Bulk Burning Rate in Passive - Reactive Diffusion.

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Abstract

We consider a passive scalar that is advected by a prescribed mean zero divergence-free velocity field, diffuses, and reacts according to a KPP-type nonlinear reaction. We introduce a quantity, the bulk burning rate, that makes both mathematical and physical sense in general situations and extends the often ill-defined notion of front speed. We establish rigorous lower bounds for the bulk burning rate that are linear in the amplitude of the advecting velocity for a large class of flows. These "percolating" flows are characterized by the presence of tubes of streamlines connecting distant regions of burned and unburned material and generalize shear flows. The bound contains geometric information on the velocity streamlines and degenerates when these oscillate on scales that are finer than the width of the laminar burning region. We give also examples of very different kind of flows, cellular flows with closed streamlines, and rigorously prove that these can produce only sub-linear enhancement of the bulk burning rate.

1 Introduction

Quite often mixtures of reactants interact in a burning region that has a rather complicated spatial structure but is thin across. This reaction region moves towards the unburned reactants leaving behind the burned ones. When the reactants are carried by an ambient fluid then the burning rate may be enhanced. The physical reason for this observed speed-up is believed to be that fluid advection tends to increase the area available for reaction.

Many important engineering applications of combustion operate in the presence of turbulent advection, and therefore the influence of advection on burning has been studied extensively by physicists, engineers and mathematicians. In the physics literature one can find a number of models and approaches that yield different predictions – relations between the turbulent intensity and the burning rate ([8, 21, 20, 41]). These results are usually obtained using heuristic models and physical reasoning. For a recent review of some of the physics literature we refer to [28, 30].

The key question we wish to address is: what characteristics of the ambient fluid flow are responsible for burning rate enhancement? The question needs first to be made precise, because the reaction region may be complicated and, in general, may move with an ill-defined velocity.

In this paper we will define in an unambiguous fashion a quantity V representing the bulk burning rate. V makes both mathematical and physical sense in general; we study its relation to the advecting velocity field in a simple model. We provide explicit estimates of V in terms of the magnitude of the advecting velocity and the geometry of streamlines. We are mostly interested in the regime where the advection is strong but our estimates are valid for all values of physical parameters, and do not involve any passage to limit. They are also valid for certain advection velocities without symmetry. In situations where traveling waves are known to exist, the estimates we derive provide automatically bounds for the speed of the traveling waves.

The main result of this paper is the identification of a class of flows that are particularly effective in speeding up the bulk burning rate. We call these "percolating flows" because their main feature is the presence of tubes of streamlines connecting distant regions of burned and unburned material. For such flows we obtain an optimal linear enhancement bound

$$V \geq KU$$

where U represents the magnitude of the advecting velocity and K is a proportionality factor that depends on the geometry of streamlines but not the speed of the flow. Other flows and in particular cellular flows, which have closed streamlines, on the other hand, may produce a weaker enhancement.

We will take an analytic approach. Numerical work concerning our results will be published elsewhere. We consider a well-established simple model, the passive-reactive diffusive scalar equation

$$\frac{\partial T}{\partial t} + u(x,y,t)\cdot\nabla T = \kappa\Delta T + \frac{v_0^2}{4\kappa} f(T). \tag{1}$$

In this equation $0 \leq T \leq 1$ represents normalized temperature, u the advecting velocity, κ thermal diffusivity, f the reaction and v_0 the laminar front speed. The advecting velocity is divergence-free and prescribed. No feedback of T on u is allowed in this simple model: T is passive. The normalization is such that the reaction rate is $\frac{v_0^2}{4\kappa}$. This is chosen so that, in the absence of advection ($u = 0$) and given (2) below, there will exist reaction-diffusive laminar traveling wave fronts that move with speed at least v_0 [23].

The equation (1), derived under assumptions of approximately constant density [8] and approximately unity Lewis number (e.g. [5]) is also used to model problems in biology ([13]), chemistry, and has other applications ([28]) but certainly does not capture all the physical instabilities present in turbulent combustion ([32, 33]).

The type of nonlinearity $f(T)$ we consider in this paper is concave KPP:

$$f \in C^2,\ \ f(0) = f(1) = 0,\ \ f'(0) = 1,\ \ f''(x) < 0. \tag{2}$$

The prototype non-linearity of the KPP type is $f(T) = T(1-T)$, called KPP after pioneering work of Kolmogorov, Petrovskii and Piskunov [23].

Other important types of nonlinearity $f(T)$ are the Arrhenius-type

$$f(T) = (1-T)e^{-\frac{A}{T}}$$

and ignition type

$$f(T) = 0 \text{ for } T \notin (\theta, 1),\quad f(T) > 0 \text{ for } T \in (\theta, 1),\quad \theta \in (0,1).$$

The mathematical study of equation (1) concentrated mainly on two issues: existence of traveling waves and asymptotic speed, and the homogenization regime $\kappa \to 0$. Traveling waves in one dimension with $u = 0$ were studied in the classical works [23] and [14] with their global asymptotic analysis addressed later in [19] for the ignition nonlinearity, in [1] in higher dimensions, and in [37] for variable diffusivity and ignition nonlinearity. Traveling waves with $u \neq 0$ were shown to exist for shear flows for KPP as well as for a more general class of nonlinearities [4, 6, 7]. Their stability was established in [5, 31, 25]. Finally, traveling waves for periodic flows $u(x, y)$ and ignition nonlinearity as well as their stability were studied in [38, 39]. Probabilistic methods were applied for the analysis of the KPP fronts and proof of the existence of the asymptotic speed of propagation was given in [18] for periodic $u \neq 0$. To the best of our knowledge, until now there have been no explicit estimates on the speed of propagation of traveling waves or asymptotic speed of propagation with $u \neq 0$, except for the perturbative small u result of [27].

The homogenization regime $\kappa \to 0$, when the front width goes to zero, was extensively studied for KPP-type nonlinearity and for advection velocity that is periodic and varies either on the integral scale [15, 16, 17] or on a small scale that is larger or comparable to that of the front width [18, 24, 10, 11, 26]. Recently a similar result was established for random statistically homogeneous ergodic advection velocities [34]. A thorough review of most of these results, both for traveling waves and homogenization techniques is given in [40]. The result of homogenization procedures is an effective equation valid in the limit $\kappa \to 0$. The effective equation is typically a non-trivial Hamilton-Jacobi equation [10, 11, 24]. Homogenization is usually very efficient when a mean field captures the essence of the question asked. When that is not the case important information is lost in the limit.

For simplicity of exposition we will consider the reaction-diffusion-advection equation (1) in a two-dimensional strip

$$\Omega = \{x, y : \ x \in (-\infty, \infty), \ y \in [0, H]\},$$

but the methods we introduce work in any dimension and for more general classes of domains. The boundary conditions are either Neumann

$$T_y(x, 0) = T_y(x, H) = 0, \tag{3}$$

or periodic in y:

$$T(x, y, t) = T(x, y + H, t). \tag{4}$$

The flow $u = (u_1, u_2)$ is incompressible:

$$\nabla \cdot u = 0 \tag{5}$$

and has zero normal component at the boundary

$$u \cdot n = 0 \quad \text{on} \quad \partial\Omega \tag{6}$$

in the case of Neumann boundary conditions. Furthermore, we assume that the total flow through the strip is zero:

$$\int_0^H u_1(x, y) dy = 0 \tag{7}$$

to eliminate the drift caused by the mean flow. We assume $1 \geq T \geq 0$, $T = 1$ being the stable (burned) state of the system, while $T = 0$ is unstable (unburned) state. The equation has maximum principle [29], so if the initial data is in the $[0, 1]$ range then the solution remains in the same range for all times. We also assume that the solution is localized, that is

$$T(x, y, t) = 1 - O(e^{\lambda x}) \ \text{ for } \ x < 0, \ \ T(x, y, t) = O(e^{-\lambda x}) \ \text{ for } \ x > 0, \tag{8}$$

$$|\nabla T| = O(e^{-\lambda |x|}) \ \text{ for some } \ \lambda > 0. \tag{9}$$

If such conditions are satisfied initially then they are valid for all subsequent times (see Section 2).

We are interested in a general situation, when traveling waves solutions may not be relevant or not exist.

We introduce a natural quantity that measures the typical burning rate

$$V(t) = \int\limits_{\Omega} T_t(x,y,t)\frac{dxdy}{H}.$$

We call $V(t)$ the (instantaneous) "bulk burning rate" and its time average

$$\langle V\rangle_t = \frac{1}{t}\int\limits_0^t V(s)ds, \quad \langle V\rangle_\infty = \liminf_{t\to\infty}\langle V\rangle_t$$

simply "bulk burning rate". Because T is non-dimensional, V has units of length per time, i.e. of velocity. Note that when $T(x,y,t)$ is a traveling wave front-like solution $T(x-ct,y,t)$, then V is the speed of the front, $V(t)=c$. But V is defined for very general initial data and more general equations and does not require assumptions about the nature of the burning region. The word "bulk" refers to the fact that we take a space (or space-time) average, capturing only bulk or large scale effects.

Our first result shows that no matter what the advecting velocity is, it cannot slow down the bulk burning rate below a universal lower bound, of the same order of magnitude as the laminar front speed. Let us denote

$$\alpha = -\inf_{0\le T\le 1} f'(T) > 0, \quad \beta = -\sup_{0\le T\le 1} f''(T) > 0.$$

(Note $\alpha=1, \beta=2$ for KPP).

Theorem 1 *There exists a constant $C>0$ such that for any advecting velocity $u(x,y,t)$ satisfying (5), (6) and (7), any solution of the passive-reactive diffusion equation (1) with boundary conditions (3) and (8) or (4), (8), the bulk burning rate $V(t)$ obeys the lower bound*

$$V(t) \ge C v_0\sqrt{\frac{\beta}{4\alpha}}(1-e^{-\alpha v_0^2 t/2\kappa}). \tag{10}$$

One of the applications of this theorem is in a homogenization regime, where the reaction is very weak (see Appendix A).

As far as the general upper bound is concerned, it is easy to show (see Section 2) for a very general class of velocities $u(x,y,t)$ that if the initial data $T_0(x,y)$ satisfies (8) with $\lambda = v_0/2\kappa$, then

$$\langle V\rangle_t \le \frac{L_0}{t} + \|u_1\|_\infty + v_0, \tag{11}$$

with the constant length L_0 depending on the initial data T_0 only. Here $\|u_1\|_\infty$ denotes, as usual, the supremum of $|u_1|$ over the whole domain. Therefore, the bulk burning rate may not exceed a linear bound in the amplitude of the advecting velocity. For a large class of flows we prove lower bounds on the bulk burning rate that are linear in the magnitude of advection. For instance, a corollary to Theorem 4 of Section 4 concerning mean zero shear flow of the form

$$u(x,y) = (u(y),0), \quad \int_0^H u(y)dy = 0$$

can be stated simply as

Theorem 2 *There exists a constant $C > 0$ that depends only on the nonlinearity f but not upon $u(y)$ and T_0 such that, for any solution $T(x,y,t)$ of the passive-reactive diffusion equation (1) with boundary conditions (3), (8) or (4), (8) and any $\tau \geq \tau_0 = max[\frac{\kappa}{v_0^2}, \frac{H}{v_0}]$ the bulk burning rate obeys*

$$\langle V \rangle_\tau \geq C \left(1 + \frac{\kappa^2}{v_0^2 h_u^2}\right)^{-1} \frac{\|u\|_1^2}{\|u\|_\infty} \tag{12}$$

where $\|u\|_1 = \int\limits_0^H |u(y)| \frac{dy}{H}$ *and* $h_u = \dfrac{\|u\|_1}{\|u'\|_\infty}$.

Recall that the normalization of the reaction rate in (1) is chosen so that the laminar traveling wave front speed is v_0 no matter what κ is. If we allow κ to vary while keeping the coefficient M in front of the reaction term fixed, we find that the bound is still independent of κ, however the time τ_0 in which it is reached behaves as $\kappa^{-1/2}$.

