify that this inequality holds for $\xi = \xi_0$, that is, whether

$$1 - \frac{K}{\alpha^{\delta/\gamma}}\frac{\Psi_{\lambda+\delta}}{\Psi_\lambda}(g(\delta)+1) \le 0.$$

However, the latter is true since $\delta < \gamma$, α is chosen so that $\alpha > 1$ and because of the choice (2.38) of the constant K.

Finally, we observe that we have shown that the temperature $T(t,x,y)$ satisfies

$$\Psi_\lambda(x,y)e^{-\lambda(x-ct)} - K\Psi_{\lambda+\delta}(x,y)e^{-(\lambda+\delta)(x-ct)} \le T(t,x,y) \le Me^{-\lambda(x-ct)}\Psi_\lambda(x,y).$$

Observe that the function on the left side is positive on an open set. Hence the conclusion of Theorem 2.3 holds. □

Proof of Proposition 2.2. We now prove Proposition 2.2. Let us recall the eigenvalue problem (2.11)

$$\begin{aligned}
&\mathcal{L}_\lambda\Psi_\lambda := -\Delta\Psi_\lambda + u(\mathbf{x})\cdot\nabla\Psi_\lambda - \lambda u_1\Psi_\lambda + 2\lambda\frac{\partial\Psi_\lambda}{\partial x} = \mu(\lambda)\Psi_\lambda \text{ in } [0,L]\times\Omega \\
&\frac{\partial\Psi_\lambda}{\partial n} + q\Psi_\lambda = 0 \text{ on } \partial\Omega, \\
&\Psi_\lambda \text{ is periodic in } x,\ \Psi_\lambda \ge 0 \text{ in } \Omega
\end{aligned} \tag{2.39}$$

and relation (2.12) between the speed c and decay rate λ:

$$c\lambda = \lambda^2 + g'(0) - \mu(\lambda). \tag{2.40}$$

We will show that there exists c_* such that (2.40) has no solutions for $c < c_*$, one solution if $c = c_*$ and two solutions $\lambda_1 < \lambda_* < \lambda_2$ for $c > c_*$ by showing that the right side of (2.40) is a convex function, that is positive at $\lambda = 0$. Hence c_* is the slope of the line tangent to the graph of $\lambda^2 + g'(0) - \mu(\lambda)$

that passes through the point $(0,0)$. To this end we define $h(\lambda) = \mu(\lambda) - \lambda^2$ and show that $h(\lambda)$ is concave.

In order to show that the function $h(\lambda)$ is concave we first establish a min-max principle for $h(\lambda)$: let

$$E_\lambda = \left\{ \phi \in C^2(\bar{\Omega}) : \ \phi(x,y) > 0, \ \ \psi = e^{\lambda x}\phi \text{ is } L\text{-periodic in } x, \ \ \frac{\partial \phi}{\partial n} + q\phi = 0 \text{ on } \partial\Omega \right\}, \tag{2.41}$$

we claim that

$$h(\lambda) = \max_{\phi \in E_\lambda} \inf_{\bar{D}_L} \frac{\mathcal{M}\phi}{\phi} = \max_{\phi \in E_\lambda} \inf_{\bar{D}_L} \frac{-\Delta\phi + u \cdot \nabla\phi}{\phi}. \tag{2.42}$$

Here $D_L = [0,L] \times \Omega$ is a flow period cell – the ratio in (2.42) does not depend on the choice of the period cell. Indeed, using $\theta_\lambda = \Psi_\lambda e^{-\lambda x}$ we obtain immediately that $\mathcal{M}\theta_\lambda = h(\lambda)\theta_\lambda$ so that

$$h(\lambda) \equiv \frac{\mathcal{M}\theta_\lambda}{\theta_\lambda} = \inf_{\bar{D}_L} \frac{\mathcal{M}\theta_\lambda}{\theta_\lambda} \le \max_{\phi \in E_\lambda} \inf_{\bar{D}_L} \frac{\mathcal{M}\phi}{\phi}.$$

since the function $\theta_\lambda \in E_\lambda$. Assume now that there exists a function $\phi \in E_\lambda$ so that

$$\mu(\lambda) < \inf_{\bar{\Omega}} \frac{\mathcal{M}\phi}{\phi},$$

that is

$$\mathcal{M}\phi > (h(\lambda) + \eta)\phi$$

with $\eta > 0$. As both functions ϕ and θ_λ are positive and continuous over the flow period cell, we may choose $\tau > 0$ so that $\phi \ge \tau\theta_\lambda$ for all $(x,y) \in D$ and there exists (x_0,y_0) so that $\phi(x_0,y_0) = \tau\Psi_\lambda(x_0,y_0)$. Let $w = \phi - \tau\theta_\lambda$, then the function w satisfies $w \ge 0$ and $w(x_0,y_0) = 0$, while

