The Bramson delay in the non-local Fisher-KPP equation

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Abstract

We consider the non-local Fisher-KPP equation modeling a population with individuals competing with each other for resources with a strength related to their distance, and obtain the asymptotics for the position of the invasion front starting from a localized population. Depending on the behavior of the competition kernel at infinity, the location of the front is either $2t - (3/2)\log t + O(1)$, as in the local case, or $2t - O(t^{\beta})$ for some explicit $\beta \in (0, 1)$. Our main tools here are a local-in-time Harnack inequality and an analysis of the linearized problem with a suitable moving Dirichlet boundary condition. Our analysis also yields, for any $\beta \in (0, 1)$, examples of Fisher-KPP type non-linearities f_β such that the front for the local Fisher-KPP equation with reaction term f_β is at $2t - O(t^{\beta})$.

Key-Words: Reaction-diffusion equations, Logarithmic delay, Parabolic Harnack inequality
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1 Introduction

The Fisher-KPP equation

$$u_t = u_{xx} + u(1 - u) \tag{1.1}$$

{sep2704

is one of the simplest models for population spreading, accounting for a competition for resources. However, (1.1) only accounts for a local competition between individuals. When this competition is non-local, one is led to the non-local Fisher-KPP equation

$$\begin{aligned} &u_t - u_{xx} = u(1 - \phi \star u), \qquad t > 0,\ x \in \mathbb{R}, \\ &u(0, x) = u_0(x). \end{aligned} \tag{1.2}$$

{eq:nonl

Here, ϕ is a probability density that represents the strength of the competition between individuals a given distance apart. Equation (1.2) has garnered much interest recently, mostly for two reasons. First, it does not admit a comparison principle, leading to inherent technical difficulties – even proving a uniform upper bound on u is non-trivial [17]. Second, unusual behavior may occur, such as the existence of oscillating wave trains behind the front [11, 12, 13, 19].

Our interest is in the spreading of the solutions of (1.2) when the initial density $u_0(x) \geq 0$ is localized. To motivate our work, we recall the known results for the local Fisher-KPP equation (1.1).

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Going back to the work of Bramson, it is known that if $u_0(x)$ is compactly supported, the front of u is located at

$$X(t) = 2t - \frac{3}{2}\log t + s_0, \tag{1.3}$$

{sep2702

where s_0 is a shift depending only on u_0 [4, 5], with less precise asymptotics obtained earlier by Uchiyama [25]. These proofs have been simplified in recent years [15, 24], with some refinements in [21, 22], and also extended to the spatially periodic case [16]. One may think of $\bar{X}(t) = 2t$ as the position of a traveling wave, and $d(t) = (3/2)\log t$ as the delay due to the fact that the initial condition $u_0(x)$ is compactly supported, so that the solution lags behind the traveling wave.

In the non-local case considered in the present paper, we show that the front position depends on the rate of decay of the kernel ϕ at infinity. When ϕ decays fast enough, solutions of (1.2) spread as those of the local equation: the front is at a position as in (1.3), up to a constant order error. However, when ϕ decays slowly, and the competition at large distances is relatively strong, the delay behind the traveling wave position $2t$ is not logarithmic but algebraic, of the order $O(t^\beta)$, with β that depends only on the rate of decay of ϕ.

We now make our assumptions more precise. First, we assume that $\phi(x) \geq 0$ is an even, continuous, and bounded probability density:

$$\int_{\mathbb{R}} \phi(x)dx = 1, \quad \text{and } \phi(x) = \phi(-x) \text{ for all } x \in \mathbb{R},$$

such that

$$A_\phi^{-1}(1+|x|)^{-r} \leq \phi(x) \leq A_\phi(1+|x|)^{-r}, \tag{1.4}$$

{eq:phi}

for all $x \in \mathbb{R}$, with some positive constants $r \in (1,\infty)$ and $A_\phi > 0$. The assumption $r > 1$ is necessary to ensure integrability for ϕ. The assumption of continuity of ϕ, along with the pointwise estimate on the decay of ϕ can be replaced with weaker assumptions requiring only decay of integrals of the type

$$\int_x^\infty \phi(y)dy$$

and a non-zero mass centered around the origin. We adopt the assumptions in (1.4) in order to simplify the statements and to avoid too technical considerations. Our analysis applies equally well to competition kernels ϕ that decay faster than algebraically.

Second, we assume that u_0 is localized to be to the left of some point x_0:

$$0 \leq u_0 \leq 1, \qquad \exists x_0 \text{ such that } u_0(x) = 0 \text{ for all } x \geq x_0, \qquad \text{and} \qquad \liminf_{x\to-\infty} u_0(x) > 0. \tag{1.5}$$

{eq:u_0}

One may allow $u_0(x)$ to have a "fast" exponential decay rather than be compactly supported on the right but we recall that the front position asymptotics for solutions of (1.1) with u_0 that has a sufficiently slow exponential tail on the right is different from (1.3), see [4, 5].

The main result of this paper is the following.

:main_delay **Theorem 1.1.** *Suppose that u satisfies (1.2) and (1.5) with ϕ satisfying (1.4). If $r > 3$, then the solution u propagates with a logarithmic delay:*

$$\lim_{L\to\infty} \limsup_{t\to\infty} \sup_{x\geq L} u\Big(t, 2t - \frac{3}{2}\log t + x\Big) = 0, \tag{1.6}$$

{eq:log_

and

$$\liminf_{t\to\infty} \inf_{x\leq 0} u\Big(t, 2t - \frac{3}{2}\log t + x\Big) > 0. \tag{1.7}$$

{eq:log_

If $r = 3$, then the solution u propagates with a weak logarithmic delay: (1.6) *holds and, for all $\epsilon > 0$,*

$$\liminf_{t\to\infty} \inf_{x\le 0} u\Big(t, 2t - \Big(\frac{3}{2} + \epsilon\Big)\log t + x\Big) > 0. \tag{1.8}$$

{eq:weak

If $r \in (1, 3)$, then the delay is algebraic: there exist $0 < c_\phi < C_\phi$, depending only on ϕ, such that

$$\lim_{t\to\infty} \sup_{x\ge 0} u\Big(t, 2t - c_\phi t^{\frac{3-r}{1+r}} + x\Big) = 0. \tag{1.9}$$

{eq:alg_

and

$$\liminf_{t\to\infty} \inf_{x\le 0} u\Big(t, 2t - C_\phi t^{\frac{3-r}{1+r}} + x\Big) > 0. \tag{1.10}$$

{eq:alg_

As we discuss later in greater detail, heuristically, the competition term $\phi\star u$ acts on the scale t^γ, with $\gamma = 2/(1+r)$. Note that

$$\frac{3-r}{1+r} = 2\gamma - 1, \tag{1.11}$$

{sep2724

and that, when $r > 3$, $\gamma < 1/2$, which, in turn, suggests that the competition scale is smaller than the diffusive scale $\sqrt{t}$. This is one way to see that there is a phase transition at $r = 3$.

As a by-product of our analysis, we also obtain results for the local Fisher-KPP equation

$$u_t = u_{xx} + f(u). \tag{1.12}$$

{sep2706

Let us assume that f is of the KPP class: $f(u)/u$ is decreasing in u near 0, $f \in C^1$, and $f'(0) = 1$. A natural question is whether these assumptions are sufficient to ensure that the front location is given by the logarithmic Bramson correction in (1.3). We show, roughly, the following: if

$$1 - \frac{f(u)}{u} \sim \log(1/u)^{1-r} \text{ with } r > 1,$$

then the conclusion of Theorem 1.1 holds, with the logarithmic delay for $r \ge 3$ and an algebraic delay of the order $O(t^{(3-r)/(1+r)})$ for $1 < r < 3$. These non-linearities are not purely mathematical curiosities: they are regularly used in biology and are known as Gompertz models, see [7] and the vast body of literature around it. The statement and proof of this result are contained in Section 6.

Let us mention a few related works. The model (1.2) considered here was first introduced by Britton [6] and has a quite involved history, see the introduction of [1] for a brief overview. The non-local term $\phi \star u$ has different effects depending on whether one is studying the behavior of u behind the front or at the front. Behind the front, there is a possible Turing instability of the steady state of the local Fisher-KPP equation $u \equiv 1$, which complicates the behavior. For example, wave trains have been constructed by Faye and Holzer [11] and, in a related setting, in [19]. Such wave trains have also been observed numerically by Genieys, Volpert, and Auger in [12]. As a result, without finer assumptions on ϕ, one cannot hope for a stronger result than the lower bounds in Theorem 1.1. As far as the behavior at the front is concerned, the main result in this direction is that traveling waves of speed $c = 2$ exist [9, 13] and solutions to the Cauchy problem with compact initial data or which satisfy (1.5) propagate with speed $c(t) = 2 + o(1)$ as $t \to +\infty$ [17]. While in the final stages of preparing this paper, we learned of a very recent probabilistic study of the delay term by Penington [23]. In our notation, she obtains the log delay up to an error term $O(\log\log(t))$, when $r > 3$, and an algebraic delay $t^{(3-r)/(1+r)\pm\epsilon}$ for any $\epsilon > 0$ when $r \in (1, 3)$. Since her work is probabilistic, the approaches are quite different from one another.

As far as algebraic delays are concerned, we point to the work of Fang and Zeitouni [10] and Maillard and Zeitouni [18], as well as [20] where a Fisher-KPP model with a diffusivity that changes slowly in time was studied, and a delay, roughly, of order $t^{1/3}$ was obtained. However, both the set-up and the mechanism for the large delay are quite different in these papers than in the present work. Finally, we also mention the recent paper of Ducrot [8] in which he constructs a class of non-linearities $f(x,u)$, which tend to $u(1-u)$ as $|x| \to \infty$, such that if the nonlinearity $u(1-u)$ in (1.1) is replaced by $f(x,u)$, then the front is at $2t - \lambda \log(t)$ for any $\lambda \geq 3/2$.

Heuristics and methods of proof

The upper bound (1.6) is obtained by a rather direct adaptation of the arguments in [15]. Let us outline a heuristic argument leading to the upper bound (1.9) for $r \in (1,3)$. It also explains how the exponent $(3-r)/(1+r)$ comes about. Let the front have a delay $d(t)$ behind $2t$, so that

$$\inf_{x \leq 2t-d(t)} u(t,x) \geq \delta_0, \tag{1.13}$$

{sep2710

with some $\delta_0 > 0$. We expect that the solution looks like an exponential to the right of $x = 2t - d(t)$ and until the "front edge" at $x = 2t + e(t)$:

$$u(t,x) \sim \exp\{-(x - 2t + d(t))\}, \quad \text{for } x \in (2t - d(t), 2t + e(t)). \tag{1.14}$$

{sep2714

The diffusive Gaussian decay dominates the exponential "traveling wave" decay for $x > 2t + e(t)$. Using (1.4) and (1.13), one may estimate $\phi \star u$ as

$$\phi \star u(t,x) \gtrsim (e(t) + d(t))^{1-r}, \quad \text{for } x \in (2t - d(t), 2t + e(t)).$$

Thus, in order for the exponential in (1.14) to be a super-solution to (1.2) inside $(2t - d(t), 2t + e(t))$, we need

$$(e(t) + d(t))^{1-r} \gtrsim d'(t). \tag{1.15}$$

{sep2716

We also need the exponential to be above $u(t,x)$ at the front edge. Using the linearized version of equation (1.2) to control $u(t,x)$, this condition is satisfied if

$$\exp\left\{t - \frac{(2t + e(t))^2}{4t}\right\} \lesssim \exp\{-(e(t) + d(t))\},$$

that is,

$$e(t)^2 \geq 4td(t). \tag{1.16}$$

{sep2720

Since $e(t)$ should be $o(t)$, we get

$$\lim_{t \to +\infty} \frac{d(t)}{e(t)} = 0. \tag{1.17}$$

{sep2718

Combining (1.15), (1.16) and (1.17) gives, for t large,

$$d'(t) \lesssim e(t)^{1-r} \lesssim t^{\frac{1-r}{2}} d(t)^{\frac{1-r}{2}},$$

and thus necessarily

$$d(t) \lesssim t^{(3-r)/(1+r)}.$$

We deduce also $e(t) \gtrsim t^{\gamma}$, with γ as in (1.11).