Actually, we prove a far more general geometric estimate (Section 4), from which (12) follows. This general estimate provides a non-trivial lower bound also for the case when the ratio of L^1 and L^∞ norms of $u(y)$ becomes small. An important feature of the bound (12) is the presence of the ratio of the characteristic scale h_u of variations of the advection velocity and the reaction scale $l = \kappa/v_0$. The estimate degenerates in the case $h_u \ll l$ which is expected from physical considerations since additional wrinkling on the scales smaller than the reaction scale should not accelerate the front.

Furthermore, we consider time dependent shear flows, and obtain a similar lower bound on the bulk burning rate:

$$\langle V \rangle \geq K(h_u, \tau_*, \frac{u}{|u|})\|u\|_\infty.$$

Here τ_* is a typical time scale of the flow, defined similarly (but not identically) to h_u. The prefactor K becomes smaller when either h_u becomes smaller than the reaction scale $l = \kappa/v_0$, or the time scale τ_* is faster than the reaction time $\tau_c = \kappa/v_0^2$ or the time $\tau_H = H/v_0$ it takes the reaction to traverse the cross-section. The precise formulation of our result for the time dependent shear flows is given in Section 5.

Finally, in Section 6 we consider a generalization of the time independent shear flows. Namely, we consider "percolating flows", flows that have two or more (sufficiently regular) tubes of streamlines connecting $x = -\infty$ and $x = +\infty$. These flows are not necessarily spatially periodic and can have completely arbitrary features outside the tubes of streamlines. We show that the bulk burning rate is still linear in the magnitude of the advecting velocity, no matter what kind of behavior (closed streamlines, areas of still fluid, etc.) the flow has outside the tubes. The proportionality coefficient depends on the geometry of the flow. Thus, we identify a broad class of the flows which increase the bulk burning rate linearly with the amplitude of the flow, the fastest possible rate of increase. We also show that in general the dependence of the bulk burning rate $V(t)$ on the magnitude of the advecting velocity may be sub-linear. An extreme example is provided by shear flow perpendicular to the front (in periodic boundary conditions) where there is no significant enhancement: the bulk burning rate remains uniformly bounded as the magnitude of the advecting velocity tends to infinity. Other examples are certain cellular flows (flows with closed streamlines): for every $\alpha \in (0,1)$ we construct cellular flows for which $V(t)$ is bounded above by $C\|u\|_\infty^\alpha$ (Section 7). A comparison of the results of this paper with extensive numerical studies will be presented in a companion paper with Fausto Cattaneo, Andrea Malagoli and Natalia Vladimirova [9].

For the rest of this paper, we assume that the reaction $f(T)$ satisfies (2), the initial data satisfy (8), (9), and the advection velocity satisfies $\|u_1\|_\infty < \infty$, $\|\nabla u\|_\infty < \infty$. As we will see in the next section, such assumptions on u ensure that the localization (8), (9) of the initial data is preserved during the time evolution. Throughout the paper we denote by C various (not necessarily equal) constants which may depend only on reaction $f(T)$.

2 Preliminaries and an upper bound on the bulk burning rate

Our considerations in this section follow the general ideas of [7]. We show in this section that the boundary conditions (8) are conserved by evolution and establish the simple upper bound (11).

Lemma 1 *Assume that the initial data $T_0(x,y)$ satisfies the following bounds:*

$$T(x,y) \leq C_0 e^{-\lambda x}, \quad 1-T(x,y) \leq C_0 e^{\lambda x}, \quad |\nabla T| \leq \frac{C_0}{H} e^{-\lambda |x|}, \quad C_0 > 0, \ \lambda > 0. \tag{13}$$

Let $c_1 \geq \|u_1\|_\infty + \kappa\lambda + \frac{v_0^2}{4\kappa\lambda}$, and $c_2 \geq \|u_1\|_\infty + \kappa\lambda + \frac{v_0^2}{2\kappa\lambda} + \frac{4\|\nabla u\|_\infty}{\lambda}$. Then

$$T(t,x,y) \leq C_0 e^{-\lambda_0(x-c_1 t)}, \quad 1-T(x,y) \leq C_0 e^{\lambda_0(x+c_1 t)}, \tag{14}$$

$$|\nabla T| \leq \frac{C_0}{H} e^{\mp\lambda(x \mp c_2 t)} \quad for \quad any \quad t. \tag{15}$$

Proof. The proof is an application of the maximum principle. Note that T satisfies an inequality

$$T_t + u\cdot\nabla T - \kappa\Delta T - \frac{v_0^2}{4\kappa} T \leq 0.$$

Introduce $\phi(x,t) = e^{-\lambda(x-c_1 t)}$, then

$$\phi_t + u\cdot\nabla\phi - \kappa\Delta\phi - \frac{v_0^2}{4\kappa}\phi = \lambda(c - u_1 - \kappa\lambda - \frac{v_0^2}{4\kappa\lambda})\phi \geq 0$$

by our assumptions on c. Applying the maximum principle to the function $w = C_0\phi - T$ we obtain the first estimate in (14). To get the second estimate we note that $G = 1-T$ satisfies the inequality

$$G_t + u\cdot\nabla G - \kappa\Delta G \leq 0.$$

We let then $\psi(x,t) = e^{\lambda_0(x+c_1 t)}$ and proceed as before applying maximum principle to $w_1 = C_0\psi - G$.

The decay of $|\nabla T|$ is obtained as follows. Let $P = |\nabla T|^2$, then P satisfies the equation

$$P_t + u\cdot\nabla P - \kappa\Delta P + 2(|\nabla T_x|^2 + |\nabla T_y|^2) = \frac{v_0^2}{2\kappa} f'(T)P - 2(T_x u_x\cdot\nabla T + T_y u_y\cdot\nabla T).$$

Therefore if we let $K = \frac{v_0^2}{2\kappa} + 4\|\nabla u\|_\infty$, then we get

$$P_t + u\cdot\nabla P - \kappa\Delta P - KP \leq 0$$

and then (15) follows as before.□

Lemma 1 implies that the bulk burning rate cannot be larger than $c_1 = v_0 + \|u_1\|_\infty$ provided that the initial data decays fast enough.

Theorem 3 *Assume that $T_0(x,y) \leq C_0 e^{-\lambda x}$ and $1 - T_0(x,y) \leq C_0 e^{\lambda x}$ with $\lambda = \dfrac{v_0}{2\kappa}$, then*

$$\langle V\rangle_t \leq \frac{4C_0\kappa}{v_0 t} + v_0 + \|u_1\|_\infty. \tag{16}$$

Proof. We have

$$\langle V\rangle(t)=\frac{1}{t}\int_0^t ds\int\frac{dxdy}{H}T_s(s,x,y)=\frac{1}{t}\int\frac{dxdy}{H}[T(x,y,t)-T_0(x,y)].$$

Lemma 1 implies that $T(x,y,t)$ satisfies the bound

$$T(x,y,t)\le C_0e^{-\lambda(x-c_1t)},$$

with $c_1=\|u_1\|_\infty+\kappa\lambda+\frac{v_0^2}{4\kappa\lambda}$. Then we have

$$\begin{aligned}\langle V\rangle(t) &\le \frac{1}{t}\int_0^H\frac{dy}{H}\int_{-\infty}^0((1-T_0)-(1-T))\,dx+\frac{1}{t}\int_0^H\frac{dy}{H}\int_0^{c_1t}[T(x,y,t)-T_0(x,y)]\\ &+ \frac{1}{t}\int_0^H\frac{dy}{H}\int_{c_1t}^\infty T\,dx\le\frac{2C_0}{t\lambda}+c_1\le\frac{4C_0\kappa}{tv_0}+\|u_1\|_\infty+v_0\end{aligned}$$

with our choice of $\lambda=\frac{v_0}{2\kappa}$.$\Box$

A simple corollary of Theorem 3 is that a shear flow in the direction perpendicular to the front propagation does not enhance the bulk burning rate.

3 A universal lower bound for bulk burning rate

In this section we prove Theorem 1. Let us integrate (1) over the set Ω and obtain

$$V(t)=\frac{v_0^2}{4\kappa}\int\limits_\Omega f(T)\frac{dxdy}{H}. \tag{17}$$

Integration by parts is justified for any t by Lemma 1. A straightforward computation that uses (1), the boundary conditions (3) or (4), (8), (9), and the incompressibility of $u(x,y)$ shows that

$$\frac{dV}{dt}\ge\frac{\beta v_0^2}{4}\int\limits_\Omega|\nabla T|^2\frac{dxdy}{H}-\frac{\alpha v_0^2}{4\kappa}V. \tag{18}$$

The two simple equations (17) and (18) are the basis of our technique for deriving lower bounds on the burning velocity. The temperature drops from 1 on the left to 0 on the right. The reaction $f(T)$ is large where T takes values in some region strictly between 0 and 1. If T varies slowly, we get a good lower bound for $V(t)$ from (17). On the other hand, if T varies fast, then the $|\nabla T|^2$ term in (18) is large, which will give a lower bound on the bulk burning rate. The equations (17) and (18) are thus complementary in this respect. The following Lemma gives precise meaning to the statement that the reactive term in (17) and the gradient term in (18) cannot be simultaneously small:

Lemma 2 *Let $f(T)$ be a function of concave KPP type (2) and assume that the continuously differentiable function $T(x,y)$ satisfies the following assumptions:*

(i) $0\le T(x,y)\le 1$,

(ii) $\lim\limits_{x\to-\infty}T(x,y)=1,\ \lim\limits_{x\to+\infty}T(x,y)=0$ *for every* $y\in[0,1]$,

Then there exists a constant $C>0$ (independent of the function T) such that

$$\int\limits_\Omega f(T)dxdy\int\limits_\Omega|\nabla T|^2dxdy\ge CH^2. \tag{19}$$

Proof. We may assume that $\int\limits_\Omega |\nabla T|^2 dxdy < \infty$ and $\int\limits_\Omega f(T(x,y))dxdy < \infty$, otherwise (19) is trivial. Let $y \in (0,1)$ be such that

$$\int\limits_{-\infty}^{\infty} |\nabla T(x,y)|^2 dx \le 3 \int\limits_\Omega |\nabla T|^2 \frac{dxdy}{H}$$

and

$$\int\limits_{-\infty}^{\infty} f(T(x,y))dx \le 3 \int\limits_\Omega f(T(x,y)) \frac{dxdy}{H}.$$

Then there exist x_1, x_2 such that $|T(x_1,y) - T(x_2,y)| \ge 1-\epsilon$ while $f(T(x,y)) \ge C\epsilon$ for all $x \in (x_1,x_2)$ because of the boundary conditions (ii) and property (2) of the non-linearity. Then we have

$$C\epsilon |x_1 - x_2| \le 3 \int\limits_\Omega f(T(x,y)) \frac{dxdy}{H}$$

and

$$\frac{(1-\epsilon)^2}{|x_1 - x_2|} \le 3 \int\limits_\Omega |\nabla T|^2 \frac{dxdy}{H}.$$

Multiplying these two equations we obtain

$$\int\limits_\Omega f(T)dxdy \int\limits_\Omega |\nabla T|^2 dxdy \ge \frac{C\epsilon(1-\epsilon)^2 H^2}{9},$$

which proves Lemma 2. □

Lemma 2, (17), and (18) imply that

$$\frac{dV}{dt} + \frac{\alpha v_0^2}{4\kappa} V \ge C \frac{\beta v_0^4}{16\kappa V}.$$

Therefore

$$V^2(t) \ge \frac{C\beta v_0^2}{4\alpha} + e^{-\alpha v_0^2 t/(2\kappa)} \left[V^2(0) - \frac{C\beta v_0^2}{4\alpha} \right],$$

from which Theorem 1 follows.

We remark that a simple variation of Lemma 2 allows to prove Theorem 1 for more general domains than a strip.