$$-\Delta w + u \cdot \nabla w > hw + \eta\phi.$$

Then the function $\tilde{w} = e^{\lambda x} w$ is periodic, has a minimum equal to zero at (x_0,y_0) and satisfies

$$-\Delta\tilde{w} + u \cdot \nabla\tilde{w} - \lambda^2\tilde{w} + 2\lambda\frac{\partial\tilde{w}}{\partial x} - \lambda u_1\tilde{w} > h\tilde{w} + \eta\tilde{\phi}$$

with a positive periodic function $\tilde{\phi} = e^{\lambda x}\phi$. Hence if the point (x_0,y_0) is an internal minimum of $\tilde{w}$, then $-\Delta\tilde{w}(x_0,y_0) > \eta\tilde{\phi} > 0$ which is impossible. However, $\tilde{w}$ may not attain a minimum equal to zero on the boundary $\partial\Omega$ because of the Hopf lemma and the boundary condition in the definition of E_λ. Hence $w \equiv 0$ which implies that $\phi = \eta\theta_\lambda$ and hence $\eta = 0$, which contradicts the assumption that $\eta > 0$. Hence (2.42) holds.

We now use (2.42) to show that the function $h(\lambda)$ is concave. We will show that

$$h(t\lambda_1 + (1-t)\lambda_2) \ge th(\lambda_1) + (1-t)h(\lambda_2) \text{ for all } 0 \le t \le 1.$$

The min-max principle (2.42) implies that it suffices to show that given any pair of functions $f_1 \in E_{\lambda_1}$ and $f_2 \in E_{\lambda_2}$ there exists a function $\phi \in E_\lambda$, $\lambda = t\lambda_1 + (1-t)\lambda_2$, so that

$$\frac{\mathcal{M}\phi}{\phi} \ge t\frac{\mathcal{M}f_1}{f_1} + (1-t)\frac{\mathcal{M}f_2}{f_2}. \tag{2.43}$$

We claim that (2.43) holds with

$$\phi = f_1^t f_2^{1-t}. \tag{2.44}$$

Indeed, if $f_1 \in E_{\lambda_1}$, $f_2 \in E_{\lambda_2}$ it is straightforward to check both that the function ϕ satisfies the correct boundary conditions in (2.41) and that $e^{\lambda x}\phi$ is L-periodic in x, so that $\phi \in E_\lambda$. We first verify that

$$\begin{aligned}
\Delta(f_1^t f_2^{1-t}) &= t f_1^{t-1} f_2^{1-t} \Delta f_1 + (1-t) f_1^t f_2^{-t} \Delta f_2 - t(1-t) f_1^t f_2^{1-t} \left(\frac{|\nabla f_1|}{f_1} - \frac{|\nabla f_2|}{f_2} \right)^2 \\
&\le f_1^t f_2^{1-t} \left[t \frac{\Delta f_1}{f_1} + (1-t) \frac{\Delta f_2}{f_2} \right].
\end{aligned}$$

Furthermore, using the above inequality and the function ϕ as in (2.44), we obtain

$$\begin{aligned}
\frac{\mathcal{M}\phi}{\phi} &= \frac{1}{f_1^t f_2^{1-t}} \left[-\Delta(f_1^t f_2^{1-t}) + u \cdot \nabla(f_1^t f_2^{1-t}) \right] \\
&\ge -t \frac{\Delta f_1}{f_1} - (1-t) \frac{\Delta f_2}{f_2} + t \frac{u \cdot \nabla f_1}{f_1} + (1-t) \frac{u \cdot \nabla f_2}{f_2} = t \frac{\mathcal{M} f_1}{f_1} + (1-t) \frac{\mathcal{M} f_2}{f_2}.
\end{aligned}$$

Thus (2.43) holds and thus the function $h(\lambda)$ is concave. This finishes the proof of Proposition 2.2. □

3 Flame extinction and blow off

We consider in this section the possibility of blow-off and extinction of solutions of the system (1.1) with the boundary conditions (1.2). We are interested in the effect of a strong flow – hence we replace the flow u in (1.1) by Au with the flow amplitude $A \gg 1$. Accordingly, we denote by $\mu(s, A)$ and $\Psi^A(x, y)$ the principal eigenvalue and eigenfunction of (2.11) with u replaced by Au and λ by s/A:

$$-\Delta\psi^A - s u_1 \psi^A + A u \cdot \nabla\psi^A + \frac{2s}{A} \frac{\partial \psi^A}{\partial x} = \mu(s, A)\psi^A, \quad \text{in} \quad [0, L] \times \Omega \tag{3.1}$$

$$\frac{\partial \psi^A}{\partial n} + q\psi^A = 0, \quad \text{on} \quad \partial\Omega. \tag{3.2}$$

The function ψ^A is L-periodic in x.