A way to estimate the solution from below, to get the lower bounds, is to study the linearized Fisher-KPP equation with a Dirichlet boundary condition at $2t+e(t)$, as in [15]. The problem that comes up after removing the exponential factor is

$$
\begin{aligned}
&z_t = z_{xx} + e'(t)(z_x - z), \quad t>0, x>0,\\
&z(t,0)=0.
\end{aligned}
\tag{1.18}
$$
{eq:self

Once again, the case $r > 3$ is treated similarly to [15]. In particular, while the term $e'(t)z$ is important and is responsible for the $3/2$ pre-factor in the logarithmic correction, the drift $e'(t)z_x$ is negligible. Roughly, we estimate $z(t,x)$ at $x \sim \sqrt{t}$, and use a "tracing back to a shifted traveling wave" argument, to construct a sub-solution for u.

When $r < 3$, we choose $e(t) = t^\gamma$. Since now $\gamma > 1/2$, the drift $e'(t)z_x$ can no longer be neglected, and the choice of the exact exponent γ is necessary to get matching asymptotics. We explicitly construct a sub-solution of u to estimate the solution at the far edge, and then perform a "tracing back" argument with a travelling wave.

Lastly, in the case when $r = 3$, the diffusive scale and the induced drift have the same order. We make the influence of the shift to the moving frame small by considering $e(t) = \epsilon t^\gamma = \epsilon\sqrt{t}$ with $\epsilon \ll 1$. This is the reason why the bound (1.8) is less precise than the upper bound (1.6).

The local in time Harnack inequality

The main tool that allows us to get "reasonably sharp" asymptotics for the front position is a local-in-time Harnack inequality that is of an independent interest.

lem:Harnack **Proposition 1.2.** *Suppose that $u \in L^\infty([0,T]\times\mathbb{R})$ is a non-negative function that solves*

$$
u_t = u_{xx} + c(t,x)u,
$$

on $[0,T]\times\mathbb{R}$ with $c \in L^\infty([0,T]\times\mathbb{R})$ and $T > 0$. Then, for any $p \in (1,\infty)$, there exist positive constants α, β, and C, that depend only on $\|c\|_{L^\infty([0,T]\times\mathbb{R})}$ and p, such that, for all $x, y \in \mathbb{R}$ and $t \in (0,T]$, we have

$$
u(T,x+y) \le C\|u\|_{L^\infty([t,T])\times\mathbb{R}}^{1-\frac{1}{p}} u(T,x)^{\frac{1}{p}} e^{\alpha t + \frac{\beta y^2}{t}}.
\tag{1.19}
$$
{sep2902

We have used a less precise form of this inequality to obtain the logarithmic delay for solutions of the cane toads equation in [3]. As far as we know, this is the only other non-local context where a delay asymptotics has been established, and the Harnack inequality is an indispensable tool to obtain "reasonably sharp" results. It allows us to bound solutions of the non-local Fisher-KPP equation (1.2) in terms of the solutions of a local Fisher-KPP equation with a local time-dependent nonlinearity $g(t,u)$, that is logarithmic in u (Gompertz type). This equation has inherent difficulties coming from the time dependence and the logarithmic behavior near zero, but it is much more tractable because it admits a comparison principle.

The rest of the paper is organized as follows. In Section 2, we present the proofs of the upper bounds (1.6) and (1.9). Section 3 is where the proofs of the lower bounds (1.7), (1.8) and (1.10) are given. In order to complete the proof of the lower bounds, some estimates on linearized problems with moving Dirichlet boundary conditions are obtained in Section 4 and Section 5. In Section 6, we state and prove the result concerning the local KPP equation with logarithmic nonlinearity. The Harnack inequality is proved in Section 7.

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2 Upper bounds on the location of the front

:upperbound

In this section, we prove the upper bounds (1.6) and (1.9) in Theorem 1.1.

2.1 The upper bound when $r \geq 3$

The case $r \geq 3$ is very close to the local Fisher-KPP equation. The $(3/2)\log t$ delay is the best case scenario – in fact, the delay has to be at least that large for any r, so the bound is a quite straightforward application of bounds obtained in [15].

Proof of (1.6). Take $t_0 > 0$ to be determined later. Working in the moving frame with the logarithmic correction, the function

$$u_{\mathrm{mov}}(t,x) = u\Big(t, 2t - \frac{3}{2}\log\Big(1 + \frac{t}{t_0}\Big) + x\Big),$$

satisfies

$$\begin{aligned}
&(u_{\mathrm{mov}})_t \leq \Big(2 - \frac{3}{2}\frac{1}{t+t_0}\Big)(u_{\mathrm{mov}})_x + (u_{\mathrm{mov}})_{xx} + u_{\mathrm{mov}}, \qquad t > 0,\ x \in \mathbb{R},\\
&u_{\mathrm{mov}}(0,x) = u_0(x).
\end{aligned}$$

We construct a super-solution $\overline{u}$ as in [15]. Let $\overline{v}$ be the solution to the boundary value problem

$$\begin{aligned}
&\overline{v}_t = \Big(2 - \frac{3}{2}\frac{1}{t+t_0}\Big)\overline{v}_x + \overline{v}_{xx} + \overline{v},\ \ t > 0 \text{ and } x > 0,\\
&\overline{v}(t,0) = 0, \qquad t > 0,\\
&\overline{v}(0,x) = \mathbb{1}_{(0,2)}(x).
\end{aligned}$$

Then [15, Lemma 2.1] implies that, provided that t_0 is sufficiently large, there exists $A_0 \geq 1$ such that for all $t \geq 0$, we have

$$\overline{v}(t,1) \geq A_0^{-1}.$$

We also have the following uniform bound on the solutions to (1.2).

upper_bound

Lemma 2.1. *[17, Theorem 1.2] Suppose that u satisfies* (1.2) *with initial data u_0 satisfying* (1.5)*. Then there exists $M > 0$ such that, $u(t,x) \leq M$ for all $t > 0$ and $x \in \mathbb{R}$.*

Let us now define $\bar{u}(t,x)$ as

$$\overline{u}(t,x) = M\Big(\mathbf{1}_{x\le x_0} + \min\Big(1, A_0\overline{v}(t, x - x_0 + 1)\Big)\mathbf{1}_{x\ge x_0}\Big),$$

where M is as in Lemma 2.1. By construction, $\overline{u}$ is a super-solution to u_{mov}, and by our assumptions on u_0 (1.5), we also have $\overline{u}(0,x) \ge u_{\text{mov}}(0,x)$ for all $x \in \mathbb{R}$. In addition, [15, Lemma 2.1] implies that there exists T_0 such that, for all x and all $t \ge T_0$,

$$\overline{v}(t,z) \le A_0 z e^{-z}. \tag{2.1}$$

{eq:chri

We are now in a position to conclude the proof. Indeed, as $u \le \overline{u}$, the upper bound in (2.1) implies

$$\begin{aligned}\limsup_{L\to\infty}\limsup_{t\to\infty}\sup_{x\ge L} u\Big(t, 2t - \frac{3}{2}\log t + x\Big) &= \limsup_{L\to\infty}\limsup_{t\to\infty}\sup_{x\ge L} u_{\text{mov}}(t,x)\\ &\le \limsup_{L\to\infty}\limsup_{t\to\infty}\sup_{x\ge L}\overline{u}(t,x) \le \lim_{L\to\infty} MA_0Le^{-L} = 0,\end{aligned} \tag{2.2}$$

{eq:c101

which concludes the proof. □

2.2 The upper bound when $r \in (1,3)$

In this section, we show how to derive the upper bound on the location of the front from the lower bound on the location of the front. In other words, we prove (1.9) assuming (1.10), which we prove in the next section.

Proof of the upper bound (1.9) *assuming the lower bound* (1.10). Note that, by (2.2), we have

$$\lim_{t\to\infty}\sup_{x\ge 2t+t^\gamma} u(t,x) = 0.$$

As a consequence, taking into account (1.11), it suffices to show that

$$\lim_{t\to\infty}\sup_{x\in(2t-c_\phi t^{2\gamma-1}, 2t+t^\gamma)} u(t,x) = 0.$$

We do this by creating a relevant super-solution to u on the interval $(2t - c_\phi t^{2\gamma-1}, 2t+t^\gamma)$. Note that the constant c_ϕ is still to be determined at this stage. Define, for any $T > 0$ and C_ϕ as in (1.10), the space-time domain (recall that $\gamma > 1/2$ for $1 < r < 3$):

$$\mathcal{P}_T := \Big\{(t,x) : t \in (T,\infty), x \in (2t - C_\phi t^{2\gamma-1}, 2t + t^\gamma)\Big\},$$

and, for $(t,x) \in \mathcal{P}_T$, the function

$$\overline{v}(t,x) := B\exp\Big\{-\Big(x - 2t + 2c_\phi t^{2\gamma-1}\Big)\Big\}.$$

On $\mathcal{P}_T$, the function $\overline{v}$ satisfies

$$\overline{v}_t = \overline{v}_{xx} + \overline{v}\Big(1 - 2c_\phi(2\gamma-1)t^{\gamma(1-r)}\Big). \tag{2.3}$$

{oct102}

The rest of the proof is devoted to showing that u is, indeed, a subsolution to (2.3) when the various constants above are suitably chosen: specifically, we show that

$$u_t - u_{xx} - u(1 - 2c_\phi(2\gamma - 1)t^{\gamma(1-r)}) \le 0 \text{ in } \mathcal{P}_T, \tag{2.4}$$

{oct104}

and

$$u(t,x) \le \overline{v}(t,x), \quad \text{on } \partial\mathcal{P}_T. \tag{2.5}$$

{oct106}

First, we show that (2.4) holds. It follows from (1.10) that there exist C_ϕ and δ_ϕ, depending only on ϕ, and T_0 such that, for all $t \ge T_0$,

$$\inf_{x \le 2t - C_\phi t^{2\gamma-1}} u(t,x) \ge \delta_\phi. \tag{2.6}$$

{eq:chri

Using (2.6), we can estimate $\phi \star u$ from below, for $t \ge T_0$ and $x > 2t - C_\phi t^{2\gamma-1}$:

$$\begin{aligned}
\phi \star u(t,x) = \int_{\mathbb{R}} \phi(x-y)u(t,y)\,dy &\ge \int_{-\infty}^{2t - C_\phi t^{2\gamma-1}} \phi(x-y)u(t,y)\,dy \\
&\ge \delta_\phi \int_{-\infty}^{2t - C_\phi t^{2\gamma-1}} \phi(x-y)\,dy = \delta_\phi \int_{x-2t+C_\phi t^{2\gamma-1}}^{+\infty} \phi(z)\,dz \\
&\ge \delta_\phi A_\phi^{-1} \int_{x-2t+C_\phi t^{2\gamma-1}}^{+\infty} z^{-r}\,dz = \frac{\delta_\phi}{A_\phi(r-1)}\Big(x - 2t + C_\phi t^{2\gamma-1}\Big)^{1-r}.
\end{aligned} \tag{2.7}$$

{eq:chri

Note that, as $r > 1$, we have

$$2\gamma - 1 = \frac{3-r}{1+r} = \gamma + \frac{1-r}{1+r} < \gamma.$$

Further increasing T, if necessary, the right side in (2.7) can be estimated, for $t \ge T$, as

$$\begin{aligned}
\frac{\delta_\phi}{A_\phi(r-1)}\Big(x - 2t + C_\phi t^{2\gamma-1}\Big)^{1-r} &\ge \frac{\delta_\phi}{A_\phi(r-1)}\Big(t^\gamma + C_\phi t^{2\gamma-1}\Big)^{1-r} \\
&= \frac{\delta_\phi}{A_\phi(r-1)}\Big(1 + C_\phi t^{\frac{1-r}{1+r}}\Big)^{1-r} t^{(1-r)\gamma} \ge \frac{\delta_\phi}{A_\phi(r-1)}\Big(1 + C_\phi T^{\frac{1-r}{1+r}}\Big)^{1-r} t^{(1-r)\gamma} \\
&\ge \frac{2^{1-r}\delta_\phi}{A_\phi(r-1)} t^{(1-r)\gamma} \ge 2c_\phi(2\gamma-1)t^{\gamma(1-r)},
\end{aligned} \tag{2.8}$$

{eq:chri

as long as c_ϕ is sufficiently small. Now, (2.4) follows from (1.2), (2.7) and (2.8).

