Fronts in reactive convection: bounds, stability and instability

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Abstract

This paper examines a simplified active combustion model in which the reaction influences the flow. We consider front propagation in a reactive Boussinesq system in an infinite vertical strip. Nonlinear stability of planar fronts is established for narrow domains when the Rayleigh number is not too large. Planar fronts are shown to be linearly unstable with respect to long wavelength perturbations if the Rayleigh number is sufficiently large. We also prove uniform bounds on the bulk burning rate and the Nusselt number in the KPP reaction case.

1 Introduction

The presence of fluid flow can have a strong effect on reaction processes in many situations of interest [19, 28]. Numerous recent studies have focused on the influence of passive advection on combustion. During passive advection the reacting material is carried by a flow, but the flow is not influenced by the reaction. Traveling waves and pulsating fronts in periodic flows have been studied in [3, 4, 2, 25, 26]; the thin front limit was considered in [11, 12, 17] using homogenization techniques. Estimates on the speed of front propagation for different classes of flows have been derived in [1, 5, 16, 6, 13]. Numerical studies of the effect of a passive flow on combustion were carried out in [14, 15, 8, 9, 23]. This list is far from complete; for further references, see the recent reviews [1, 27].

There have been few rigorous works on models with feedback of the chemical reaction on the flow field. In this paper we study a simplified active combustion model [18, 22, 20, 21] in which the reaction does influence the flow. The feedback of the flame on the fluid is taken in a Boussinesq approximation. The model thus couples an advection-diffusion-reaction equation for the temperature with an incompressible Navier-Stokes system driven by temperature differences. We study this problem in a two dimensional strip of infinite vertical height and finite horizontal width. The vertical direction is the direction of gravity. The system admits planar fronts as particular solutions. These fronts correspond to traveling solutions of the

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one dimensional reaction-diffusion system without horizontal variation. We study them in the context of the larger reactive Boussinesq system. Coupling with the Boussinesq system introduces at least two new interesting effects. One is symmetry breaking: gravity breaks the vertical symmetry of the reaction diffusion systems. This has a dynamical effect: fronts connecting low regions of hot fluid to high regions of cold fluid are susceptible to the Rayleigh-Taylor instability [7]. The second effect is due to the introduction of new horizontal degrees of freedom: new length scales are introduced. When one ignores the fluid advection, the planar fronts have a characteristic thickness δ (2.11), which is determined by the thermal diffusivity and the characteristic reaction time t_c (2.13). Using these as length and time units, three significant nondimensional parameters emerge. One is the aspect ratio λ (2.23), the ratio of the horizontal width of the strip to the thickness of the planar front. The second parameter is the Prandtl number σ (2.19), the ratio of kinematic viscosity to thermal diffusivity. The third important parameter is the Rayleigh number ρ (2.20) which measures the relative strength of the buoyancy force on the scale of the front thickness.

We prove three kinds of results: stability, instability and uniform bounds. Stability is analyzed in Section 4. If the aspect ratio λ is small, then the only traveling solutions are planar fronts, and all solutions become eventually planar. This is a consequence of the fact that diffusion acts rapidly across a narrow strip. This stability mechanism is quite robust and operates for all kinds of nonlinearities. If the Rayleigh number is large enough, then the planar front loses stability to longwave perturbations, which are present if the aspect ratio is large enough. The instability is of Rayleigh-Taylor type and is also expected to be quite robust. We note that nonplanar traveling waves have been recently shown to exist ([20, 21]) in reactive Boussinesq systems with bistable nonlinearity at large enough Rayleigh numbers. Our results on planar front instability agree qualitatively with recent numerical results [22]. The proof appears in Section 5 and is based on the intuitive idea that, in the presence of gravity, the neutral mode - corresponding to the broken vertical translation symmetry - misaligns over long horizontal distances, and gives rise to a longwave unstable mode. Technically, one needs to deal with a variable coefficient linear operator in which the instability of heavy fluid on top of light fluid is exploited using the monotonicity of the front. In Section 3 we prove bounds for arbitrary solutions of front type. The results state that the bulk burning rate, speed of front, gradients of temperature, and fluid quantities, are all bounded. The bounds do depend on the aspect ratio. The proof applies only to concave KPP nonlinearities. The results imply that there are no accelerating fronts in this system. The proof of the upper bound passes via a lower bound: the temperature gradients squared bound from below the bulk burning rate [5]. On the other hand, the bulk burning rate is bounded above by the sum of laminar burning rate and temperature gradients to power one. This implies that the temperature gradients are bounded, and the rest follows.

2 Reactive Boussinesq fronts

The reactive Boussinesq equations are

$$\frac{\partial v}{\partial t} + v \cdot \nabla v + \nabla p - \nu \Delta v = gATe_z, \tag{2.1}$$

$$\nabla \cdot v = 0, \tag{2.2}$$

$$\frac{\partial T}{\partial t} + v \cdot \nabla T - \kappa \nabla T = \frac{v_0^2}{4\kappa} f(T). \tag{2.3}$$

Here $v(\mathbf{x}, t) = (u(\mathbf{x}, t), w(\mathbf{x}, t))$ and $T(\mathbf{x}, t)$ are the velocity and the (normalized) temperature. The vector e_z is the unit vector pointing in the direction opposite to the pull of gravity. The constants ν, κ, A, v_0, g are all positive, and represent, respectively, the kinematic viscosity, thermal diffusivity, thermal expansion coefficient multiplied by the temperature variation scale, speed of purely reactive-diffusive front and acceleration of gravity. The variables $\mathbf{x} = (x, z)$ belong to a strip, with $z \in \mathbb{R}$ and $x \in [0, L]$. The boundary conditions for the normalized temperature are front conditions,

$$T(x, z, t) \to 1 \quad \text{as } z \to -\infty, \quad T(x, z, t) \to 0 \quad \text{as } z \to \infty. \tag{2.4}$$

We assume throughout the paper that the initial condition $T_0(x, z)$ satisfies $0 \leq T_0(x, z) \leq 1$. The velocity vanishes at the two ends of the strip:

$$v(x, z, t) \to 0 \quad \text{as } |z| \to \infty. \tag{2.5}$$

The lateral boundary conditions are periodic:

$$T(x + L, z, t) = T(x, z, t), \quad v(x + L, z, t) = v(x, z, t). \tag{2.6}$$

We consider the vorticity

$$\omega(x, z, t) = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \tag{2.7}$$

where u is the component of v in the x direction and w is the component of v in the z direction. The momentum equation (2.1) implies

$$\frac{\partial \omega}{\partial t} + v \cdot \nabla \omega - \nu \Delta \omega = gA \frac{\partial T}{\partial x}. \tag{2.8}$$