Similarly, we denote by $\tilde{\mu}(s)$ and $\tilde{\phi}$ the principal eigenvalue and eigenfunction of the problem

$$\begin{aligned}
&-\Delta\tilde{\phi} - s u_1 \tilde{\phi} = \tilde{\mu}(s)\tilde{\phi} \quad (x, y) \in \Omega \\
&\frac{\partial \tilde{\phi}}{\partial n} + q\tilde{\phi} = 0, \quad y \in \partial\Omega
\end{aligned} \tag{3.3}$$

with $\tilde{\phi}$ being L-periodic in x.

The eigenfunctions ψ^A and $\tilde{\phi}$ are positive and normalized so that

$$\int_\Omega |\psi^A|^2 d\Omega = 1, \quad \int_\Omega \tilde{\phi}^2 d\Omega = 1 \tag{3.4}$$

Moreover, with this normalization we have

$$\mu(s, A) = \int_\Omega |\nabla\psi^A|^2 d\Omega + q \int_{\partial\Omega} |\psi^A|^2 dS_y - s \int_\Omega u_1 |\psi^A|^2 d\Omega \tag{3.5}$$

and

$$\tilde{\mu}(s) = \int_\Omega |\nabla\tilde{\phi}|^2 d\Omega + q \int_{\partial\Omega} \tilde{\phi}^2 dS_y - s \int_\Omega u_1 \tilde{\phi}^2 d\Omega \tag{3.6}$$

The eigenvalue $\tilde{\mu}(s)$ satisfies a variational principle

$$\tilde{\mu}(s) = \inf_{\psi \in H_q, ||\psi||_2=1} \left(\int_\Omega |\nabla\psi|^2 d\Omega + q \int_{\partial\Omega} \psi^2 dS_y - s \int_\Omega u_1 \psi^2 d\Omega \right) \tag{3.7}$$

with

$$H_q = \{\psi \in H^2(\Omega) : \partial_n\psi + q\psi = 0, \quad y \in \partial\Omega, \quad \text{and } L\text{-periodic in } x\}.$$

Expression (3.5) together with the variational principle (3.7) imply that

$$\mu(s,A) \geq \tilde{\mu}(s) \quad \text{for all} \quad s, A \tag{3.8}$$

Theorem 3.1 *Let the initial data T_0, Y_0 satisfy (2.13)-(2.15). (a) Blow-off: assume that there exists s_0 so that $\tilde{\mu}(s_0) > g'(0)$. Then there exist $A_0 \in [0, \max(s_0/\lambda, s_0/\sqrt{\tilde{\mu}(s_0) - g'(0)})]$ so that for all $A > A_0$ we have $T(x,y,t) \leq Ce^{-\eta(x+\gamma t)}$ for all $t \geq 0$. The positive constants η and γ depend on A, and there exists a constant $C > 0$ so that the blow-off speed $\gamma(A) \geq CA$. (b) Extinction: moreover, if $\mu(0,0) = \tilde{\mu}(0) > g'(0)$, then $T(x,y,t) \leq Ce^{-\gamma_0 t}$ with $\gamma_0 > 0$ independent of A.*

Proof. We observe that $0 \leq Y(x,y,t) \leq 1$ for all $t \geq 0$, as follows from the maximum principle. Therefore, $0 \leq T(x,y,t) \leq \Psi(x,y,t)$, where $\Psi(x,y,t)$ is any solution of

$$\Psi_t + Au \cdot \nabla\Psi \geq \Delta\Psi + g'(0)\Psi \quad (x,y) \in D \tag{3.9}$$

provided that $T_0(x,y) \leq \Psi(x,y,0)$. We seek the super-solution in the form

$$\Psi(x,y,t) = Ce^{-\eta(x+\gamma t)}\psi^A(x,y).$$

Inserting this expression into (3.9) we obtain

$$-\Delta\psi^A - \eta A u_1 \psi^A + Au \cdot \nabla\psi^A + 2\eta\psi_x^A \geq (g'(0) + \eta^2 + \eta\gamma)\psi^A \quad \text{in} \quad [0,L] \times \Omega. \tag{3.10}$$