4 Bulk burning rate in shear flows

Let us consider the passive-reactive diffusion equation in a shear flow in two dimensions:

$$T_t + u(y)T_x = \kappa \Delta T + \frac{v_0^2}{4\kappa} f(T). \tag{20}$$

The boundary and initial conditions are as in (3) or (4) and (8), (9). The flow $u(y)$ is continuously differentiable and has mean zero (a non-zero mean can be taken into account by a simple change of variables):

$$\int_0^H u(y)dy = 0.$$

We prove now an estimate for the bulk burning rate which is more general than the one presented in Theorem 2.

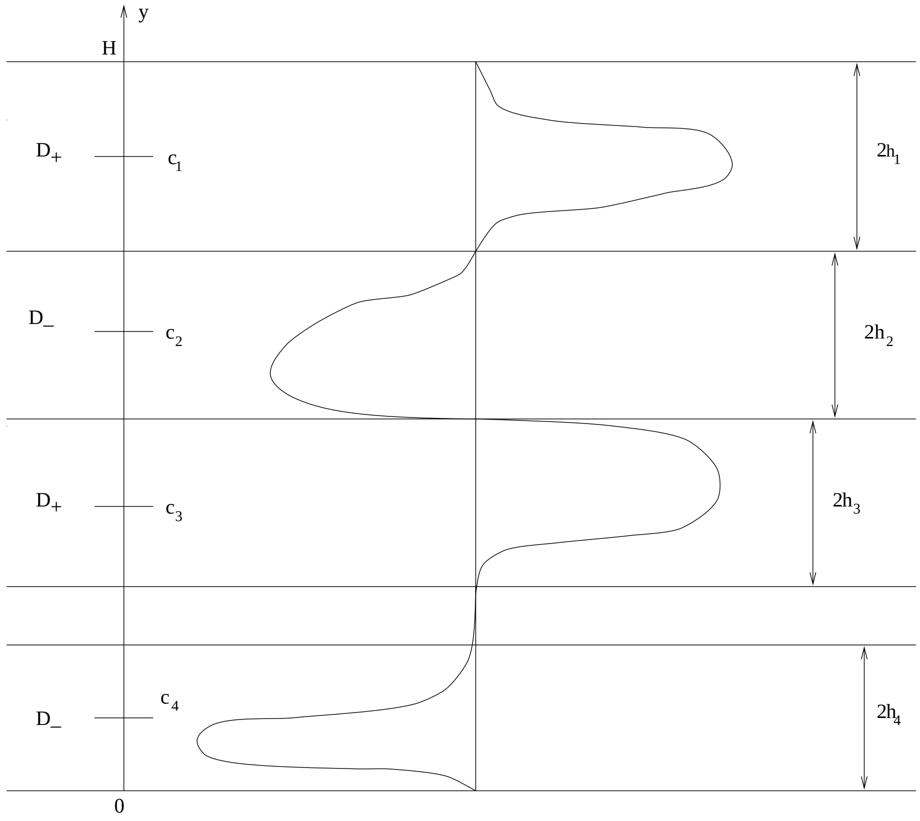

Figure 1: The structure of the shear flow.

Theorem 4 *Let us consider an arbitrary partition of the interval $[0, H]$ into subintervals $I_j = [c_j - h_j, c_j + h_j]$ on which $u(y)$ does not change sign. Denote by D_-, D_+ the unions of intervals I_j where $u(y) > 0$ and $u(y) < 0$ respectively (see figure 1). Then there exists a constant $C > 0$, independent of the partition, $u(y)$, and the initial data $T_0(x, y)$, so that the average burning rate $\langle V \rangle_\tau$ satisfies the following estimate:*

$$\langle V \rangle_\tau \geq C \left(c_+ \sum_{I_j \subset D_+} \left(1 + \frac{l^2}{h_j^2}\right)^{-1} \int\limits_{c_j - \frac{h_j}{2}}^{c_j + \frac{h_j}{2}} |u(y)| \frac{dy}{H} + c_- \sum_{I_j \subset D_-} \left(1 + \frac{l^2}{h_j^2}\right)^{-1} \int\limits_{c_j - \frac{h_j}{2}}^{c_j + \frac{h_j}{2}} |u(y)| \frac{dy}{H} \right) \qquad (21)$$

for any $\tau \geq \tau_0 = max \left[\frac{\kappa}{v_0^2}, \frac{H}{v_0}\right]$. Here $l = \kappa / v_0$. The constants $c_\pm$ are defined by

$$c_\pm = \left(\sum_{I_j \subset D_\mp} \frac{h_j^3}{h_j^2 + l^2} \right) \left(\sum_{I_j} \frac{h_j^3}{h_j^2 + l^2} \right)^{-1}.$$

Remarks. 1. The exact choice of the unions of intervals $D_\pm$ is left to us. Given a shear flow, we should attempt to pick $D_\pm$ in a way to maximize the bound (21). We provide a simple example after the proof which illustrates how this bound works.
2. The proof of this bound becomes much easier, and also extends to more general types of reaction, if we assume $T_t(x, y, t) \geq 0$. This is the case for traveling waves, the existence of which has been proven for shear flows and various types of reaction in [7]. The minimal speed of traveling waves

also provides lower bounds for asymptotic speed of propagation of any front-like data [25, 39]. We plan to address the results which one can prove along these lines in subsequent publications. The disadvantage of an a priori assumption $T_t \geq 0$, however, is that we cannot get an estimate on the time required to reach the lower bound from any initial data, and cannot extend the results to time-dependent flows, as we do in Section 5.

Proof. The plan of the proof is as follows. We know that the bulk burning rate $V(t)$ satisfies the inequalities

$$V(t) + \frac{4\kappa}{\alpha v_0^2}\frac{dV}{dt} \geq \frac{\beta\kappa}{\alpha}\int_\Omega |\nabla T|^2 dx dy. \tag{22}$$

and

$$V(t) = \frac{v_0^2}{4\kappa}\int_\Omega f(T)\frac{dxdy}{H}. \tag{23}$$

We will be able to get a bound from below in terms of u for the combination of the terms on the right-hand sides of these equations, which then provide a bound for $\langle V\rangle_\tau$. The key will be to integrate in x along the streamlines of the flow and then to average several times in y and t to bring T_{yy} and T_t terms to a form convenient for the estimate.

Consider one interval of D_+, $I_j = [c_j - h_j, c_j + h_j]$, and so that $u(y) > 0$ for $y \in (c_j - h_j, c_j + h_j)$. Integrating (20) over $x \in \mathbb{R}$, we obtain

$$\int_{\mathbb{R}} T_t\, dx - \kappa \int_{\mathbb{R}} T_{yy}\, dx - \frac{v_0^2}{4\kappa}\int_{\mathbb{R}} f(T)\, dx = u(y). \tag{24}$$

Therefore

$$\int_{\mathbb{R}} T_t\, dx - \kappa\int_{\mathbb{R}} T_{yy}\, dx \geq u(y). \tag{25}$$

(It might appear that we make the estimate rather crude by dropping the $f(T)$ term. There is however heuristic and numerical evidence that in many situations this term is insignificant in the regions where $u(y) > 0$, and that the front is quite sharp in these regions. In contrast, in the regions where $u(y) < 0$ the burning region is wider and the $f(T)$ term will not be discarded there.) Now we estimate both terms on the left hand side of (25). Let us begin with the second derivative term. To reduce the order of differentiation, we employ the following averaging in y :

$$\int_0^{h_j/2} d\gamma \int_{h_j/2-\gamma}^{h_j/2+\gamma} d\delta \int_{c_j-\delta}^{c_j+\delta} \cdot\, dy = \int_{c_j-h_j}^{c_j+h_j} G(h_j, y - c_j)\cdot\, dy, \tag{26}$$

where the kernel $G(h,\xi)$ can be computed explicitly as

$$G(h,\xi) = \begin{cases} \frac{1}{2}(h-|\xi|)^2 - \left(\frac{h}{2} - |\xi|\right)^2, & |\xi| < h/2 \\ \frac{1}{2}(h-|\xi|)^2, & h/2 \leq |\xi| < h \end{cases} \tag{27}$$

Observe that the function $G(h,\xi)$ has the following properties

$$\begin{aligned} &G(h,\xi) \geq 0 \ \text{ for all } \ \xi \in [-h,h] \\ &G(h,\xi) \leq \frac{h^2}{4} \ \text{ for } \ \xi \in [-h,h] \\ &G(h,\xi) \geq \frac{h^2}{8} \ \text{ for } \ \xi \in [-\frac{h}{2},\frac{h}{2}]. \end{aligned} \tag{28}$$

Let us apply the averaging procedure (26) to the equation (25). We have

Lemma 3

$$\left|\int_{\mathbb{R}} dx \int_{c_j-h_j}^{c_j+h_j} G(h_j, y-c_j) T_{yy}(x,y)\, dy\right| \le C\left(h_j^2 \int_{\mathbb{R}} dx \int_{c_j-h_j}^{c_j+h_j} |\nabla T|^2\, dy + \int_{\mathbb{R}} dx \int_{c_j-h_j}^{c_j+h_j} f(T)\, dy\right). \qquad (29)$$

Proof. We are going to split $x \in \mathbb{R}$ into two sets. In one set, the L^2_y norm of the gradient will be large and we will use it to estimate the second derivative term. In the other set, the variation of the temperature will be small, and we will use the reactive term to bound the second derivative term. More precisely, let ρ be a number such that $\sqrt{2\rho}h_j = 1/3$ and define the set $\mathcal{D}_{j\rho} \subset \mathbb{R}$ by

$$\mathcal{D}_{j\rho} = \left\{x \in \mathbb{R} : \int_{c_j-h_j}^{c_j+h_j} |\nabla T|^2(x,y) dy \ge \rho h_j\right\}$$

so that

$$\int_{c_j-h_j}^{c_j+h_j} dy |T_y(x,y)| \le \sqrt{\frac{2}{\rho}} \int_{c_j-h_j}^{c_j+h_j} |\nabla T(x,y)|^2 dy = 6h_j \int_{c_j-h_j}^{c_j+h_j} |\nabla T(x,y)|^2 dy \qquad (30)$$

for $x \in \mathcal{D}_{j\rho}$. Notice that for such x, according to (26),

$$\int_{c_j-h_j}^{c_j+h_j} G(h_j, y-c_j) T_{yy}(x,y)\, dy = \int_0^{h_j/2} d\gamma \int_{h_j/2-\gamma}^{h_j/2+\gamma} d\delta (T_y(x, c_j+\delta) - T_y(x, c_j-\delta)) \qquad (31)$$

$$\le 3h_j^2 \int_{c_j-h_j}^{c_j+h_j} |\nabla T(x,y)|^2 dy.$$

For x outside $\mathcal{D}_{j\rho}$, we use the representation

$$\int_{c_j-h_j}^{c_j+h_j} G(h_j, y-c_j) T_{yy}(x,y)\, dy = \int_{h_j/2<|y-c_j|\le h_j} T(x,y) dy - \int_{|y-c_j|\le h_j/2} T(x,y) dy. \qquad (32)$$

We need the following crucial observation:

Lemma 4 *Assume that*

$$|T(x,y_1) - T(x,y_2)| \le \frac{1}{3},$$

then we have

$$|T(x,y_1) - T(x,y_2)| \le C(f(T(x,y_1)) + f(T(x,y_2))$$

for any $y_1, y_2 \in (c_j - h_j, c_j + h_j)$.

