Existence and Non-existence of Fisher-KPP Transition Fronts

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January 12, 2011

Abstract

We consider Fisher-KPP-type reaction-diffusion equations with spatially inhomogeneous reaction rates. We show that a sufficiently strong localized inhomogeneity may prevent existence of transition-front-type global in time solutions while creating a global in time bump-like solution. This is the first example of a medium in which no reaction-diffusion transition front exists. A weaker localized inhomogeneity leads to existence of transition fronts but only in a finite range of speeds. These results are in contrast with both Fisher-KPP reactions in homogeneous media as well as ignition-type reactions in inhomogeneous media.

1 Introduction and main results

Fisher-KPP traveling fronts in homogeneous media

Traveling front solutions of the reaction-diffusion equation

$$u_t = u_{xx} + f(u) \tag{1.1}$$

are used to model phenomena in a range of applications from biology to social sciences, and have been studied extensively since the pioneering papers of Fisher [6] and Kolmogorov-Petrovskii-Piskunov [12]. The Lipschitz nonlinearity f is said to be of *KPP-type* if

$$f(0) = f(1) = 0 \qquad \text{and} \qquad 0 < f(u) \le f'(0)u \ \text{ for } u \in (0,1), \tag{1.2}$$

and one considers solutions $0 < u(t,x) < 1$. A *traveling front* is a solution of (1.1) of the form $u(t,x) = \phi_c(x-ct)$, with the function $\phi_c(\xi)$ satisfying

$$\phi_c'' + c\phi_c' + f(\phi_c) = 0, \quad \phi_c(-\infty) = 1, \quad \phi_c(+\infty) = 0. \tag{1.3}$$

Here c is the *speed* of the front and traveling fronts exist precisely when $c \ge c_* \equiv 2\sqrt{f'(0)}$. For the sake of convenience we will assume that $f'(0) = 1$, which can be achieved by a simple rescaling of space or time.

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The traveling front profile $\phi_c(\xi)$ satisfies $\phi_c(\xi) \sim e^{-r(c)\xi}$ as $\xi \to +\infty$. The decay rate $r(c)$ can be obtained from the linearized problem $v_t = v_{xx} + v$, and is given by

$$r(c) = \frac{c - \sqrt{c^2 - 4}}{2}. \tag{1.4}$$

It is the root of both $r^2 - cr + 1 = 0$ and $r^2 + r\sqrt{c^2 - 4} - 1 = 0$ and for $c \gg 1$ we have $r(c) = c^{-1} + O(c^{-3})$, whence $\lim_{c\to+\infty} cr(c) = 1$.

Fisher-KPP transition fronts in inhomogeneous media and bump-like solutions

In this paper we consider the *inhomogeneous* reaction-diffusion equation

$$u_t = u_{xx} + f(x, u) \tag{1.5}$$

with $x \in \mathbb{R}$ and a KPP reaction f. That is, we assume that f is Lipschitz, $f_u(x, 0)$ exists,

$$f(x, 0) = f(x, 1) = 0, \qquad \text{and} \qquad 0 < f(x, u) \le f_u(x, 0)u \;\text{ for } (x, u) \in \mathbb{R} \times (0, 1). \tag{1.6}$$

We let $a(x) \equiv f_u(x, 0) > 0$ and assume that for some $C, \delta > 0$ we have

$$f(x, u) \ge a(x)u - Cu^{1+\delta} \;\text{ for } (x, u) \in \mathbb{R} \times (0, 1). \tag{1.7}$$

Finally, we will assume here

$$0 < a_- \le a(x) \le a_+ < +\infty \;\text{ for } x \in \mathbb{R} \tag{1.8}$$

and

$$\lim\nolimits_{|x|\to\infty} a(x) = 1. \tag{1.9}$$

That is, we will consider media which are localized perturbations of the homogeneous case.

In this case traveling fronts with a constant-in-time profile cannot exist in general, and one instead considers *transition fronts*, a generalization of traveling fronts introduced in [3, 13, 17]. In the present context, a global in time solution of (1.5) is said to be a transition front if

$$\lim_{x\to-\infty} u(t, x) = 1 \qquad \text{and} \qquad \lim_{x\to+\infty} u(t, x) = 0 \tag{1.10}$$

for any $t \in \mathbb{R}$, and for any $\varepsilon > 0$ there exists $L_\varepsilon < +\infty$ such that for any $t \in \mathbb{R}$ we have

$$\operatorname{diam}\{x \in \mathbb{R} \,|\, \varepsilon \le u(t, x) \le 1 - \varepsilon\} < L_\varepsilon. \tag{1.11}$$

That is, a transition front is a global in time solution connecting $u = 0$ and $u = 1$ at any time t, which also has a uniformly bounded in time width of the transition region between ε and $1 - \varepsilon$.

Existence of transition fronts has been previously established for a class of time-dependent spatially homogeneous *bistable* nonlinearities in [17], and for spatially inhomogeneous *ignition* nonlinearities in [14, 15, 18]. The results in these papers, while non-trivial, are similar in spirit to the situation for such nonlinearities in homogenous media: there exists a unique (up to a time shift) transition front, and it is asymptotically stable for the Cauchy problem. In the present paper we will demonstrate that the situation can be very different for KPP-type nonlinearities, even in the case of a spatially localized inhomogeneities.

Before we do so, let us define another type of a solution of (1.5). We say that a global in time solution $0 < u(t,x) < 1$ of (1.5) is *bump-like* if $u(t,\cdot) \in L^1(\mathbb{R})$ for all $t \in \mathbb{R}$. We will show that bump-like solutions can exist for inhomogeneous KPP-type nonlinearities. What makes such solutions special is that they do not exist in many previously studied settings, as can be seen from the following proposition.

Proposition 1.1. *Assume that either $f(x,u) \geq 0$ is an ignition reaction (i.e., $f(x,u) = 0$ if $u \in [0,\theta(x)] \cup \{1\}$, with $\theta \equiv \inf_{x\in\mathbb{R}} \theta(x) > 0$; see [14, 15, 18]) or $f(x,u) = f(u)$ is a spatially homogeneous KPP reaction satisfying* (1.2) *and*

$$f(u) \equiv u \quad \textit{for } u \in [0,\theta] \tag{1.12}$$

for some $\theta \in (0,1)$. Then (1.5) *does not admit global in time bump-like solutions.*

Remark. Hypothesis (1.12) is likely just technical but we make it for the sake of simplicity.

Non-existence of transition fronts for strong KPP inhomogeneities

Our first main result shows that a localized KPP inhomogeneity can create global in time bump-like solutions of (1.5) as well as prevent existence of any transition front solutions. This is the first example of a medium in which no reaction-diffusion transition fronts exist. Moreover, in the case $a(x) \geq 1$ and $a(x) - 1$ compactly supported, Theorems 1.2 and 1.3 together provide a sharp criterion for the existence of transition fronts. Namely, transition fronts exist when $\lambda < 2$ and do not exist when $\lambda > 2$, with $\lambda \equiv \sup \sigma(\partial_{xx} + a(x))$ the supremum of the spectrum of the operator $L \equiv \partial_{xx} + a(x)$ on $\mathbb{R}$. One can consider these to be the main results of this paper.

Note that (1.9) implies that the essential spectrum of L is $(-\infty, 1]$ and so $\lambda \geq 1$. Hence if $\lambda > 1$ then λ is the principal eigenvalue of L and

$$\psi'' + a(x)\psi = \lambda\psi \tag{1.13}$$

holds for the positive eigenfunction $0 < \psi \in L^2(\mathbb{R})$ satisfying also $\|\psi\|_\infty = 1$. We note that $\psi(x)$ decays exponentially as $x \to \pm\infty$ due to (1.9).

Theorem 1.2. *Assume that $f(x,u)$ is a KPP reaction satisfying* (1.6)–(1.9) *with $a_- = 1$. If $\lambda > 2$, then any global in time solution of* (1.5) *such that $0 < u(t,x) < 1$ satisfies (with $C_c > 0$)*

$$u(t,x) \leq C_c\, e^{-|x|+ct} \tag{1.14}$$

for any $c < \lambda/\sqrt{\lambda - 1}$ and all $(t,x) \in \mathbb{R}^- \times \mathbb{R}$. In particular, no transition front exists.

Moreover, bump-like solutions do exist, and if there is $\theta > 0$ such that

$$f(x,u) \equiv a(x)u \quad \textit{for all } (x,u) \in \mathbb{R} \times [0,\theta], \tag{1.15}$$

then there is a unique (up to a time-shift) global in time solution $0 < u(t,x) < 1$. This solution satisfies $u(t,x) = e^{\lambda t}\psi(x)$ for $t \ll -1$.

Existence and non-existence of transition fronts for weak KPP inhomogeneities

We next show that transition fronts do exist when $\lambda < 2$, albeit in a bounded range of speeds. If u is a transition front, let $X(t)$ be the rightmost point x such that $u(t,x) = 1/2$. If

$$\lim_{t-s\to+\infty} \frac{X(t)-X(s)}{t-s} = c,$$

then we say that u has *global mean speed* (or simply *speed*) c. Recall that in the homogeneous KPP case with $f'(0) = 1$, traveling fronts exist for all speeds $c \geq 2$.

Theorem 1.3. *Assume that $f(x,u)$ is a KPP reaction satisfying* (1.6)–(1.9) *and $a(x) - 1$ is compactly supported. If $\lambda \in (1,2)$, then for each $c \in (2, \lambda/\sqrt{\lambda-1})$ equation* (1.5) *admits a transition front solution with global mean speed c. Moreover, bump-like solutions also exist.*

Remarks. 1. In fact, the constructed fronts will satisfy $\sup_{t\in\mathbb{R}} |X(t) - ct| < \infty$.

2. Fisher-KPP equations in homogeneous media also admit global in time solutions that are mixtures of traveling fronts moving with different speeds, constructed in [7, 8]. Such global in time mixtures of transition fronts constructed in Theorem 1.3 also exist, but this problem will be considered elsewhere in order to keep this paper concise. Existence of transition fronts with the critical speeds $c_* = 2$ and $c^* \equiv \lambda/\sqrt{\lambda-1}$ is a delicate issue and will also be left for a later work.

Finally, we show that the upper limit $\lambda/\sqrt{\lambda-1}$ on the front speed in Theorem 1.3 is not due to our techniques being inadequate. Indeed, we will prove non-existence of fronts with speeds $c > \lambda/\sqrt{\lambda-1}$, at least under additional, admittedly somewhat strong, conditions on f.

Theorem 1.4. *Assume that $f(x,u) = a(x)f(u)$ where a is even, satisfies* (1.8) *with $a_- = 1$, and $a(x) - 1$ is compactly supported, and f is such that* (1.2) *and* (1.12) *hold for some $\theta \in (0,1)$. In addition assume that* (1.13) *has a unique eigenvalue $\lambda > 1$. Then there are no transition fronts with global mean speeds $c > \lambda/\sqrt{\lambda-1}$.*

Let us indicate here the origin of the threshold $\lambda/\sqrt{\lambda-1}$ for speeds of transition fronts. In the homogeneous case $f(x,u) = f(u)$ with $f(u) = u$ for $u \leq \theta$, the traveling front with speed $c \geq 2$ satisfies $u(t,x) = e^{-r(c)(x-ct)}$ (up to a time shift) for $x \gg ct$. This means that u increases at such x at the exponential rate $cr(c)$ in t. We have $\lim_{|x|\to\infty} f_u(x,0) = 1$, so it is natural to expect a similar behavior of a transition front u (with speed c) at large x. On the other hand, any non-negative non-trivial solution of (1.5) majorizes a multiple of $e^{\lambda_M t}\psi_M(x)$ for $t \ll -1$, with λ_M and ψ_M the principal eigenvalue and eigenfunction of $\partial_{xx} + a(x)$ on $[-M,M]$ with Dirichlet boundary conditions (extended by 0 outside $[-M,M]$). So u has to increase at least at the rate λ_M, and since $\lim_{M\to\infty} \lambda_M = \lambda$, it follows that one needs $cr(c) \geq \lambda$ in order to expect existence of a transition front with speed c. Using (1.4), this translates into $c \leq \lambda/\sqrt{\lambda-1}$.