This is true provided that

$$\eta\gamma \leq \mu(\eta A, A) - g'(0) - \eta^2. \tag{3.11}$$

The last inequality holds provided that

$$\eta\gamma \leq \tilde{\mu}(\eta A) - g'(0) - \eta^2. \tag{3.12}$$

since $\mu(s,A) > \tilde{\mu}(s)$. Therefore, such a super-solution exists if we may find a constant η such that $\tilde{\mu}(\eta A) - g'(0) - \eta^2 > 0$ with $0 < \eta \leq \lambda$. This indeed holds for $\eta = s_0/A$ as long as $A > A_0$. The blow-off speed γ then may be chosen as

$$\gamma = \frac{A}{s_0}(\tilde{\mu}(s_0) - g'(0) - s_0^2/A^2). \tag{3.13}$$

This completes the proof of part (a).

The second part is proved similarly: the function $\Psi(x,y,t) = Ce^{-(\tilde{\mu}(0)-g'(0))t}\psi^A(x,y)$ is a super solution of (3.9) (since $\tilde{\mu}(0) \leq \mu(0,A)$) and $\tilde{\mu}(0) > g'(0)$. □

An interesting consequence of Theorem 3.1 is that a sufficiently strong periodic flow may blow off the initial data that propagate to the right in the absence of the flow. This happens for a flow u and a heat-loss parameter $q > 0$ such that $\mu(0,0) < g'(0)$ while there exist $A_0 > 0$ such that $\mu(s_0, A_0) > g'(0)$. An example of such a flow can be constructed as follows. First, we observe that

$\mu(0,0) = \tilde{\mu}(0)$. Therefore, it is sufficient to construct a flow $u_1(x,y)$ such that $\tilde{\mu}(0) < g'(0)$ while $\tilde{\mu}(s_0) > g'(0)$ and then use the inequality (3.8).

Let us show that such a flow can be constructed. Fix $s = 0$, then $\tilde{\mu}_0(q) \to \tilde{\mu}_D$, the first eigenvalue of problem (3.3) with Dirichlet boundary conditions in the domain Ω, as $q \to \infty$. Let us choose the domain Ω such that $\tilde{\mu}_D > g'(0)$, so that there exist a heat loss parameter $q_0 > 0$ sufficiently large so that $\tilde{\mu}_0(q_0) = g'(0)$, because $\tilde{\mu}_0(q)$ is a continuous increasing function of q with $\mu_0(0) = 0$. We also choose the first component of the flow u_1 so that

$$\frac{d\tilde{\mu}_s(0,q_0)}{ds} = -\int_\Omega u_1(x,y)\tilde{\phi}_0^2(y;q_0)dxdy > 0. \tag{3.14}$$

This is possible since $\tilde{\phi}_0$ is not a constant and is independent of the flow u. There exist then $s_0 > 0$ so that $\tilde{\mu}_{s_0} > g'(0)$. Continuity and monotonicity of $\tilde{\mu}_s(q)$ as a function of q imply that there exists $q_1 < q_0$ so that $\tilde{\mu}_0(q_1) < g'(0)$ while $\tilde{\mu}_{s_0}(q_1) > g'(0)$. Thus $\mu_{0,0}(q_1) < g'(0)$ and $\mu_{s_0,A}(q_1) > g'(0)$.

The homogenization regime. Let us comment briefly on the homogenization regime when the flow has the form

$$u(\mathbf{x}) = \frac{A}{\varepsilon}u\left(\frac{\mathbf{x}}{\varepsilon}\right).$$

We assume that $A \gg 1$ and $\varepsilon \ll 1$ with $\varepsilon \ll 1/A \ll 1$. Then we may first pass to the limit $\varepsilon \to 0$ in the KPP system

$$\begin{aligned} T_t + \frac{A}{\varepsilon}u\left(\frac{\mathbf{x}}{\varepsilon}\right)\cdot\nabla T &= \Delta T + TY \\ T_t + \frac{A}{\varepsilon}u\left(\frac{\mathbf{x}}{\varepsilon}\right)\cdot\nabla T &= \Delta T - TY \end{aligned} \tag{3.15}$$

with the heat-loss boundary conditions (1.2). This leads to a homogenized eigenvalue problem

$$-\kappa_A\Delta\Psi = \mu_A\Psi \text{ in } \Omega \tag{3.16}$$

$$\frac{\partial\Psi}{\partial n} + q\Psi = 0 \text{ on } \Omega. \tag{3.17}$$

Here κ_A is the effective diffusivity [1, 12] corresponding to the flow u.