Proof. Let us denote $T_1 = T(x,y_1)$, $T_2 = T(x,y_2)$, then we have using (2)

$$f(T_1) + f(T_2) \ge \inf_{T\in(|T_1-T_2|, 1-|T_1-T_2|)} f(T) \ge C|T_1 - T_2|.$$

This proves Lemma 4. □

Notice that if $x \notin \mathcal{D}_{j\rho}$ we have

$$\int_{c_j-h_j}^{c_j+h_j} |T_y(x,y)| dy \le \sqrt{2\rho} h = \frac{1}{3}$$

and therefore $|T(x,y_1)-T(x,y_2)|<1/3$ for any $y_1,y_2\in(c_j-h_j,c_j+h_j)$. Applying Lemma 4 to (32), we get

$$\left|\int\limits_{\mathbb{R}} dx \int\limits_{c_j-h_j}^{c_j+h_j} G(h_j,y-c_j)T_{yy}(x,y)\,dy\right| \le C\int\limits_{\mathbb{R}} dx \int\limits_{c_j-h_j}^{c_j+h_j} f(T)\,dy.$$

This completes the proof of Lemma 3. □

Averaging equation (25) and applying Lemma 3 we obtain

$$\begin{aligned}\int\limits_{c_j-h_j}^{c_j+h_j} dyu(y)G(h_j,y-c_j) &\le \int\limits_{\mathbb{R}} dx \int\limits_{c_j-h_j}^{c_j+h_j} dyG(h_j,y-c_j)T_t(x,y) \\ &+C\left(h_j^2\kappa\int\limits_{\mathbb{R}} dx\int\limits_{c_j-h_j}^{c_j+h_j} dy|\nabla T(x,y)|^2+\kappa\int\limits_{\mathbb{R}} dx\int\limits_{c_j-h_j}^{c_j+h_j} dyf(T(x,y))\right).\end{aligned} \tag{33}$$

Next we consider the intervals where velocity $u(y)\le 0$. These are analyzed similarly, except that now we do not discard the last term on the right side in (24). The estimate analogous to (33) is

$$\begin{aligned}\int\limits_{c_j-h_j}^{c_j+h_j} dy|u(y)|G(h_j,y-c_j) &\le \frac{v_0^2}{4\kappa}\int\limits_{\mathbb{R}} dx \int\limits_{c_j-h_j}^{c_j+h_j} dyG(h_j,y-c_j)f(T(x,y)) \\ &-\int\limits_{\mathbb{R}} dx \int\limits_{c_j-h_j}^{c_j+h_j} dyG(h_j,y-c_j)T_t(x,y) \\ &+C\left(h_j^2\kappa\int\limits_{\mathbb{R}} dx\int\limits_{c_j-h_j}^{c_j+h_j} dy|\nabla T(x,y)|^2+\kappa\int\limits_{\mathbb{R}} dx\int\limits_{c_j-h_j}^{c_j+h_j} dyf(T(x,y))\right).\end{aligned} \tag{34}$$

Thus we succeeded in replacing the second order derivative term with expressions directly linked to the burning rate. Now it remains to estimate the time derivative term. Summing (33) and (34) over $I_j\subset D_+$ and $I_j\subset D_-$ respectively, and using properties (28) of the kernel G, we get

$$\begin{aligned}&\kappa\int\limits_{D_+} dy\int\limits_{\mathbb{R}} dx|\nabla T(x,y,t)|^2+\frac{v_0^2}{4\kappa}\int\limits_{D_+} dy\int\limits_{\mathbb{R}} dxf(T(t,x,y)) \\ &+\sum_{I_j\subset D_+}\int\limits_{I_j} dy\int\limits_{\mathbb{R}} dx\frac{G(h_j,y-c_j)}{h_j^2+l^2}T_t(x,y,t)\ge C\sum_{I_j\subset D_+}\left(1+\frac{l^2}{h_j^2}\right)^{-1}\int\limits_{c_j-\frac{h_j}{2}}^{c_j+\frac{h_j}{2}}|u(y)|dy\end{aligned} \tag{35}$$

and

$$\begin{aligned}&\kappa\int\limits_{D_-} dy\int\limits_{\mathbb{R}} dx|\nabla T(x,y,t)|^2+\frac{v_0^2}{4\kappa}\int\limits_{D_-} dy\int\limits_{\mathbb{R}} dxf(T(t,x,y)) \\ &-\sum_{I_j\subset D_-}\int\limits_{I_j} dy\int\limits_{\mathbb{R}} dx\frac{G(h_j,y-c_j)}{h_j^2+l^2}T_t(x,y,t)\ge C\sum_{I_j\subset D_-}\left(1+\frac{l^2}{h_j^2}\right)^{-1}\int\limits_{c_j-\frac{h_j}{2}}^{c_j+\frac{h_j}{2}}|u(y)|dy\end{aligned} \tag{36}$$

(recall $l = \kappa/v_0$). Let us choose the weights m_+ and m_- according to

$$m_\pm = \sum_{I_j \subset D_\mp} \int_{I_j} dy \frac{G(h_j, y - c_j)}{h_j^2 + l^2} \tag{37}$$

Set also $M = \max(m_+, m_-)$. Notice that by the properties of G, we have

$$\frac{1}{16} c_\pm \le \frac{m_\pm}{M} \le \frac{1}{4} c_\pm \tag{38}$$

for constants $c_\pm$ in the formulation of the Theorem. Let us define measures

$$d\nu_\pm = \sum_{I_j \subset D_\pm} \frac{m_\pm \chi_{I_j}(y) G(h_j, y - c_j)}{MH(h_j^2 + l^2)} dy. \tag{39}$$

Here we denote, as usual, by χ_S the characteristic function of the set S. Multiplying (35) and (36) by m_+ and m_- respectively, and adding them together we obtain

$$\kappa \int_\Omega |\nabla T|^2 \frac{dxdy}{H} + \frac{v_0^2}{4\kappa} \int_\Omega f(T) \frac{dxdy}{H} + \int_{I\!R} dx \int_0^H d\nu_+(y) T_t - \int_{I\!R} dx \int_0^H d\nu_-(y) T_t \tag{40}$$
$$\ge C \left(\frac{m_+}{M} \sum_{I_j \subset D_+} \left(1 + \frac{l^2}{h_j^2}\right)^{-1} \int_{c_j - \frac{h_j}{2}}^{c_j + \frac{h_j}{2}} |u(y)| dy + \frac{m_-}{M} \sum_{I_j \subset D_-} \left(1 + \frac{l^2}{h_j^2}\right)^{-1} \int_{c_j - \frac{h_j}{2}}^{c_j + \frac{h_j}{2}} |u(y)| dy \right).$$

We have the following

Lemma 5 *For any τ_1, τ_2,*

$$\int_{\tau_1}^{\tau_2} dt \left(\int_{I\!R} dx \int_0^H d\nu_+(y) T_t - \int_{I\!R} dx \int_0^H d\nu_-(y) T_t \right)$$
$$\le C \sum_{i=1}^{2} \left(\frac{H\kappa}{v_0} \int_\Omega |\nabla T(x, y, \tau_i)|^2 \frac{dxdy}{H} + \left(1 + \frac{Hv_0}{\kappa}\right) \int_\Omega f(T(x, y, \tau_i)) \frac{dxdy}{H} \right).$$

Proof. By definition (39) of the measures $\nu_\pm$, their total weights are equal:

$$\int_0^H d\nu_+(y) = \int_0^H d\nu_-(y) < 1. \tag{41}$$

It is easy to construct a measure preserving bijective map $\Phi(y) : D_+ \to D_-$, so that $\nu_+(S) = \nu_-(\Phi(S))$. Then we can write

$$\int_{\tau_1}^{\tau_2} dt \left(\int_{I\!R} dx \int_0^H d\nu_+(y) T_t - \int_{I\!R} dx \int_0^H d\nu_-(y) T_t \right) \tag{42}$$
$$= \int_{I\!R} dx \int_0^H d\nu_+(y) (T(x, \Phi(y), \tau_1) - T(x, y, \tau_1)) + \int_{I\!R} dx \int_0^H d\nu_+(y) (T(x, y, \tau_2) - T(x, \Phi(y), \tau_2)).$$

Consider the first term on the left hand side of (42). We split all $x \in I\!R$ into two sets, saying $x \in S$ if there exists y such that

$$|T(x,y,\tau_1) - T(x,\Phi(y),\tau_1)| > \frac{1}{3}.$$

Using the same argument we applied in the proof of Lemma 2, one can show that

$$\int_0^H |\nabla T(x,y)|^2\,dy \int_0^H f(T(x,y))\,dy \geq C\left[\sup_{y_1,y_2\in[0,H]}|T(x,y_1)-T(x,y_2)|\right]^3,$$

where C is some universal constant, depending only on f. Therefore, for every $x \in S$,

$$\left(\int_0^H |\nabla T(x,y)|^2\,dy \int_0^H f(T(x,y))\right)^{\frac{1}{2}} \geq C,$$

and hence, using the fact that the total weight of ν_+ does not exceed 1 by (41), we have

$$\int_0^H d\nu_+(y)|(T(x,\Phi(y),\tau_1) - T(x,y,\tau_1))| \leq C\left(\frac{\kappa}{v_0}\int_0^H |\nabla T(x,y)|^2\,dy + \frac{v_0}{\kappa}\int_0^H f(T(x,y))\,dy\right). \quad (43)$$

For $x \notin S$, we have

$$\int_0^H d\nu_+(y)|T(x,y,\tau_1) - T(x,\Phi(y),\tau_1)| \leq C\int_0^H d\nu_+(y)[f(T(x,y)) + f(T(x,\Phi(y)))] \leq C\int_0^H f(T(x,y))\frac{dy}{H} \quad (44)$$

by Lemma 4 and (39). Equations (43) and (39) together imply Lemma 5. □

Theorem 4 now follows from Lemma 5, relations (22) and (23), and inequality (40). Given time interval $[0,\tau]$, apply the following averaging to the both sides of (40):

$$\frac{1}{\tau^3}\int_0^{\frac{\tau}{4}} d\gamma \int_{\frac{\tau}{4}-\gamma}^{\frac{\tau}{4}+\gamma} d\delta \int_{\frac{\tau}{2}-\delta}^{\frac{\tau}{2}+\delta} dt = \frac{1}{\tau^3}\int_0^{\tau} G(\frac{\tau}{2}, t-\frac{\tau}{2})\,dt. \quad (45)$$

A direct computation using (22) and (23) shows that the left hand side of (40) after averaging does not exceed

$$\frac{C}{\tau}\int_0^{\tau} V(t)\,dt\left(1 + \frac{H/v_0 + \kappa/v_0^2}{\tau} + \frac{H\kappa}{v_0^3\tau^2}\right). \quad (46)$$

On the other hand, the right hand side of (40) is independent of time, and averaging (45) results in multiplication by a constant independent of τ. Hence from (46) we see that if

$$\tau \geq \tau_0 = \max[\frac{\kappa}{v_0^2}, \frac{H}{v_0}],$$

then

$$\langle V\rangle_\tau \geq C\left(\frac{m_+}{M}\sum_{I_j\subset D_+}\left(1+\frac{l^2}{h_j^2}\right)^{-1}\int_{c_j-\frac{h_j}{2}}^{c_j+\frac{h_j}{2}} |u(y)|dy + \frac{m_-}{M}\sum_{I_j\subset D_-}\left(1+\frac{l^2}{h_j^2}\right)^{-1}\int_{c_j-\frac{h_j}{2}}^{c_j+\frac{h_j}{2}} |u(y)|dy\right).$$

This proves Theorem 4 (recall that $m_\pm$ and $c_\pm$ are related by (38)). $\square$

Remark. At the expense of making the proof slightly more technical, the total width of the strip H in the formula for τ_0 can be replaced by an often smaller value $\tilde{H}$, which is introduced as follows. Consider the function $g(y) = \nu_+[0,y] - \nu_-[0,y]$ on $[0,H]$. Then $\tilde{H}$ is defined as a maximal distance between two neighboring roots of g. It is straightforward to generalize the proof of Lemma 5 to yield this result, by taking a specific measure preserving function Φ which maps D_+ to D_- only within the intervals between the neighboring roots of g. The characteristic time $\frac{\tilde{H}}{v_0}$ has a clear intuitive physical meaning: this is the time needed to burn across the scale on which the shear flow u wrinkles the front.