The nonlinearity $f(T)$ satisfies $f(0) = f(1) = 0$, is a smooth function and is assumed to be of one of the following three types: bistable, ignition type or concave KPP. Bistable nonlinearities are such that there exists $\theta \in (0, 1)$ so that $f(T) < 0$ for $T \in (0, \theta)$ and $f(T) > 0$ for $T \in (\theta, 1)$. Ignition type nonlinearities are such that there exists $\theta \in (0, 1)$ so that $f(T) = 0$ for $T \in [0, \theta]$ and $f(T) > 0$ for $T \in (\theta, 1)$. Concave KPP type nonlinearities have $f(T) > 0$ for $T \in (0, 1)$ and $f''(T) < -M$ for $T \in [0, 1]$ with $M > 0$. In their case we assume that the slope at zero is positive and normalized, $f'(0) = 1$, while the slope at 1 is negative, $f'(1) = -m < 0$. We will use the KPP properties only in the proof of a uniform upper bound for the burning rate presented in Section 3. We summarize the essential properties of the KPP reaction used in that proof:

$$0 \leq f(T) \leq T, \ -f''(T) \geq M > 0, \ -f'(T) \leq m, \ \text{for } 0 \leq T \leq 1. \tag{2.9}$$

The Boussinesq system has flat traveling wave solutions

$$T_{fr} = \tau(z - ct), v_{fr} = 0. \tag{2.10}$$

The momentum equation (2.1) holds in this case because the pressure can balance a temperature that depends on z and t alone. The speed c takes all values $c \geq v_0$ in the KPP case, and is unique in the bistable and ignition case. The profile $\tau(z)$ is monotonically decreasing in all three cases, and obeys

$$\kappa\tau'' + c\tau' + \frac{v_0^2}{4\kappa} f(\tau) = 0,$$

where $\tau' = \dfrac{d\tau}{dz}$. We choose now dimensional units. First we note that the function τ varies on length scale of the order

$$\delta = \frac{2\kappa}{v_0}, \tag{2.11}$$

that is, the function τ has the form $\tau(z) = P(z/\delta)$, where P obeys the equation

$$P'' + 2\bar{c}P' + f(P) = 0. \tag{2.12}$$

Here $\bar{c} = c/v_0$ is a nondimensional constant that is determined solely by the form of the nonlinearity $f(T)$, and that in turn determines the traveling front speed $c = \bar{c}v_0$ in the original dimensional variables. We choose reaction units: we take (2.11) for length unit and we take

$$t_c = \frac{4\kappa}{v_0^2} \tag{2.13}$$

for time unit. The velocity fluctuations are written as

$$v(\mathbf{x}, t) = \frac{\delta}{t_c}\tilde{v}\left(\frac{\mathbf{x}}{\delta}, \frac{t}{t_c}\right) \tag{2.14}$$

and the temperature fluctuations are written as

$$\theta(\mathbf{x}, t) = \tilde{\theta}\left(\frac{\mathbf{x}}{\delta}, \frac{t}{t_c}\right). \tag{2.15}$$

Using these units, rescaling, using $\mathbf{x} = (x, z) = (x_{new}, z_{new}) = (x_{old}/\delta, z_{old}/\delta)$ and $t = t_{new} = t_{old}/t_c$, and dropping tildes, we derive the nonlinear equations

$$\frac{\partial \omega}{\partial t} + v \cdot \nabla\omega - \sigma\Delta\omega = \sigma\rho\frac{\partial T}{\partial x} \tag{2.16}$$

and

$$\frac{\partial T}{\partial t} + v \cdot \nabla T - \Delta T = f(T), \tag{2.17}$$

where $v = (u, w)$, with

$$\Delta u = -\frac{\partial \omega}{\partial z}, \quad \Delta w = \frac{\partial \omega}{\partial x} \tag{2.18}$$

which follows from (2.7) and incompressibility. The nondimensional parameters are the Prandtl number

$$\sigma = \frac{\nu}{\kappa} \tag{2.19}$$

and the Rayleigh number across a laminar front width

$$\rho = \frac{gA\delta^3}{\kappa\nu}. \tag{2.20}$$

The boundary conditions in z are of front type

$$\begin{cases} T(x,z,t) \to 1 \text{ as } z \to -\infty, \\ T(x,z,t) \to 0 \text{ as } z \to +\infty, \\ v(x,z,t) \to 0, \text{ as } |z| \to \infty. \\ \omega(x,z,t) \to 0 \text{ as } |z| \to \infty. \end{cases} \tag{2.21}$$

The boundary conditions in x are periodic

$$T(x+\lambda,z,t) = T(x,z,t), \;\; v(x+\lambda,z,t) = v(x,z,t), \;\; \omega(x+\lambda,z,t) = \omega(x,z,t) \tag{2.22}$$

with period

$$\lambda = \frac{L}{\delta}. \tag{2.23}$$

3 General bounds in the KPP case

We study general solutions of the system (2.16, 2.17, 2.18) with a KPP nonlinearity that satisfies assumptions (2.9), and with front-like initial data: functions that approach the values 0 as $z \to \infty$ and, respectively 1 as $z \to -\infty$ at least at an exponential rate. We denote $D = [0,\lambda] \times \mathbb{R}$. We will use the notation

$$\|g\|_{L^2}^2 = \frac{1}{\lambda}\int_D |g(x,z)|^2 dxdz \tag{3.1}$$

for the *normalized* L^2 norm. We consider the bulk burning rate

$$V(t) = \frac{1}{\lambda}\int_D \frac{\partial T(x,z,t)}{\partial t} dxdz \tag{3.2}$$

as a measure of the reaction rate. Using equation (2.17), incompressibility of u and the boundary conditions for u and T as $z \to \pm\infty$, we have also

$$V(t) = \frac{1}{\lambda}\int_D f(T(x,z,t))dxdz. \tag{3.3}$$

The time average of V is denoted $\overline{V}$

$$\overline{V}(t) = \frac{1}{t}\int_0^t V(s)ds. \tag{3.4}$$

Lemma 1 *Assume that there exists a constant $\alpha \in \mathbb{R}$ so that the front-like initial data $T_0(x,z)$ obeys*

$$T_0(x,z) \le \exp(\alpha - z) \tag{3.5}$$

and

$$(1 - T_0(x,z)) \le \exp(\alpha + z). \tag{3.6}$$

Then the solution of (2.17) obeys the bounds

$$T(x,z,t) \leq \exp\left[\alpha - z + 2t + \int_0^t \|w(\cdot,s)\|_{L^\infty} ds\right] \tag{3.7}$$

and

$$(1 - T(x,z,t)) \leq \exp\left[\alpha + z + t - \int_0^t \|w(\cdot,s)\|_{L^\infty} ds\right] \tag{3.8}$$

for all $t \geq 0$.