It is well known [9, 12, 16, 17] that the effective diffusivity in a cellular flow behaves as $\kappa_A \sim \sqrt{A}$ for $A \gg 1$. That means that the eigenvalue μ_A grows with A. Hence, when A is sufficiently large, we have $\mu_A > 1$ and the flow becomes extinct. For this to happen, however, cells have to be sufficiently small so that we are in a homogenization regime. This is similar to the quenching problem in a cellular flow with an ignition type non-linearity: for the flow to quench a flame one needs both a large flow amplitude and a sufficiently small cell size [20].

Let us consider for simplicity the case when the cross-section Ω is an interval $[0, L_y]$. Then, if L_y is sufficiently large, the flame will not become extinct in the absence of the flow. The homogenization result shows that in the limit $\varepsilon \to 0$, the leading eigenvalue is $\mu_A \sim C\sqrt{A}/L_y^2$. Hence the flow amplitude required to extinct a flame in a channel of width L_y in the homogenization regime is of the order $A_c \sim L_y^4$. Some of the scaling predictions of the homogenization theory, such as the adiabatic flame speed and quenching amplitude, have been previously shown to hold outside of their theoretical regime of validity [20]. One would expect that the scaling $A_c \sim L_y^4$ may also hold for cells that are not infinitesimally small but rather just smaller than a critical size l_c beyond which flame does not become extinct no matter how large the flow amplitude is.

4 The numerical simulations

The goal of the numerical simulations is to verify the main theoretical conclusions of this paper: (i) that the travelling front speed is independent of the Lewis number, (ii) that the flame is extinguished in a narrow strip (L sufficiently small), and (iii) that a flame may be extinguished by a periodic flow in a strip that is sufficiently wide to support flame propagation without a flow if the cells are sufficiently small.

We present here the numerical simulations of the KPP system with a heat loss of the form

$$\begin{aligned} T_t + Au \cdot \nabla T &= \Delta T + \frac{1}{4} YT, \\ Y_t + Au \cdot \nabla Y &= \frac{1}{\mathrm{Le}} \Delta Y - \frac{1}{4} YT \end{aligned} \tag{4.1}$$

in a two-dimensional strip, of width L in the y-variable, and the length $2mL$ with an integer $m \gg 1$ in the x-direction. The Dirichlet boundary conditions for temperature and the Neumann condition for the concentration

$$T = 0, \quad \frac{\partial Y}{\partial y} = 0 \quad \text{on} \quad y = 0, L \quad \text{or} \quad x = \pm mL. \tag{4.2}$$

are imposed on the boundaries of the strip for simplicity. The periodic cellular flow u has the form

$$u = \left(\sin \frac{x}{4\pi l} \cos \frac{y}{4\pi l}, - \cos \frac{x}{4\pi l} \sin \frac{y}{4\pi l} \right),$$

while the parameter A measures the flow amplitude. The size of the cell, $l = L/n$, was an integer fraction of L, so that the strip always contains integer number of cells.

The initial concentration was set to $Y = 1$ everywhere in the domain; while the initial temperature was set to $T = 0$ everywhere except for a hot spot in the middle of the domain. The interfaces between hot and cold fluid were smoothed at $t = 0$ to match the laminar flame thickness. More precisely, we approximated the initial temperature by $T(x, y, 0) = \frac{1}{2}\left(\tanh \frac{x+a}{\lambda} - \tanh \frac{x-a}{\lambda}\right)$ where $\lambda = 16$ is of the order of laminar flame thickness. The initial length of the hot spot was typically the doubled strip width, $a = L$, but we have performed simulations with larger a and found no difference in the asymptotic behavior.

Equations (4.1) have been solved using an explicit finite difference scheme of fourth order in space and a third-order Adams-Bashforth integration in time. The grid size, $\Delta x = \Delta y = 0.25$, was chosen to resolve both temperature distribution across the interface, and the flow. The computational domain extended a considerable distance upstream and downstream from the burning front, $mL = \max(64L, 256\lambda)$, so that boundary effects were negligible.