Example. Let $u(y) = u_0 \sin \frac{2\pi n y}{H}$. We can take intervals I_j as half-periods of u where it does not change sign. Factors $c_\pm$ are equal in this case. Then Theorem 4 implies that for any $\tau \geq \tau_0 = \max[\frac{\kappa}{v_0^2}, \frac{H}{v_0}]$ we have

$$\langle V \rangle_\tau \geq C \left(1 + \frac{n^2 l^2}{H^2}\right)^{-1} u_0.$$

According to the above Remark, it is easy to see that in this example, H in definition of τ_0 can be replaced with $\tilde{H} = H/n$. The map Φ in this example can be taken to map half-periods where u is positive on the neighboring half-periods where u is negative.

Theorem 2 is a direct corollary of Theorem 4.
Proof. Let us denote by $|S|$ the Lebesgue measure of set S. Define the sets

$$\mathcal{F}_\pm = \left\{ y \in [0,H] : \ \pm u(y) \geq \frac{1}{4} \int_0^H |u(y)| \frac{dy}{H} = \frac{1}{4}\|u\|_1 \right\}.$$

Notice that since u is mean zero, $\|u_\pm\|_1 = \frac{1}{2}\|u\|_1$ (here $u_\pm$ are the positive and the negative parts of u). Therefore

$$\frac{1}{4}\|u\|_1 (H - |\mathcal{F}_\pm|) + \|u\|_\infty |\mathcal{F}_\pm| \geq \frac{1}{2}\|u\|_1 H,$$

and so

$$|\mathcal{F}_\pm| \geq \frac{\|u\|_1 H}{4\|u\|_\infty}. \tag{47}$$

Let $h_u = \dfrac{\|u\|_{L^1}}{\|u'\|_{L^\infty}}$, then for any $y \in \mathcal{F}_\pm$ and any $y' \in (y - h_u/8, y + h_u/8)$ we have $|u(y')| \geq \dfrac{1}{8}\|u\|_{L^1}$. It is easy to construct unions $D_\pm = \cup_j I_j^\pm$ of non-overlapping intervals $I_j^\pm = (y_j^\pm - h_u/8, y_j^\pm + h_u/8)$ with $y_j^\pm \in \mathcal{F}_\pm$, such that $|D_\pm| \geq \dfrac{1}{2}|\mathcal{F}_\pm|$. Then we have:

$$\int_{D_\pm} |u(y)| \frac{dy}{H} \geq \frac{1}{16} \frac{\|u\|_1}{H} |\mathcal{F}_\pm|. \tag{48}$$

Combining (47), (48), and (21) we get

$$\langle V \rangle_\tau \geq C \left(1 + \frac{l^2}{h_u^2}\right)^{-1} \frac{\|u\|_1^2}{\|u\|_\infty}. \qquad \square$$

5 Time dependent shear flows

One may expect linear growth of the bulk burning rate in the amplitude of the flow, but the temporal characteristic scale τ_* of the variations of the flow will play a role similar to that of the scale h_u in Theorem 2. That is, too rapid oscillations will diminish the enhancement of the bulk burning rate. Consider two systems of intervals I_j in $[0, H]$, D_+ and D_-. At this point we do not make any assumptions regarding the behavior of $u(y, t)$ on $D_\pm$. Let us introduce the notation

$$\begin{aligned} J(t,u) &= c_+ \sum_{I_j \in D_+} \left(1 + \frac{l^2}{h_j^2}\right)^{-1} \int\limits_{c_j - \frac{h_j}{2}}^{c_j + \frac{h_j}{2}} u(t,y) \frac{dy}{H} \\ &\quad - c_- \sum_{I_j \in D_-} \left(1 + \frac{l^2}{h_j^2}\right)^{-1} \int\limits_{c_j - \frac{h_j}{2}}^{c_j + \frac{h_j}{2}} u(t,y) \frac{dy}{H}, \end{aligned} \tag{49}$$

where $c_\pm$ are defined as in Theorem 4. Given a starting time t_0 and length of the time interval τ, we define

$$J(t_0, \tau, u) = \frac{1}{\tau^3} \int\limits_{t_0}^{t_0+\tau} G(\frac{\tau}{2}, t - t_0 - \frac{\tau}{2}) J(t,u)\, dt. \tag{50}$$

We also denote $\langle V \rangle_{t_0,\tau}$ the average of $V(t)$ over an interval of time of duration τ starting at time t_0 :

$$\langle V \rangle_{t_0,\tau} = \frac{1}{\tau} \int\limits_{t_0}^{t_0+\tau} V(t)\, dt.$$

We have

Theorem 5 *For any choice of the intervals $I_j \subset [0, H]$ in $D_\pm$ and any τ, t_0*

$$\langle V \rangle_{t_0,\tau} \geq C \left(1 + \left(\frac{\tau_0}{\tau}\right)^2\right)^{-1} J(t_0, \tau, u), \tag{51}$$

where $\tau_0 = max[\frac{\kappa}{v_0^2}, \frac{H}{v_0}]$.

Proof. The proof is a direct corollary of the proof of Theorem 4. We remark that the choice of the averaging in time (45) in the last stage of the proof is not the only one possible. Given a particular time dependent flow at some initial moment t_0, one can try to adjust the averaging procedure to get a better lower bound. □

Remark. Similarly to the remark after the proof of Theorem 4, we can replace H in the definition of τ_0 by a smaller value $\tilde{H}$ (defined in that remark).

To clarify the meaning of Theorem 5 we make several observations and consider two examples. The general way one can apply this theorem is as follows. Given a moment of time t_0, we try to choose τ and $D_\pm$ so as to maximize the lower bound (51). There is a certain tradeoff involved in choosing τ. If we take τ to be small, it is likely that we can find $D_\pm$ so that velocity $u(y, t)$ does not change sign there during time interval $[t_0, t_0 + \tau]$, staying positive on D_+ and negative on D_-. Then there is no cancellation in equation (50) defining $J(t_0, \tau, u)$. However, the factor $(1 + \frac{\tau_0^2}{\tau^2})^{-1}$

may become very small if $\tau \ll \tau_0$. If we take τ large, some cancellation is likely to occur in (50), making the bound weaker, unless the shear flow varies on time scales still larger than τ. In the flows which oscillate in time on the scale smaller than τ_0, we will not be able to avoid either cancellation in (50) or small factor $(\tau/\tau_0)^2$ in the bound (51), and will end up with weaker lower bound than we would have gotten if the flow varied slower in time. Notice that in any case the bound grows linearly with the amplitude of the flow, and there are factors reflecting the moderating effect of fast oscillations both in space and in time.

If we want to know the average of the bulk burning rate over a long period of time, much larger than the typical time scale of the flow, we can use Theorem 5 by splitting this long time period into appropriately chosen smaller ones and getting lower bounds on the averages over these smaller time intervals. Combined, they will also give us an estimate on the long time average.

Example 1. Consider a flow

$$u(y,t) = u_0 \sin 2\pi\omega t \sin \frac{2\pi n y}{H}.$$

To get an estimate on long-time average of the burning rate for such flow, consider $t_0 = 0$. Set $\tau = \frac{1}{2\omega}$, and take

$$D_+ = \bigcup_{j=1}^{n} I_{2j-1}, \quad D_- = \bigcup_{j=1}^{n} I_{2j},$$

where $I_j = (\frac{(j-1)H}{2n}, \frac{jH}{2n})$. Then we get

$$\begin{aligned} J(0,\tau,u) &= u_0(1+4(\tau_0\omega)^2)^{-1}\left(1+\frac{n^2 l^2}{H^2}\right)^{-1} \\ &\times \int_0^\tau dt \frac{G(\tau/2, t-\tau/2)}{\tau^3} \sin 2\pi\omega t \sum_{j=1}^{2n} \int_{\frac{(j-3/4)H}{2n}}^{\frac{(j-1/4)H}{2n}} |\sin \frac{2\pi n y}{H}|\, dy \\ &\geq C(1+4(\tau_0\omega)^2)^{-1}\left(1+\frac{n^2 l^2}{H^2}\right)^{-1} u_0, \end{aligned}$$

with constant C dependent only on reaction f. A similar estimate is valid for $t_0 = \frac{1}{2\omega}$, we only need to switch $D_\pm$. Therefore, we get that for any t_0 and any $\tau_1 \geq \frac{1}{\omega}$,

$$\langle V \rangle_{t_0,\tau_1} \geq C(1+4(\tau_0\omega)^2)^{-1}\left(1+\frac{n^2 l^2}{H^2}\right)^{-1} u_0. \tag{52}$$

It is not difficult to obtain estimates for averages over times smaller than $\frac{1}{\omega}$, but these would generally (and naturally) depend on the choice of starting time t_0.

Example 2. Consider

$$u(y,t) = u_0 \sin \frac{2\pi n(y-ct)}{H}.$$

This is a flow which shifts in y direction. We assume for simplicity that the boundary conditions for T are periodic in y. Given any t_0, pick $\tau = \frac{H}{8cn}$. Time τ is chosen so that during this time, the regions where u is positive and negative do not shift completely; there are regions where velocity stays positive or negative during $[0,\tau]$. Take

$$D_+ = \bigcup_{j=1}^{n} I_{2j-1}, \quad D_- = \bigcup_{j=1}^{n} I_{2j-1}$$

with $I_j = (\frac{(4j-3)H}{8n}, \frac{(4j-1)H}{8nH})$. A direct computation of $J(t_0, \tau, u)$ shows the following bound:

$$\langle V\rangle_{t_0,\tau} \geq C\left(1+\left(\frac{8cn\tau_0}{H}\right)^2\right)^{-1}\left(1+\frac{n^2l^2}{H^2}\right)^{-1}u_0, \tag{53}$$

where C may depend only on reaction function f. We note that in this example, it is to easy to show that (53) extends to any averaging time τ, independently of the starting time t_0 :

$$\langle V\rangle_{t_0,\tau} \geq C\left(1+\frac{\tau_0^2}{\tau^2}\right)^{-1}\left(1+\frac{n^2l^2}{H^2}\right)^{-1}u_0.$$

6 Percolating flows

We now consider a more general class of flows, which we call "percolating". By this we mean that there exist at least two tubes of streamlines of the advecting velocity $u(x,y)$, one of which connects $x=-\infty$ and $x=+\infty$, and the other one goes from $x=+\infty$ to $x=-\infty$. More precisely, let us assume that there exist regions D_j^+ and D_j^-, $j=1,\ldots N$ such that each of them is bounded by the streamlines of $u(x,y)$, and the projection of each streamline of $u(x,y)$, contained in either D_j^+ or D_j^-, onto the x-axis covers the whole real line (these projections need not be one-to-one, however). As before, we denote $D_\pm$ the union of all $D_j^\pm$ respectively.

Our considerations in this section will follow closely the ideas of the shear flow case. However, there are two natural geometries in the problem. The Laplace operator is best described in Euclidean coordinates, while for the advection term the geometry of streamlines imposed by the flow is most natural. In the case of the shear flows these geometries coincide, but generally they are at odds. Due to this fact, additional technical difficulties arise when we consider percolating flows.

We assume that the streamlines in $D_j^\pm$ are sufficiently regular, so that inside each $D_j^\pm$ there exists a one-to-one C^2 change of coordinates $(x,y)\to(\rho,\theta)$, such that ρ is constant on the streamlines, while θ is an orthogonal coordinate for ρ (with a slight abuse of notation we shall use the same notation (ρ,θ) in all $D_j^\pm$, although these coordinates are not defined globally). Moreover, $u\cdot\nabla\theta>0$ in D_j^+, while $u\cdot\nabla\theta<0$ in each D_j^-. On $D_j^\pm$, ρ varies in $[c_j^\pm-h_j^\pm, c_j^\pm+h_j^\pm]$, while θ varies in $(-\infty,\infty)$. See figure 2 for a sketch of coordinates (ρ,θ). The square of the length element inside each set $D_j^\pm$ is given by

$$dx^2+dy^2 = E_1^2(\rho,\theta)d\rho^2 + E_2^2(\rho,\theta)d\theta^2.$$

We assume that the functions $E_{1,2}$ satisfy the following conditions. They are bounded from above and below:

$$C^{-1}\leq E_{1,2}(\rho,\theta)\leq C \tag{54}$$

uniformly on all $D_j^\pm$. Moreover, the function

$$\omega(\rho,\theta) = \frac{E_2(\rho,\theta)}{E_1(\rho,\theta)} \tag{55}$$

satisfies the following bounds:

$$C^{-1}\leq|\omega(\rho,\theta)|\leq C, \quad \left|\frac{\partial\omega}{\partial\rho}(\rho,\theta)\right|\leq\frac{C}{h_j^\pm} \text{ on } D_j^\pm \text{ respectively}, \tag{56}$$

with $2h_j^\pm$ being the absolute value of the difference of the values of ρ on the two components of the boundary $\partial D_j^\pm$ (recall that $D_j^\pm$ are bounded by two streamlines of $u(x,y)$).