Proof. For the bound of T we seek a supersolution of the form:

$$\theta_+(z,t) = \exp\left[-az + a\int_0^t \|w(\cdot,s)\|_{L^\infty} ds + bt + \alpha\right], \quad a > 0.$$

Using (2.9) we get

$$\frac{\partial \theta_+}{\partial t} + v \cdot \nabla\theta_+ - \Delta\theta_+ - f(\theta_+) \geq 0$$

if $b \geq a^2 + 1$. We chose for simplicity of exposition $a = 1$, and took the most economic $b = 2$. For the bound of $1 - T$ we seek a subsolution for T of the form

$$\theta_-(z,t) = 1 - \exp\left[az - a\int_0^t \|w(\cdot,s)\|_{L^\infty} ds + bt + \alpha\right]$$

and, using the fact that $f \geq 0$ on $[0,1]$, the condition

$$\frac{\partial \theta_-}{\partial t} + v \cdot \nabla\theta_- - \Delta\theta_- - f(\theta_-) \leq 0$$

follows if

$$b \geq a^2.$$

Again, we chose $a = 1$ for simplicity, and took the best $b = 1$. The proof shows that similar inequalities hold if the exponential rate of decay at infinity of the initial data is different than $a = 1$, or if the initial data approaches its limit at negative infinity at a different exponential rate than $a = 1$.

Let us consider the average quantities

$$W(t) = \frac{1}{t}\int_0^t \|w(\cdot,s)\|_{L^\infty} ds \tag{3.9}$$

and

$$N(t) = \frac{1}{t}\int_0^t \|\nabla T(\cdot,s)\|_{L^2}^2 ds \tag{3.10}$$

representing the average maximum vertical velocity and temperature gradient squared.

Lemma 2 *Consider front-like initial data that satisfy (3.5). Then the solutions of (2.17) obey*

$$\overline{V}(t) \leq W(t) + 2 + \frac{\gamma}{t} \tag{3.11}$$

for all $t \geq 0$ *with* γ *depending on the initial data.*

Proof. First we write

$$\overline{V}(t) = \frac{1}{\lambda t}\int_D (T(x,z,t) - T_0(x,z))\, dxdz.$$

which we bound as

$$\overline{V}(t) \leq \frac{1}{\lambda t}\int_0^\lambda dx \left[\int_{-\infty}^0 (1 - T_0(x,z))\, dz + \int_0^\infty T(x,z,t)dz\right], \tag{3.12}$$

using the fact that $T(t,x,z) \leq 1$. Now, denoting

$$B_1(t) = \alpha + 2t + \int_0^t \|w(\cdot,s)\|_{L^\infty} ds, \tag{3.13}$$

we have from (3.7) that

$$\int_{B_1(t)}^\infty T(x,z,t)dz \leq 1,$$

while, because $T \leq 1$, we have

$$\int_0^{B_1(t)} T(x,z,t)dz \leq B_1(t).$$

We use these bounds in (3.12) and obtain (3.11). Now we bound $W(t)$ below by $N(t)$:

Lemma 3 *Consider front-like initial data that satisfy (3.5). Then the solutions of (2.17) obey*

$$N(t) \leq \frac{m+2}{M}W(t) + \frac{2m+3}{M} + \frac{\Gamma}{t} \tag{3.14}$$

with m and M given in (2.9) and with Γ depending on the initial data.

Proof. We start by computing

$$\frac{d}{dt}V(t) = \frac{1}{\lambda}\int_D f'(T(x,z,t))\frac{\partial T(x,z,t)}{\partial t} dxdz.$$

Using (2.17) and integrating by parts, using incompressibility of u and the boundary conditions, we obtain

$$\frac{dV}{dt} - \frac{1}{\lambda}\int_D f'(T(x,z,t))f(T(x,z,t))dxdz = -\frac{1}{\lambda}\int_D f''(T(x,z,t))|\nabla T(x,z,t)|^2 dxdz.$$

Using (2.9) we deduce

$$\frac{dV}{dt} + mV(t) \geq M\|\nabla T(\cdot,t)\|_{L^2}^2.$$

Taking a time average we get

$$\frac{1}{t}\left(V(t) - V(0)\right) + m\overline{V}(t) \geq MN(t). \tag{3.15}$$

We observe that

$$V(t) = \frac{1}{\lambda}\int_D f(T(x,z,t))dxdz = \int_0^\lambda \frac{dx}{\lambda}\int_{-\infty}^{-B_2(t)} f(T(x,z,t))dz \tag{3.16}$$
$$+ \int_0^\lambda \frac{dx}{\lambda}\int_{-B_2(t)}^{B_1(t)} f(T(x,z,t))dz + \int_0^\lambda \frac{dx}{\lambda}\int_{B_1(t)}^{\infty} f(T(x,z,t))dz,$$

where $B_1(t)$ is given in (3.13) and

$$B_2(t) = \alpha + t + \int_0^t \|w(\cdot,s)\|_{L^\infty}ds.$$

We use the inequality $f(T) \le m(1-T)$ which follows from (2.9). Then we have

$$\int_0^\lambda \frac{dx}{\lambda}\int_{-\infty}^{-B_2(t)} f(T(x,z,t))dz \le m,$$

as follows from (3.8). Similarly,

$$\int_0^\lambda \frac{dx}{\lambda}\int_{B_1(t)}^{\infty} f(T(x,z,t))dz \le 1$$

follows from (3.7) and (2.9). The second term on the right in (3.16) is bounded by $B_1(t) + B_2(t)$. Thus, returning to (3.15), we have

$$MN(t) \le m\overline{V}(t) + 3 + 2W(t) + \frac{c}{t}.$$

In view of (3.11) of the previous lemma, (3.14) is proven.

The next step consists of bounding the quantity $W(t)$ in terms of $N(t)$, using the vorticity equation (2.16).

Lemma 4 *There exists an absolute constant C so that for all $t > 0$ one has*

$$W(t) \le C\lambda^{3/2}\left\{\rho\lambda\sqrt{N(t)} + \frac{1}{\sqrt{\sigma t}}\|\omega_0\|_{L^2}\right\}, \tag{3.17}$$

where $\omega_0(x,z)$ is the initial data for $\omega(x,z,t)$.