In the rest of the paper we prove Proposition 1.1 and Theorems 1.2, 1.3, 1.4 (in Sections 2, 3, 4, and 5–7, respectively).

Acknowledgment. JN was supported by NSF grant DMS-1007572, JMR by ANR grant 'PREFERED', LR by NSF grant DMS-0908507, and AZ by NSF grant DMS-0901363 and an Alfred P. Sloan Research Fellowship.

2 Nonexistence of bump-like solutions for ignition reactions and homogeneous KPP reactions: The proof of Proposition 1.1.

Assume, towards contradiction, that there exists a bump-like solution. We note that parabolic regularity and f Lipschitz then yield for each $t \in \mathbb{R}$,

$$u, u_x \to 0 \qquad \text{as } |x| \to \infty.$$

This will guarantee that differentiations in t of integrals over $\mathbb{R}$ and integration by parts below are valid. Let us define

$$I(t) \equiv \int_{\mathbb{R}} u(t,x)\,dx \qquad \text{and} \qquad J(t) \equiv \frac{1}{2}\int_{\mathbb{R}} u(t,x)^2\,dx.$$

Integration of (1.5) and of (1.5) multiplied by u over $x \in \mathbb{R}$ yields

$$I'(t) = \int_{\mathbb{R}} f(x,u)\,dx \geq 0 \qquad \text{and} \qquad J'(t) = \int_{\mathbb{R}} f(x,u)u\,dx - \int_{\mathbb{R}} |u_x|^2\,dx \leq I'(t) - \int_{\mathbb{R}} |u_x|^2\,dx.$$

So $\lim_{t\to-\infty} I(t) = C \geq 0$ and then $\lim_{t\to-\infty} \int_{\mathbb{R}} |u_x|^2\,dx = 0$. Parabolic regularity again gives

$$u, u_x \to 0 \qquad \text{as } t \to -\infty, \text{ uniformly in } x.$$

Thus $u(x,t) \leq \theta$ for all $t < t_0$ and all $x \in \mathbb{R}$. Then u in the ignition case ($v(t,x) \equiv e^{-t}u(t,x)$ in the KPP case) solves the heat equation for $t \leq t_0$. Since $u \geq 0$ ($v \geq 0$) and it is L^1 in x, it follows that $u = 0$ ($v = 0$), a contradiction.

3 The case $\lambda > 2$: The proof of Theorem 1.2

We obviously only need to consider $c \in (2, \lambda/\sqrt{\lambda-1})$, so let us assume this. We will first assume, for the sake of simplicity, that $a(x) - 1$ is compactly supported and (1.15) holds. At the end of this section we will show how to accommodate the proof to the general case.

Let us shift the origin by a large enough M so that in the shifted coordinate frame $a(x) \equiv 1$ for $x \notin [0, 2M]$, and the principal eigenvalue λ_M of $\partial_{xx} + a(x)$ on $(0, 2M)$ with Dirichlet boundary conditions satisfies $\lambda_M > 2$. This is possible since

$$\lim_{M\to+\infty} \lambda_M = \lambda.$$

We let ψ_M be the corresponding L^∞-normalized principal eigenfunction, that is, $\|\psi_M\|_\infty = 1$ and

$$\psi_M'' + a(x)\psi_M = \lambda_M\psi_M, \quad \psi_M > 0 \text{ on } (0,2M), \quad \psi_M(0) = \psi_M(2M) = 0. \tag{3.1}$$

It is easy to show that any entire solution $u(t,x)$ of (1.5) such that $0 < u(t,x) < 1$ satisfies $\lim_{t\to-\infty} u(t,x) = 0$ and $\lim_{t\to+\infty} u(t,x) = 1$ for any $x \in \mathbb{R}$, so after a possible translation of u forward in time by some t_0, we can assume

$$\sup_{t\leq 0} u(t,M) < \theta\psi_M(M) \leq \theta. \tag{3.2}$$

In that case (1.14) for this translated u yields $u(t,x) \leq Ce^{-|x-M|+c(t-t_0)}$ when $t < t_0$ for the original u, but then the result follows for a larger C from the fact that $Ce^{-|x-M|+(1+\|a\|_\infty)(t-t_0)}$ is a supersolution of (1.5) on $(-t_0, 0) \times \mathbb{R}$.

Non-existence of transition fronts

Assume that u is a global in time solution of (1.5). Non-existence of transition fronts obviously follows from (1.14). The following lemma is the main step in the proof of (1.14).

Lemma 3.1. *For any $c, c' \in (2, \lambda_M/\sqrt{\lambda_M - 1})$ with $c < c'$, there is $C_0 > 0$ (depending only on a, θ, c, c') and $\tau_0 > 0$ (depending also on $u(0, M)$) such that*

$$u(t, x) \leq C_0 u(0, M) e^{x+ct} \tag{3.3}$$

holds for all $t \leq -1$ and $x \in [0, c'(-t-1)]$, as well as for all $t \leq -\tau_0$ and $x \geq 0$.

Remark. This is a one-sided estimate but by symmetry of the arguments in its proof, the same estimate holds for $u(-t, 2M - x)$.

Let us show how this implies (1.14), despite the fact that (3.3) seemingly goes in two wrong directions. First, the estimate holds for $x \geq 0$ but the exponential on the right side grows as $x \to +\infty$. Second, this exponential is moving to the left as time progresses in the positive direction, while we are estimating u to the right of $x = 0$. The point of (3.3) is that the speed c at which the exponential moves is larger than 2, the latter being the minimal speed of fronts when $a(x) = 1$ everywhere. Thus, when looking at large negative times, this gives us a much smaller than expected upper bound on u at $|x| \leq c|t|$. Using this bound and then going forward in time towards $t = 0$, we will find that u cannot become $O(1)$ at $(0, M)$.

Given $c \in (2, \lambda/\sqrt{\lambda - 1})$, pick M such that $c < \lambda_M/\sqrt{\lambda_M - 1})$ and then $c' > c$ as in Lemma 3.1. Let $\tau_1 \equiv 1 + 2M/c'$ (so τ_1 depends on a, θ, c but not on u). By the first claim of Lemma 3.1 we have

$$u(t, 2M) \leq C_0 u(0, M) e^{2M+ct} \tag{3.4}$$

for all $t \leq -\tau_1$ because then $2M \leq c'(-t-1)$.

Next, for any $t_0 \leq -\tau_0$, we let

$$v_{t_0}(t, x) \equiv C_0 u(0, M) e^{x+ct_0+2(t-t_0)} + C_0 u(0, M) e^{4M-x+ct}.$$

Then v_{t_0} is a super-solution for (1.5) on $(t_0, \infty) \times (2M, \infty)$ since $a(x) \equiv 1$ for $x > 2M$. Moreover, the second claim of Lemma 3.1 and $t_0 \leq -\tau_0$ imply that at the "initial time" t_0 we have

$$u(t_0, x) \leq C_0 u(0, M) e^{x+ct_0} \leq v_{t_0}(t_0, x)$$

for all $x > 2M$. Since $c > 2$, it follows from (3.4) that $u(t, 2M) \leq v_{t_0}(t, 2M)$ for all $t \in (t_0, -\tau_1)$. Since the super-solution v_{t_0} is above u initially (at $t = t_0$) on all of $(2M, \infty)$ and at $x = 2M$ for all $t \in (t_0, -\tau_1)$, the maximum principle yields

$$u(t, x) \leq v_{t_0}(t, x) \tag{3.5}$$

for all $t \in [t_0, -\tau_1]$ and $x \geq 2M$. Since $c > 2$, taking $t_0 \to -\infty$ in (3.5) gives

$$u(t, x) \leq C_0 u(0, M) e^{4M-x+ct}, \tag{3.6}$$

for $t \leq -\tau_1$ and $x \geq 2M$. Note that unlike our starting point (3.3), the estimate (3.6) actually goes in the right direction, since the exponential is decaying as $x \to +\infty$.

An identical argument gives $u(t,x) \le C_0 u(0,M) e^{2M+x+ct}$ for $t \le -\tau_1$ and $x \le 0$, so

$$u(t,x) \le C_0 e^{2M} u(0,M) e^{-|x|+ct} \tag{3.7}$$

for $t \le -\tau_1$ and $x \in \mathbb{R} \setminus (0, 2M)$. Harnack inequality extends this bound to all $t \le -\tau_1 - 1$ and $x \in \mathbb{R}$, with some C_1 (depending only on a and θ) in place of $C_0 e^{2M}$:

$$u(t,x) \le C_1 u(0,M) e^{-|x|+ct} \tag{3.8}$$

for all $t \le -\tau_1 - 1$ and $x \in \mathbb{R}$. Finally, it follows from (3.8) that

$$u(t,x) \le C_1 u(0,M) e^{-|x|+c(-\tau_1-1)} e^{(1+\|a\|_\infty)(t-(-\tau_1-1))}$$

for $t \ge -\tau_1 - 1$ because the right-hand side is a super-solution of (1.5). Since τ_1 only depends on a, θ, c (once M, c' are fixed) and not on u, and since $a_1 \ge 1$, it follows that

$$u(t,x) \le C_2 u(0,M) e^{-|x|+ct} \tag{3.9}$$

for all $t \le 0$ and $x \in \mathbb{R}$, with C_2 depending only on a, θ, c. This is (1.14), proving non-existence of transition fronts when $\lambda > 2$ under the additional assumptions of $a(x) - 1$ compactly supported and (1.15) (except for the proof of Lemma 3.1 below).

Bump-like solutions and uniqueness of a global in time solution

Existence of a bump-like solution is immediate from (1.15). Indeed, it is obtained by continuing the solution of (1.5), given by $u(t,x) = e^{\lambda t}\psi(x)$ for $t \ll -1$, to all $t \in \mathbb{R}$.

In order to prove the uniqueness claim, we note that the same argument as above, with $u(0,M)$ replaced by $u(s,M)$ and $t \le s \le 0$, gives (with the same C_2)

$$u(t,x) \le C_2 u(s,M) e^{-|x|-2(s-t)}. \tag{3.10}$$

We also have $\|u(t,\cdot)\|_\infty \le \theta$ for all $t \le t_0 \equiv -\frac{1}{2}\log C_2$. Therefore, the function $v(t,x) \equiv u(t,x)e^{-2t}$ solves the linear equation

$$v_t = v_{xx} + (a(x) - 2)v \tag{3.11}$$

on $(-\infty, t_0) \times \mathbb{R}$. It can obviously be extended to an entire solution of (3.11) by propagating it forward in time. Taking $t = s$ in (3.10) gives $v(t,x) \le C_2 v(t,M)$ for $(t,x) \in (-\infty, t_0) \times \mathbb{R}$. Moreover, it is well known that since λ is an isolated eigenvalue (because $\lambda > 1$ and the essential spectrum is $(-\infty, 1]$), the function $e^{-(\lambda-2)t}v(t,x)$ converges uniformly to $\psi(x)$ as $t \to \infty$. It follows that

$$v(t,x) \le C_3 v(t,M) \tag{3.12}$$

holds for some $C_3 > 0$ and all $(t,x) \in \mathbb{R}^2$.