Conditions (54) and (56) are satisfied, for instance, in the following examples:

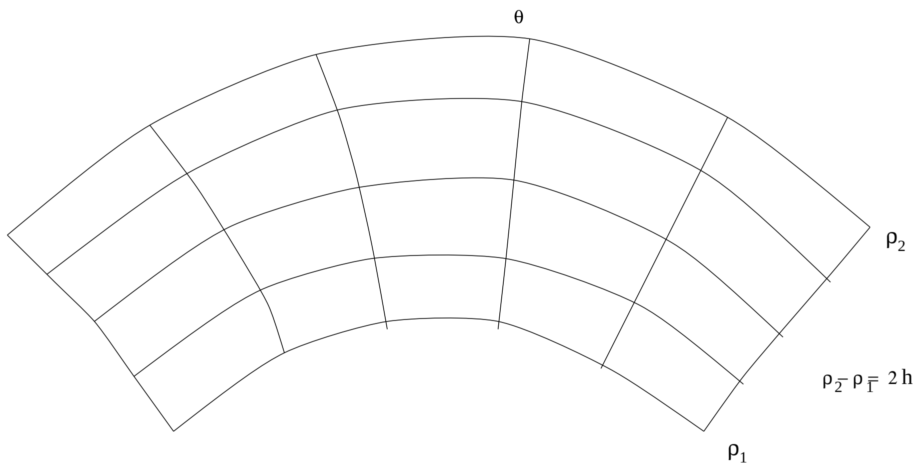

Figure 2: Curvilinear coordinates (ρ, θ).

(1) A flow $u(x, y) = U\tilde{u}(x, y)$ with U being a scalar, and the flow $\tilde{u}(x, y)$ satisfying on $D_j^{\pm}$

$$C^{-1} \leq |\tilde{u}(x, y)| \leq C, \quad |\nabla \times \tilde{u}(x, y)| \leq \frac{C}{d}, \tag{57}$$

where d is the maximum length of a level set of θ inside $D_j^{\pm}$.

(2) More generally, it is enough to ask that in each $D_j^{\pm}$ there exists a function ψ constant on the streamlines of u such that

$$C^{-1} \leq |\nabla\psi| \leq C, \quad |\Delta\psi| \leq \frac{C}{d}. \tag{58}$$

We remark that (1) is a particular case of (2) where ψ is taken to be a stream function of the flow $\tilde{u}$.

We do not make any assumptions on the behavior of the streamlines of $u(x, y)$ outside the regions D_+ and D_-. In particular, there may be pockets of still fluid, streamlines may be closed, etc. (see Figure 3).

Another assumption concerns the relative measure of the sets $D_\pm$, which we assume to oscillate not too wildly. More precisely, let R_{ab} denote the rectangle $R_{ab} = [a, b]_x \times [0, H]_y$, and let $D_\pm^{ab} = D_\pm \cap R_{ab}$. We define the measures

$$\mu_\pm[a, b] = \int\limits_{D_\pm^{ab}} d\rho d\theta \sum_j E_1(\rho, \theta) E_2(\rho, \theta) \frac{G(h_j, \rho - c_j)}{H(h_j^2 + l^2)}$$

with $l = \kappa/v_0$ and the function $G(h, \rho)$ defined by (27). Then we assume that there exists a partition of the real axis

$$\ldots < x_{-n} < \ldots < x_{-1} < x_0 < x_1 < \ldots x_n < \ldots, \quad x_{i+1} - x_i \leq L \text{ for all } i \in Z, \tag{59}$$

and a number m_0 so that the ratio

$$m_0 = \frac{\mu_+[x_i, x_{i+1}]}{\mu_-[x_i, x_{i+1}]} \quad \text{is independent of } i \in Z. \tag{60}$$

This assumption is not the weakest necessary assumption and we make it for clarity of exposition. The simplest example where this assumption is satisfied is periodic percolating flows for which (57) or (58) is satisfied.

Then we have the following Theorem.

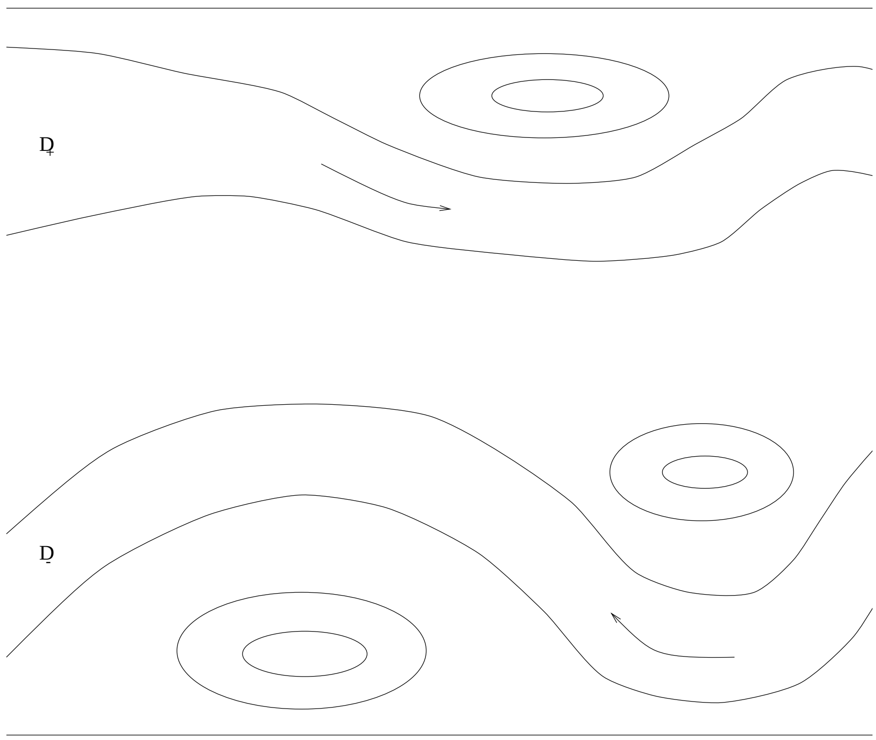

Figure 3: Streamlines of $u(x,y)$.

Theorem 6 *Let each of the sets* $D_j^{\pm}$ *be of the form* $D_j^{\pm} = \{\rho \in [c_j - h_j, c_j + h_j]\}$. *Then under the assumptions made above, we have*

$$\langle V \rangle_\tau \;\geq\; C\left(\frac{1}{1+m_0}\sum_{D_j^+}\left(1+\frac{l^2}{h_j^2}\right)^{-1}\int\limits_{c_j-\frac{h_j}{2}}^{c_j+\frac{h_j}{2}} |u(\rho,\theta)|E_1(\rho,\theta)\frac{d\rho}{H}\right. \tag{61}$$

$$\left. +\; \frac{1}{1+m_0^{-1}}\sum_{D_j^-}\left(1+\frac{l^2}{h_j^2}\right)^{-1}\int\limits_{c_j-\frac{h_j}{2}}^{c_j+\frac{h_j}{2}} |u(\rho,\theta)|E_1(\rho,\theta)\frac{d\rho}{H}\right)$$

for every

$$\tau \geq \tau_0 = max\left[\frac{\kappa}{v_0^2}, \frac{H+L}{v_0}\right].$$

Here $l = \kappa/v_0$, L *and* m_0 *are as in (59) and (60), and the constant* C *in (61) depends on the function* $f(T)$ *and the constants appearing in (54) and (56).*

Remark. Notice that the integrals on the right-hand side are independent of θ and give fluxes of the fluid through the middles of the tubes of streamlines.

Proof. The proof of this theorem follows the steps of the proof of Theorem 4 for the shear flow. We will again utilize the differential inequality (18), and the expression (17) for the bulk burning rate, as well as multiple averaging over regions bounded by the streamlines of the advecting velocity $u(x,y)$.

Let us consider one region $D_j^+ = \{(\rho,\theta):\ \rho\in[c_j-h_j,c_j+h_j]\}$. Let us also denote the tube of streamlines $D_\delta = \{(\rho,\theta):\ \rho\in[c_j-\delta,c_j+\delta],\theta\in(-\infty,\infty)\}$, $\delta<h_j$ and integrate (1) over the set $D_\delta\subset D_j^+$:

$$\int\limits_{D_\delta} T_t dxdy - \int\limits_{-\delta}^{\delta} d\rho u(\rho,\theta)E_1(\rho,\theta) + \kappa\int\limits_{D_\delta}\Delta T dxdy = \frac{v_0^2}{4\kappa}\int\limits_{D_\delta} f(T)dxdy. \tag{62}$$

We used here the relation

$$\int\limits_{D_\delta} u\cdot\nabla T dxdy = -\int\limits_{-\delta}^{\delta} u(\rho,\theta)E_1(\rho,\theta,)d\rho \tag{63}$$

which follows from the fact that D_δ is the tube of streamlines, and the boundary conditions (8). Moreover, the quantity on the right side of (63) is independent of θ because $u(x,y)$ is incompressible (5).

We first estimate the term that involves the Laplacian in (62). The following analog of Lemma 3 holds:

Lemma 6

$$\left|\int\limits_0^{h_j/2} d\gamma \int\limits_{h_j/2-\gamma}^{h_j/2+\gamma} d\delta \int\limits_{D_\delta}\Delta T(x,y)dxdy\right| \le C\int\limits_{D_j^+}\left[h_j^2|\nabla T|^2+f(T)\right]dxdy.$$

Proof. Notice that

$$\begin{aligned}\int\limits_{D_\delta}\Delta T(x,y)dxdy &= \int_{-\infty}^{\infty} d\theta\left[\frac{E_2}{E_1}\frac{\partial T}{\partial\rho}(\delta,\theta)-\frac{E_2}{E_1}\frac{\partial T}{\partial\rho}(-\delta,\theta)\right] \\ &= \int_{-\infty}^{\infty} d\theta\left[\omega(\delta,\theta)\frac{\partial T}{\partial\rho}(\delta,\theta)-\omega(-\delta,\theta)\frac{\partial T}{\partial\rho}(-\delta,\theta)\right]\end{aligned} \tag{64}$$

with $\omega(\theta,\rho)$ defined in (55). Next, following the general procedure in the proof of Theorem 4 we fix $\theta\in I\!R$ and average (64) over $\delta\in[h_j/2-\gamma,h_j/2+\gamma]$ with $\gamma\in[0,h_j/2]$, and then also average in γ. Then (64) becomes

$$\int\limits_0^{h_j/2} d\gamma\int\limits_{h_j/2-\gamma}^{h_j/2+\gamma} d\delta\left[T_\rho(\delta,\theta)\omega(\delta,\theta)-T_\rho(-\delta,\theta)\omega(-\delta,\theta)\right]. \tag{65}$$