Proof. We multiply (2.16) by ω and integrate. After one integration by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\int_D |\omega(x,z,t)|^2\frac{dxdz}{\lambda} + \sigma\int_D |\nabla\omega(x,z,t)|^2\frac{dxdz}{\lambda} = \sigma\rho\int_D \omega(x,z,t)\frac{\partial T(x,z,t)}{\partial x}\frac{dxdz}{\lambda}. \tag{3.18}$$

We introduce

$$\overline{T}(z,t) := \int_0^\lambda T(x,z,t)\frac{dx}{\lambda}$$

and note the obvious fact that

$$\frac{\partial T(x,z,t)}{\partial x} = \frac{\partial\big(T(x,z,t) - \overline{T}(z,t)\big)}{\partial x}.$$

Inserting this expression into the right hand side of (3.18) and integrating by parts on the right side we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_D |\omega(x,z,t)|^2\frac{dxdz}{\lambda} + \sigma\int\limits_D |\nabla\omega(x,z,t)|^2\frac{dxdz}{\lambda}\\
&= -\sigma\rho\int\limits_D \frac{\partial\omega(x,z,t)}{\partial x}\left(T(x,z,t)-\overline{T}(z,t)\right)\frac{dxdz}{\lambda}.
\end{aligned}
$$

Using Young's inequality together with the inequality

$$
\int_D \left|T(x,z,t)-\overline{T}(z,t)\right|^2\frac{dxdz}{\lambda} \le \lambda^2\int_D |\nabla T(x,z,t)|^2\frac{dxdz}{\lambda}
$$

we deduce

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_D |\omega(x,z,t)|^2\frac{dxdz}{\lambda} + \sigma\int_D |\nabla\omega(x,z,t)|^2\frac{dxdz}{\lambda}\\
&\le \frac{\sigma}{2}\int_D \left|\frac{\partial\omega(x,z,t)}{\partial x}\right|^2\frac{dxdz}{\lambda} + \frac{\sigma\rho^2\lambda^2}{2}\int_D |\nabla T(x,z,t)|^2\frac{dxdz}{\lambda}.
\end{aligned}
\tag{3.19}
$$

Integrating (3.19) in time we deduce

$$
\frac{1}{t}\int_0^t ds\int_D |\nabla\omega(x,z,s)|^2\frac{dxdz}{\lambda} \le \rho^2\lambda^2 N(t) + \frac{1}{\sigma t}\|\omega_0\|_{L^2}^2. \tag{3.20}
$$

Let us represent the function w in terms of its Fourier series

$$
w(x,z,t) = \sum_{k\in\frac{2\pi}{\lambda}\mathbf{Z}} w_k(z,t)e^{ikx}
$$

and note that, in view of incompressibility, $w_0(z,t)$ is independent of z, and hence the boundary conditions at $z \pm \infty$ imply that

$$
w_0(z,t) = 0.
$$

In view of (2.18), the embedding inequality

$$
\|w(\cdot,t)\|_{L^\infty} \le C\lambda^{3/2}\|\nabla\omega(\cdot,t)\|_{L^2} \tag{3.21}
$$

follows. The constant C is absolute (recall that the L^2 norm is normalized by λ (3.1)); it can be computed either by using Parseval's identity or using (5.16) below. From (3.21) and (3.20) we deduce (3.17), using the Cauchy-Schwartz inequality for integration in time.

Theorem 1 *Solutions of (2.16), (2.17), (2.18) with front-like initial data obey*

$$
N(t) \le \frac{C^2(m+2)^2}{M^2}\rho^2\lambda^5 + \frac{4m+6}{M} + \left(\frac{K_1}{\sqrt{t}} + \frac{K_2}{t}\right), \tag{3.22}
$$

$$
W(t) \le \frac{C^2(m+2)}{M}\rho^2\lambda^5 + C\sqrt{\frac{4m+6}{M}}\rho\lambda^{5/2} + \frac{K_3}{t^{\frac{1}{4}}} + \frac{K_4}{t^{\frac{1}{2}}} \tag{3.23}
$$

and

$$\limsup_{t\to\infty} \overline{V}(t) \leq 2 + \frac{C^2(m+2)}{M}\rho^2\lambda^5 + C\sqrt{\frac{4m+6}{M}}\rho\lambda^{5/2} \tag{3.24}$$

with M, m given in (2.9), C the absolute constant of (3.21) and with K_1, K_2, K_3, K_4 depending on the initial data .

Proof. We insert (3.17) into the right hand side of (3.14). We obtain

$$N(t) \leq \frac{m+2}{M}C\rho\lambda^{5/2}\sqrt{N(t)} + \frac{C\lambda^{3/2}(m+2)}{M\sqrt{\sigma t}}\|\omega_0\|_{L^2} + \frac{2m+3}{M} + \frac{\Gamma}{t}.$$

Now we use Young's inequality

$$\frac{m+2}{M}C\rho\lambda^{5/2}\sqrt{N(t)} \leq \frac{N(t)}{2} + \frac{C^2(m+2)^2}{2M^2}\rho^2\lambda^5$$

and deduce (3.22). Then (3.23) follows from (3.22) and (3.17), while (3.24) follows from (3.23) and (3.11).

Remarks. 1. In terms of the original dimensional variables x_{old} of (2.1),(2.2),(2.3) the bound on the wave number (3.22) means that the Nusselt number on scale L

$$Nu = \limsup_{t\to\infty} \frac{1}{t}\int_0^t\int_0^L\int_{-\infty}^{\infty} |\nabla T(x,z,s)|^2 dxdzds = \lambda \limsup_{t\to\infty} N(t)$$

is bounded as

$$Nu \leq \frac{C^2(m+2)^2}{M^2}Ra^2 + \frac{4m+6}{M}\lambda \tag{3.25}$$

with Ra the Rayleigh number on scale L:

$$Ra = \frac{gAL^3}{\nu\kappa}. \tag{3.26}$$

2. Numerical evidence [22] shows that the $\overline{V}(t)$ is indeed an increasing function of λ and ρ.

4 Nonlinear stability of planar fronts in narrow domains

In this section we consider the reaction-diffusion Boussinesq problem (2.16) - (2.18) in a narrow domain, i.e. for small aspect ratio λ. The nonlinearity f is of either one of the three types: KPP, ignition or bistable. We keep the time and space units chosen above (2.11, 2.13, 2.14, 2.15). We prove two results. The first one concerns traveling solutions of the form

$$T(x,z,t) = T(x, z-ct), \quad v(x,z,t) = v(x, z-ct). \tag{4.1}$$

The result states that, if the Rayleigh number ρ is sufficiently small, then such solutions must be planar fronts. Planar fronts are solutions of the form

$$T(x,z,t) = P(z-ct), \quad v(x,z,t) = 0 \tag{4.2}$$

for which T does not depend on x and v vanishes. Planar fronts do exist for $c \geq 2$ in the KPP nonlinearity case and for a unique front speed c_* in the ignition and bistable cases.