We can now apply Proposition 2.5 from [9] to (3.11). More precisely, as $a(x) \equiv 1$ outside of a bounded interval, Hypothesis A of this proposition is satisfied, while $\lambda > 2$ ensures that Hypothesis H1 of [9] holds for the solution $w(t,x) = e^{(\lambda-2)t}\psi(x)$ of (3.11). Finally, (3.12) guarantees that condition (2.12) of [9] holds, too. It then follows from the aforementioned proposition that $w(t,x)$ is the unique (up to a time shift) global in time solution of (3.11), proving the uniqueness claim in Theorem 1.2.

It remains now only to prove Lemma 3.1 in order to finish the proof of Theorem 1.2 in the case when $a(x) - 1$ is compactly supported and (1.15) holds.

The proof of Lemma 3.1

We will prove Lemma 3.1 using the following lemma.

Lemma 3.2. *For every $\varepsilon \in (0,1)$ there exists $C_\varepsilon \geq 1$ (depending also on a, θ, and λ_M) such that*

$$u(t,x) \leq C_\varepsilon u(0,M)\sqrt{|t|}\, e^{\sqrt{\lambda_M - 1}\,x + (\lambda_M - \varepsilon)t} \tag{3.13}$$

holds for all $t \leq -1$ and $x \in [0, c_\varepsilon(-t-1)]$, with $c_\varepsilon \equiv (\lambda_M - \varepsilon)/\sqrt{\lambda_M - 1}$.

Let us first explain how Lemma 3.2 implies Lemma 3.1. Pick $\varepsilon > 0$ such that $c_\varepsilon = c'$. Then there is $C_0 > 0$ depending only on a, θ, c (via $\varepsilon, \lambda_M, C_\varepsilon$) such that for all $t \leq -1$ and $x \in [0, c'(-t-1)]$ we have

$$u(t,x) \leq C_\varepsilon u(0,M)\sqrt{|t|}\, e^{\sqrt{\lambda_M - 1}(x + c't)} \leq C_\varepsilon u(0,M)\sqrt{|t|}\, e^{x + c't} \leq C_0 u(0,M) e^{x+ct}, \tag{3.14}$$

the first claim of Lemma 3.1 .

Next let

$$\tau_0 \equiv \frac{|\log(C_0 u(0,M)e^{-c})|}{c' - c} + 1, \tag{3.15}$$

so that $C_0 u(0,M)e^{x+ct} \geq 1$ for $t \leq -\tau_0$ and $x \geq c'(-t-1)$. Since $u(t,x) \leq 1$, this means that (3.3) also holds for all $t \leq -\tau_0$ and $x \geq 0$, the second claim of Lemma 3.1.

Thus we are left with the proof of Lemma 3.2. This, in turn, relies on the following lemma.

Lemma 3.3. *For each $m \in \mathbb{R}$ and $\varepsilon > 0$ there is $k_\varepsilon > 0$ such that if $u \in [0,1]$ solves (1.5) with $u(0,x) \geq \gamma\chi_{[l-1,l]}(x)$ for some $\gamma \leq \theta/2$ and $l \in \mathbb{R}$, then for $t \geq 0$ and $x \leq l + m - 2t$,*

$$u(t,x) \geq k_\varepsilon \gamma e^{(1-\varepsilon)t} \int_{l-1}^{l} \frac{e^{-|x-z|^2/4t}}{\sqrt{4\pi t}}\, dz.$$

Proof. The result, with 1 in place of $1-\varepsilon$, clearly holds when $f(x,u) \geq u$ for all x, u. Since $f(x,u) \geq u$ only for $u \leq \theta$, we will have to be a little more careful.

It is obviously sufficient to consider $l = 0$. Let g be a concave function on $[0,1]$ such that $g(w) = w$ for $w \in [0,1/2]$ and $g(1) = 0$ and define $g_\gamma(w) \equiv 2\gamma g(w/2\gamma)$ (hence $g_\gamma(w) = w$ for $w \in [0,\gamma]$, and $g_\gamma \leq f$). The comparison principle implies that $u(x) \geq w(x)$, where $w(x)$ solves

$$w_t = w_{xx} + g_\gamma(w) \tag{3.16}$$

with initial condition $w(0,x) = \gamma\chi_{[-1,0]}(x)$. It follows from standard results on spreading of solutions to KPP reaction-diffusion equations (see, for instance, [2]) that for each $\varepsilon > 0$ there exists $t_\varepsilon \geq (m+1)/2\sqrt{1-\varepsilon}$ such that for all $t \geq t_\varepsilon$ we have $w(t, -2\sqrt{1-\varepsilon}(t - t_\varepsilon) - 1) \geq \gamma$. The time t_ε is independent of γ because w/γ is independent of γ.

Note that the function

$$v(t,x) = e^{-2t_\varepsilon}\gamma e^{(1-\varepsilon)t} \int_{-1}^{0} \frac{e^{-|x-z|^2/4t}}{\sqrt{4\pi t}}\, dz$$

solves $v_t = v_{xx} + (1-\varepsilon)v$, so v is a sub-solution of (3.16) on any domain where $v(t,x) \le \gamma$. We have $\|v(t,\cdot)\|_\infty \le e^{-(1-\varepsilon)t_\varepsilon}\gamma \le \gamma$ for $t \le t_\varepsilon$, as well as

$$v(t, -2\sqrt{1-\varepsilon}(t-t_\varepsilon)-1) \le e^{-2t_\varepsilon+(1-\varepsilon)t-\frac{4(1-\varepsilon)(t-t_\varepsilon)^2}{4t}}\gamma \le \gamma$$

for $t \ge t_\varepsilon$. Since $v(t,\cdot)$ is obviously increasing on $(-\infty,-1)$, it follows that v is a sub-solution of (3.16) on the domain

$$D \equiv ([0,t_\varepsilon)\times\mathbb{R}) \cup \{(t,x) \,|\, t_\varepsilon \ge t \text{ and } x < -2\sqrt{1-\varepsilon}(t-t_\varepsilon)-1\}. \tag{3.17}$$

Moreover, w is a solution of (3.16),

$$v(0,x) = e^{-2t_\varepsilon}\gamma\chi_{[-1,0]}(x) \le w(0,x),$$

and

$$v(t,-2\sqrt{1-\varepsilon}(t-t_\varepsilon)-1) \le \gamma \le w(t,-2\sqrt{1-\varepsilon}(t-t_\varepsilon)-1)$$

for $t \ge t_\varepsilon$. Thus $v \le w \le u$ on $\bar{D}$. Since $-2\sqrt{1-\varepsilon}(t-t_\varepsilon)-1 \ge m-2t$ implies $(t,x)\in\bar{D}$ whenever $x \le m-2t$, the result now follows with $k_\varepsilon \equiv e^{-2t_\varepsilon}$. □

Proof of Lemma 3.2. Assume that

$$u(t',x) \ge C_\varepsilon u(0,M)\sqrt{|t'|}\,e^{\sqrt{\lambda_M-1}\,x+(\lambda_M-\varepsilon)t'}$$

for some $t' \le -1$ and $x \in [0, c_\varepsilon(-t'-1)]$, let $t \equiv t'+1 \le 0$, and define

$$\beta \equiv \frac{x}{2|t|\sqrt{\lambda_M-1}} \le \frac{\lambda_M-\varepsilon}{2(\lambda_M-1)} < 1.$$

By the Harnack inequality and parabolic regularity that there exists $c_0 \in (0, e^{-\lambda_M}\theta/2)$ (depending on a,θ) such that

$$u(t,z) \ge c_0 C_\varepsilon u(0,M)\sqrt{|t|+1}\,e^{\sqrt{\lambda_M-1}\,x+(\lambda_M-\varepsilon)t} \tag{3.18}$$

for all $z \in (x-1,x)$. Note that the right side of (3.18) is below $\theta/2$ since $u(t,x)\le 1$. Then Lemma 3.3 with $l \equiv x$ and $m \equiv 2M$ shows that for $y \in [0,2M]$ and $C'_\varepsilon \equiv k_\varepsilon c_0 C_\varepsilon$ (with k_ε from that lemma and using $\sqrt{\lambda_M-1} > 1$) we have

$$\begin{aligned} u(t+\beta|t|,y) &\ge C'_\varepsilon u(0,M)\sqrt{|t|+1}\,e^{\sqrt{\lambda_M-1}\,x+(\lambda_M-\varepsilon)t}e^{(1-\varepsilon)\beta|t|}\int_{x-1}^{x}\frac{e^{-|y-z|^2/4\beta|t|}}{\sqrt{4\pi\beta|t|}}\,dz \\ &\ge \frac{C'_\varepsilon u(0,M)}{\sqrt{4\pi}}e^{\sqrt{\lambda_M-1}\,x+\lambda_M t-\frac{x^2}{4\beta|t|}+\beta|t|}. \end{aligned}$$

The normalization $\|\psi_M\|_\infty = 1$ and the comparison principle then give

$$u(0,z) \ge \min\left\{\theta, e^{\lambda_M(1-\beta)|t|}\frac{C'_\varepsilon u(0,M)}{\sqrt{4\pi}}e^{\sqrt{\lambda_M-1}\,x-\lambda_M|t|-\frac{x^2}{4\beta|t|}+\beta|t|}\right\}\psi_M(z) = \min\left\{\theta, \frac{C'_\varepsilon u(0,M)}{\sqrt{4\pi}}\right\}\psi_M(z)$$

for any $z\in\mathbb{R}$. Taking $z = M$ and $C_\varepsilon = 4\sqrt{\pi}/k_\varepsilon c_0\psi_M(M)$, it follows that

$$u(0,M) \ge \min\{\theta\psi_M(M), 2u(0,M)\},$$

which contradicts (3.2) and $u(0,M) > 0$. Thus, (3.13) holds for this C_ε. □

The case of general inhomogeneities

We now dispense with the assumptions of $a(x)-1$ compactly supported and (1.15). The proof of (1.14) easily extends to the case of (1.7) and (1.9). First, pick $\varepsilon \in (0, c-2)$ (recall that $c > 2$) such that $(\lambda - 2\varepsilon)/\sqrt{\lambda - 1} > c$ and then $\theta > 0$ such that $f(x,u) \geq (a(x) - \varepsilon/2)u$ for $u \leq \theta$. Next, choose M large enough so that $a(x) \leq 1 + \varepsilon$ outside $(0, 2M)$ (after a shift in x as before) and the principal eigenvalue λ_M $(< \lambda - \varepsilon/2)$ of the operator

$$\partial_{xx} + a(x) - \varepsilon/2$$

on $(0, 2M)$ with Dirichlet boundary conditions satisfies $\lambda_M > \lambda - \varepsilon$. Thus $c_\varepsilon \equiv (\lambda_M - \varepsilon)/\sqrt{\lambda_M - 1} > c$, so we can again let $c' \equiv c_\varepsilon > c$.