We show how to estimate the first term in (65), with the second term treated in the same way. We integrate it by parts to get

$$\begin{aligned}&\int\limits_0^{h_j/2} d\gamma\int\limits_{h_j/2-\gamma}^{h_j/2+\gamma} d\delta T_\rho(\delta,\theta)\omega(\delta,\theta) = \int\limits_0^{h_j/2} d\gamma\left[T(\frac{h_j}{2}+\gamma,\theta)\omega(\frac{h_j}{2}+\gamma,\theta)\right. \\ &\left.-T(\frac{h_j}{2}-\gamma,\theta)\omega(\frac{h_j}{2}-\gamma,\theta)\right] - \int\limits_0^{h_j/2} d\gamma\int\limits_{h_j/2-\gamma}^{h_j/2+\gamma} d\delta T(\delta,\theta)\omega_\rho(\delta,\theta) \\ &= \int\limits_0^{h_j/2} d\gamma\left[\left(T(\frac{h_j}{2}+\gamma,\theta)-1\right)\omega(\frac{h_j}{2}+\gamma,\theta)-\left(T(\frac{h_j}{2}-\gamma,\theta)-1\right)\omega(\frac{h_j}{2}-\gamma,\theta)\right]\end{aligned} \tag{66}$$

$$-\int_0^{h_j/2} d\gamma \int_{h_j/2-\gamma}^{h_j/2+\gamma} d\delta\,(T(\delta,\theta)-1)\,\omega_\rho(\delta,\theta).$$

Consider the set of θ such that

$$\int_0^{h_j} |T_\rho(\rho,\theta)|^2 d\rho \geq \frac{1}{4h_j}.$$

We have for such θ

$$\int_0^{h_j/2} d\gamma \int_{h_j/2-\gamma}^{h_j/2+\gamma} d\delta |T_\rho(\delta,\theta)\omega(\delta,\theta)| \leq C \int_0^{h_j/2} d\gamma \sqrt{2\gamma} \left(\int_0^{h_j} d\delta T_\rho^2(\delta,\theta)\right)^{1/2} \leq C h_j^2 \int_0^{h_j} T_\rho^2 d\rho. \tag{67}$$

Next we look at θ such that

$$\int_0^{h_j} |T_\rho|^2 d\rho \leq \frac{1}{4h_j}.$$

In this case for any $\rho_1, \rho_2 \in [0, h_j]$ we have

$$|T(\rho_1,\theta) - T(\rho_2,\theta)| \leq \sqrt{h_j} \left(\int_0^{h_j} T_\rho^2 d\rho\right)^{1/2} \leq \frac{1}{2}.$$

Therefore, either $T(\rho,\theta) \geq 1/4$ for all ρ, or $|T(\rho,\theta)-1| \geq 1/4$ for all ρ. Then we have, $1-T \leq Cf(T)$, or $T \leq Cf(T)$, respectively. We use one of these bounds and the corresponding part of (66) to get for such θ

$$\left|\int_0^{h_j/2} d\gamma \int_{h_j/2-\gamma}^{h_j/2+\gamma} d\delta T_\rho(\delta,\theta)\omega(\delta,\theta)\right| \leq C \int_0^{h_j} f(T(\rho,\theta)) d\rho + \frac{C}{h_j} \int_0^{h_j/2} d\gamma \int_{h_j/2-\gamma}^{h_j/2+\gamma} d\delta f(T(\delta,\theta))$$
$$\leq C \int_0^{h_j} f(T(\rho,\theta)) d\rho. \tag{68}$$

Now we put together the estimates (67) and (68) to obtain for all $\theta \in I\!\!R$:

$$\left|\int_0^{h_j/2} d\gamma \int_{h_j/2-\gamma}^{h_j/2+\gamma} d\delta \int_{-\infty}^{\infty} d\theta T_\rho(\delta,\theta)\omega(\delta,\theta)\right| \tag{69}$$
$$\leq C \left[h_j^2 \int_{-\infty}^{\infty} d\theta \int_0^{h_j} d\rho T_\rho^2 + \int_{-\infty}^{\infty} d\theta \int_0^{h_j} d\rho f(T(\rho,\theta))\right]$$
$$\leq C \int_{D_j^+} \left[h_j^2 |\nabla T|^2 + f(T(x,y))\right] dxdy. \quad \Box$$

Similarly to the shear case, Lemma 6 and (62) imply the inequality

$$
-\int\limits_{D_j^+} \frac{d\rho d\theta}{H} E_1(\rho,\theta)E_2(\rho,\theta)\frac{G(h_j,\rho_j-c_j)}{(h_j^2+l^2)}T_t(\rho,\theta)+\int\limits_{c_j-h_j}^{c_j+h_j}\frac{d\rho}{H}\frac{G(h_j,\rho-c_j)}{h_j^2+l^2}|u(\rho,\theta)|E_1(\rho,\theta)
$$
$$
\le C\left[\kappa\int\limits_{D_j^+}|\nabla T|^2\frac{dxdy}{H}+\frac{v_0^2}{4\kappa}\int\limits_{D_j^+}f(T(x,y))\frac{dxdy}{H}\right] \tag{70}
$$

where $G(h,\xi)$ is defined as before by (27). An estimate similar to (70) holds in the regions D_j^-, where the flow is going backwards, except that the time derivative term in (70) enters now with the opposite sign:

$$
\int\limits_{D_j^-} \frac{d\rho d\theta}{H} E_1(\rho,\theta)E_2(\rho,\theta)\frac{G(h_j,\rho_j-c_j)}{h_j^2+l^2}T_t(\rho,\theta)+\int\limits_{c_j-h_j}^{c_j+h_j}\frac{d\rho}{H}\frac{G(h_j,\rho-c_j)}{h_j^2+l^2}|u(\rho,\theta)|E_1(\rho,\theta) \tag{71}
$$
$$
\le C\left[\kappa\int\limits_{D_j^-}|\nabla T|^2\frac{dxdy}{H}+\frac{v_0^2}{4\kappa}\int\limits_{D_j^-}f(T(x,y))\frac{dxdy}{H}\right].
$$

Let us choose the weights

$$
m_+=\frac{1}{1+m_0},\quad m_-=\frac{1}{1+m_0^{-1}},
$$

(recall m_0 is defined by (60)) so that

$$
m_+\mu_+[x_i,x_{i+1}]=m_-\mu_-[x_i,x_{i+1}] \tag{72}
$$

for any two points x_i, x_{i+1} of the partition (59) of the x-axis, similarly to what we did in the proof of Theorem 4 (see (37)). In order to finish the proof of Theorem 6 we multiply equations (70) and (71) by m_+ and m_-, respectively, and add them. It remains now to estimate the time derivative term, and the following general Lemma provides us with the analog of Lemma 5 in the shear case.

Lemma 7 *Let Ω_0 be a rectangle $\Omega_0=L\times H$, and let $\Omega_{1,2}\subset\Omega_0$ be two open subsets of Ω_0. Consider two continuous non-negative functions $\phi_{1,2}:\Omega_{1,2}\to I\!R$ such that $0\le\phi_{1,2}(x,y)\le C$ and*

$$
\int\limits_{\Omega_1}dxdy\phi_1(x,y)=\int\limits_{\Omega_2}dxdy\phi_2(x,y). \tag{73}
$$

Let $T:\Omega_0\to I\!R$ be a continuously differentiable function, $0\le T\le 1$, then for any $\varepsilon>0$ we have

$$
\left|\int\limits_{\Omega_1}dxdy\phi_1(x,y)T(x,y)-\int\limits_{\Omega_2}dxdy\phi_2(x,y)T(x,y)\right| \tag{74}
$$
$$
\le C\left[(L+H)\left[\varepsilon\int\limits_{\Omega_0}dxdy|\nabla T|^2+\frac{1}{\varepsilon}\int\limits_{\Omega_0}f(T(x,y))dxdy\right]+\int\limits_{\Omega_0}f(T(x,y))dxdy\right].
$$

We postpone the proof of Lemma 7 till the end of this section. Using Lemma 7 in each rectangle $R_{x_i,x_{i+1}}$ with $\varepsilon = \frac{\kappa}{v_0}$, $\Omega_1 = D^+ \cap R_{x_i,x_{i+1}}$, $\Omega_2 = D^- \cap R_{x_i,x_{i+1}}$, and functions $\phi_{1,2}$ given by

$$\phi_1 = m_+ \sum_{D_j^+} \frac{G(h_j, \rho - c_j)}{H(h_j^2 + l^2)} E_1(\rho,\theta) E_2(\rho,\theta) \chi_{D_j^+}(\rho,\theta)$$

$$\phi_2 = m_- \sum_{D_j^-} \frac{G(h_j, \rho - c_j)}{H(h_j^2 + l^2)} E_1(\rho,\theta) E_2(\rho,\theta) \chi_{D_j^-}(\rho,\theta),$$

we arrive at the analog of Lemma 5:

$$\int_{\tau_1}^{\tau_2} dt \Bigg(\int_{D_j^+} m_+ \sum_{D_j^+} \frac{G(h_j, \rho - c_j)}{H(h_j^2 + l^2)} E_1(\rho,\theta) E_2(\rho,\theta) \chi_{D_j^+}(\rho,\theta) T_t(\rho,\theta) d\rho d\theta \qquad (75)$$

$$- \int_{D_j^-} m_- \sum_{D_j^+} \frac{G(h_j, \rho - c_j)}{H(h_j^2 + l^2)} E_1(\rho,\theta) E_2(\rho,\theta) \chi_{D_j^-}(\rho,\theta) T_t(\rho,\theta) d\rho d\theta \Bigg)$$

$$\leq C \sum_{i=1}^{2} \left(\frac{(H+L)\kappa}{v_0} \int_\Omega |\nabla T(x,y,\tau_i)|^2 \frac{dxdy}{H} + \left(1 + \frac{(H+L)v_0}{\kappa}\right) \int_\Omega f(T(x,y,\tau_i)) \frac{dxdy}{H} \right).$$

The rest of the proof of Theorem 6 is completely analogous to the proof of Theorem 4. We average in time according to (45) and use (17) and (18) to conclude the proof. □

We now give the proof of Lemma 7.

Proof. We define the measures $\nu_{1,2}$ by

$$d\nu_{1,2}(x,y) = \phi_{1,2}(x,y) \chi_{\Omega_{1,2}}(x,y) dxdy.$$

Let $\mathcal{A} \subset \Omega_1$ be the set of points where $T(x,y) > 5/8$, and let the open set $\mathcal{B} \subset \Omega_2$ be such that $\nu_1(\mathcal{A}) = \nu_2(\mathcal{B})$. Then we have

$$\int_{\Omega_1} d\nu_1(x,y) T(x,y) - \int_{\Omega_2} d\nu_2(x,y) T(x,y) \leq C \int_{\Omega_1 \setminus \mathcal{A}} dxdy f(T(x,y)) \qquad (76)$$

$$+ \int_{\mathcal{A}} d\nu_1(x,y) T(x,y) - \int_{\mathcal{B}} d\nu_2(x,y) T(x,y).$$

Let us decompose further $\mathcal{B} = \mathcal{B}' \cup \mathcal{B}''$, where $\mathcal{B}' = \{(x,y) \in \mathcal{B} : \, T(x,y) > 3/4\}$. We also consider an open set $\mathcal{A}' \subset \mathcal{A}$ such that $\nu_1(\mathcal{A}') = \nu_2(\mathcal{B}')$, and write $\mathcal{A} = \mathcal{A}' \cup \mathcal{A}''$. Then we obtain

$$\int_{\mathcal{A}} dnu_1 T - \int_{\mathcal{B}} d\nu_2 T = \int_{\mathcal{A}'} d\nu_1 T - \int_{\mathcal{B}'} d\nu_2 T + \int_{\mathcal{A}''} d\nu_1 T - \int_{\mathcal{B}''} d\nu_2 T \qquad (77)$$

and, moreover, Lemma 4 implies that

$$\int_{\mathcal{A}'} d\nu_1 T(x,y) - \int_{\mathcal{B}'} d\nu_2 T(x,y) \leq C \int_{\mathcal{A}'} dxdy f(T(x,y)) + \int_{\mathcal{B}'} dxdy f(T(x,y)). \qquad (78)$$