To show (2.5), first, we consider the right spatial boundary $x = 2t + t^\gamma$, $t \ge T$. As this point is at the far edge of the front, it is natural to use the linearized problem

$$\begin{aligned}
&\overline{u}_t = \overline{u}_{xx} + \overline{u}, \quad t > 0, x \in \mathbb{R}, \\
&\overline{u}(t = 0, x) = u_0(x).
\end{aligned}$$

Then, with x_0 as in (1.5), we can write for $t \ge T$:

$$\begin{aligned}
u(t, 2t + t^\gamma) \le \overline{u}(t, 2t + t^\gamma) &= \frac{e^t}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(2t+t^\gamma-y)^2}{4t}} u_0(y)dy \le \frac{e^t}{\sqrt{4\pi t}} \int_{-\infty}^{x_0} e^{-\frac{(2t+t^\gamma-y)^2}{4t}} dy \\
&= \frac{e^t}{\sqrt{\pi}} \int_{\frac{2t+t^\gamma-x_0}{2\sqrt{t}}}^{+\infty} e^{-y^2} dy \le \frac{Ce^t\sqrt{t}}{2t + t^\gamma - x_0} e^{-\frac{(2t+t^\gamma-x_0)^2}{4t}} \\
&\le C_0 \exp\Big\{ -t^\gamma - \frac{1}{4}t^{2\gamma-1}\Big\} \le B\exp\Big\{ -t^\gamma - 2c_\phi t^{2\gamma-1}\Big\} = \overline{v}(t, 2t + t^\gamma),
\end{aligned} \tag{2.9}$$

{eq:chri

so long as $B \geq C_0$. Above, we have increased T and decreased c_ϕ if necessary. The constant C_0 depends only on γ and x_0. Thus, (2.5) holds at $x = 2t + t^\gamma$ for all $t \geq T$ as long as $B \geq C_0$.

At the left boundary $x = 2t - C_\phi t^{2\gamma-1}$, we have

$$\overline{v}(t, 2t - C_\phi t^{2\gamma-1}) = B \exp\Big\{(C_\phi - 2c_\phi)t^{2\gamma-1}\Big\} \geq M \geq u\Big(t, 2t - C_\phi t^{2\gamma-1}\Big), \tag{2.10}$$

{eq:chri

as long as $2c_\phi \leq C_\phi$ and $B \geq M$. Here, M is the upper bound in Lemma 2.1.

Lastly, we check that (2.5) holds at $t = T$, for $2T - C_\phi T^{2\gamma-1} \leq x \leq 2T + T^\gamma$:

$$\overline{v}(T, x) = B \exp\Big\{ - \Big(x - 2T + 2c_\phi T^{2\gamma-1}\Big)\Big\} \geq B \exp\Big\{ - T^\gamma - 2c_\phi T^{2\gamma-1}\Big\}.$$

As long as $B \geq M \exp\Big\{T^\gamma + 2c_\phi T^{2\gamma-1}\Big\}$, we have that, for all $x \in [2T - C_\phi T^{2\gamma-1}, 2T + T^\gamma]$

$$\overline{v}(T, x) \geq M \geq u(T, x), \tag{2.11}$$

{eq:chri

and (2.5) holds on all of $\partial \mathcal{P}_T$.

It follows from (2.4) and (2.5) that, with T and B sufficently large, and c_ϕ sufficiently small, we have

$$\lim_{t\to\infty} \sup_{x \geq 2t - c_\phi t^{2\gamma-1}} u(t, x) \leq \lim_{t\to\infty} \sup_{x \geq 2t - c_\phi t^{2\gamma-1}} \overline{v}(t, x) \leq \lim_{t\to\infty} B \exp\Big\{ - \Big(2c_\phi - c_\phi\Big)t^{2\gamma-1}\Big\} = 0,$$

which finishes the proof of the upper bound. □

lower_bound

3 Lower bounds on the location of the front

The proofs of the lower bounds in Theorem 1.1 are much more involved. They hinge on estimating $\phi \star u$ in terms of u in a local way, and then deriving precise heat kernel type estimates on the resulting local equation.

3.1 Estimating the non-local term by a local counterpart

To begin, we estimate the convolution term $\phi \star u$ in terms of u under the assumptions of Theorem 1.1. The assumptions of these two theorems differ only in the range of r. In this section, we assume only that $r > 1$ so our computations apply to all cases.

estphistaru

Lemma 3.1. *There exists $C_{\text{conv}} > 0$, depending only on ϕ, such that, for all $t \geq 1$ and all $x \in \mathbb{R}$,*

$$\phi \star u(t, x) \leq C_{\text{conv}} \max\Big\{1, \Big[\frac{1}{t}\log\Big(\frac{M}{u(t, x)}\Big)\Big]^{\frac{r-1}{2}}\Big\} \log\Big(\frac{M}{u(t, x)}\Big)^{1-r}. \tag{3.1}$$

{eq:phi_

Proof. It is here that the local-in-time Harnack inequality is used crucially. Fix any time $t \geq 1$ and $x, y \in \mathbb{R}$. Proposition 1.2 with $p = 2$ implies that there exists $\alpha > 0$ so that

$$u(t, x + y) \leq C\sqrt{u(t, x)} \exp\Big\{\alpha t' + \frac{\alpha y^2}{t'}\Big\}, \quad \text{for all } t' \in (0, t], . \tag{3.2}$$

{eq:Harn

Above, we absorbed the uniform bound M of $\|u\|_\infty$ given by Lemma 2.1 into the constant C. By increasing M if necessary, we may assume that $M \geq \|u\|_\infty + 1$, which allows us to simplify notation in the sequel. Using (1.4) and (3.2), we obtain, for $R > 0$ and $t' \in (0, t]$ to be determined,

$$
\begin{aligned}
\phi \star u(t,x) &\leq \int_{\mathbb{R}} \phi(y) u(t, x-y)\, dy \leq C \int_{B_R} \phi(y) \sqrt{u(t,x)} e^{\alpha t' + \frac{\alpha R^2}{t'}} dy + M \int_{B_R^c} \phi(y) dy \\
&\leq C \sqrt{u(t,x)} R \exp\Big\{ \alpha t' + \frac{\alpha R^2}{t'} \Big\} + CMR^{-r+1}.
\end{aligned} \tag{3.3}
$$

{oct110}

The constant C changes line-by-line for the remainder of the proof and depends only on ϕ and α.

We now optimize the right side in (3.3) with respect to $t' \in (0, t]$ and $R > 0$. If $t' = R$, then

$$
\phi \star u(t,x) \leq C \sqrt{u(t,x)} R e^{2\alpha R} + CMR^{-r+1}. \tag{3.4}
$$

{eq:phi_

To roughly balance the two terms in the right side of (3.4), we choose

$$
R = \frac{1}{8\alpha} \log\Big(\frac{M}{u(t,x)} \Big), \tag{3.5}
$$

{oct112}

the most important point being that R should be of order $\log u$. As we have set $t' = R$ in (3.4), and we need to have $0 \leq t' \leq t$, the choice (3.5) is possible only if

$$
t \geq \frac{1}{8\alpha} \log\Big(\frac{M}{u(t,x)} \Big). \tag{3.6}
$$

{oct114}

With this, we find, from (3.4):

$$
\begin{aligned}
\phi \star u(t,x) &\leq C \sqrt{u(t,x)} \log\Big(\frac{M}{u(t,x)} \Big) \exp\Big\{ -\frac{1}{4} \log\Big(\frac{u(t,x)}{M} \Big) \Big\} + C\Big(\log\Big(\frac{M}{u(t,x)} \Big) \Big)^{1-r} \\
&\leq C\Big(u(t,x)^{1/4} \Big(\log\Big(\frac{M}{u(t,x)} \Big) \Big)^r + 1 \Big) \Big(\log\Big(\frac{M}{u(t,x)} \Big) \Big)^{1-r} \leq C\Big(\log\Big(\frac{M}{u(t,x)} \Big) \Big)^{1-r}.
\end{aligned} \tag{3.7}
$$

{eq:phi_

When (3.6) does not hold, so that

$$
t \leq \frac{1}{8\alpha} \log\Big(\frac{M}{u(t,x)} \Big). \tag{3.8}
$$

{oct116}

we choose $t' = t$ and set

$$
R = \Big(\frac{t}{8\alpha} \log\Big(\frac{M}{u(t,x)} \Big) \Big)^{1/2},
$$

in (3.3), leading to

$$
\begin{aligned}
\phi * u(t,x) &\leq C \sqrt{u(t,x)} \sqrt{t} \log^{1/2}\Big(\frac{M}{u(t,x)} \Big) \exp\Big\{ \alpha t + \frac{1}{8} \log\Big(\frac{M}{u(t,x)} \Big) \Big\} + C\Big(t \log\Big(\frac{M}{u(t,x)} \Big) \Big)^{-\frac{(r-1)}{2}} \\
&\leq C \sqrt{u(t,x)} \log\Big(\frac{M}{u(t,x)} \Big) \exp\Big\{ \frac{1}{4} \log\Big(\frac{M}{u(t,x)} \Big) \Big\} + C\Big(t \log\Big(\frac{M}{u(t,x)} \Big) \Big)^{\frac{-(r-1)}{2}} \\
&\leq C u(t,x)^{1/4} \log\Big(\frac{M}{u(t,x)} \Big) + C\Big(t \log\Big(\frac{M}{u(t,x)} \Big) \Big)^{\frac{-(r-1)}{2}} \\
&\leq C\Big(1 + u(t,x)^{1/4} \log\Big(\frac{M}{u(t,x)} \Big) \Big(t \log\Big(\frac{M}{u(t,x)} \Big) \Big)^{\frac{r-1}{2}} \Big) \Big(t \log\Big(\frac{M}{u(t,x)} \Big) \Big)^{\frac{-(r-1)}{2}} \\
&\leq C\Big(u(t,x)^{1/4} \Big(\log\Big(\frac{M}{u(t,x)} \Big) \Big)^r + 1 \Big) \Big(t \log\Big(\frac{M}{u(t,x)} \Big) \Big)^{\frac{-(r-1)}{2}} \leq C\Big(t \log\Big(\frac{M}{u(t,x)} \Big) \Big)^{\frac{1-r}{2}}.
\end{aligned} \tag{3.9}
$$

{eq:phi_

We used (3.8) several times above, as well as the upper bound $u(t, x) \le M$ in the last inequality. The combination of (3.7) and (3.9) concludes the proof of the lemma.