As a measure of the reaction enhancement we use the bulk burning rate

$$V(t) = -\frac{1}{L} \int_0^L \int_{-\infty}^{\infty} Y_t(x, y, t) \, dy dx.$$

A typical temperature distribution in the flame front is shown in Fig. 4.1 for different cell sizes and the flow amplitude well below critical. The maximal temperature is attained in the middle of the strip while the temperature is lower near the boundary, as one would expect. One notices that the spatial variations of the temperature distribution are more pronounced when the cell size is comparable to the front thickness – the pattern with the small cells (in the bottom of Figure 4.1) is very similar (almost indistinguishable) to that in the absence of the flow ($A = 0$). The spatial pattern is less pronounced in the concentration that seems to have no interesting spatial features.

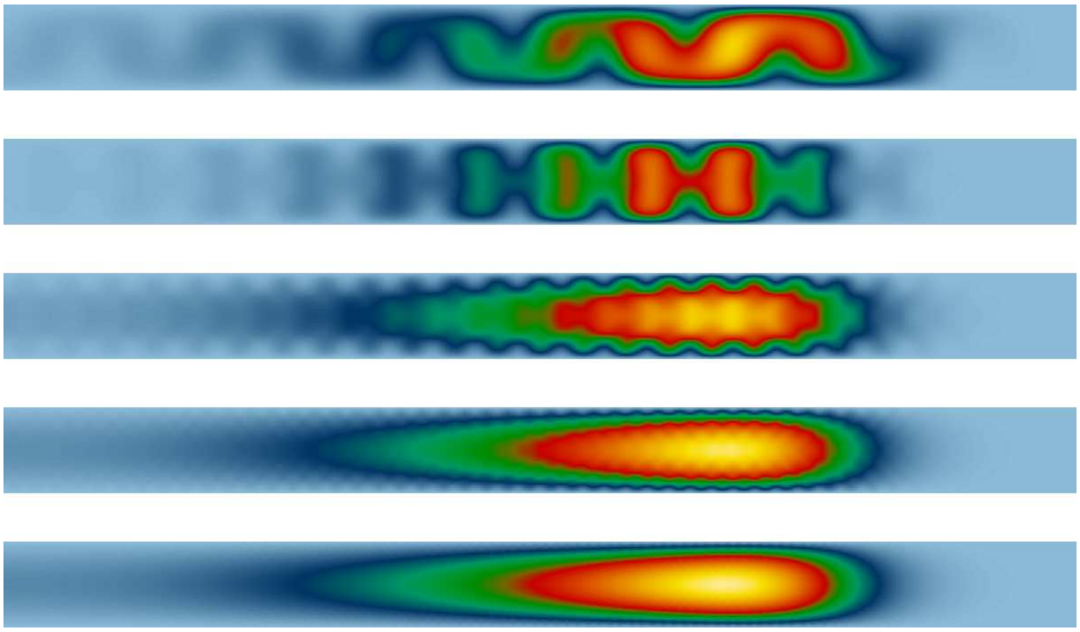

Figure 4.1: Temperature distribution in the domain of width $L = 16$ for different roll sizes, $l = 16$, 8, 4, 2, 1 (from top to bottom) and velocity amplitude $A = 1$ and Le $= 1$. Snapshots are taken at time $t = 100$.

The concentration changes in the horizontal direction on the scale of the distributed flame front with minor variations in the vertical direction.

Figure 4.2 presents the dependence of the bulk burning rate on the flow amplitude (rescaled according to the homogenization scaling). We observe two phenomena: first, the front speed is indeed independent of the Lewis number. Second, as the flow cells are taken to be small, the increase in the flow amplitude leads to a decrease in the bulk burning rate. In particular, we observe the flame extinction at a sufficiently high flow amplitude.

One may also observe a certain dichotomy between the numerical simulations and the predictions of the homogenization theory – if the cells were sufficiently small for the homogenization prediction for the critical amplitude necessary to extinct the flame to be valid, all the graphs in Figure 4.2 would reach extinction at the same value of the flow amplitude (rescaled as on the graph axis). However, the plot of the maximal temperature as a function of the flow amplitude in Figure 4.3 shows a remarkable agreement with the homogenization scaling even though the previous discussion shows that the homogenization regime has not yet set in for those cell sizes.

The dependence of the qualitative properties of flame propagation on the cell size may be seen in Figure 4.4 that presents the front speed as the function of the normalized flow amplitude for a strip of width $L = 12$. Note that while the flow amplitude increase speeds up propagation when the period cell is $l = 12$, that is, there is one cell per strip width, the front speed is diminished by an increase in the flow amplitude for cells of a smaller size. This reflects the qualitative idea that the small cells more effectively improve mixing in the system, increasing the effect of the boundary heat-loss.