Then Lemma 3.3 holds for the chosen ε, θ without a change in the proof, even though now we have only $f(x,u) \geq (1 - \varepsilon/2)u$ for $u \leq \theta$. Lemmas 3.2 and 3.1 are also unchanged. The only change in the proof of non-existence of fronts in Theorem 1.2 is that one has to take

$$v_{t_0}(t,x) \equiv C_0 u(0,M) e^{x - ct_0 + (2+\varepsilon)(t+t_0)} + C_0 u(0,M) e^{4M - x + ct}.$$

Since $c > 2 + \varepsilon$, we again obtain

$$u(t,x) \leq C_2 u(0,M) e^{-|x| + ct}$$

for $t \leq 0$ and $x \in \mathbb{R}$, so (1.14) as well as non-existence of fronts follow.

A bump-like solution is now obtained as a limit of solutions $u_n(t,x)$ defined on $(-n, \infty) \times \mathbb{R}$ with initial data $u(-n,x) = C_n \psi(x)$. Here $0 < C_n \to 0$ are chosen so that $u_n(0,0) = 1/2$, and parabolic regularity ensures that a global in time solution u of (1.5) can be obtained as a locally uniform limit on $\mathbb{R}^2$ of u_n, at least along a subsequence. Since $C_n e^{\lambda(t-n)} \psi(x)$ is a supersolution of (1.5), we have $C_n e^{\lambda} \geq C_{n-1}$. Since $C_n e^{(\lambda - \varepsilon_n)(t-n)} \psi(x)$ is a subsolution of (1.5) on $[-n, -n+1]$ provided

$$\varepsilon_n \equiv \sup_{(x,u) \in \mathbb{R} \times (0, C_n e^{\lambda})} \left[a(x) - \frac{f(x,u)}{u} \right] \quad (\leq C C_n^{\delta} e^{\lambda \delta} \text{ by } (1.7))$$

and using $\|\psi\|_\infty = 1$, we have $C_n e^{\lambda - \varepsilon_n} \leq C_{n-1}$. Thus C_n decays exponentially and then so does ε_n. As a result, $C_n e^{\lambda n} \to C_\infty \in (0, \infty)$ and so $u_n(t,x) \leq 2 C_\infty e^{\lambda t} \psi(x)$ for all large n and all (t,x). Thus the limiting solution u also satisfies this bound and it is therefore bump-like.

The proof of uniqueness of global solutions also extends to (1.9), but this time (1.15) is necessary in order to obtain (3.11) and to then apply Proposition 2.5 from [9].

4 Fronts with speeds $c \in (2, \lambda/\sqrt{\lambda - 1})$: The proof of Theorem 1.3

First note that the proof of existence of bump-like solutions from Theorem 1.2 works for any $a_- > 0$ and extends to $\lambda < 2$, so we are left with proving existence of fronts.

Assume that $a(x) = 1$ outside $[-M, M]$ and also (for now) that (1.15) holds. Consider any $c \in (2, \lambda/\sqrt{\lambda - 1})$. We will construct a positive solution v and a sub-solution w to the PDE

$$u_t = u_{xx} + a(x)u,$$

such that $w \le \min\{v, \theta\}$ and both move to the right with speed c (in a sense to be specified later). It follows that v and w are a supersolution and a subsolution to (1.5), and we will see later that this ensures the existence of a transition front $u \in (w, v)$ for (1.5).

For any $\gamma \in (\lambda, 2)$ let ϕ_γ be the unique solution of

$$\phi_\gamma'' + a(x)\phi_\gamma = \gamma\phi_\gamma, \tag{4.1}$$

with $\phi_\gamma(x) = e^{-\sqrt{\gamma-1}\,x}$ for $x \ge M$. We claim that then

$$\phi_\gamma > 0. \tag{4.2}$$

Indeed, assume $\phi_\gamma(x_0) = 0$ and let ψ_γ be the solution of (4.1) with $\psi_\gamma(x) = e^{\sqrt{\gamma-1}\,x}$ for $x \ge M$. Then $\phi_\gamma - \varepsilon\psi_\gamma$ would have at least two zeros for all small ε (near x_0 and at some $x_1 \gg M$). Since $\gamma > \lambda = \sup\sigma(\partial_{xx}^2 + a(x))$, this would contradict the Sturm oscillation theory, so (4.2) holds. Since there are $\alpha_\gamma, \beta_\gamma$ such that

$$\phi_\gamma(x) = \alpha_\gamma e^{-\sqrt{\gamma-1}\,x} + \beta_\gamma e^{\sqrt{\gamma-1}\,x}$$

for $x \le -M$, it follows that $\alpha_\gamma > 0$.

This means that the function

$$v(t,x) \equiv e^{\gamma t}\phi_\gamma(x) > 0$$

is a supersolution of (1.5) (if we define $f(x,u) \equiv 0$ for $u > 1$). Notice that in the domain $x > M$, the graph of v moves to the right at *exact* speed $\gamma/\sqrt{\gamma-1}$ as time increases. This is essentially true also for $x \ll -M$ (since $\phi_\gamma(x) \approx \alpha_\gamma e^{-\sqrt{\gamma-1}\,x}$ there), so v is a supersolution moving to the right at speed $\gamma/\sqrt{\gamma-1}$ in the sense of Remark 1 after Theorem 1.3.

Next let $0 < \varepsilon' \le \varepsilon$ and $A > 0$ be large, and define

$$w(t,x) \equiv e^{\gamma t}\phi_\gamma(x) - Ae^{(\gamma+\varepsilon)t}\phi_{\gamma+\varepsilon'}(x).$$

Then w satisfies

$$w_t = w_{xx} + a(x)w - (\varepsilon - \varepsilon')Ae^{(\gamma+\varepsilon)t}\phi_{\gamma+\varepsilon'}(x). \tag{4.3}$$

If we define $f(x,u) \equiv 0$ for $u < 0$, then w will be a subsolution of (1.5) if $\sup_{(t,x)} w(t,x) \le \theta$, due to (1.15). We will now show that we can choose $\varepsilon, \varepsilon', A$ so that this is the case.

For large t such that $\operatorname{supp} w_+ \subseteq (M, \infty)$ (namely, $t > \varepsilon^{-1}(\sqrt{\gamma+\varepsilon'-1}\,M - \sqrt{\gamma-1}\,M - \log A)$), the maximum $\max_x w(t,x)$ is attained at x such that

$$\sqrt{\gamma-1}\,e^{\gamma t}e^{-\sqrt{\gamma-1}\,x} = A\sqrt{\gamma+\varepsilon'-1}\,e^{(\gamma+\varepsilon)t}e^{-\sqrt{\gamma+\varepsilon'-1}\,x}, \tag{4.4}$$

that is, at

$$x_t \equiv \frac{1}{\sqrt{\gamma+\varepsilon'-1} - \sqrt{\gamma-1}}\left[\varepsilon t + \log\left(A\frac{\sqrt{\gamma+\varepsilon'-1}}{\sqrt{\gamma-1}}\right)\right]. \tag{4.5}$$

If we define

$$\kappa = \kappa(\varepsilon', \gamma) \equiv \frac{\sqrt{\gamma-1}}{\sqrt{\gamma+\varepsilon'-1} - \sqrt{\gamma-1}} > 0,$$

then we have

$$w(t,x_t) = e^{(\gamma-\varepsilon\kappa)t}A^{-\kappa}\left(\frac{\sqrt{\gamma+\varepsilon'-1}}{\sqrt{\gamma-1}}\right)^{-\kappa-1}\left(\frac{\sqrt{\gamma+\varepsilon'-1}}{\sqrt{\gamma-1}} - 1\right) \tag{4.6}$$

for $t \gg 1$. So if $\varepsilon \geq \varepsilon'$ are chosen so that $\varepsilon\kappa = \gamma$ (this is possible because $\gamma > 2(\gamma - 1)$), then $\max_x w(t, x)$ is constant for $t \gg 1$.

The same argument works for $t \ll -1$, with $A\alpha_{\gamma+\varepsilon'}/\alpha_\gamma$ in place of A in (4.4)—(4.6), as well as with all three equalities holding only approximately due to the term $\beta_\gamma e^{\sqrt{\gamma-1}\,x}$. Nevertheless, the equalities hold in the limit $t \to -\infty$, and $\max_x w(t, x)$ has a positive limit as $t \to -\infty$. Therefore $\max_x w(t, x)$ is uniformly bounded in t, and this bound converges to 0 as $A \to \infty$, due to (4.6). We can therefore pick A large enough so that $\sup_{(t,x)} w(t, x) \leq \theta$, so that w is now a subsolution of (1.5). Note that $\varepsilon\kappa = \gamma$ also implies that x_t (and hence w) moves to the right with speed

$$\frac{\varepsilon}{\sqrt{\gamma + \varepsilon' - 1} - \sqrt{\gamma - 1}} = \frac{\gamma}{\sqrt{\gamma - 1}}$$

(in the sense of $\sup_t |x_t - \gamma t/\sqrt{\gamma - 1}| < \infty$).

So given $c \in (2, \lambda/\sqrt{\lambda - 1})$ let us pick $\gamma \in (\lambda, 2)$ such that $c = \gamma/\sqrt{\gamma - 1}$ (and then choose $\varepsilon, \varepsilon', A$ as above). Then we have a subsolution w and a supersolution v of (1.5) with $v > \max\{w, 0\}$, $\max_x w(t, x)$ bounded below and above by positive constants, with the same decay as $x \to \infty$, and with $v \to \infty$ and $w \to -\infty$ as $x \to -\infty$. Moreover, v and w are moving at the same speed c to the right, in the sense that points where $\max_x w(t, x)$ is achieved and where, say, $v(t, x) = 1/2$, both move to the right with speed c (exact for $t \gg 1$ and almost exact for $t \ll -1$).

A standard limiting argument (see, for instance, [5]) now recovers a global in time solution to (1.5) that is sandwiched between v and w. Indeed, we obtain it as a locally uniform limit (along a subsequence if needed) of solutions u_n of (1.5) defined on $(-n, \infty) \times \mathbb{R}$, with initial condition $u_n(-n, x) \equiv \min\{v(-n, x), 1\}$, so that $u \in (\max\{w, 0\}, \min\{v, 1\})$ by the strong maximum principle. Another standard argument based on the global stability of the constant solution 1 (on the set of solutions $u \in (0, 1)$), same speed c of v and w, and uniform boundedness below of $\max_x w(t, x)$ in t shows that u has to be a transition front moving with speed c, in the sense of Remark 1 after Theorem 1.3.