□

3.2 A local equation and related bounds

In view of Lemma 3.1, it is natural to introduce the following nonlinearity. Fix $r > 1$, and for any positive constants θ_g and A_g, set $\Theta_g := \theta_g \exp\{-A_g^{1/(r-1)}\}$ and define $g \in C^{0,1}$ on $(0, \Theta_g)$ as

$$g(t,u) := A_g \max\left\{1, \left[\left(t + A_g^{\frac{1}{r-1}}\right)^{-1} \log\left(\frac{\theta_g}{u}\right)\right]^{\frac{r-1}{2}}\right\} \log\left(\frac{\theta_g}{u}\right)^{1-r}, \quad \text{if } u \in \Big(0, \Theta_g\Big). \qquad (3.10)$$ {eq:g}

Outside $[0, \Theta_g]$ we set $g(t,u) = 0$ for $u < 0$ and $g(t,u) = 1$ for $u > \Theta_g$. By construction, $g(t, \cdot)$ is continuous. The "$A_g^{1/(r-1)}$" term in the second part of the maximum in the definition of g does not affect the analysis in any way. In fact, any other choice of g that preserves the asymptotics as u and t tend to zero would have the desired properties that we prove in the sequel.

We will make use of the local equation with a moving boundary at the front edge:

$$\begin{aligned} &w_t = w_{xx} + w(1 - g(t,w)), && \text{in } \mathcal{P}_{g,\gamma} := \Big\{(t,x) : t > 0, x > 2t + (t + t_0)^\gamma - t_0^\gamma\Big\},\\ &w(t, 2t + (t+t_0)^\gamma - t_0^\gamma) = 0, && \text{for all } t > 0,\\ &w(0,x) = w_0(x). \end{aligned} \qquad (3.11)$$ {eq:w}

The following proposition contains the crucial lower bounds for the solutions of (3.11) we will need.

prop:g **Proposition 3.2.** *Assume that there exists $\delta_w > 0$ and $x_w \in \mathbb{R}^+$ such that the initial condition $w_0(x)$ for (3.11) satisfies $w_0(x) \ge \delta_w 1_{(0,x_w)}(x)$.*

1. *If $r > 3$, then there exists X_w and T_0 such that if $x_w \ge X_w$ and $t_0 \ge T_0$ then there exists a positive constant B_1, depending only on x_w, δ_w, t_0, γ, and g, such that, for all t sufficiently large, we have*
$$w(t, 2t + t^\gamma + \sqrt{t}) \ge B_1^{-1} t^{-1} e^{-\sqrt{t} - t^\gamma}.$$

2. *If $r = 3$, then set $t_0 = 1$. For all $\epsilon > 0$, there exists a positive constant λ_ϵ such that $\lambda_\epsilon \to 0$ as $\epsilon \to 0$ and if $x_w \ge 1$ then there exists a positive constant B_2, depending only on ϵ, δ_w, γ, and g, such that, for all t sufficiently large, we have*
$$w(t, 2t + (1+\epsilon)\sqrt{t}) \ge B_2^{-1} t^{-3/2 - \lambda_\epsilon - \epsilon} e^{-(1+\epsilon)\sqrt{t}}.$$

3. *If $r \in (1,3)$, then set $t_0 = 1$. There exists $B_3 > 0$, depending only on δ_w and g, such that if $x_w \ge 1$ then, for all $t \ge 1$, we have*
$$w(t, 2t + t^\gamma + \sqrt{t}) \ge B_3^{-1} e^{-t^\gamma - B_3 t^{2\gamma - 1}}.$$

We delay the proof of this proposition until Section 4 and now continue the proof of the lower bounds of Theorem 1.1. Having reduced the problem to estimating a delay for a local equation, we now transfer known bounds of Theorem 1.1 on w to bounds on u.

3.3 From a bound on w to a bound on u

Let us take $\theta_g = M$ and $A_g = C_{\rm conv}$ in the definition (3.10) of $g(t,u)$ and let the initial condition in (3.11) be $w_0(x) = e^{-M}u_0(x)$. A combination of Lemma 3.1 and Proposition 3.2 implies that u is a super-solution for w for $t \geq 1$. Further, it follows from considerations as in [3, Section 3], that $w(1,x) \leq u(1,x)$ for all $x \in \mathbb{R}$ due to the e^{-M} pre-factor in the definition of w_0. The maximum principle then implies that $w(t,x) \leq u(t,x)$ for all $t \geq 1$ and all $x \in \mathbb{R}$.

Using the assumptions on the initial data (1.5), we can, up to translating u_0, and thus w_0 as well, assume that w_0 satisfies the hypothesis $x_w = x_0 \geq X_w$ in Proposition 3.2. Translating further and using parabolic regularity we may remove the dependence on t_0. As a direct consequence, we have established:

prop:estu **Corollary 3.3.** *Suppose that u satisfies (1.2) and (1.5) with ϕ satisfying (1.4). Then there exists S_0, depending only on u_0 and ϕ, such that:*

1. *If $r > 3$, then there exists a positive constant B_1, depending only on u_0 and ϕ such that, for all t sufficiently large, we have*
$$u(t, 2t + t^\gamma + \sqrt{t} - S_0) \geq B_1^{-1} t^{-1} e^{-\sqrt{t} - t^\gamma}.$$
2. *If $r = 3$, then for all $\epsilon > 0$, there exists positive constants λ_ϵ and B_2 such that $\lambda_\epsilon \to 0$ as $\epsilon \to 0$ and such that, for all t sufficiently large, we have*
$$u(t, 2t + (1+\epsilon)\sqrt{t} - S_0) \geq B_2^{-1} t^{-(1+\lambda_\epsilon)} e^{-(1+\epsilon)\sqrt{t}}.$$
3. *If $r \in (1,3)$, then there exists a positive constant B_3, depending only on u_0 and ϕ, such that, for all $t \geq 1$, we have*
$$u(t, 2t + t^\gamma + \sqrt{t} - S_0) \geq B_3^{-1} e^{-t^\gamma - B_3 t^{2\gamma - 1}}.$$

3.4 From a bound on u on the right to the location of the front

We are now in a position to obtain the lower bounds (1.7), (1.8), and (1.10). Thanks to Corollary 3.3, we fit a suitable translate of a traveling wave solution for (3.11) underneath u, for $x \leq 2t + t^\gamma + \sqrt{t}$ when $r \neq 3$, and for $x \geq 2t + (1+\varepsilon)\sqrt{t}$ when $r = 3$.

Fix any $A_V > C_{\rm conv}$ and let V be a traveling wave solution of

$$-2V' = V'' + V\Big(1 - A_V \log\Big(\frac{M}{V}\Big)^{1-r}\Big), \quad V(-\infty) = M\exp\Big\{-A_V^{1/(r-1)}\Big\} \text{ and } V(+\infty) = 0.$$

The existence, uniqueness up to translation, and monotonicity of V is given by, for example, [2]. We also recall the fact that there exists $\kappa > 0$ such that, see [14]:

$$V(\xi) \sim \kappa \xi e^{-\xi}, \text{ as } \xi \to \infty. \tag{3.12}$$ {oct202}

Define v as

$$\begin{aligned}
v(t,x) &= V\left(x - 2t + \frac{3}{2}\log t + s_0\right), && \text{if } r > 3,\\
v(t,x) &= V\left(x - 2t + \Big(\frac{3}{2} + \lambda_\epsilon + \epsilon\Big)\log t + s_0\right), && \text{if } r = 3,\\
v(t,x) &= V\left(x - 2t + 2B_3 t^{2\gamma-1} + s_0\right), && \text{if } r \in (1,3),
\end{aligned}$$

where the shift s_0 is to be determined below and λ_ε is as in Proposition 3.2 and Corollary 3.3.

lem:compTW **Lemma 3.4.** *There exists $T_1 > 0$ and $\underline{s}_0$ such that if $s_0 \geq \underline{s}_0$, then $v(t,x) \leq u(t,x)$ if either:*

- *$r \neq 3$, $t \geq T_1$, and $x \leq 2t + t^\gamma + \sqrt{t} - S_0$,* –or–
- *$r = 3$, $t \geq T_1$, and $x \leq 2t + (1+\varepsilon)\sqrt{t} - S_0$,*

where S_0 is the shift given in Corollary 3.3.

Proof. We prove the lemma for $r > 3$, so that $\gamma < 1/2$, the proof being the same in the other cases up to situational modifications. We use the parabolic maximum principle. First, we note that, up to increasing s_0 and A_V, we may ensure that

$$v(T_1, x) \leq u(T_1, x) \quad \text{for all } x \leq 2T_1 + \sqrt{T}_1 + T_1^\gamma - S_0.$$

Second, we claim that, up to increasing s_0, we have

$$v(t, 2t + \sqrt{t} + t^\gamma - S_0) \leq u(t, 2t + \sqrt{t} + t^\gamma - S_0) \quad \text{for all } t \geq T_1.$$

Indeed, for t sufficiently large, (3.12) implies, as $\gamma < 1/2$:

$$\begin{aligned} v(t, 2t + \sqrt{t} + t^\gamma - S_0) &= V\Big(\sqrt{t} + t^\gamma + \frac{3}{2}\log t + s_0 - S_0\Big) \\ &\leq 2\kappa\Big(\sqrt{t} + t^\gamma + \frac{3}{2}\log t + s_0 - S_0\Big) \exp\Big\{ -\Big(\sqrt{t} + t^\gamma + \frac{3}{2}\log t + s_0 - S_0\Big)\Big\} \\ &\leq 4\kappa\sqrt{t}t^{-3/2} \exp\Big\{ -\sqrt{t} - t^\gamma - s_0\Big\} \leq 4\kappa t^{-1} \exp\Big\{ -\sqrt{t} - t^\gamma - s_0 + S_0\Big\}. \end{aligned} \tag{3.13}$$

It follows that

$$v(t, 2t + \sqrt{t} + t^\gamma - S_0) \leq 4\kappa e^{S_0 - s_0} B_1 u(t, 2t + \sqrt{t} + t^\gamma - S_0) \leq u(t, 2t + \sqrt{t} + t^\gamma - S_0),$$

for T_1 sufficiently large and all $s_0 \geq S_0 + \log(4\kappa B_1)$.

Third, up to increasing A_V, the ordering holds true near $-\infty$. Indeed, using Lemma 3.1 and the assumptions (1.5) on u_0, it is easy to see that there exists $\delta > 0$, depending only on u_0 and ϕ such that, for any $\overline{x} < 0$ with $|\overline{x}|$ is sufficiently large, the function

$$\underline{u}(x) = \delta \cos((x - \overline{x})/100)$$

is a sub-solution for u for all $t \geq 1$, so that $\delta = \underline{u}(\overline{x}) \leq u(t, \overline{x})$ for all $t \geq 1$. Thus, increasing A_V, if necessary, we have that, for all $t > 0$,

$$\lim_{x \to -\infty} v(t,x) < M e^{-A_V^{1/(r-1)}} < \delta \leq \inf_{t \geq 1} \liminf_{x \to -\infty} u(t,x).$$

Now, assume for the sake of a contradiction that there exists a first touching time $(t_{\rm ft}, x_{\rm ft})$ such that

$$t_{\rm ft} \geq T_1,\ x_{\rm ft} \leq 2t_{\rm ft} + \sqrt{t_{\rm ft}} + t_{\rm ft}^\gamma - S_0,$$

and

$$u(t_{\rm ft}, x_{\rm ft}) = v(t_{\rm ft}, x_{\rm ft}),$$

and $u(t,x) > v(t,x)$ for all $t \in [T_1, t_{\rm ft})$ and $x < 2t + \sqrt{t} + t^\gamma - S_0$. Our goal is to obtain a contradiction by estimating $\phi \star u$ and looking at the equation satisfied by $u - v$.