The dependence of the critical flow amplitude A_{cr} necessary to extinct a flame in a strip of width L is presented in Figure 4.5. The homogenization regime prediction is $A_{cr} \sim L^4$ for small cells. We see that the numerical results show an exponent slightly higher than 4 for $l = 2$ and $l = 4$. Still, we observe a reasonable agreement with this particular scaling prediction even for cell sizes that, as we discussed above, do not provide a complete agreement with the homogenization theory.

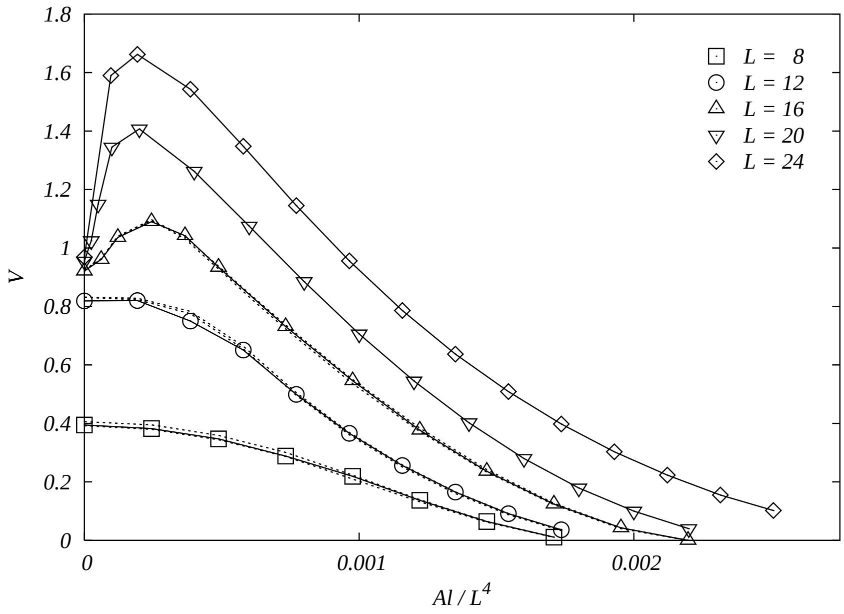

Figure 4.2: The bulk burning rate dependence on the flow amplitude for Le = 1 (the sold line) and Le = 1/2 and Le = 2 (the dashed lines) and various strip widths L.

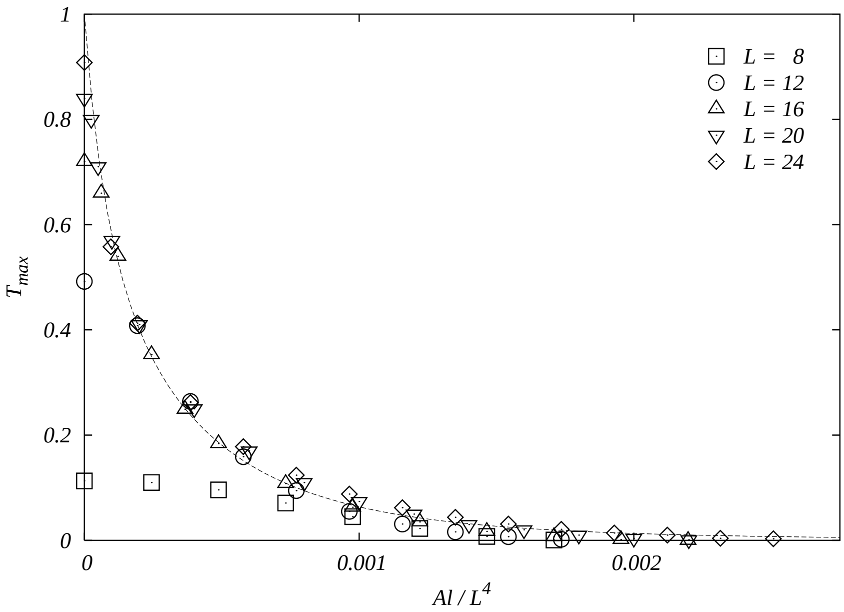

Figure 4.3: The maximal temperature dependence on the flow amplitude for Le = 1 and various strip widths L.