Theorem 2 *There exist constants $C_1 > 0$ and $C_2 > 0$ such that if $\lambda < C_1$, and $\rho < C_2/\lambda^3$, then the only solutions of (2.16)-(2.18), (2.21, 2.22) of traveling front type (4.1), are planar fronts of the form (4.2).*

The second result in this section is about arbitrary solutions. We show that all solutions of the Boussinesq system in a narrow domain eventually become planar:

Theorem 3 *There exist constants $C_1 > 0$ and $C_2 > 0$ so that if $\lambda < C_1$ and $\rho < C_2/\lambda^3$, then*

$$\|\omega(\cdot,t)\|_{L^2} + \|T_x(\cdot,t)\|_{L^2} \to 0 \quad \text{as } t \to +\infty. \tag{4.3}$$

Moreover, the front speed is uniformly bounded:

$$\limsup_{t\to+\infty} \bar{V}(t) \le 2. \tag{4.4}$$

Proof of Theorem 2. Let $T(x, z-ct)$, $v(x, z-ct)$ be a traveling front solution of (2.16)-(2.18), then

$$\begin{aligned} -cT_z + v\cdot\nabla T &= \Delta T + f(T) \\ -cv_z + v\cdot\nabla v + \nabla p &= \sigma\Delta v + \sigma\rho T\hat{e}_z \\ -c\omega_z + v\cdot\nabla\omega &= \sigma\Delta\omega + \sigma\rho T_x. \end{aligned} \tag{4.5}$$

We multiply the second equation in (4.5) by v and integrate to obtain

$$\sigma\|\nabla v\|^2 = \sigma\rho\int T(x,z)w(x,z)dxdz.$$

In this section we will use the plain L^2 norm

$$\|g\|^2 = \int_D |g(x,z)|^2 dxdz.$$

The right side may be bounded as in the proof of Lemma 4 to obtain

$$\sigma\|\nabla v\|^2 = \sigma\rho\int T(x,z)w(x,z)dxdz \le \sigma\rho\|T - \overline{T}\|\|w\| \le \sigma\rho\lambda\|T_x\|\|w\|, \tag{4.6}$$

where

$$\overline{T}(z) = \int_0^\lambda T(x,z,t)\frac{dx}{\lambda}.$$

We used above the fact that $w(x,z)$ has mean zero in x because of incompressibility and boundary conditions at $z \to \infty$. Furthermore, it satisfies the Poincaré inequality so that $\|w\| \le C\lambda\|\nabla v\|$, and the incompressibility of v implies that $\|\nabla v\| = \|\omega\|$. Therefore (4.6) implies that

$$\|w\| \le C\rho\lambda^3\|T_x\|$$

and

$$\|\omega\| \le C\rho\lambda^2\|T_x\|. \tag{4.7}$$

We differentiate the first equation in (4.5) in x, multiply by T_x and integrate to obtain:

$$-\int T_x(v_x \cdot \nabla T)dxdz = \int |\nabla T_x|^2 dxdz - \int f'(T)T_x^2 dxdz. \tag{4.8}$$

We use the Poincaré inequality for T_x to obtain a bound for the right side:

$$\int |\nabla T_x|^2 dxdz - \int f'(T)T_x^2 dxdz \geq \left(1 - C\lambda^2\right) \int |\nabla T_x|^2 dxdz \geq \frac{1}{2}\|\nabla T_x\|^2.$$

if $\lambda \leq C_1$ is sufficiently small, with C_1 an absolute constant that may depend only on $m_1 = \max_{0 \leq T \leq 1} |f'(T)|$. The left side of (4.8) is bounded by

$$\left|\int T_x(v_x \cdot \nabla T)dxdz\right| = \left|\int T(v_x \cdot \nabla T_x)dxdz\right| \leq \|v_x\|\|\nabla T_x\| \leq \|\omega\|\|\nabla T_x\|.$$

The last two bounds imply that $\|\nabla T_x\| \leq 2\|\omega\|$ for a sufficiently small λ. Then the inequality (4.7) and the Poincaré inequality for T_x imply that

$$\|\nabla T_x\| \leq C\rho\lambda^2\|T_x\| \leq C\rho\lambda^3\|\nabla T_x\|.$$

Therefore there exists a constant $C_0 > 0$ such that no x-dependent traveling front exists if

$$\rho \leq \frac{C_0}{\lambda^3}. \tag{4.9}$$

This finishes the proof of Theorem 2. We now prove Theorem 3.

Proof of Theorem 3. We multiply the evolution equation for $v(x,z,t)$ by $v = (u,w)$ and integrate to obtain

$$\frac{1}{2}\frac{d}{dt}\|v(\cdot,t)\|^2 + \sigma\|\nabla v(\cdot,t)\|^2 = \sigma\rho\int T(x,z,t)w(x,z,t)dxdz. \tag{4.10}$$

The right side of the above equation may be bounded as in (4.6). We integrate (4.10) in time, and use (4.6) and the Poincaré inequality for $w(t,x,z)$ to get

$$\begin{aligned}
&\|w(\cdot,t)\|^2 + \frac{C\sigma}{\lambda^2}\int_0^t \|w(\cdot,s)\|^2 ds \leq \sigma\rho\lambda\int_0^t \|T_x(\cdot,s)\|\|w(\cdot,s)\|ds + \|v(\cdot,0)\|^2 \qquad (4.11)\\
&\leq \sigma\rho\lambda\left(\int_0^t \|T_x(\cdot,s)\|^2 ds\right)^{1/2}\left(\int_0^t \|w(\cdot,s)\|^2 ds\right)^{1/2} + \|v(\cdot,0)\|^2\\
&\leq \frac{\sigma\rho\lambda}{2}\left(\frac{C}{\rho\lambda^3}\int_0^t \|w(\cdot,s)\|^2 ds + \frac{\rho\lambda^3}{C}\int_0^t \|T_x(\cdot,s)\|^2 ds\right) + \|v(\cdot,0)\|^2
\end{aligned}$$

Therefore we have

$$\int_0^t \|w(\cdot,s)\|^2 ds \leq C\rho^2\lambda^6\int_0^t \|T_x(\cdot,s)\|^2 ds + \frac{C\lambda^2}{\sigma}\|v(\cdot,0)\|^2.$$

Furthermore, once again, after integrating (4.10) in time we obtain

$$\int_0^t \|\omega(\cdot,s)\|^2 ds \leq \rho\lambda\int_0^t \|T_x(\cdot,s)\|\|w(s)\|ds \leq C\rho^2\lambda^4\int_0^t \|T_x(s)\|^2 ds + \frac{C}{\sigma}\|v(\cdot,0)\|^2. \tag{4.12}$$

Furthermore, we differentiate the first equation in (2.17) in x and obtain

$$T_{xt} + v \cdot \nabla T_x + v_x \cdot \nabla T = \Delta T_x + f'(T) T_x. \tag{4.13}$$

Then, as in the proof of Theorem 2, provided that $\lambda < C_1$ is sufficiently small, if we multiply (4.13) by T_x, and integrate, we obtain, using the Poincaré inequality for T_x:

$$\frac{1}{2}\frac{d}{dt}\left(\|T_x\|^2\right) + C\|\nabla T_x\|^2 \le \|v_x\|\|\nabla T_x\|.$$

This implies that

$$\frac{1}{2}\frac{d}{dt}\left(\|T_x\|^2\right) + \frac{C'}{2\lambda^2}\|T_x\|^2 \le \frac{C''}{2}\|\omega\|^2$$

and hence

$$\|T_x(\cdot,t)\|^2 \le \|T_x(\cdot,0)\|^2 e^{-C't/\lambda^2} + C''\int_0^t \|\omega(\cdot,s)\|^2 e^{-C'(t-s)/\lambda^2} ds. \tag{4.14}$$