First, we estimate $\phi \star u(t_{\rm ft}, x_{\rm ft})$ using Lemma 3.1. By increasing s_0 if necessary, we obtain

$$
\begin{aligned}
v(t, 2t + \sqrt{t} + t^\gamma - S_0) &= V\Big(\sqrt{t} + t^\gamma + \frac{3}{2}\log(t) + s_0 - S_0\Big) \\
&\geq \frac{\kappa}{2}\Big(\sqrt{t} + t^\gamma + \frac{3}{2}\log t + s_0\Big)\exp\Big\{-\Big(\sqrt{t} + t^\gamma + \frac{3}{2}\log t + s_0\Big)\Big\} \qquad (3.14)\\
&\geq \frac{\kappa}{2t^{3/2}}\left(\sqrt{t} + t^\gamma + \frac{3}{2}\log t + s_0 - S_0\right)\exp\Big\{-\sqrt{t} - t^\gamma - s_0 + S_0\Big\}.
\end{aligned}
$$

{eq:chri

Since V is monotonic, $\gamma < 1$, and $x_{\rm ft} \leq 2t_{\rm ft} + t_{\rm ft}^{\frac{1}{2}} + t_{\rm ft}^\gamma - S_0$, it follows that up to increasing T_1, we have that

$$u(t_{\rm ft}, x_{\rm ft}) = v(t_{\rm ft}, x_{\rm ft}) \geq M e^{-t_{\rm ft}},$$

which, in turn, implies that

$$\log\Big(\frac{M}{u(t_{\rm ft}, x_{\rm ft})}\Big)^{-(r-1)} \geq t^{-\frac{r-1}{2}}\log\Big(\frac{M}{u(t_{\rm ft}, x_{\rm ft})}\Big)^{-\frac{r-1}{2}}.$$

In view of the bound on $\phi \star u$ obtained in Lemma 3.1, we have that, at $(t_{\rm ft}, x_{\rm ft})$,

$$u_t - u_{xx} - u\Big(1 - A_V\log\Big(\frac{M}{u}\Big)^{-(r-1)}\Big) \geq \Big(A_V - C_{\rm conv}\Big)\Big(\log\Big(\frac{M}{u}\Big)\Big)^{1-r} > 0, \qquad (3.15)$$

{eq:chri

where we used the fact that $A_V > C_{\rm conv}$ in the last inequality. In addition, we note that

$$
\begin{aligned}
v_t - v_{xx} &- u\Big(1 - A_V\log\Big(\frac{M}{v}\Big)^{-(r-1)}\Big) \\
&= \Big(\frac{3}{2(t+1)} - 2\Big)V' - V'' - V\Big(1 - A_V\log\Big(\frac{M}{V}\Big)^{-(r-1)}\Big) = \frac{3}{2(t+1)}V' \leq 0.
\end{aligned} \qquad (3.16)
$$

{eq:chri

Hence, setting $\psi = u - v$, (3.15) and (3.16), imply that

$$\psi_t - \psi_{xx} > 0.$$

On the other hand, using that $t_{\rm ft}$ is the first time that ψ touches zero and $x_{\rm ft}$ is the location of a minimum of ψ, we have that

$$\psi_t - \psi_{xx} \leq 0.$$

This yields a contradiction, finishing the proof. □

The lower bounds now follow easily.

Proof of (1.7), (1.8), *and* (1.10). We conclude the proof by noticing that, for all $t \geq T_1$,

$$
\begin{aligned}
\inf_{x \leq 2t - (3/2)\log t} u(t, x) &= \inf_{x \leq 0} u\Big(t, x + 2t - \frac{3}{2}\log t\Big) \geq \inf_{x \leq 0} v\Big(t, x + 2t - \frac{3}{2}\log t\Big) \\
&= \inf_{x \leq 0} V\Big(x + s_0\Big) = V(s_0),
\end{aligned} \qquad (3.17)
$$

which means that (1.7) holds. The proof of (1.10) is similar, as is the proof of (1.8), except one needs to recall that $\lambda_\epsilon \to 0$ to conclude the proof.

□

4 Proof of Proposition 3.2

sec:estw

To obtain estimates on the solution of (3.11), we consider the corresponding linearized problem with the Dirichlet boundary condition:

$$\begin{aligned} &\tilde{v}_t = \tilde{v}_{xx} + \tilde{v}, && \text{on } \{(t,x) : t > 0, x > 2t + (t+t_0)^\gamma - t_0^\gamma\}, \\ &\tilde{v}(t, 2t + (t+t_0)^\gamma - t_0^\gamma) = 0, && \text{for all } t > 0, \\ &\tilde{v}(0,x) = w_0(x), && \text{for all } x > 0, \end{aligned} \tag{4.1}$$

{eq:self

where w_0 is as in Proposition 3.2.

4.1 The case $r > 3$

The following key lemma about solutions to (4.1) will allow is to prove Proposition 3.2 when $r > 3$. We prove this lemma in Section 5.1.

elf_similar

Lemma 4.1. *Assume $r > 3$. If t_0 and x_w are sufficiently large, depending only on γ, there exist positive constants T and B, depending only on w_0 and t_0, such that, for all $t \geq T$, we have $\|\tilde{v}(t,\cdot)\|_\infty \leq Be^{-t^\gamma}$ and*

$$\tilde{v}(t, 2t + t^\gamma + \sqrt{t}) \geq B^{-1} t^{-1} \exp\Big\{ -\sqrt{t} - t^\gamma \Big\}.$$

We now finish the proof of Proposition 3.2. Let $\tilde{v}$ be as in Lemma 4.1. We may assume, without loss of generality, that $T \geq 1$, and set

$$\delta = \min \Big\{ B^{-1}, B^{-1}\theta_g e^{-A_g^{1/(r-1)}}, e^{-T} \Big\}.$$

We also take a continuous function $a(t) \leq 1$ for all $t \geq 0$, to be determined, and set

$$\underline{v}(t,x) = \delta a(t) \tilde{v}(t,x). \tag{4.2}$$

{oct212}

Using (4.1), we obtain

$$\underline{v}_t - \underline{v}_{xx} - \underline{v}(1 - g(t,\underline{v})) = \delta a' \tilde{v} + \delta a \tilde{v}_t - \delta a \tilde{v}_{xx} - \delta a \tilde{v} + \delta a \tilde{v} g(t, \delta a \tilde{v}) = \delta \tilde{v} \Big(a' + a g(t, \delta a \tilde{v}) \Big). \tag{4.3}$$

{eq:chri

Thus, $\underline{v}$ is a sub-solution of w for $t \geq T$ as long as

$$a' + a g(t, \delta a \tilde{v}) \leq 0.$$

Using the upper bound on $\tilde{v}$ along with the definition of δ, we see that this inequality would hold if

$$a' + a A_g \max \Big\{ 1, \Big[\Big(t + A_g^{\frac{1}{r-1}} \Big)^{-1} \log\Big(\frac{1}{a e^{-t^\gamma}} \Big) \Big]^{\frac{r-1}{2}} \Big\} \log\Big(\frac{1}{a e^{-t^\gamma}} \Big)^{1-r} \leq 0. \tag{4.4}$$

{oct210}

A lengthy but straightforward computation using, in particular, that $A_g \geq 1$, shows that (4.4) is satisfied if we take

$$a(t) = \exp\Big\{ \beta \Big[(t+1)^{2\gamma - 1} - 1 \Big] \Big\},$$

with a suitable $\beta > 0$.

Hence $\underline{v}$ is a sub-solution of w. Further, arguing as in [3, Section 3] and using the choice of δ and a, we have that $\underline{v}(T,x) \leq w(T,x)$ for all $x \geq 2t + (t+t_0)^\gamma - t_0^\gamma$. The maximum principle then implies that $\underline{v}(t,x) \leq w(t,x)$ for all $t > T$ and $x > 2t + (t+t_0)^\gamma - t_0^\gamma$. The conclusion of the proposition follows immediately from Lemma 4.1 since $2t + t^\gamma \geq 2t + (t+t_0)^\gamma - t_0^\gamma$.

4.2 The case $r = 3$

We follow here the same strategy as for $r > 3$, but the estimates on $\tilde{v}$ are obtained differently.

ar_critical **Lemma 4.2.** *For $r = 3$ and t sufficiently large, there exists $B > 0$ such that*

$$\|\tilde{v}(t, \cdot)\|_\infty \leq B t^{-3/2-\lambda_\epsilon} e^{-\epsilon t^{1/2}},$$

and

$$\tilde{v}(t, 2t + (1+\epsilon)\sqrt{t}) \geq B^{-1} t^{-1-\lambda_\epsilon} \exp\Big\{ - (1+\epsilon)\sqrt{t}\Big\},$$

where λ_ε tends to 0 as ϵ tends to 0.

With this lemma, proved in Section 5.2, one may repeat the argument for $r > 3$, building a sub-solution $\underline{v}(t,x)$ as in (4.2), with $\delta > 0$ sufficiently small, and $a(t)$ such that

$$a' + aA_g \max\Big\{1, \Big(t + \sqrt{A_g}\Big)^{-1} \log\Big(\frac{t^{3/2+\lambda_\epsilon} e^{\epsilon\sqrt{t}}}{C\delta a}\Big)\Big\} \log\Big(\frac{t^{3/2+\lambda_\epsilon} e^{\epsilon\sqrt{t}}}{C\delta a}\Big)^{-2} \leq 0.$$

The above inequality is satisfied with $a(t) = (t + \sqrt{A_g})^{-\epsilon}$ for all $t \geq 1$ so long as δ is chosen small enough, depending only on A_g, C, and ϵ.

4.3 The estimate when $r \in (1,3)$

Here we directly construct a sub-solution of w. We seek a sub-solution $\tilde{v}$ solving

$$\begin{aligned} &\tilde{v}_t \leq \tilde{v}_{xx} + \tilde{v}, && \text{for } t > 0, x > 2t + (t+1)^\gamma - 1,\\ &\tilde{v}(t, 2t + (t+1)^\gamma - 1) = 0, && \text{for } t > 0. \end{aligned} \tag{4.5}$$ {eq:line

Recall that $t_0 = 1$ in parts 2 and 3 of Proposition 3.2. Given $a > 0$, set

$$v(t,x) = \frac{x}{(1+t)^3} \exp\Big\{-x - \frac{\gamma}{2}x\Big(1+t\Big)^{\gamma-1} - \Big(1+t\Big)^{\gamma} - \Big[\frac{\gamma^2}{4(2\gamma-1)} + a\Big](1+t)^{2\gamma-1} - \frac{x^2}{2(1+t)}\Big\}. \tag{4.6}$$ {eq:sub-

Here, the key computation is the following:

ub-solution **Lemma 4.3.** *There exists $a_0 > 0$ such that if $a \geq a_0$ then then $\tilde{v}(t,x) = v(t, x - (2t + (t+1)^\gamma - 1))$ solves* (4.5)*.*

We delay the proof of Lemma 4.3 until Section 5.3 and proceed with the proof of Proposition 3.2.

A bound for small times. Unfortunately, v is not compactly supported at $t = 0$, so we need to "fit it under" w at a later time. To do this, we first obtain a preliminary, lower bound on w at time 1 by using the infinite speed of propagation of the heat equation. Recall that $w_0 \geq \delta_w \mathbb{1}_{(-\infty, x_w)}$ and $1 - g(t,w) \geq 0$. Hence, we have

$$w_t - w_{xx} \geq 0,$$

so that w is a super-solution to the heat equation with Dirichlet boundary conditions fixed at

$$\overline{x}_0 := 4 + (2 + t_0)^\gamma - t_0^\gamma = 3^\gamma + 3,$$

on the time interval $[0,2]$. It follows that

$$\begin{aligned} w(2,x) &\ge \frac{1}{\sqrt{8\pi}}\int_0^\infty w_0(y+\overline{x}_0)\Big[e^{-|x-y|^2/8}-e^{-|x+y|^2/8}\Big]dy \\ &\ge \frac{\delta_w e^{-x^2/8}}{\sqrt{8\pi}}\int_0^{x_w-\overline{x}_0} e^{-y^2/8}\Big[e^{xy/4}-e^{-xy/4}\Big]dy \\ &\ge \frac{\delta_w e^{-x^2/8-(x_w-\overline{x}_0)^2/8}}{\sqrt{8\pi}}\frac{2}{x}\Big(\cosh\Big(\frac{x(x_w-\overline{x}_0)}{4}\Big)-1\Big) \ge \frac{1}{C}\delta_w e^{-x^2/8}, \end{aligned} \tag{4.7}$$

{eq:w1_l

for some C independent of all parameters, as long as $x_w \ge \overline{x}_0+1$. From the explicit expression (4.6) for v, we get

$$v(2,x-\bar{x}_0) \le C(x-\overline{x}_0)\exp\Big\{-x-x\frac{\gamma}{2}3^{\gamma-1}-\frac{x^2}{6}+\frac{x\overline{x}_0}{3}\Big\}.$$

It is also straightforward to obtain a lower bound on $w(1,x)/(x-\overline{x}_0)$ as $x\to\overline{x}_0$ by using the above formula. Thus, there exists $\epsilon>0$ such that

$$\epsilon\tilde{v}(1,x)=\epsilon v(1,x-\bar{x}_0)\le w(1,x) \text{ for } x\ge\bar{x}_0=3^\gamma+3.$$

The subsolution. We now follow the same strategy as before, constructing a sub-solution of the form $\underline{v}(t,x)=\delta a(t)\tilde{v}(t,x)$ on

$$\mathcal{P}:=\{(t,x): t\ge 1, x>2t+(1+t)^\gamma-1\}.$$

Another lengthy but straightforward computation shows that $\underline{v}(t,x)$ is a sub-soltuon for w on $\mathcal{P}$ if we choose $a(t)=\exp\Big\{-\beta t^{2\gamma-1}\Big\}$ for a suitable β.