5 Conclusions

We have considered the qualitative behavior of a reaction-diffusion system of the KPP type in a periodic flow and with a heat-loss boundary condition for temperature. In the absence of the flow a reaction front is formed and propagates if the strip is wide enough, while the flame becomes extinct in a sufficiently narrow channel. We show that in the presence of a periodic flow the qualitative

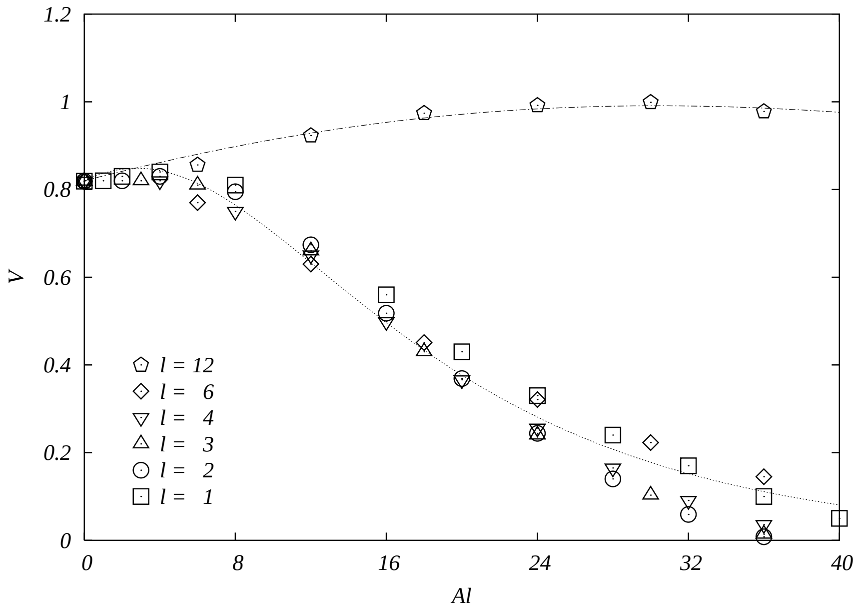

Figure 4.4: The bulk burning rate dependence on the flow amplitude for $\mathrm{Le} = 1$, $L = 12$ and various cell sizes l.

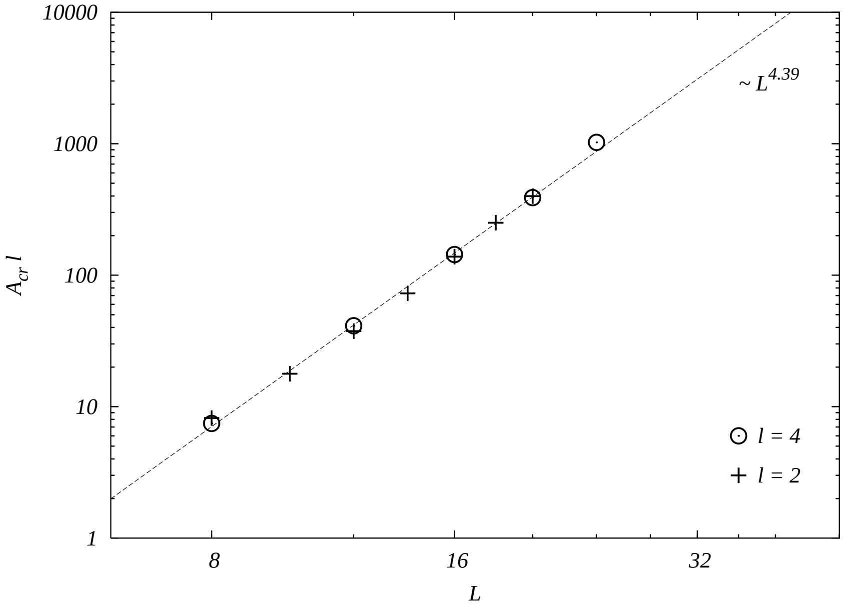

Figure 4.5: The critical flow amplitude necessary to extinguish a flame in a strip of width L with the cell sizes $l = 2$ and $l = 4$.

behavior is governed by the eigenvalue problem that arises after linearization of the problem ahead of the front. Depending on the behavior of the leading eigenvalue the flame either propagates with a speed that is independent of the Lewis number, or is either blown-off by a sufficiently strong flow or is extinct. We show numerically in the general case and analytically in the homogenization limit that when the period cells are sufficiently small, a sufficiently strong vortical flow may extinguish the flame in a channel that is sufficiently wide to support flame propagation in the absence of the flow. We observe a reasonable agreement with the homogenization scaling $A_{cr} \sim L^4$ for the critical flow amplitude necessary to extinguish a flame in strip of width L. On the other hand, a strong vortical flow with large cells would speed up flame propagation.

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