This proves the existence-of-front part of Theorem 1.3 when (1.15) holds. In that case we could even have chosen $\varepsilon' = \varepsilon$ so that $\varepsilon\kappa = \gamma$ because then $\lim_{\varepsilon\to 0} \varepsilon\kappa = 2\sqrt{\gamma - 1} < \gamma < \infty = \lim_{\varepsilon\to\infty} \varepsilon\kappa$. If we only have (1.7), we need to pick $\varepsilon' < \varepsilon$ such that $\varepsilon\kappa = \gamma$ and the last term in (4.3) to be larger than $Cw(t, x)^{1+\delta}$ where $w(t, x) > 0$, so that w stays a subsolution of (1.5). For the latter it is sufficient if

$$(\varepsilon - \varepsilon')Ae^{(\gamma+\varepsilon)t}e^{-\sqrt{\gamma+\varepsilon'-1}\,x} \geq C_1 e^{-(1+\delta)\sqrt{\gamma-1}\,x} \tag{4.7}$$

where $w(t, x) > 0$, with some large C_1 depending on C, ϕ_γ, $\phi_{\gamma+\varepsilon'}$. If we let $y \equiv x - ct = x - \gamma t/\sqrt{\gamma - 1}$ and use $\varepsilon\kappa = \gamma$, this boils down to

$$\sqrt{\gamma + \varepsilon' - 1}\, y < (1 + \delta)\sqrt{\gamma - 1}\, y + \log \frac{(\varepsilon - \varepsilon')A}{C_1} \tag{4.8}$$

when $w(t, ct + y) > 0$. Notice that for say $A = 1$, the leftmost point where $w(x, t) = 0$ stays uniformly (in t) close to ct (say distance $d(t) \leq d_0$), and only moves to the right if we increase A. Therefore we only need to pick $\varepsilon' < \varepsilon$ such that $\sqrt{\gamma + \varepsilon' - 1} \leq (1 + \delta)\sqrt{\gamma - 1}$ and $\varepsilon\kappa = \gamma$, and then $A > 1$ large enough so that (4.8) holds for any $y \geq -d_0$. The rest of the proof is unchanged. $\square$

5 Nonexistence of fronts with speeds $c > \lambda/\sqrt{\lambda-1}$: The proof of Theorem 1.4

Assume $a(x) \equiv 1$ outside $[-M_0, M_0]$ and let us denote the roots of $r^2 - cr + 1 = 0$ by

$$r_\pm(c) = \frac{c \pm \sqrt{c^2-4}}{2}.$$

Notice that if $\lambda \leq 2$ and $c > \lambda/\sqrt{\lambda-1}$, then

$$0 < r_-(c) < \sqrt{\lambda-1} \qquad \text{and} \qquad r_+(c) > \frac{1}{\sqrt{\lambda-1}}. \tag{5.1}$$

Also recall that we denote by $X(t)$ the right-most point x such that $u(t,x) = 1/2$. The proof of Theorem 1.4 relies on the following upper and lower exponential bounds on the solution ahead of the front (at $x \geq X(t)$).

Lemma 5.1. *Let $c > 2$ and $u(t,x)$ be a transition front for* (1.5) *moving with speed c. Then for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$u(t,x) \leq C_\varepsilon e^{-(r_-(c)-\varepsilon)(x-X(t))} \quad \textit{for } x \geq X(t). \tag{5.2}$$

Lemma 5.2. *Assume that the function $a(x)$ is even and that* (1.13) *has a unique eigenvalue $\lambda > 1$. Let $c > \lambda/\sqrt{\lambda-1}$ and $u(t,x)$ be a transition front for* (1.5) *moving with speed c. Then for all $\varepsilon > 0$, there is $C_\varepsilon > 0$ and $T > 0$ such that:*

$$u(t,x) \geq C_\varepsilon e^{-(r_-(c)+\varepsilon)(x-X(t))} \quad \textit{for } t \geq T \textit{ and } x \geq X(t).$$

Proof of Theorem 1.4. Let us assume $\lambda \in (1,2]$ since the cas $\lambda > 2$ has already been proved in Theorem 1.2. Assume that there exists a transition front $u(t,x)$ with speed

$$c > \lambda/\sqrt{\lambda-1}. \tag{5.3}$$

We first wish to prove the following estimate: for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$u(t,x) \leq C_\varepsilon e^{(\lambda-\varepsilon)t - \sqrt{\lambda-\varepsilon-1}\,x} \quad \text{for all } x \geq 0 \text{ and } t \leq 0. \tag{5.4}$$

From Lemma 3.2, the estimate is true for $x = 0$ and, more generally, on every bounded subset of $\mathbb{R}_+$, so let us extend it to the whole half-line. For this, we notice that, for all $t \leq 0$, we have

$$u(t,x) \leq Ce^t, \quad \text{for } x \geq 0. \tag{5.5}$$

Indeed, the function

$$\alpha(t) = \int_{M_0}^{+\infty} u(t,x)\, dx,$$

which is finite due to Lemma 5.1, solves

$$\alpha' - \alpha = -u_x(t, M_0) - \int_{M_0}^{+\infty} (u(t,x) - f(u(t,x))\, dx.$$

From parabolic regularity and (5.4) for x on compact intervals, we have $|u_x(t,M_0)| \le Ce^{(\lambda-\varepsilon)t}$ for $t \le 0$. From Lemma 5.1, the fact that u travels with a positive speed, and $a(x) = 1$ for $x \ge M_0$, we have $f(u(t,x)) = u(t,x)$ for $x \ge M_0$ and $t \ll -1$. Hence we have

$$\alpha' - \alpha = O(e^{(\lambda-\varepsilon)t})$$

for $t \ll -1$, which implies $\alpha(t) = O(e^t)$ for $t \le 0$ since $\lambda > 1$. Estimate (5.5) then follows from parabolic regularity.

Then, we set

$$w(t,x) = e^{-t}u(t,x) - C_\varepsilon e^{(\lambda-\varepsilon-1)t-\sqrt{\lambda-\varepsilon-1}(x-M-1)}.$$

Since (5.4) holds on compact subsets of $\mathbb{R}_+$, we have

$$\begin{aligned} &w_t - w_{xx} \le 0 \text{ for } t \le 0,\ x \ge M_0,\\ &w(t,M_0) \le 0 \text{ for } t \le 0. \end{aligned}$$

From (5.5) (and $\lambda > 1$) the function w is bounded on $\mathbb{R}_- \times [M_0,+\infty)$. Consequently, it cannot attain a positive maximum, and there cannot be a sequence (t_n,x_n) such that $w(t_n,x_n)$ tends to a positive supremum. This implies that w is negative, hence estimate (5.4) for $x \ge M_0$ follows. It also holds on $[0,M_0]$ due to parabolic regularity.

Let us now turn to positive times. The function $v(t,x) = u(t,x+ct)$ solves

$$\begin{aligned} &v_t - v_{xx} - cv_x \le v \text{ for } t \ge 0,\ \ x \ge M_0,\\ &v(t,M_0) \le 1 \text{ for } t \ge 0,\\ &v(0,x) \le C_\varepsilon e^{-\sqrt{\lambda-1-\varepsilon}\,x}, \end{aligned}$$

the last inequality due to (5.4). Since for small enough $\varepsilon > 0$ we have $r_-(c) < \sqrt{\lambda-\varepsilon-1} < r_+(c)$, the stationary function $e^{-\sqrt{\lambda-1-\varepsilon}\,x}$ is a super-solution to

$$v_t - v_{xx} - cv_x = v.$$

This in turn implies $v(t,x) \le C_\varepsilon e^{-\sqrt{\lambda-1-\varepsilon}\,x}$ for small $\varepsilon > 0$. Using the fact that the front travels with speed c, we get

$$u(t,x) \le Ce^{-\sqrt{\lambda-1-2\varepsilon}(x-X(t))}$$

with a new C. This contradicts Lemma 5.2 since $r_-(c) < \sqrt{\lambda-1}$. □

The rest of the paper contains the proofs of Lemmas 5.1 and 5.2.

6 An upper bound for fronts with speed $c > \lambda/\sqrt{\lambda-1}$: The proof of Lemma 5.1

It is obviously sufficient to prove that for any $\varepsilon > 0$ there exists x_ε such that for any $t \in \mathbb{R}$ we have

$$u(t,x) \le e^{-(r_-(c)-\varepsilon)(x-X(t))} \quad \text{for } x \ge X(t) + x_\varepsilon. \tag{6.1}$$

Therefore assume, towards contradiction, that there exists $\varepsilon > 0$ and $T_n \in \mathbb{R}$, $x_n \to +\infty$ such that

$$u(T_n, X(T_n) + x_n) \geq e^{-(r_-(c)-\varepsilon)x_n}.$$

By the Harnack inequality, there is a constant $\delta > 0$ such that

$$u(T_n - 1, X(T_n) + x) \geq \delta e^{-(r_-(c)-\varepsilon)x_n} \quad \text{for } x \in [x_n, x_n + 1]. \tag{6.2}$$

As u satisfies (1.11) and moves with speed c, we know that for every $\alpha > 0$ we have

$$\lim_{s \to +\infty} \sup_{T \in \mathbb{R},\ x \geq X(T)+(c+\alpha)s} u(T + s, x) = 0.$$

Therefore, for every $\alpha > 0$ there is $x_\alpha > 0$ such that for any $T \in \mathbb{R}$,

$$f(u(t,x)) = u(t,x) \quad \text{for } t \geq T \text{ and } x \geq X(T) + (c+\alpha)(t-T) + x_\alpha$$

Then from $u \leq 1$ we have for $t \geq T$

$$u_t - u_{xx} = a(x)u + a(x)\left(f(u) - u\right) \geq u - C\mathbf{1}_{x \leq X(T)+(c+\alpha)(t-T)+x_\alpha}$$

with $C = \|a\|_\infty$. Thus we have

$$u(t,x) \geq e^t \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{4(t-T)}}}{\sqrt{4\pi(t-T)}} u(T,y)\, dy - C \int_T^t \int_{-\infty}^{x_\alpha+(c+\alpha)s} \frac{e^{-\frac{(x-y)^2}{4(t-s)}+(t-s)}}{\sqrt{4\pi(t-s)}}\, dyds =: I(t,x) - I\!I(t,x)$$

We are going to evaluate $I(t,x)$ and $I\!I(t,x)$ for $T = T_n - 1$ at

$$(t,x) = (t_n, z_n) := \left(T_n - 1 + \frac{x_n}{\sqrt{c^2-4}}, X(T_n) + \frac{cx_n}{\sqrt{c^2-4}}\right),$$

and show that $I(t_n, z_n) \to +\infty$ faster than $I\!I(t_n, z_n)$ provided $\alpha > 0$ is small enough, giving a contradiction with $u(t,x) \leq 1$.

Fix n and for the sake of simplicity assume $T_n = 1$ and $X(T_n) = 0$ (this can be achieved by a translation in space and time). So $T = 0$ and by (6.2) we have

$$I(t_n, z_n) \geq e^{t_n} \int_{x_n}^{x_n+1} \frac{e^{-\frac{(z_n-y)^2}{4t_n}}}{\sqrt{4\pi t_n}} u(0,y)\, dy \geq \frac{\delta}{\sqrt{4\pi t_n}} e^{t_n - (r_-(c)-\varepsilon)x_n} \int_0^1 e^{-\frac{(z_n-x_n-z)^2}{4t_n}}\, dz.$$

Note that for $z \in [0,1]$ we have

$$\frac{(z_n - x_n - z)^2}{t_n} = \frac{(z_n - x_n)^2}{t_n} + O(1),$$

thus with some n-independent $q > 0$ we have

$$I(t_n, z_n) \geq \frac{q\delta}{\sqrt{4\pi t_n}} e^{-\frac{(z_n-x_n)^2}{4t_n} + t_n - (r_-(c)-\varepsilon)x_n}.$$

The exponent is easily evaluated using the relations $x_n = \sqrt{c^2-4}\,t_n$, $z_n - x_n = 2r_-(c)t_n$, and $r_-(c)^2 + \sqrt{c^2-4}\,r_-(c) - 1 = 0$, leading to

$$I(t_n, z_n) \geq \frac{q\delta}{\sqrt{4\pi t_n}} e^{(\varepsilon\sqrt{c^2-4}-\alpha)t_n}. \tag{6.3}$$

To estimate $II(t_n, z_n)$, notice that we have (using $z_n = ct_n$ and with $z := y - z_n$)

$$\begin{aligned}
II(t_n, z_n) &\leq C\int_0^{t_n}\int_{-\infty}^{x_\alpha+(c+\alpha)s-z_n} \frac{e^{t_n-s-\frac{z^2}{4(t_n-s)}}}{\sqrt{4\pi(t_n-s)}}dzds \\
&= C\int_0^{t_n}\left(\int_{x_\alpha-c(t_n-s)}^{x_\alpha-c(t_n-s)+\alpha s} + \int_{-\infty}^{x_\alpha-c(t_n-s)}\right)\frac{e^{t_n-s-\frac{z^2}{4(t_n-s)}}}{\sqrt{4\pi(t_n-s)}}dzds \\
&=: II_1(t_n, z_n) + II_2(t_n, z_n).
\end{aligned}$$