Note also that $\tilde{v}$ and w satisfy the same boundary conditions at $x=2t+(1+t)^\gamma-1$. Finally, choosing $\delta=\epsilon$ and using the computation (4.7) and the discussion following it, we see that

$$\underline{v}(2,x)\le w(2,x) \text{ for all } x>3+3^\gamma.$$

The conclusion of the proposition when $r\in(1,3)$ follows by simply using the explicit form of $\underline{v}(t,x)$.

5 Estimates on the linearized KPP equation

ec:estlinKPP

5.1 The case $r>3$: the proof of Lemma 4.1

elf_similar

The key observation is that $\gamma<1/2$ when $r>3$. Thus, the t^γ term is of a lower order than the diffusive scale $\sqrt{t}$. This allows us to use the strategy in [15], obtaining energy estimates in self-similar variables. Since the present proof is similar to that in [15], we provide a rather brief treatment.

Proof of Lemma 4.1. We begin by removing an exponential factor from $\tilde{v}$ and changing to the moving frame, and setting

$$z(t,x):=e^x\tilde{v}(t,2t+(t+t_0)^\gamma-t_0^\gamma+x),\quad x>0.$$

This function satisfies

$$\begin{aligned} &z_t=z_{xx}+\gamma(t+t_0)^{\gamma-1}\Big(z_x-z\Big),\quad t>0,x>0,\\ &z(t,0)=0,\\ &z(0,x)=e^x w_0(x). \end{aligned} \tag{5.1}$$

{eq:self

We now turn to self-similar variables, which are natural for the diffusive process. Let

$$\tau = \log\Big(1+\frac{t}{t_0}\Big), \qquad y = (t+t_0)^{-1/2}x,$$

and $\zeta(\tau,y) = z\Big(t_0(e^\tau - 1), t_0^{1/2}e^{\tau/2}y\Big)$. Then ζ satisfies the equation

$$\zeta_\tau = \zeta_{yy} + \frac{y}{2}\zeta_y + 1 + \gamma(t_0e^\tau)^{\gamma-1/2}\zeta_y - \Big(1+\gamma(t_0e^\tau)^\gamma\Big)\zeta.$$

We remove the integrating factor above, setting

$$\zeta(\tau,y) = e^{-(\tau+t_0^\gamma(e^{\gamma\tau}-1))}\bar\zeta(\tau,y),$$

so that $\bar\zeta$ satisfies

$$\bar\zeta_\tau = L\bar\zeta + \gamma t_0^{\gamma-1/2}e^{(\gamma-1/2)\tau}\bar\zeta_y, \tag{5.2}$$ {oct402}

with

$$L := \partial_y^2 + \frac{y}{2}\partial_y + 1. \tag{5.3}$$ {oct408}

It is now heuristically clear that the last term in (5.2) should be not important since $\gamma < 1/2$. The following lemma is proved in Appendix A.

perturbative **Lemma 5.1.** *Let $\bar\zeta$ solve*

$$\bar\zeta_\tau = L\bar\zeta + \varepsilon e^{(\gamma-1/2)\tau}\bar\zeta_y,$$

with initial data $\bar\zeta(\tau=0,\cdot) = \bar\zeta_0$. There exists $\varepsilon_0 > 0$ such that for all compact subsets $K \subset \mathbb{R}_+$ there exists $C_K > 0$ such that for all $\varepsilon < \varepsilon_0$,

$$\bar\zeta(\tau,y) = y\Big(\frac{e^{-y^2/4}}{2\sqrt{\pi}}\Big(\int_0^\infty \xi\bar\zeta_0(\xi)d\xi + O(\varepsilon)\Big) + e^{(\gamma-1/2)\tau}\bar h(\tau,y)\Big),$$

for all $y > 0$, $\tau > 0$, and such that $|\bar h(\tau,y)| \le C_K$ for all $\tau > 0$ and $y \in K$.

Undoing the various changes of variable, we get

$$\begin{aligned}\tilde v(t, 2t + (t+t_0)^\gamma - t_0^\gamma + x) &= e^{-x}z(t,x) = e^{-x}\zeta\Big(\log\Big(1+\frac{t}{t_0}\Big), \frac{x}{(t+t_0)^{1/2}}\Big) \qquad (5.4)\\ &= \frac{xe^{-x}t_0e^{-((t+t_0)^\gamma - t_0^\gamma)}}{(t+t_0)^{3/2}}\Big(\frac{e^{-\frac{x^2}{4(t+t_0)}}}{2\sqrt{\pi}}\Big(\int_0^\infty \xi e^\xi w_0(\sqrt{t_0}\xi)d\xi + O\Big(t_0^{\gamma-\frac12}\Big)\Big) + \Big(1+\frac{t}{t_0}\Big)^{\gamma-\frac12}h(t,x)\Big),\end{aligned}$$

where $h(t,x) = \bar h\Big(\log\Big(1+\frac{t}{t_0}\Big), (t+t_0)^{-\frac12}x\Big)$.

First, notice that the L^∞ bound on $\tilde v$ follows immediately from the expression above on sets of the form $[2t+t^\gamma, 2t+t^\gamma+\sigma\sqrt{t}]$. To obtain bounds on sets of the form $[2t+t^\gamma+\sigma\sqrt{t},\infty)$, we simply use that $e^{-t}\tilde v$ is a sub-solution to the heat equation on $\mathbb{R}$. Hence, we obtain that, for $x \ge 0$,

$$\tilde v(t, 2t+t^\gamma+\sigma\sqrt{t}+x) \le \frac{Ce^t}{\sqrt{t}}\exp\Big\{-\frac{(2t+t^\gamma+\sigma\sqrt{t}+x)^2}{4t}\Big\} \le Ce^{-\sqrt{t}-t^\gamma}, \tag{5.5}$$ {eq:c102

where C is some constant depending only on the initial data and γ. Second, we have

$$\int_0^\infty \xi e^\xi w_0(t_0^{1/2}\xi)d\xi + O(t_0^{\gamma-\frac12}) \ge \delta_w\int_0^{x_w/t_0^{1/2}} \xi e^\xi d\xi + O(t_0^{\gamma-\frac12})$$

choosing first $x_w \geq \sqrt{t_0}$ and $t_0 \gg 1$ so that the first two terms in the parentheses in (5.4) are positive and then choosing T_0 large depending on t_0 and α, we have that, for all $0 \leq x \leq \sqrt{t+t_0}$ and $t \geq T_0$,

$$\tilde{v}(t, 2t + (t+t_0)^\gamma - t_0^\gamma + x) \geq \frac{x}{C} \frac{e^{-x-((t+t_0)^\gamma - t_0^\gamma)}}{(t+t_0)^{3/2}}.$$

The lower bound on $\tilde{v}(t, 2t + t^\gamma + \sqrt{t})$ is immediate after evaluating at $x = t^\gamma - \Big((t+t_0)^\gamma - t_0^\gamma\Big) + \sqrt{t}$. This concludes the proof.

□

5.2 The case $r = 3$: the proof of Lemma 4.2

c:subsolr=3

Note that in this case $\gamma = 1/2$. As a consequence, the drift induced by the moving boundary has the same order as the diffusion. It is thus useful to modify the t^γ term in the moving boundary by a small multiplicative factor.

Proof of Lemma 4.2. To begin, fix $\epsilon > 0$. Work in the moving frame $2t + \epsilon[(t+1)^{1/2} - 1]$ and remove an exponential factor, as previously:

$$z(t,x) := e^x \tilde{v}(t, x + 2t + \epsilon[(1+t)^{1/2} - 1]).$$

Passing then to self-similar coordinates

$$\tau = \log(t+1) \quad \text{and} \quad y = (t+1)^{-1/2} x,$$

so that

$$\zeta(\tau, y) := z\Big(e^\tau - 1, e^{\tau/2} y\Big),$$

we see that ζ satisfies

$$\zeta_\tau = L\zeta + \frac{\epsilon}{2}\zeta_y - \Big(1 + \frac{\epsilon}{2} e^{\tau/2}\Big)\zeta,$$

with L as in (5.3). Finally, pulling out the zeroth order factor

$$\zeta(\tau, y) = e^{-\tau - \epsilon(e^{\tau/2} - 1)} \bar{\zeta}(\tau, y),$$

we see that $\bar{\zeta}$ solves

$$\bar{\zeta}_\tau = L\zeta + \frac{\epsilon}{2}\zeta_y. \tag{5.6}$$ {eq:c001

We finish using the following perturbative lemma, proved in Appendix A. This result falls outside of [15] and Lemma 5.1 because the $\bar{\zeta}_y$ term in (5.6) is no longer a remainder term.

urbativeeps

Lemma 5.2. *Let $\bar{\zeta}$ solve (5.6), then it can be represented as*

$$\bar{\zeta}(\tau, y) = \exp\Big\{-\frac{y^2}{8} - \frac{\epsilon y}{4}\Big\}\Big(\Big(\int_{\mathbb{R}+} \psi_\varepsilon(y) e^{\frac{y^2}{8} + \frac{\epsilon y}{4}} \bar{\zeta}(0, y)\, dy\Big)\psi_\varepsilon(y) e^{-\lambda_\varepsilon \tau} + y\bar{h}(\tau, y) e^{-\mu_\varepsilon \tau}\Big). \tag{5.7}$$ {eq:c002

Here, $\mu_\epsilon > 1/2$, $\psi_\epsilon(y)$ and $\lambda_\epsilon > 0$ are such that $\lambda_\epsilon \to 0$ and $\psi_\epsilon(y) \to ye^{-y^2/8}/\sqrt{2\sqrt{\pi}}$ uniformly on compact sets as $\epsilon \to 0$, ψ_ϵ is uniformly bounded in ϵ, $\|\psi_\epsilon\|_2 = 1$, and $\overline{h}(\tau, y)$ is bounded on all compact subsets of $[0, \infty)$.

Returning to the original variables, we first note that, for all $t > 0$ and all $x \leq 2t + (1+\varepsilon)\sqrt{t}$, it follows directly from (5.7) that

$$|\tilde{v}(t,x)| \leq Ct^{-3/2-\lambda_\epsilon} e^{-\epsilon t^{1/2}}. \tag{5.8}$$

{eq:c3}

In fact, this estimate holds for all x since, as above, $\tilde{v}$ may be estimated on $[2t+(1+\varepsilon)\sqrt{t}, \infty)$ using the same approach as in (5.5). Second, taking t sufficiently large and evaluating at $x = 2t+(1+\varepsilon)\sqrt{t}$, we see that

$$\tilde{v}(t, 2t + (1+\varepsilon)\sqrt{t}) \geq \frac{\alpha_\epsilon}{t^{1+\lambda_\epsilon}} e^{-(1+\epsilon)\sqrt{t}},$$

for some α_ϵ depending only on u_0 and ϵ. This concludes the proof.