We integrate (4.14) and obtain

$$\int_0^t \|T_x(\cdot,s)\|^2 ds \le C\lambda^2\|T_x(\cdot,0)\|^2 + C\int_0^t\int_0^s \|\omega(\cdot,s_1)\|^2 e^{-C'(s-s_1)/\lambda^2} ds_1 ds$$
$$\le C\lambda^2\|T_x(\cdot,0)\|^2 + C\lambda^2\int_0^t \|\omega(\cdot,s)\|^2 ds.$$

We use now the bound (4.12) to get

$$\int_0^t \|T_x(\cdot,s)\|^2 ds \le C\lambda^2\|T_x(\cdot,0)\|^2 + \frac{C\lambda^2}{\sigma}\|v(\cdot,0)\|^2 + C\rho^2\lambda^6\int_0^t \|T_x(s)\|_2^2 ds. \tag{4.15}$$

Therefore there exists a constant $C_0 > 0$ so that if (4.9) holds, then

$$\int_0^t \|T_x(\cdot,s)\|^2 ds \le C\lambda^2\|T_x(\cdot,0)\|^2 + \frac{C\lambda^2}{\sigma}\|v(\cdot,0)\|^2.$$

Then (4.9) and (4.12) imply that

$$\int_0^t \|\omega(\cdot,s)\|^2 ds \le C\rho^2\lambda^4\left[C\lambda^2\|T_x(\cdot,0)\|^2 + \frac{C\lambda^2}{\sigma}\|v(\cdot,0)\|^2\right] + \frac{C}{\sigma}\|v(\cdot,0)\|^2 \tag{4.16}$$
$$\le C\|T_x(\cdot,0)\|^2 + \frac{C}{\sigma}\|v(\cdot,0)\|^2.$$

The bounds (4.14) and (4.16) together imply in an elementary way that

$$\|T_x(\cdot,t)\| \to 0 \text{ as } t \to +\infty.$$

We multiply the vorticity equation (2.16) by ω and integrate in space:

$$\frac{1}{2}\frac{d}{dt}\left(\|\omega(\cdot,t)\|^2\right) + \|\nabla\omega(\cdot,t)\|^2 \le \sigma\rho\|\omega(\cdot,t)\|\|T_x(\cdot,t)\|. \tag{4.17}$$

Then we have

$$\|\omega(\cdot,t)\|^2 \le \|\omega(\cdot,\tau)\|^2 + 2\sigma\rho \int_\tau^t \|\omega(\cdot,s)\|\|T_x(\cdot,s)\|ds$$
$$\le \|\omega(\cdot,\tau)\|^2 + 2\sigma\rho \left(\int_\tau^\infty \|\omega(\cdot,s)\|^2 ds\right)^{1/2} \left(\int_\tau^\infty \|T_x(\cdot,s)\|^2 ds\right)^{1/2}.$$

The uniform bound (4.16) implies that there exists a sequence of times $\tau_k \to +\infty$ so that $\|\omega(\cdot,\tau_k)\| \to 0$. Then we have for $t > \tau_k$:

$$\|\omega(\cdot,t)\|^2 \le \|\omega(\cdot,\tau_k)\|^2 + 2\sigma\rho \left(\int_{\tau_k}^\infty \|\omega(\cdot,s)\|^2 ds\right)^{1/2} \left(\int_{\tau_k}^\infty \|T_x(\cdot,s)\|^2 ds\right)^{1/2}$$

and hence $\|\omega(\cdot,t)\| \to 0$ as $t \to +\infty$ because of (4.15) and (4.16). In order to prove (4.4) we note that (4.17) implies that

$$\int_{t_1}^{t_2} \|\nabla\omega(\cdot,s)\|^2 ds \le \int_{t_1}^{t_2} \sigma\rho\|\omega(\cdot,s)\|\|T_x(\cdot,s)\|ds + \frac{1}{2}\|\omega(\cdot,t_1)\|^2$$

so that (4.3) implies that for any $\varepsilon > 0$ we have for t_1, $t_2 > t_1 + 1$ sufficiently large we have

$$\int_{t_1}^{t_2} \|\nabla\omega(\cdot,s)\|^2 ds \le \varepsilon(t_2 - t_1).$$

Therefore, (3.11), (3.21) and the Cauchy-Schwartz inequality imply that

$$\bar{V}(t) \le 2 + \frac{1}{t}\int_0^t \|w(s)\|_\infty ds + o(1) \le 2 + \frac{C}{\sqrt{t}}\left(\int_0^t \|\nabla\omega(\cdot,s)\|^2 ds\right)^{1/2} + o(1) \le 2 + \varepsilon + o(1)$$

and (4.4) follows. This finishes the proof of Theorem 3.

5 Linear instability

In the previous section we established the stability of planar fronts with respect to short wavelength perturbations. We analyze now the linear instability of planar front with respect to large wavelength perturbations.

We perform a Galilean transformation $z \mapsto z - v_0 t$ in (2.1)-(2.3) following the flat front. We write $T(x,z,t) = \tau(z - v_0 t) + \theta(x, z - v_0 t, t)$, and, with a slight abuse of notation, $v(x,z,t) = v(x, z - v_0 t, t)$. We also linearize equations (2.1)-(2.3), dropping the terms that are quadratic in θ and v. We obtain the linearized system

$$\frac{\partial\theta}{\partial t} - v_0\partial_z\theta - \kappa\Delta\theta - \frac{v_0^2}{4\kappa}f'(\tau(z))\theta = -w\tau' \tag{5.1}$$
$$\frac{\partial u}{\partial t} - v_0\partial_z u - \nu\Delta u + \nabla p = gA\theta e_z \tag{5.2}$$

and for the vorticity we get the equation

$$\frac{\partial\omega}{\partial t} - v_0\partial_z\omega - \nu\Delta\omega = gA\partial_x\theta. \tag{5.3}$$

Using the same reaction units (2.11)-(2.15) as before, equations (5.1), (5.3) become

$$\frac{\partial \theta}{\partial t} - 2\frac{\partial \theta}{\partial z} - \Delta\theta - f'(P(z))\theta = -wP'(z) \tag{5.4}$$

and

$$\frac{\partial \omega}{\partial t} - 2\frac{\partial \omega}{\partial z} - \sigma\Delta\omega = \sigma\rho\frac{\partial \theta}{\partial x} \tag{5.5}$$

The nonlinear equations, in the same units and frame of reference, are

$$\frac{\partial \omega}{\partial t} + v\cdot\nabla\omega - 2\frac{\partial \omega}{\partial z} - \sigma\Delta\omega = \sigma\rho\frac{\partial T}{\partial x} \tag{5.6}$$

and

$$\frac{\partial T}{\partial t} + v\cdot\nabla T - 2\frac{\partial T}{\partial z} - \Delta T = f(T). \tag{5.7}$$