Using the estimate

$$\int_{-\infty}^{x_\alpha-c(t_n-s)} \frac{e^{-\frac{z^2}{4(t_n-s)}}}{\sqrt{t_n-s}}dz \leq C_\alpha \frac{e^{-\frac{c^2(t_n-s)}{4}}}{\sqrt{t_n-s}}$$

and $c > 2$, we have $II_2(t_n, z_n) = O(1)$ as $n \to +\infty$. In order to estimate $II_1(t_n, z_n)$, we represent $\zeta := z + c(t_n - s) \in [x_\alpha, x_\alpha + \alpha s]$ so that

$$t_n - s - \frac{z^2}{4(t_n-s)} = t_n - s - \frac{c^2(t_n-s)^2 + \zeta^2 - 2c(t_n-s)\zeta}{4(t_n-s)} \leq \frac{c\zeta}{2} \leq cx_\alpha + c\alpha t_n.$$

It follows that

$$II_1(t_n, z_n) \leq \alpha t_n e^{cx_\alpha + c\alpha t_n}\int_0^{t_n}\frac{ds}{\sqrt{4\pi(t_n-s)}} \leq C\alpha t_n^{3/2} e^{cx_\alpha+c\alpha t_n} \leq C_\alpha e^{2c\alpha t_n}.$$

We now choose $\alpha > 0$ so that $\varepsilon\sqrt{c^2-4} - \alpha > 2c\alpha$. Using (6.3), it follows that $u(t_n, z_n) = I(t_n, z_n) - II(t_n, z_n) > 1$ for all large n, a contradiction. This finishes the proof of Lemma 5.1. $\square$

7 A lower bound for fronts with speed $c > \lambda/\sqrt{\lambda-1}$: The proof of Lemma 5.2

7.1 A heat kernel estimate

We will need a rather precise information on the behavior, for large x and t, of the solutions of the Cauchy problem

$$\begin{aligned}
&u_t - u_{xx} - A(x)u = 0, \quad t > 0,\ x \in \mathbb{R}, \\
&u(0, x) = u_0(x).
\end{aligned} \tag{7.1}$$

The function $B(x) = A(x) - 1$ is assumed to be nonnegative and to have compact support, in an interval $[L - M_0, L + M_0]$. Basically, A should be thought of as a translate of the function a:

in the proof of Lemma 5.2 below, the number M_0 will be of fixed size, the number L will vary arbitrarily. A lot – most probably, including our estimate below – is known about solutions of (7.1). See, for instance, [16] and the references therein. However we were not able to find in the literature an estimate of the type (7.3) below. Moreover, the proof is short, so it is worth presenting it in reasonable detail. Denote by $G(t,x,y)$ the heat kernel of (7.1), i.e. the function such that the solution $u(t,x)$ is

$$u(t,x) = \int_{-\infty}^{+\infty} G(t,x,y)u_0(y)\, dy.$$

Let us also denote by $H(t,z)$ the standard heat kernel:

$$H(t,z) = \frac{e^{-z^2/4t}}{\sqrt{4\pi t}}.$$

Proposition 7.1. *Assume the function $B(x-L)$ to be even and nonnegative, and that the eigenvalue problem*

$$\phi_0'' + (1 + B(x-L))\phi_0 = \lambda\phi_0$$

has a unique eigenvalue $\lambda > 1$. Let $\phi_0 > 0$ be the eigenfunction with $\|\phi_0\|_2 = 1$. Then we have

$$G(t,x,y) \geq e^t H(t,x-y). \tag{7.2}$$

for all $x, y \in \mathbb{R}$. Conversely, if $x < L - M_0$ and $y > L + M_0$, or $y < L - M_0$ and $x > L + M_0$, then there is a smooth function ψ_0 such that $\psi_0(x) = O(e^{-\sqrt{\lambda-1}|x|})$ for $|x-L| \geq 2M_0$, and such that, for all $\varepsilon > 0$ we have

$$|G(t,x,y) - \Big(e^{\lambda t}\phi_0(x)\phi_0(y) + e^t(H(t,.) * \psi_0)(x-y)\Big)| \leq Ce^{t+C|x-y|/t}H(t,x-y). \tag{7.3}$$

Also, there is $C > 0$, depending on M_0 but not on L, such that if $x, y < L - M_0$ or $x, y > L + M_0$, we have

$$G(t,x,y) - \Big(e^{\lambda t}\phi_0(x)\phi_0(y) + e^t(H(t,.) * \psi_0)(x+y-2L)\Big) \leq Ce^{t+C|x+y-2L|/t}H(t,x+y-2L). \tag{7.4}$$

Proof. The lower bound (7.2) is obvious, because $A(x) \geq 1$. So, let us examine the upper bound. First, we may without loss of generality assume $L = 0$, the result will just follow by translating x and y by the amount L. Also, it is enough to replace $A(x)$ by $B(x)$ (thus we deal with a compactly supported potential), at the expense of multiplying the final result by e^t. Our proof will use some basic facts of eigenfunction expansions, see [11], that we recall now. For $k \in \mathbb{R}^*$, let us denote by $f(x,k)$ the solution of

$$-\phi'' = (B(x) + k^2)\phi, \quad x \in \mathbb{R} \tag{7.5}$$

satisfying

$$f(x,k) = e^{ikx} \quad \text{for } x \geq M_0 \tag{7.6}$$

and let us denote by $g(x,k)$ the solution of (7.5) such that

$$g(x,k) = e^{-ikx} \quad \text{for } x \leq -M_0. \tag{7.7}$$

Denoting by $W(u(x), v(x))$ the Wronskian of two solutions u and v of (7.5), let us set

$$a(k) = -\frac{1}{2ik} W(f(x,k), g(x,k)), \qquad b(k) = \frac{1}{2ik} W(f(x,k), g(x,-k))$$

and

$$c(k) = -b(-k), \quad d(k) = a(k). \tag{7.8}$$

We have

$$\begin{aligned} f(x,k) = &\ a(k)g(x,-k) + b(k)g(x,k) \\ g(x,k) = &\ c(k)f(x,k) + d(k)f(x,-k), \end{aligned} \tag{7.9}$$

and $|a(k)|^2 = 1 + |b(k)|^2$, $b(-k) = \overline{b(k)}$, and $a(-k) = \overline{a(k)}$. The following decompositions hold:

$$\delta(x-y) = \phi_0(x)\phi_0(y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x,k)\overline{f(y,k)}\ dk - \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x,k)f(y,k)\frac{b(-k)}{a(k)}\ dk, \tag{7.10}$$

and

$$\delta(x-y) = \phi_0(x)\phi_0(y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(x,k)\overline{g(y,k)}\ dk + \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(x,k)g(y,k)\frac{b(k)}{a(k)}\ dk. \tag{7.11}$$

These decompositions may also be viewed as a consequence of Agmon's limiting absorption principle, see [1], Theorem 4.1. Consequently, we have the representation

$$\begin{aligned} G(t,x,y) = &\ e^{(\lambda-1)t}\phi_0(x)\phi_0(y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-tk^2} f(x,k)\overline{f(y,k)}\ dk \\ &-\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-tk^2} f(x,k)f(y,k)\frac{b(-k)}{a(k)}\ dk \\ = &\ e^{(\lambda-1)t}\phi_0(x)\phi_0(y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-tk^2} g(x,k)\overline{g(y,k)}\ dk \\ &+\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-tk^2} g(x,k)g(y,k)\frac{b(k)}{a(k)}\ dk. \end{aligned} \tag{7.12}$$

Now we prove (7.3). If $y < -M_0$ and $x > M_0$, the identity (7.9) and the first equality in (7.12) implies that

$$\begin{aligned} G(t,x,y) = &\ e^{(\lambda-1)t}\phi_0(x)\phi_0(y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{-tk^2}}{a(-k)} e^{ik(x-y)}\ dk \\ = &\ e^{(\lambda-1)t}\phi_0(x)\phi_0(y) + (H(t,\cdot) * F_1)(x-y) \end{aligned} \tag{7.13}$$

where F_1 is the inverse Fourier Transform of $\frac{1}{a(-k)}$. By using the second equality in (7.12), we see that the same holds for $y > M_0$ and $x < -M_0$. This function F_1 may be estimated by (7.10) and (7.9) if $y < -M_0$ and $x > M_0$:

$$\begin{aligned} -\phi_0(x)\phi_0(y) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x,k)\left(\overline{f(y,k)} - f(y,k)\frac{b(-k)}{a(k)}\right) dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(a(k)e^{ikx} + b(k)e^{-ikx}\right)\left(e^{-iky} - e^{iky}\frac{b(-k)}{a(k)}\right) dk \end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{|a(k)|^2 - |b(k)|^2}{a(-k)} e^{ik(x-y)} dk$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{ik(x-y)}}{a(-k)} dk.$$

The same is true for $y < -M_0$ and $x > M_0$, one just has to use (7.11) and (7.9). Therefore,

$$F_1 = \psi_0 + T_0, \tag{7.14}$$

where $\psi_0(x) = c_0 e^{-\sqrt{\lambda-1}|x|}$ for $|x| \geq 2M_0$, T_0 is a compactly supported distribution, and where we have made the abuse of notation consisting in using the argument x in a distribution. Combining this with (7.13) we obtain

$$G(t,x,y) = \ e^{(\lambda-1)t}\phi_0(x)\phi_0(y) + (H(t,.) * \psi_0)(x-y) + (H(t,.) * T_0)(x-y)$$

and estimate (7.3) is concluded by a standard distributional computation. Now we prove (7.4). If x and y are on the same side, say $x \geq M$ and $y \geq M$, then (7.12) implies

$$\begin{aligned} G(t,x,y) = &\ e^{(\lambda-1)t}\phi_0(x)\phi_0(y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-tk^2+ik(x-y)}\ dk - \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-tk^2+ik(x+y)}\frac{b(-k)}{a(k)}\ dk \\ = &\ e^{(\lambda-1)t}\phi_0(x)\phi_0(y) + H(t,x-y) + (H(t,\cdot) * F_2)(x+y), \quad \text{for } x \geq M,\ \ y \geq M \end{aligned} \tag{7.15}$$

where F_2 is the inverse Fourier transform of the function $b(-k)/a(k)$. Similarly,

$$\begin{aligned} G(t,x,y) = &\ e^{(\lambda-1)t}\phi_0(x)\phi_0(y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-tk^2-ik(x-y)}\ dk - \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-tk^2-ik(x+y)}\frac{b(k)}{a(k)}\ dk \\ = &\ e^{(\lambda-1)t}\phi_0(x)\phi_0(y) + H(t,x-y) + (H(t,\cdot) * F_3)(x+y), \quad \text{for } x \leq -M,\ \ y \leq -M, \end{aligned} \tag{7.16}$$

where F_3 is the Fourier transform of the function $b(k)/a(k)$. It follows from [11], that F_2 and F_3 are $W^{1,1}$ functions. From the relations (7.9) and decomposition (7.10), we find that

$$F_2(x+y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x+y)}\frac{b(-k)}{a(k)}\ dk = \phi_0(x)\phi_0(y), \quad \text{for } x \geq M_0,\ \ y \geq M_0. \tag{7.17}$$

Consequently, $F_2(z) = c_1 e^{-\sqrt{\lambda-1}|z|}$ for $z > 2M_0$. In the same fashion we have, from the decomposition (7.11),

$$F_3(x+y) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ik(x+y)}\frac{b(k)}{a(k)}\ dk = \phi_0(x)\phi_0(y) \quad \text{for } x \leq -M_0,\ y \leq -M_0. \tag{7.18}$$

From the evenness of B and the relations (7.8), the function $b(k)$ is purely imaginary, so $b(-k)/a(k) = \overline{b(k)}/a(k) = -b(k)/a(k)$. Thus, $F_3(z) = F_2(-z)$. And so, similarly to (7.14) there holds

$$F_i = \psi_0 + T_i, \quad i \in 2, 3$$

where T_2 and T_3 are $W^{1,1}$ functions supported in $(-\infty, 2M_0)$ and $(-2M_0, \infty)$, respectively. So, for $x \geq M_0$ and $y \geq M_0$, estimate (7.4) now follows from (7.15), since

$$|(H(t,\cdot) * T_2)(x+y)| = \left|\int_{-\infty}^{2M_0} H(t, x+y-z)T_2(z)\, dz\right| \leq H(t, x+y-2M_0)\|T_2\|_1$$

The same argument is valid for $x \le -M_0$ and $y \le -M_0$ using (7.16). $\square$

Proposition 7.1 admits the following corollary, which takes care of what happens when y is in the support of B.