□

5.3 The case $r \in (1,3)$: the proof of Lemma 4.3

sub-solution

To motivate some of the steps in the following proof, we briefly discuss a heuristic. In the stationary frame, we may always estimate $\tilde{v}$ above by removing the Dirichlet boundary condition and using the fact that, up to a e^t integrating factor, $\tilde{v}$ solves the heat equation:

$$\tilde{v}(t, x+2t+t^\gamma) \lesssim t^{-1/2} \exp\Big\{t - \frac{(x+2t+t^\gamma)^2}{4t}\Big\} = t^{-1/2} \exp\Big\{-x - \frac{x^2}{4t} - \frac{x}{\sqrt{t}}\frac{t^{\gamma-1/2}}{2} - t^\gamma - \frac{t^{2(\gamma-1/2)}}{4}\Big\}.$$

Recalling that $\gamma > 1/2$, we see that on the diffusive scale $x \sim \sqrt{t}$, the Gaussian term $x^2/4t$ and the $t^{-1/2}$ in front are (much) lower order and, thus, negligible, but all other terms are large. Hence, our sub-solution should contain all such terms to be reasonably sharp. In particular, while the $xt^{\gamma-1}$ term appears small at first glance since $\gamma < 1$, it is not negligible in the diffusive scale $x \sim \sqrt{t}$. While the terms depending only on t show up as obvious integrating factors, this term will not. Hence, the key to the proof below is in carefully taking account of this term. Note that here we see the effect of $\gamma > 1/2$.

Proof of Lemma 4.3. We first show how to "guess" the form of the sub-solution v. We begin by removing an exponential from $\tilde{v}$ and changing to the moving frame. Define, for $x \in \mathbb{R}^+$,

$$z(t,x) := e^x \tilde{v}(t, 2t + (t+1)^\gamma - 1 + x),$$

so that (4.5) becomes

$$\begin{aligned} &z_t \leq z_{xx} + \gamma(t+1)^{\gamma-1}(z_x - z), \quad t > 0, x > 0, \\ &z(t,0) = 0, \\ &z(0,x) = e^x w_0(x). \end{aligned} \tag{5.9}$$

{eq:self

Turning to self-similar variables,

$$\tau = \log(1+t), \quad y = (t+1)^{-1/2}x, \quad \text{and} \quad \zeta(\tau,y) = z(e^\tau - 1, e^{\tau/2}y),$$

we wish to construct ζ that satisfies the inequality

$$\zeta_\tau \leq \zeta_{yy} + \frac{y}{2}\zeta_y + \gamma e^{(\gamma-1/2)\tau}\zeta_y - \gamma e^{\gamma\tau}\zeta. \tag{5.10}$$

{oct412}

As $\gamma > 1/2$, the drift in (5.10) is not a perturbation anymore. The heuristic discussion preceding this proof indicates that we should consider

$$\zeta(\tau,y) = e^{-\alpha y e^{(\gamma-1/2)\tau}} \psi(\tau,y),$$

with $\alpha \in \mathbb{R}$ to be determined:

$$\psi_\tau \le L\psi + (\gamma - 2\alpha)e^{(\gamma-1/2)\tau}\psi_y - \Big(1 + \alpha(\gamma-\alpha)e^{(2\gamma-1)\tau} + \gamma e^{\tau\gamma}\Big)\psi - \alpha(1-\gamma)ye^{(\gamma-1/2)\tau}\psi. \quad (5.11)$$ {eq:chri

with L as in (5.3). To remove the drift term, we set $\alpha = \gamma/2$. Then (5.11) becomes

$$\psi_\tau - L\psi + \Big(1 + \frac{\gamma^2}{4}e^{(2\gamma-1)\tau} + \gamma e^{\tau\gamma}\Big)\psi + \frac{\gamma}{2}(1-\gamma)ye^{(\gamma-1/2)\tau}\psi \le 0.$$

Further, writing

$$\psi(\tau, y) = \exp\Big\{ -\tau - e^{\gamma\tau} - \frac{\gamma^2}{4(2\gamma-1)}e^{(2\gamma-1)\tau}\Big\}\Psi(\tau, y),$$

we arrive at

$$\Psi_\tau - L\Psi + \frac{1}{2}\gamma(1-\gamma)ye^{(\gamma-1/2)\tau}\Psi \le 0. \quad (5.12)$$ {eq:chri

To deal with the last term in (5.12), let $a, b > 0$ be constants to be determined and define

$$\Psi(\tau, y) = y\exp\Big\{ -ae^{\tau(2\gamma-1)} - a'\tau - \frac{y^2}{b}\Big\}.$$

By a direct computation, we see that

$$\begin{aligned}\Psi_\tau - &L\Psi + \frac{\gamma}{2}(1-\gamma)ye^{(\gamma-1/2)\tau}\Psi \\ &= \Big[-a' - a(2\gamma-1)e^{\tau(2\gamma-1)} - \frac{y^2}{b}\Big(\frac{4}{b} - 1\Big) + \Big(\frac{6}{b} - \frac{3}{2}\Big) + \frac{\gamma(1-\gamma)}{2}ye^{(\gamma-1/2)\tau}\Big]\Psi.\end{aligned} \quad (5.13)$$ {oct416}

It is clear that to have (5.12), we must choose $b < 4$. For simplicity, we take $b = 2$, and

$$a' = \frac{6}{b} - \frac{3}{2} = \frac{3}{2},$$

so that (5.13) becomes

$$\Psi_\tau - L\Psi + \frac{\gamma}{2}(1-\gamma)ye^{(\gamma-1/2)\tau}\Psi = \Big[-a(2\gamma-1)e^{\tau(2\gamma-1)} + \frac{\gamma(1-\gamma)}{2}ye^{(\gamma-1/2)\tau} - \frac{y^2}{2}\Big]\Psi.$$

The choice

$$a \ge \frac{\gamma^2(1-\gamma)^2}{8(2\gamma-1)}$$

ensures that (5.12) holds. Returning to our original variables, we see that

$$\begin{aligned} v(t,x) &= \tilde{v}(t, 2t + (t+1)^\gamma - 1 + x) = e^{-x}\zeta(\log(1+t), (t+1)^{-1/2}x) \\ &= e^{-x}e^{-\alpha(t+1)^{-1/2}x(1+t)^{\gamma-1/2}}\psi(\log(1+t), (t+1)^{-1/2}x) \\ &= \frac{1}{1+t}\exp\Big\{ -x - \frac{\gamma}{2}x\Big(1+t\Big)^{\gamma-1} - (1+t)^\gamma - \frac{\gamma^2(1+t)^{2\gamma-1}}{4(2\gamma-1)}\Big\}\Psi(\log(1+t), (t+1)^{-1/2}x) \\ &= \frac{x}{(1+t)^3}\exp\Big\{ -x - \frac{\gamma}{2}x(1+t)^{\gamma-1} - (1+t)^\gamma - \Big[\frac{\gamma^2}{4(2\gamma-1)} + a\Big](1+t)^{2\gamma-1} - \frac{x^2}{2(1+t)}\Big\}. \end{aligned}$$

This concludes the proof.

□

6 The Fisher-KPP equation with a Gompertz non-linearity

al_equation A side effect of our analysis gives the asymptotics for a related local equation:

$$u_t - \Delta u = f_r(u). \tag{6.1}$$ {eq:loca

Here, we assume that $f_r \in C^1$, $r \in (1, \infty)$, and there exist positive constants θ_f, δ_f, and A_f such that

$$f_r(0) = 0, \quad f_r(u) > 0 \text{ for all } u \in (0, \theta_f), \quad f_r(\theta_f) = 0, \quad f_r(u) = 0 \text{ for all } u \geq \theta_f, \tag{6.2}$$

and

$$u\Big(1 - A_f \log\left(\frac{1}{u}\right)^{1-r}\Big) \leq f_r(u) \leq u\Big(1 - A_f^{-1} \log\left(\frac{1}{u}\right)^{1-r}\Big), \tag{6.3}$$ {eq:loca

for $u \in (0, \delta_f)$.

local_delay **Theorem 6.1.** *Suppose that the initial condition $u_0(x)$ for (6.1) is as in (1.5). If $r > 3$, then the solution $u(t, x)$ propagates with a logarithmic delay:*

$$\lim_{L \to \infty} \limsup_{t \to \infty} \sup_{x \geq L} u\Big(t, 2t - \frac{3}{2} \log t + x\Big) = 0, \tag{6.4}$$ {eq:loca

and

$$\lim_{L \to \infty} \limsup_{t \to \infty} \sup_{x \leq -L} \Big| u\Big(t, 2t - \frac{3}{2} \log t + x\Big) - \theta_f \Big| = 0. \tag{6.5}$$ {eq:loca

If $r = 3$, $u(t, x)$ propagates with a weak logarithmic delay: (6.4) holds and, for all $\epsilon > 0$,

$$\liminf_{t \to \infty} \sup_{x \leq 0} \Big| u\Big(t, 2t - \Big(\frac{3}{2} + \epsilon\Big) \log t + x\Big) - \theta_f \Big| = 0. \tag{6.6}$$ {eq:weak

If $r \in (1, 3)$, then the delay is algebraic: there exist $C_f > c_f > 0$, depending only on f_r, such that

$$\lim_{t \to \infty} \sup_{x \geq 0} u\Big(t, 2t - c_f t^{\frac{3-r}{1+r}} + x\Big) = 0, \tag{6.7}$$ {eq:loca

and

$$\lim_{t \to \infty} \sup_{x \leq 0} \Big| u\Big(t, 2t - C_f t^{\frac{3-r}{1+r}} + x\Big) - \theta_f \Big| = 0. \tag{6.8}$$ {eq:loca

The proof of (6.4) follows directly from Section 3. The proofs of (6.5), (6.6), and (6.8) follow from what was done in Section 4, combined with a standard argument saying that the convergence is necessarily to the steady state θ_f (see, e.g., [15]). The bound (6.7) needs an additional ingredient. Indeed, since our non-linearity is local, we cannot "pull" information from the front as we did above when we used the value of u at the front to bound $\phi \star u$ far ahead of the front. In order to get around this, we state a weak lower bound on u.

l_alg_lower **Lemma 6.2.** *Let the hypotheses of Theorem 6.1 be satisfied. Then there exists $\delta_f > 0$, depending only on f, such that*

$$u(t, x) \geq \exp\{-\delta_f t^\gamma\}$$

for all t sufficiently large and all $x \leq 2t + t^\gamma$, where we again define $\gamma = 2/(1 + r)$.

Such a bound follows from the analysis of the lower bound in part (3) of Proposition 3.2 and requires no new ideas. As such, we omit the proof.

The main point in the proof of Theorem 6.1 is to use the lower bound in Lemma 6.2 on u along with the form of the non-linearity to replace the estimate of $\phi \star u$ that we used in the proof of the upper bound in Theorem 1.1 when $r \in (1, 3)$.