We wish to show that flat profiles cannot remain linearly stable with respect to sufficiently large wavelength infinitesimal perturbations. We start for simplicity with the case of the infinite Prandtl number. We retain (5.4) but we use the infinite Prandtl number limit of (5.5),

$$-\Delta\omega = \rho\frac{\partial \theta}{\partial x}, \tag{5.8}$$

which, together with (2.18), allows us to write the active scalar rule

$$w = -\rho(\partial_x)^2(-\Delta)^{-2}\theta. \tag{5.9}$$

We express $\theta(x,z,t)$ in terms of its Fourier series:

$$\theta(x,z,t) = \sum_{k\in\frac{2\pi}{\lambda}\mathbf{Z}} g_k(z,t)e^{ikx}. \tag{5.10}$$

The linearized temperature equation (5.4) transforms into

$$\frac{\partial g_k}{\partial t} - 2\frac{\partial g_k}{\partial z} + \left(k^2 - \partial_{zz}\right)g_k - f'(P)g_k = \rho QKg_k \tag{5.11}$$

with $k = \pm\frac{2\pi}{\lambda}, \pm 2\frac{2\pi}{\lambda}, \dots$, the operator K defined by the Fourier transform of (5.9)

$$Kg = k^2\left(k^2 - \partial_{zz}\right)^{-2} g \tag{5.12}$$

and

$$Q(z) = -P'(z) > 0. \tag{5.13}$$

We take a positive wave number

$$k = \frac{2\pi n}{\lambda}. \tag{5.14}$$

The operator K defined by (5.12) is given explicitly by a convolution with a positive function

$$(Kg)(z) = \frac{1}{4k}\int_{-\infty}^{\infty}(1 + k|z-\zeta|)e^{-k|z-\zeta|}g(\zeta)d\zeta. \tag{5.15}$$

The expression (5.15) is obtained by an elementary calculation, starting from

$$\left(k^2 - \partial_{zz}\right)^{-1} g(z) = \frac{1}{2k}\int_{-\infty}^{\infty} e^{-k|z-\zeta|} g(\zeta)d\zeta \tag{5.16}$$

and iterating. It is well known that the profile P is decreasing in the case of KPP, bistable and ignition nonlinearities [24], so that the function Q is positive. Moreover

$$Q(z) \geq ae^{-b|z|} \tag{5.17}$$

holds for all z, with $a > 0$ and $b > 0$ absolute numbers that depend only on the nonlinearity $f(T)$. The positivity of Q and the positivity of the kernel of the operator K in the right hand side of (5.11) imply that the solution of the initial value problem (5.11) remains nonnegative if the initial data is nonnegative. Let us consider a function $\phi(z)$ which has the properties

$$e^{-k|z|} \leq \phi(z) \leq Ce^{-k|z|} \tag{5.18}$$

with $C > 1$ and

$$|\phi'(z)| \leq Cke^{-k|z|}, \quad |\phi''(z)| \leq Ck^2e^{-k|z|}. \tag{5.19}$$

Such a function is obtained for instance by gluing smoothly $3e^{1-z}$ on $z \geq 1$ to $3e^{1+z}$ on $z \leq -1$ by a function bounded below by 2 on the interval $[-1, 1]$. Then one rescales $z \mapsto kz$. We multiply the equation (5.11) by $\phi(z)$ and integrate. Using the properties of ϕ and the positivity of $g_k(z,t)$ we obtain

$$\frac{d}{dt}\int \phi(z)g_k(z,t)dz \geq -\nu_k \int \phi(z)g_k(z)dz + \rho \int \phi(z)Q(z)(Kg_k)(z)dz \tag{5.20}$$

This follows from integration by parts, and bounds (5.19) on the first two derivatives of ϕ by constant multiples of ϕ. The constant ν_k obeys

$$\nu_k \leq 2C(1 + k + k^2), \tag{5.21}$$

so it is uniformly bounded for bounded k, for instance for $0 \leq k \leq 1$. Now we bound from below the term

$$\rho \int \phi(z)Q(z)(Kg_k)(z)dz \geq \frac{\rho}{4k}\int\int ae^{-b|z|}\phi(z)e^{-k|z|-k|\zeta|}g_k(\zeta,t)d\zeta dz. \tag{5.22}$$

We used (5.17), (5.15) and the positivity of g_k. We neglected the nonnegative term contributed by $k|z-\zeta|$ in (5.15). Now we use (5.18):

$$\rho \int \phi(z)Q(z)(Kg_k)(z)dz \geq \frac{\rho}{4k}\frac{a}{C}\left(\int e^{-(b+2k)|z|}dz\right)\left(\int \phi(\zeta)g_k(\zeta,t)d\zeta\right). \tag{5.23}$$

We obtain the ordinary differential inequality

$$\frac{d}{dt}\int \phi(z)g_k(z)dz \geq \left(\frac{\rho}{4k}\frac{a}{2C(b+2k)} - \nu_k\right)\int \phi(z)g_k(z)dz, \tag{5.24}$$

and thus $\|g_k\|_{L^1(\mathbb{R})}$ grows exponentially in time. Therefore we have the following theorem for the infinite Prandtl number case:

Theorem 4 *Let $P(z-2t)$, $u=0$ be a planar, x-independent traveling front solution of the infinite Prandtl number Boussinesq system*

$$\frac{\partial T}{\partial t} + v\cdot\nabla T - \Delta T = f(T) \tag{5.25}$$

with

$$-\Delta v + \nabla p = \rho T e_z, \quad \nabla\cdot u = 0, \tag{5.26}$$

with front boundary conditions for T at $z=\pm\infty$, vanishing velocity at $z=\pm\infty$ and periodic boundary conditions in x of period λ. There exists a positive constant $\beta>0$ such that, if

$$\rho\lambda > \beta, \tag{5.27}$$

then the solution P is linearly unstable. This means that there exist infinitesimal perturbations which grow exponentially, when viewed in a Galilean frame of reference moving with the traveling front. Their exponential growth rate is proportional to $\rho\lambda$.

A similar instability analysis can be applied to a convective system in which the infinite Prandtl number equation (5.8) is replaced by Darcy's law. The results are consistent with the recent numerical study [10].