Corollary 7.2. *Let ψ_0 be defined as in Proposition 7.1 . There is a constant C such that if $y \in [L - M_0, L + M_0]$ and $x \notin [L - M_0, L + M_0]$, we have*

$$G(t,x,y) - \left(e^{\lambda t}\phi_0(x)\phi_0(y) + (e^t H(t,.) * \psi_0)(x-L) \right) \le C e^{t + C|x-L|/t} H(t, x-L). \tag{7.19}$$

The proof is similar to that of the proposition, and is omitted.

7.2 Proof of Lemma 5.2

Assume the conclusion of Lemma 5.2 to be false. Then there exists a sequence $T_n \to +\infty$, and a sequence $x_n \to +\infty$ such that

$$u(T_n, X(T_n) + x_n) \le e^{-(r_-(c)+\varepsilon)x_n}. \tag{7.20}$$

Extending (7.20) to a large interval

We are going to apply the Harnack inequality in the following way: if $u(t,x)$ is a global solution (in time and space) of a linear parabolic equation on $(t,x) \in \mathbb{R} \times \mathbb{R}$, there exists a universal constant $\rho \in (0,1)$ such that

$$u(t,x) \ge \rho u(t-1, x+\xi), \quad \text{for all } t, x \in \mathbb{R} \text{ and all } \xi \in [-1,1].$$

Thus, for all $\xi \in [-1,1]$ and all $t \in \mathbb{R}$ and $x \in \mathbb{R}$, and any non-negative integer $p \in \mathbb{N}$ we have

$$u(t,x) \ge \rho^p u(t-p, x+p\xi). \tag{7.21}$$

Then, assumption (7.20) on u together with (7.21) translate into

$$\begin{aligned} u(T_n - p, X(T_n) + x_n + p\xi) &\le \rho^{-p} e^{-(r_-(c)+\varepsilon)x_n} \\ &= \rho^{-p} e^{-(r_-(c)+\varepsilon)[X(T_n-p)-X(T_n)]} e^{-(r_-(c)+\varepsilon)(x_n - [X(T_n-p)-X(T_n)])}, \end{aligned} \tag{7.22}$$

for all $\xi \in [-1,1]$. Note that, as $u(t,x)$ is a front moving with the speed c, there exists a constant $B > 0$ so that

$$X(T_n) - 2c(p+B) \le X(T_n - p) \le X(T_n) + \frac{c}{2}(-p+B). \tag{7.23}$$

We are going to choose p as a small fraction of x_n, that is, $p = [\eta x_n]$ where $[x]$ denotes the integer part of x, and $\eta > 0$ is small. Then, for any $x \in [(1-\eta)x_n, (1+\eta)x_n]$ we rewrite (7.22), using also (7.23) as

$$\begin{aligned} u(T_n - p, X(T_n) + x) &\le \rho^{-p} e^{-(r_-(c)+\varepsilon)[X(T_n-p)-X(T_n)]} e^{-(r_-(c)+\varepsilon)(x - [X(T_n-p)-X(T_n)]) + (r_-(c)+\varepsilon)(x - x_n)} \\ &\le C\rho^{-p} e^{2c(r_-(c)+\varepsilon)(p+B)} e^{-(r_-(c)+\varepsilon)(x - [X(T_n-p)-X(T_n)]) + (r_-(c)+\varepsilon)p} \\ &\le C \exp\left[\left(-r_-(c) - \varepsilon + \frac{Kp}{x - [X(T_n-p) - X(T_n)]} \right) (x - [X(T_n-p) - X(T_n)]) \right], \end{aligned}$$

with a constant K that depends on c, ρ and B but not on p or x. As $p = [\eta x_n]$, $x_n \to +\infty$, and $X(T_n - p) \le X(T_n) + cB/2$, choosing $\eta = \varepsilon/(1+2K)$ so that $K\eta/(1-\eta) < \varepsilon/2$ ensures that

$$\frac{Kp}{x - [X(T_n - p) - X(T_n)]} \le \frac{\varepsilon}{2} \quad \text{for all } x \in [(1-q\varepsilon)x_n, (1+q\varepsilon)x_n],$$

for n large enough. Here we have set $q = 1/(1+2K)$.

Let us now shift the origin of time and space placing it at $(t,x) = (T_n - p, X(T_n - p))$. And thus, in the new coordinates we have

$$u_0(x) := u(0,x) \le Ce^{-(r_-(c)+\varepsilon/2)x} \quad \text{for } x \in [(1-q\varepsilon)x_n, (1+q\varepsilon)x_n]. \tag{7.24}$$

The support of $a-1$ is also shifted accordingly: it is supported in an interval $[L - M_0, L + M_0]$, with $L = -X(T_n - p) < -M_0$ for large n.

Reduction of $u(t,x)$

We start from

$$u(t,x) = \mathcal{S}_a(t)u_0(x) - \int_0^t \mathcal{S}_a(t-s)a(u - f(u))\,ds \le \mathcal{S}_a(t)u_0(x) - \int_0^t \mathcal{S}_1(t-s)a(u-f(u))\,ds.$$

and we are going to evaluate it for a well chosen $(t,x) \in \mathbb{R}_+ \times \mathbb{R}_+$. Here $\mathcal{S}_a$ denotes the semigroups generated by the operator $\partial^2_{xx} + a(x)$, and $\mathcal{S}_1$ is the semigroup generated by the operator $\partial^2_{xx} + 1$, with $a(x)$ appropriately shifted to our new coordinate frame. Because $x > 0$, it is outside of $\mathrm{supp}(a-1) = [L - M_0, L + M_0]$; we will use Proposition 7.1 and Corollary 7.2 to deal with $\mathcal{S}_a(t)u_0(x)$. We have

$$\begin{aligned}
\mathcal{S}_a(t)u_0(x) &\le e^t \int H(t, x-y)\Big((u_0 * \psi_0)(y) + Ce^{C|x-y|/t}u_0(y)\Big)\,dy \\
&\quad + e^t \int E(t,x,y))\Big((u_0 * \psi_0)(y) + Ce^{C|x-y|/t}u_0(y)\Big)\,dy + e^{\lambda t}\langle \phi_0, u_0\rangle \phi_0(x) \\
&= u_1(t,x) + u_2(t,x) + u_3(x),
\end{aligned} \tag{7.25}$$

where $E(t,x,y) = 0$ if $y < L - M_0$ (since $x > L + M_0$), while

$$E(t,x,y) = C\frac{e^{-|x+y-2L|^2/(t+1)}}{\sqrt{4\pi(t+1)}}$$

if $y > L - M_0$. We will also set

$$u_4(t,x) = \int_0^t \mathcal{S}_1(t-s)a(u - f(u))\,ds. \tag{7.26}$$

We will estimate each of u_1, u_2, u_3 and u_4 separately at an appropriately chosen point (t_n, z_n) and show that u_4 is much larger than $u_1 + u_2 + u_3$ giving a contradiction.

Estimate of $u_1(t,x)$

This is the most involved, the estimates of u_2 and $_3$ being simpler or similar. First, we anticipate that u_1 will be evaluated at a point (t,x) such that t and x are both large, and x and t of the same order of magnitude. Also, in the integral expressing u_1, the integrands will be maximized at points y such that $|x-y|$ is of orde t. Hence, from standard convolutions between exponentials (and the fact that $r_-(c) < \sqrt{\lambda-1}$, we do not lose any generality if we assume the existence of a function $w_0(x)$ and a constant $C>0$ such that

(i). the function w_0 is bounded on $\mathbb{R}$,

(ii). there is a constant $C>0$ such that (even if it means restricting q a little)

$$\text{for all } \delta>0, \text{ there is } C_\delta>0 \text{ such that } w_0(x) \leq C_\delta e^{-(r_-(c)-\delta)x} \text{ for } x>0,$$
$$w_0(x) \leq C_\delta e^{-(r_-(c)+\varepsilon)x} \text{ for } x \in [(1-q\varepsilon)x_n, (1+q\varepsilon)x_n],$$

(iii). and we have

$$\int H(t,x-y)\Big((u_0*\psi_0)(y) + Ce^{C|x-y|/t}u_0(y)\Big)\,dy \leq \int H(t,x-y)w_0(y)\,dy$$
$$\int E(t,x,y)\Big((u_0*\psi_0)(y) + Ce^{C|x-y|/t}u_0(y)\Big)\,dy \leq \int H(t,x-y)w_0(y)\,dy$$

And thus, we start with

$$u_1(t,x) \leq \frac{Ce^t}{\sqrt{t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}}w_0(y)\,dy. \tag{7.27}$$

And, as in the proof of Lemma 5.1, we are going to estimate $u_1(t,x)$ at the points

$$t_n = \frac{x_n}{\sqrt{c^2-4}}, \quad z_n = ct_n.$$

Observe that for n sufficiently large, $L+M_0<0$, so $z_n > L+M_0$. Thus $z_n \notin \operatorname{supp}(a-1)$ and the estimate (7.27) applies. Let us decompose

$$u_1(t_n,z_n) = \frac{Ce^{t_n}}{\sqrt{t_n}}\left(\int_{-\infty}^0 + \int_0^{(1-q\varepsilon)x_n} + \int_{(1-q\varepsilon)x_n}^{(1+q\varepsilon)x_n} + \int_{(1+q\varepsilon)x_n}^{+\infty}\right)e^{-\frac{(z_n-y)^2}{4t_n}}w_0(y)\,dy$$
$$:= u_{11}(t_n,z_n) + u_{12}(t_n,z_n) + u_{13}(t_n,z_n) + u_{14}(t_n,z_n).$$

As $z_n - y \geq ct_n$ for $y \leq 0$, $t_n \geq 1$, and $0 \leq w_0(y) \leq 1$, we have

$$u_{11}(t_n,z_n) \leq Ce^{(1-\frac{c^2}{4})t_n} \to 0 \text{ as } n \to +\infty, \tag{7.28}$$

since $c>2$. By Lemma 5.1 we have, for every $\delta>0$

$$u_{12}(t_n,z_n) \leq C_\delta \int_0^{(1-q\varepsilon)x_n} e^{t_n - \frac{(ct_n-y)^2}{4t_n} - (r_-(c)-\delta)y}\,\frac{dy}{\sqrt{t_n}}. \tag{7.29}$$