Proof of (6.7) *assuming Lemma 6.2.* We will use a super-solution

$$\overline{v}(t,x) := B \exp\Big\{ - \Big(x - 2t + 2c_f t^{2\gamma-1}\Big)\Big\},$$

with $c_f > 0$ to be determined. Then $\overline{v}$ satisfies

$$\overline{v}_t = \overline{v}_{xx} + \overline{v}\Big(1 - 2c_f(2\gamma-1)t^{2\gamma-2}\Big).$$

On the other hand, using the bound on f (6.3) along with Lemma 6.2, we have that, for all t sufficiently large and $x \leq 2t + t^\gamma$,

$$u_t - u_{xx} = f_r(u) \leq u\Big(1 - A_f \log\Big(\frac{1}{u(t,x)}\Big)^{1-r}\Big) \leq u\Big(1 - A_f\delta_f^{1-r}t^{\gamma(1-r)}\Big).$$

Recalling $2\gamma - 2 = \gamma(1-r)$, and choosing c_f such that $A_f\delta_f^{r-1} \geq 2c_f(2\gamma-1)$, we see that $\overline{v}$ is a super-solution for u. □

7 The local-in-time Harnack inequality: Proposition 1.2

sec:harnack

Proof of Proposition 1.2. Up to a shift in time, we may assume that $t = 0$. We may also assume that $c \equiv 0$. Indeed, let

$$u_\pm(t,x) = e^{\pm t\|c\|_{L^\infty([0,T]\times\mathbb{R})}}w(t,x),$$

where w solves the heat equation

$$w_t = w_{xx},$$

with the initial condition $w(t=0,x) = u(t=0,x)$. Then u_+ is a super-solution to u while u_- is a sub-solution to u. Hence, we have

$$\frac{u(t,x+y)}{\|u_-\|_{L^\infty}^{1-1/p}u(t,x)^{1/p}} \leq \frac{u_+(t,x+y)}{\|u_-\|_{L^\infty}^{1-1/p}u_-(t,x)^{1/p}} \leq e^{2\|c\|_{L^\infty}t}\frac{w(t,x+y)}{\|w\|_{L^\infty}^{1-1/p}w(t,x)^{1/p}}.$$

In view of this inequality, it is enough to prove the claim for w, that is, solutions to the heat equation.

Let G be the one-dimensional heat kernel $G(t,x) = (4\pi t)^{-1/2}e^{-x^2/(4t)}$. Fix $s = (p+1)/2p$, notice that $s \in (0,1)$ and $sp > 1$, and let q be the dual exponent of p. Then we have

$$\begin{aligned}
w(t,x+y) &= \int_{\mathbb{R}} w(0,z)G(T,x+y-z)dz \\
&\leq \|w\|_\infty^{1-1/p}\int_{\mathbb{R}} w(0,z)^{1/p}G(T,x+y-z)^sG(T,x+y-z)^{1-s}dz \\
&\leq \|w\|_\infty^{1-1/p}\Big(\int_{\mathbb{R}} w(0,z)G(T,x+y-z)^{sp}dz\Big)^{1/p}\Big\|G^{1-s}(T,\cdot)\Big\|_q \\
&\leq C_pT^{(1/4)(1-1/p)}\|w\|_\infty^{1-1/p}\Big(\int_{\mathbb{R}} w(0,z)G(T,x+y-z)^{sp}dz\Big)^{1/p}.
\end{aligned} \tag{7.1}$$

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We now seek a bound on $G(T, x+y-z)^{sp}$ in terms of $G(T, x-z)$. To this end, we recall that $sp > 1$, let $x' = x - z$ and we compute

$$\begin{aligned}
\frac{G(T, x'+y)^{sp}}{G(T, x')} &= (4\pi T)^{(1-sp)/2} \exp\Big\{ -\frac{sp(x'+y)^2}{4T} + \frac{|x'|^2}{4T} \Big\} \\
&= (4\pi T)^{(1-sp)/2} \exp\Big\{ -\frac{sp|x'|^2}{4T} - \frac{spx'y}{2T} + \frac{spy^2}{4T} + \frac{|x'|^2}{4T} \Big\} \\
&= (4\pi T)^{(1-sp)/2} \exp\Big\{ -\frac{(sp-1)|x'|^2}{4T} + \frac{spx'y}{2T} + \frac{spy^2}{4T} \Big\} \\
&\le (4\pi T)^{(1-sp)/2} \exp\Big\{ -\frac{(sp-1)|x'|^2}{4T} + \Big(\frac{(sp-1)|x'|^2}{4T} + \frac{(sp)^2 y^2}{4T(sp-1)} \Big) + \frac{spy^2}{4T} \Big\}.
\end{aligned}$$

Define $\beta = (s^2p^2/(sp-1) + sp)/4p$. Using the above bound in (7.1), we obtain

$$\begin{aligned}
w(t, x+y) &\le C_p e^{\beta y^2/T} T^{(1/4)(1-1/p)+(1-sp)/2p} \|w\|_\infty^{1-1/p} \Big(\int_{\mathbb{R}} w(0,z) G(T, x-z) dz \Big)^{1/p} \\
&= C e^{\beta y^2/T} \|w\|_\infty^{1-1/p} w(t,x)^{1/p}.
\end{aligned}$$

In the second line we used the explicit choice of s to simplify the exponent of T. This concludes the proof. ☐

A Proofs of Lemmas 5.1 and 5.2.

:lemestpert

Proof of Lemma 5.1. The proof of this lemma is similar to that of a corresponding estimate in [15]. However, the proof there only deals with moving boundary conditions of the form $2t + r\log(t)$. Hence, for completeness, we provide a streamlined proof. Recall that $\bar\zeta$ solves

$$\bar\zeta_\tau = L\bar\zeta + \varepsilon e^{(\gamma-1/2)\tau} \bar\zeta_y.$$

To rectify the fact that the operator L is not self-adjoint, we remove a Gaussian term. Let

$$\bar\zeta(\tau, y) = e^{-y^2/8} \zeta^*(\tau, y),$$

then z^* satisfies

$$\zeta^*_\tau + M\zeta^* = \varepsilon e^{(\gamma-1/2)\tau} \Big(\bar\zeta^*_y - \frac{y}{4} \bar\zeta^* \Big), \tag{A.1}$$

{eq:self

where

$$M\zeta^* := -\zeta^*_{yy} + \Big(\frac{y^2}{16} - \frac{3}{4} \Big) \zeta^*.$$

The principle eigenvalue of M is associated to the eigenfunction

$$\psi(y) := (2\sqrt{\pi})^{-1/2} y e^{-y^2/8}.$$

Define the non-negative quadratic form

$$Q(f) := \langle Mf, f \rangle = \int_{\mathbb{R}} \Big(f_y^2 + \Big(\frac{y^2}{16} - \frac{3}{4} \Big) f^2 \Big) dy,$$

for all $f \in H^1(0,\infty)$ such that $yf \in L^2(\mathbb{R}^+)$.

Multiplying (A.1) by ζ^* and integrating, we obtain

$$\partial_\tau \|\zeta^*\|^2_{L^2(\mathbb{R}^+)} + 2Q(\zeta^*) = -2\varepsilon e^{(\gamma-1/2)\tau} \int_0^\infty \frac{y}{4}(\zeta^*)^2 dy \leq 0.$$

Hence ζ^* is bounded uniformly in L^2 independently of τ. Next, let $\zeta_1^* = \langle \psi, \zeta^* \rangle$. We have

$$|\partial_\tau \zeta_1^*| \leq \varepsilon e^{-(\gamma-1/2)\tau}(|\langle(-\psi_y), \zeta^*\rangle| + |\zeta_1^*|) \lesssim \varepsilon e^{-(\gamma-1/2)\tau}\|\zeta^*\|_2. \qquad \text{(A.2)}$$ {eq:bar_

Integrating this inequality in τ and using the L^2 bound above, we obtain

$$|\zeta_1^*(\tau) - \zeta_1^*(0)| \lesssim \varepsilon \|\bar{\zeta}(0, \cdot)\|_2. \qquad \text{(A.3)}$$ {eq:chri

We now show that the component of ζ^* that is orthogonal to ψ decays in time. Let

$$\zeta^{*\perp} := \zeta^* - \zeta_1^* \psi,$$

then

$$Q(\zeta^{*\perp}) \geq \frac{1}{2}\|\zeta^{*\perp}\|_2^2.$$

Using (A.1), we obtain

$$\partial_\tau \|\zeta^{*\perp}\|^2_{L^2(\mathbb{R}^+)} + 2Q(\zeta^{*\perp}) \lesssim \varepsilon e^{-(\gamma-1/2)\tau}\|\zeta^*(0,\cdot)\|_2 \|\zeta^{*\perp}\|_2,$$

from which we deduce that

$$\|\bar{\zeta}^\perp(\tau,\cdot)\|_2 \lesssim e^{-(\gamma-1/2)\tau}\|\bar{\zeta}(0,\cdot)\|_2.$$

Gathering all estimates concludes the proof. □

Proof of Lemma 5.2. Recall that $\bar{\zeta}$ solves

$$\bar{\zeta}_\tau = L\bar{\zeta} + \frac{\epsilon}{2}\bar{\zeta}_y.$$

To pass to a self-adjoint form, write

$$\bar{\zeta}(\tau, y) = \exp\Big\{-\frac{y^2}{8} - \frac{\epsilon y}{4}\Big\}\bar{\zeta}^*(\tau, y),$$

so that ζ^* solves

$$\zeta^*_\tau + M_\epsilon \zeta^* = 0,$$

where,

$$M_\varepsilon \zeta^* := -\zeta^*_{yy} + \Big[\Big(\frac{y^2}{16} - \frac{3}{4}\Big) + \epsilon\Big(\frac{y}{8} + \frac{\epsilon}{16}\Big)\Big]\zeta^* = M\zeta^* + \epsilon\Big(\frac{y}{8} + \frac{\epsilon}{16}\Big)\zeta^*.$$

This operator is now self-adjoint with a compact resolvent. Let ψ_ϵ and λ_ϵ be the principal eigenfunction and eigenvalue of the operator above satisfying the boundary condition $\psi_\epsilon(0) = 0$ and the normalisation $\|\psi_\epsilon\|_{L^2(\mathbb{R}^+)} = 1$.

Observe that

$$Q_\varepsilon(f) := \langle M_\varepsilon f, f\rangle = \langle Mf, f\rangle + \Big\langle \varepsilon\Big(\frac{y}{8} + \frac{\epsilon}{16}\Big)f, f\Big\rangle \geq 0,$$

and thus $\lambda_\epsilon > 0$. By elliptic regularity, $\psi_\epsilon \to \psi$ in L^2 and locally uniformly in y and $\lambda_\epsilon \to 0$ as $\epsilon \to 0$, where ψ is the principal eigenfunction of the operator M, given in the proof of Lemma 5.1.

Write

$$\zeta^* := \langle \psi_\varepsilon, \zeta^* \rangle \psi_\varepsilon + \zeta^{*\perp},$$

so that

$$Q_\varepsilon(\zeta^{*\perp}) \geq \mu_\varepsilon \|\zeta^{*\perp}\|_2^2, \tag{A.4}$$ {oct502}

where μ_ε is the second eigenvalue of M_ε. After a time differentiation we have

$$\langle \psi_\varepsilon, \zeta^* \rangle(\tau) = \langle \psi_\varepsilon, \zeta^* \rangle(0) e^{-\lambda_\varepsilon \tau},$$

and as a consequence of (A.4), that

$$\|\zeta^{*\perp}\|_2(\tau) \leq \|\zeta^{*\perp}\|_2(0) e^{-\mu_\varepsilon \tau}.$$

Then, locally we have $\|\zeta^{*\perp}\|_\infty(\tau) \lesssim e^{-\mu_\varepsilon \tau}$ by parabolic regularity. This yields

$$\bar{\zeta} := \exp\Big\{ -\frac{y^2}{8} - \frac{\epsilon y}{4} \Big\} \Big(\Big(\int_{\mathbb{R}^+} \psi_\varepsilon(y) \exp\Big\{ \frac{y^2}{8} + \frac{\epsilon y}{4} \Big\} \bar{\zeta}(0, y)\, dy \Big) \psi_\varepsilon(y) e^{-\lambda_\varepsilon \tau} + \bar{h}(\tau, y) e^{-\mu_\varepsilon \tau} \Big), \tag{A.5}$$ {eq:c2}

where $\bar{h}$ is bounded in τ, locally in y. Moreover, there exists $C_K > 0$ such that

$$C_K^{-1} \psi(y) \leq \psi_\varepsilon(y) \leq C_K \psi(y),$$

finishing the proof.

□

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