We return to the system (5.4, 5.5) at a finite Prandtl number. We use the Fourier expansion

$$w(x,z,t) = \sum_{n\in\mathbf{Z}} w_k(z,t)e^{2\pi i n x/\lambda} \tag{5.28}$$

and using (2.18) we have, with $k = 2\pi n/\lambda$,

$$w_k(z,t) = -\frac{i}{2}\int e^{-k|z-z'|}\omega_k(z',t)dz' \tag{5.29}$$

with the obvious notation for the Fourier coefficients of ω. An elementary calculation involving solving the linear equation (5.5) starting with the zero initial data $\omega(x,z,0)=0$, and using (2.18), gives

$$w_k(z,t) = \frac{\sigma\rho k}{2}\int_0^t\int g_k(z_2,s)I_k(z-z_2+2(t-s),t-s)dz_2ds \tag{5.30}$$

where $g_k(z,t)$ are the Fourier coefficients of θ as in (5.10) and

$$I_k(\zeta,\tau) = e^{-\sigma k^2\tau}\int e^{-k|\zeta+u\sqrt{2\sigma\tau}|}e^{-\frac{|u|^2}{2}}\frac{du}{\sqrt{2\pi}}. \tag{5.31}$$

We now bound I_k at this point as follows:

$$I_k(\zeta,\tau) \ge e^{-\sigma k^2\tau}\int e^{-k|\zeta|-k|u|\sqrt{2\sigma\tau}}e^{-\frac{|u|^2}{2}}\frac{du}{\sqrt{2\pi}} \ge 2e^{-k|\zeta|}\mathrm{Erf}(k\sqrt{2\sigma\tau}) \tag{5.32}$$

with

$$\mathrm{Erf}(a) = \int_a^\infty e^{-\frac{u^2}{2}}\frac{du}{\sqrt{2\pi}}. \tag{5.33}$$

For $a \geq 1$ we will use just a piece of this integral,

$$\mathrm{Erf}(a) \geq \int_a^{2a} e^{-\frac{u^2}{2}} \frac{du}{\sqrt{2\pi}} \geq \frac{1}{\sqrt{2\pi}} e^{-a^2} \tag{5.34}$$

and for $a \leq 1$ we write

$$\mathrm{Erf}(a) \geq \mathrm{Erf}(1). \tag{5.35}$$

This allows us to deduce, for later use

$$I_k(\zeta, \tau) \geq 2(1 - \frac{1}{\sqrt{2\pi}})e^{-k|\zeta|}e^{-2\sigma k^2 \tau}. \tag{5.36}$$

The evolution equation for g_k follows from (5.4),

$$\frac{\partial g_k}{\partial t} - 2\frac{\partial g_k}{\partial z} + (k^2 - \partial_{zz})g_k - f'(P)g_k = \sigma\rho Q w_k \tag{5.37}$$

with w_k computed using (5.30) and with $Q = -P'$ as before. Because of the explicit expression (5.30) we see that if the initial data $g_k(z, 0)$ is real and positive, then $g_k(z, t)$ remains real and positive. We take such initial data. Then, using (5.36) and the triangle inequality we get,

$$w_k(z, t) \geq (1 - \frac{1}{\sqrt{2\pi}})\rho\sigma k e^{-k|z|} \int_0^t e^{-2(t-s)k(\sigma k + 2)} \left[\int g_k(z_2, s)e^{-k|z_2|}dz_2 \right] ds. \tag{5.38}$$

As in the case of the infinite Prandtl number, we take a function $\phi(z)$ with properties (5.18), (5.19), multiply (5.37) by ϕ and integrate. Let us denote

$$Y(t) = \int \phi(z)g_k(z, t)dz. \tag{5.39}$$

Integrating by parts we get

$$\frac{dY}{dt} + C_3(1 + k + k^2)Y \geq C_4\sigma\rho k \int_0^t e^{-2(t-s)k(\sigma k+2)}Y(s)ds. \tag{5.40}$$

with C_3 and C_4 positive numbers that depend on the properties of ϕ. Consider also

$$Z(t) = e^{2k(\sigma k+2)t}Y(t). \tag{5.41}$$

Multiplying by $e^{2k(\sigma k+2)t}$, we get from (5.40)

$$\frac{dZ}{dt} + \beta Z \geq \alpha \int_0^t Z(s)ds \tag{5.42}$$

with

$$\beta = C_3(1 + k + k^2) - 2k(\sigma k + 2) \tag{5.43}$$

and

$$\alpha = C_4\sigma\rho k. \tag{5.44}$$

It is clear that solutions of the differential inequality (5.42) are larger than solutions of the ODE

$$\frac{dy}{dt} + \beta y = \alpha \int_0^t y(s)ds \tag{5.45}$$

with the same initial data. This ODE is solved differentiating one more time,

$$\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} - \alpha y = 0 \tag{5.46}$$

and seeking solutions of the form $e^{\mu t}$. Equation (5.46) always has at least one exponentially growing solution because $\alpha > 0$ (irrespective of the sign of β). The general solution of (5.46) is

$$y(t) = y_1 e^{t\mu_+} + y_2 e^{t\mu_-} \tag{5.47}$$

with

$$\mu_\pm = \frac{1}{2}\left(-\beta \pm \sqrt{\beta^2 + 4\alpha}\right). \tag{5.48}$$

The solutions of (5.45) correspond to the linear subspace for which $\frac{dy}{dt}(0) + \beta y(0) = 0$. Choosing without loss of generality the coefficient of the growing exponential $y_1 = 1$, we deduce the relation

$$\mu_+ + \beta + y_2(\mu_- + \beta) = 0 \tag{5.49}$$

In order to have exponentially growing positive solutions we need the initial datum for (5.45) to be positive. This initial datum is $1 + y_2$ and is computed using (5.48) and (5.49)

$$1 + y_2 = \frac{2\sqrt{\beta^2 + 4\alpha}}{\sqrt{\beta^2 + 4\alpha} - \beta} \tag{5.50}$$

so it is always positive. Starting with this initial condition (or any positive multiple thereof) we get exponential growth for $Z(t)$. This will imply exponential growth for $Y(t)$ if

$$\mu_+ > 2k(\sigma k + 2). \tag{5.51}$$

This turns out to be the condition

$$C_4\sigma\rho > 2C_3(1 + k + k^2)(\sigma k + 2) \tag{5.52}$$

or, putting $C = \frac{2C_3}{C_4}$,

$$\rho > C\left(\frac{2}{\sigma} + k\right)\left(1 + k + k^2\right) \tag{5.53}$$

Theorem 5 *Planar reactive Boussinesq fronts (5.6, 5.7) are linearly unstable to large wavelength perturbations whenever the local Rayleigh number ρ based on the laminar front thickness is large compared to the inverse of the Prandtl number,*

$$\rho > \frac{2C}{\sigma}.$$

Perturbations with wave numbers k satisfying (5.53) grow exponentially in a frame of reference moving with the planar front. The growth rate is proportional to $\sqrt{\sigma\rho k}$.

Remark. The exponential growth rate proportional to the square root of the wave number is a signature of the Rayleigh-Taylor instability, operating here only at large scales. When the wavelength of the initial perturbation is decreased to a length comparable to the thickness of the planar front, the perturbation decays in time.

6 Conclusions

We proved that front-like solutions of the reactive Boussinesq system with concave KPP reaction in a strip have bounded bulk burning rate. Front acceleration does not occur in this system. The concavity of the nonlinearity was used in the proof. For small aspect ratios and for small Rayleigh numbers, the only traveling modes are planar, and all front-like solutions become planar. For large enough Rayleigh numbers, if the aspect ratio is large, then the planar fronts lose stability to longwave perturbations. The instability is of Rayleigh-Taylor type. The results proved here agree with the recent numerical study [22].

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