The integrand above is maximized at the point

$$y_\delta = (c - 2r + 2\delta)t_n = (\sqrt{c^2-4} + 2\delta)t_n = x_n + \frac{2\delta}{\sqrt{c^2-4}} x_n,$$

that is $O(\delta x_n)$ close to x_n – this is, indeed, why t_n was chosen as above. Here we have used (1.4). As $y_\delta > x_n$, the integrand in (7.29) on the interval $[0, (1-\varepsilon q)x_n]$ is maximized at the upper limit, leading to

$$\begin{aligned} u_{12}(t_n, z_n) &\le C \int_0^{(1-q\varepsilon)x_n} e^{(1-(r_-(c)+q\varepsilon\sqrt{c^2-4}/2)^2)t_n - (r_-(c)-\delta)(1-q\varepsilon)x_n} \frac{dy}{\sqrt{t_n}} \\ &\le C\sqrt{t_n} e^{[-q^2(c^2-4)\varepsilon^2/4 + \delta(1-q\varepsilon)\sqrt{c^2-4}]t_n}. \end{aligned}$$

Recall that $\varepsilon < 1$. Hence, if we choose $\delta \le \dfrac{q^2\varepsilon^2}{100}\sqrt{c^2-4}$ we have

$$-q^2\frac{(c^2-4)\varepsilon^2}{4} + \delta(1-q\varepsilon)\sqrt{c^2-4} \le -q^2\frac{(c^2-4)\varepsilon^2}{8},$$

and therefore

$$u_{12}(t_n, z_n) \le C_\delta \sqrt{t_n} e^{-q^2\varepsilon^2(c^2-4)t_n/8} \to 0 \text{ as } n \to +\infty. \tag{7.30}$$

Consider now $u_{14}(t_n, z_n)$:

$$\begin{aligned} u_{14}(t_n, z_n) &\le \frac{Ce^{t_n}}{\sqrt{t_n}} \int\limits_{(1+q\varepsilon)x_n}^{+\infty} e^{-\frac{(z_n-y)^2}{4t_n} - (r_-(c)-\delta)y} dy = Ce^{t_n}\left[\int_{(1+q\varepsilon)x_n}^{z_n} + \int_{z_n}^{+\infty}\right] \frac{e^{-\frac{|z_n-y|^2}{4t_n} - (r_-(c)-\delta)y}}{\sqrt{t_n}} dy \\ &= u'_{14}(t_n, z_n) + u''_{14}(t_n, z_n). \end{aligned}$$

For u''_{14} we have:

$$u''_{14}(t_n, z_n) = Ce^{t_n}\int_{z_n}^{+\infty} \frac{e^{-\frac{(y-z_n)^2}{4t_n} - (r_-(c)-\delta)y}}{\sqrt{t_n}} dy \le Ce^{t_n - (r_-(c)-\delta)ct_n} = Ce^{-(r_-(c)^2-\delta)t_n} \to 0,$$

as $n \to +\infty$, while for u'_{14} we have

$$u'_{14}(t_n, z_n) \le Ce^{t_n}\int_{(1+q\varepsilon)x_n}^{z_n} \frac{e^{-\frac{(z_n-y)^2}{4t_n} - (r_-(c)-\delta)y}}{\sqrt{t_n}} dy,$$

and this term can be estimated exactly as $u_{12}(t_n, z_n)$.

We turn to $u_{13}(t_n, z_n)$ – it is here that we use the crucial assumption (7.24). It follows from this bound on $w_0(y)$ inside the interval of integration that

$$u_{13}(t_n, z_n) \le C \int\limits_{(1-q\varepsilon)x_n}^{(1+q\varepsilon)x_n} \frac{e^{t_n - \frac{(ct_n-y)^2 - C|ct_n-y|}{4t_n} - (r_-(c)+\varepsilon/2)y}}{\sqrt{4\pi t_n}} dy \le C \int\limits_{(1-q\varepsilon)x_n}^{(1+q\varepsilon)x_n} \frac{e^{t_n - \frac{(ct_n-y)^2}{4t_n} - (r_-(c)+\varepsilon/2)y}}{\sqrt{4\pi t_n}} dy. \tag{7.31}$$

Now, the maximum of the integrand is achieved at the point

$$y_n = x_n - \frac{\varepsilon}{\sqrt{c^2-4}} x_n.$$

At the expense of possibly decreasing q so that $q < 1/\sqrt{c^2-4}$, we have $y_n < (1-q\varepsilon)x_n$. Then the integrand in (7.31) is maximized at $y = (1-q\varepsilon)x_n$, and we have, for all $y \in [(1-q\varepsilon)x_n, (1+q\varepsilon)x_n]$:

$$\begin{aligned} -\frac{(ct_n-y)^2}{4t_n} - (r_-(c)+\frac{\varepsilon}{2})y &\le -\frac{(ct_n-(1-q\varepsilon)x_n)^2}{4t_n} - (r_-(c)+\frac{\varepsilon}{2})(1-q\varepsilon)x_n \\ &\le \left(-1-\frac{\varepsilon}{2}\sqrt{c^2-4}+O(\varepsilon^2)\right)t_n. \end{aligned} \tag{7.32}$$

This gives, for $\varepsilon > 0$ sufficiently small,

$$u_{13}(t_n, z_n) \le C x_n e^{-\varepsilon t_n \sqrt{c^2-4}/4} \tag{7.33}$$

and, all in all, we have the following upper bound for $u_1(t_n, z_n)$:

$$u_1(t_n, z_n) \le C\sqrt{t_n} e^{-\varepsilon t_n \sqrt{c^2-4}/4} + C_\delta \sqrt{t_n} e^{-q^2\varepsilon^2(c^2-4)t_n/8}. \tag{7.34}$$

The estimate for $u_2(t_n, z_n)$

The quantity $L + M_0$ is bounded from above by a universal constant, so

$$\begin{aligned} u_2(t_n, z_n) &\le \frac{Ce^{t_n}}{\sqrt{t_n}} \int_{L-M_0}^{\infty} e^{-\frac{|z_n+y-2L|^2}{4t_n}} w_0(y)dy = Ce^{t_n} \int_{(z_n-(L+M_0))/\sqrt{4t_n}}^{\infty} e^{-y^2} dy \le Ce^{t_n - z_n^2/(4t_n)} \\ &\le Ce^{(1-c^2/4)t_n}. \end{aligned} \tag{7.35}$$

This will decay exponentially fast since $c > 2$.

Estimate of $u_3(t_n, z_n)$

The last term we need to consider is the eigenvalue contribution:

$$u_3(t, x) = e^{\lambda t}\phi_0(x) \int \phi_0(y) w_0(y) dy,$$

and this is also easy: we have

$$u_3(t_n, z_n) \le Ce^{\lambda t_n - \sqrt{\lambda - 1} z_n} = Ce^{(\lambda - c\sqrt{\lambda-1})t_n}, \tag{7.36}$$

and this quantity will also decay exponentially fast because $c > \lambda/\sqrt{\lambda-1}$.

The estimate for $u_4(t_n, z_n)$

We wish to show that $u_4(t_n, z_n)$ goes to 0 as $n \to +\infty$ slower than the first three terms. As the front is moving with the speed c, for any small $\delta > 0$, there exists a large $x_\delta > 0$ such that

$$u(t,x) \geq \frac{1}{2} \quad \text{for } x \leq (c-\delta)t - x_\delta \text{ and } t \geq 0.$$

By our assumption on $f(u)$ there is a constant $C > 0$ such that $u - f(u) \geq C$ for all $u \in [1/2, 1]$. Therefore, as $a(x) \geq a_0 > 0$, we have

$$\begin{aligned} u_4(t_n, z_n) &\geq a_0 \int_0^{t_n} \int_{\mathbb{R}} \frac{e^{t_n - s - \frac{(ct_n - y)^2}{4(t_n - s)}}}{\sqrt{4\pi(t_n - s)}} (u(s,y) - f(u(s,y))) ds dy \\ &\geq C \int_0^{t_n} \int_{(c-\delta)s - x_\delta - 1}^{(c-\delta)s - x_\delta} \frac{e^{t_n - s - (ct_n - y)^2/4(t_n - s)}}{\sqrt{(t_n - s)}} ds dy. \end{aligned} \tag{7.37}$$

The change of variables $y = (c-\delta)s - x_\delta + z$ in the last integral yields

$$u_4(t_n, z_n) \geq \frac{C}{\sqrt{t_n}} \int_0^{t_n} \int_{-1}^{0} e^{t_n - s - (c(t_n - s) + \delta s + x_\delta - z)^2/4(t_n - s)} ds dy.$$

We have, for $z \in (-1, 0)$ and $0 \leq s < t_n - 1$:

$$\begin{aligned} \Psi_\delta(s, t_n, z) &:= t_n - s - \frac{(c(t_n - s) + \delta s + x_\delta - z)^2}{4(t_n - s)} \\ &= (1 - \frac{c^2}{4})(t_n - s) - \frac{c\delta s}{2} - \frac{\delta^2 s^2}{4(t_n - s)} - 2(x_\delta - z)\frac{c(t_n - s) + \delta s}{4(t_n - s)} - \frac{(x_\delta - z)^2}{4(t_n - s)}. \end{aligned}$$

We evaluate the integral on the time interval $(1 - \gamma_1)t_n \leq s \leq (1 - \gamma_2)t_n$ with $0 < \gamma_2 < \gamma_1 \ll 1$ to be chosen. There is a constant $C_{\delta,\gamma}$ that depends on $\gamma_{1,2}$ and δ but not on n such that for all $z \in [-1, 0]$ and all s in this interval we have

$$\begin{aligned} \Psi_\delta(s, t_n, z) \geq &\ (1 - \frac{c^2}{4})(t_n - s) - \frac{c\delta s}{2} - \frac{\delta^2 s^2}{4(t_n - s)} - C_{\delta,\gamma} \\ \geq &\ \left((1 - \frac{c^2}{4})\gamma_1 - \frac{c}{2}\delta - \frac{\delta^2 (1 - \gamma_2)^2}{4\gamma_2}\right) t_n - C_{\delta,\gamma} := -A_{\delta,\gamma} t_n - C_{\delta,\gamma}. \end{aligned}$$

Therefore

$$u_4(t_n, z_n) \geq C\sqrt{t_n} e^{-A_{\delta,\gamma} t_n - C_{\delta,\gamma}}. \tag{7.38}$$

Gathering (7.34), (7.35), (7.36) and (7.38) we have, for a constant $C > 0$ depending only on δ:

$$u(t_n, z_n) \leq C_\delta \left(-e^{-A_{\delta,\gamma} t_n - C_{\delta,\gamma}} + e^{-\varepsilon c t_n} + e^{-\varepsilon t_n \sqrt{c^2 - 4}/4} + e^{(1 - \frac{c^2}{4} + o(1))t_n} + e^{-\frac{q^2}{2}\varepsilon^2 (c^2 - 4)t_n} + e^{(\lambda - c\sqrt{\lambda - 1})t_n} \right).$$

Choosing γ_1 and γ_2 small enough, and then $\delta = \gamma_2$ makes the constant $A_{\delta,\gamma}$ arbitrarily small. In particular, we may ensure that it is much smaller than the coefficients in front of t_n in the last five exponential terms above. This yields

$$u(t_n, z_n) < 0$$

for large n which is the contradiction. □

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