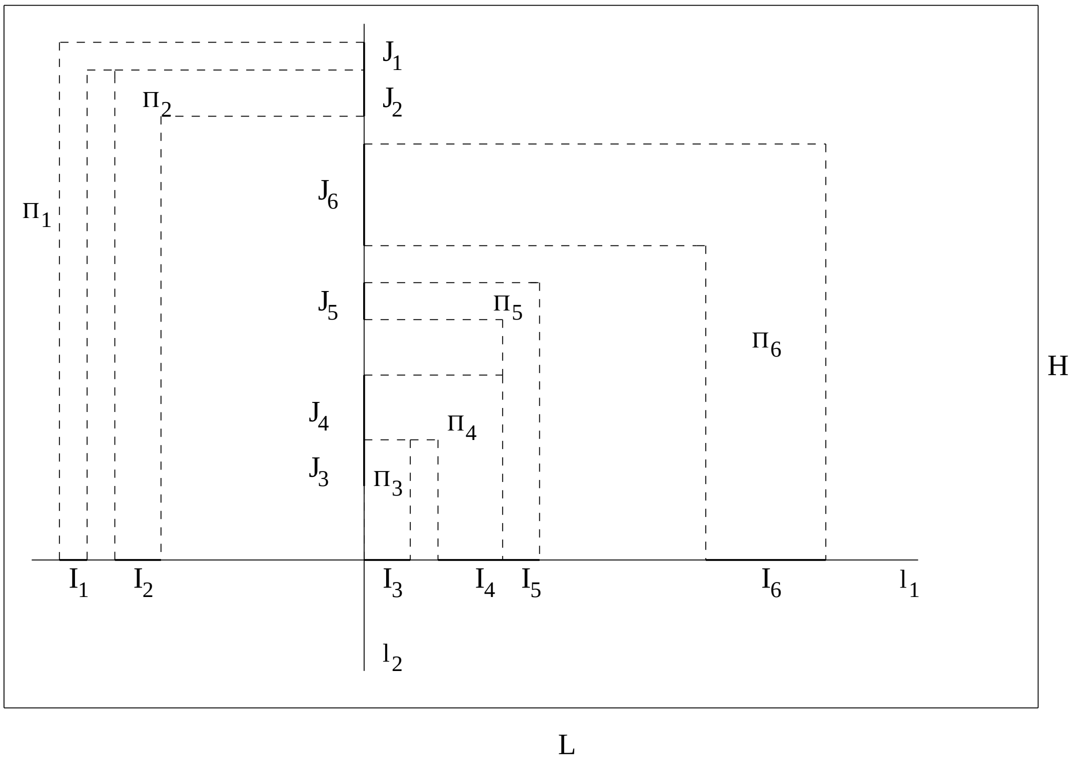

Figure 4: Staircase for I_k and J_k.

Therefore, we are done if $\nu_1(\mathcal{A}'') = \nu_2(\mathcal{B}'') = 0$. Assume now that this is not the case. Then we may find a horizontal line $l_1 :\ y = y_0$ and a vertical line $l_2 :\ x = x_0$ such that

$$|l_1 \cap \mathcal{A}''| \geq \frac{C}{L}\nu_1(\mathcal{A}''), \quad |l_2 \cap \mathcal{B}''| \geq \frac{C}{H}\nu_2(\mathcal{B}''),$$

where $|S|$ denotes the one-dimensional Lebesgue measure. Moreover, we may choose subsets $Q_1 \subset l_1 \cap \mathcal{A}''$ and $Q_2 \subset l_2 \cap \mathcal{B}''$ so that $Q_1 = \cup_{k=1}^{N} I_k$ and $Q_2 = \cup_{k=1}^{M} J_k$ are finite unions of intervals, and $|Q_1| = |Q_2| \geq \dfrac{C\nu_1(\mathcal{A}'')}{L+H}$. We may assume (possibly after subdividing into smaller intervals) that $N = M$, and $|I_k| = |J_k|$ for all k. Let us connect each pair of intervals I_k and J_k by perpendicular lines "staircase" as depicted on Figure 4. Notice that for every point $(x,y) \in I_k$, $T(x,y) > 3/4$ while for every point $(x',y') \in J_k$, $T(x',y') < 5/8$. An argument directly analogous to that in the proof of Lemma 2 shows that the following estimate holds:

$$\int\limits_{\Pi_k} f(T)dxdy \int\limits_{\Pi_k} |\nabla T|^2 dxdy \geq C|I_k|^2.$$

Therefore we have for any $\varepsilon > 0$

$$\frac{1}{\varepsilon}\int\limits_{\Omega_0} f(T)dxdy + \varepsilon \int\limits_{\Omega_0} |\nabla T|^2 dxdy \geq \frac{C}{L+H}\nu_1(\mathcal{A}'') \tag{79}$$

$$\geq \frac{C}{L+H}\left[\ \int\limits_{\mathcal{A}''} dxdy\phi_1(x,y)T(x,y) - \int\limits_{\mathcal{B}''} dxdy\phi_2(x,y)T(x,y)\right].$$

Equations (76-79) show that

$$\int_{\Omega_1} dxdy\phi_1(x,y)T(x,y) - \int_{\Omega_2} dxdy\phi_2(x,y)T(x,y) \tag{80}$$
$$\leq C\left[(L+H)\left[\varepsilon\int_{\Omega_0} dxdy|\nabla T|^2 + \frac{1}{\varepsilon}\int_{\Omega_0} f(T(x,y))dxdy\right] + \int_{\Omega_0} f(T(x,y))dxdy\right].$$

The same proof shows that this bound holds for $\int_{\Omega_2} dxdy\phi_2(x,y)T(x,y) - \int_{\Omega_1} dxdy\phi_1(x,y)T(x,y)$. This finishes the proof of Lemma 7. □

7 Examples of sub-linear growth of the bulk burning rate

We give in this section examples of flows for which bulk burning rate grows sub-linearly in the amplitude of the advecting velocity. We do not try to identify the most general class of such flows but consider rather one simple family of flows of the form

$$u(x,y) = U\nabla^{\perp}\Psi_m(x,y) = UL_y\left(\frac{\partial\Psi_m}{\partial y}, -\frac{\partial\Psi_m}{\partial x}\right) \tag{81}$$

with the stream function

$$\Psi_m(x,y) = \cos^m(\pi x/L_x)\cos^m(\pi y/L_y), \quad m \geq 1 \tag{82}$$

periodic in x and y. The scalar U has the dimension of velocity. The structure of the level sets of the functions Ψ_m is clearly the same for all m. The period cell for these flows is the rectangle $D = [-\frac{L_x}{2}, \frac{3L_x}{2}] \times [-\frac{L_y}{2}, \frac{3L_y}{2}]$, that consists of four smaller rectangles separated by separatrices $\Psi_1 = 0$. The normal component of $u(x,y)$ is equal to zero at the boundary of the period cell of $u(x,y)$, which slows down the burning as compared to percolating flows. This effect is quantified by the following Proposition.

Proposition 1 *Let $T(x,y,t)$ be the solution of the reaction diffusion equation (1) with either Neumann or periodic boundary conditions (3) and (4), respectively. Let $u(x,y)$ be given by (81), (82) with $L_y = H/2$. Moreover, assume that the initial data $T_0(x,y)$ has the property that $T_0(x,y) = 1$ for $x \leq x_0$, and $T_0(x,y) = 0$ for $x \geq x_1$. Then there exists a constant $C > 0$ such that for $U \geq v_0$ we have*

$$\frac{\langle V\rangle_\infty}{v_0} \leq C\left(1 + \frac{l}{L_x}\right)\left(\frac{U}{v_0}\right)^{2/(1+m)} + \frac{L_x}{4l}. \tag{83}$$

Proof. We will construct a function $\phi(x)$, independent of y (and hence satisfying both the Neumann and periodic boundary conditions (3) and (4)), and L_x-periodic in x, such that the function $\Phi(x,t) = e^{-\lambda(x-ct)}\phi(x)$ satisfies the inequality

$$\Phi_t + u\cdot\nabla\Phi - \kappa\Delta\Phi - \frac{v_0^2}{4\kappa}\Phi \geq 0. \tag{84}$$

Moreover, the function $\phi(x)$ will be positive, bounded, and bounded away from zero. Then maximum principle will imply that the solution $T(x,y,t)$ of (1) with the Neumann or periodic boundary conditions satisfies the inequality

$$T(x,y,t) \leq C_\lambda e^{-\lambda(x-ct)}$$

since it holds at $t=0$ for all $\lambda>0$ because of our choice of the initial data. Then we will have

$$\langle V\rangle_\infty \le c \tag{85}$$

as in Lemma 3. Therefore our goal is to find a function $\phi(x)$ and $\lambda>0$ so as to satisfy (84) with as small c as possible. The function $\phi(x)$ should obey the inequality

$$L\phi = \frac{\kappa}{\lambda}\Delta\phi - 2\kappa\phi_x - \frac{u}{\lambda}\cdot\nabla\phi + u_1\phi \le B\phi \tag{86}$$

with

$$B = c - \kappa\lambda - \frac{v_0^2}{4\kappa\lambda}. \tag{87}$$

We will define $\phi(x)$ on the interval $[-L_x/2, 3L_x/2]$, and then extend it periodically to the whole real line. In order to make use of the fact that the x-component of $u(x,y)$ is small near the lines $x=-L_x/2, 3L_x/2$ we consider a smooth cut-off function $\chi(x)$ defined as follows. Let $\eta(x)$ be a cut-off function

$$\eta(x) = \begin{cases} 1, & |x| \le \frac{1}{2}\left(\frac{U}{v_0}\right)^{-\alpha} \\ 0, & |x| \ge \left(\frac{U}{v_0}\right)^{-\alpha}, \end{cases}$$

and η decays monotonically from one to zero between those intervals, so that

$$|\eta'| \le C\left(\frac{U}{v_0}\right)^{\alpha}, \quad |\eta''| \le C\left(\frac{U}{v_0}\right)^{2\alpha}.$$

The exponent $\alpha>0$ is to be chosen later. Then we define for $x\in[-L_x/2, 3L_x/2]$

$$\chi(x) = \eta\left(\frac{x}{L_x}+\frac{1}{2}\right) + \eta\left(\frac{x}{L_x}-\frac{3}{2}\right),$$

so that the two terms have non-overlapping support, and set

$$\phi(x) = \chi(x) + (1-\chi(x))e^{\lambda x} := \chi(x) + \beta(x)$$

so that

$$e^{-\lambda L_x/2} \le \phi(x,y) \le e^{3\lambda L_x/2}.$$

First we observe that since $|u_1| \le CU\,(U/v_0)^{-m\alpha}$ on the support of $\chi(x)$, we have

$$\begin{aligned} |L\chi(x)| &= \left|\frac{\kappa}{\lambda}\chi'' - 2\kappa\chi' - \frac{u_1}{\lambda}\chi' + u_1\chi\right| \\ &\le C\left[\frac{\kappa}{\lambda L_x^2}\left(\frac{U}{v_0}\right)^{2\alpha} + \frac{\kappa}{L_x}\left(\frac{U}{v_0}\right)^{\alpha} + \frac{v_0}{\lambda L_x}\left(\frac{U}{v_0}\right)^{1-m\alpha+\alpha} + v_0\left(\frac{U}{v_0}\right)^{1-m\alpha}\right]. \end{aligned} \tag{88}$$

Moreover, we have

$$|L\beta| = \left|-(1-\chi)\kappa\lambda - \frac{\kappa}{\lambda}\chi'' + \frac{1}{\lambda}u_1\chi'\right|e^{\lambda x} \le C\left(\frac{\kappa}{\lambda L_x^2}\left(\frac{U}{v_0}\right)^{2\alpha} + \frac{v_0}{\lambda L_x}\left(\frac{U}{v_0}\right)^{1-m\alpha+\alpha}\right)e^{\lambda x} + \kappa\lambda\beta. \tag{89}$$

We put together the bounds (88) and (89), and obtain

$$\left|\frac{L\phi}{\phi}\right| \le Ce^{3\lambda L_x/2}\left[\frac{\kappa}{\lambda L_x^2}\left(\frac{U}{v_0}\right)^{2\alpha} + \frac{\kappa}{L_x}\left(\frac{U}{v_0}\right)^{\alpha} + v_0\left(\frac{U}{v_0}\right)^{1-m\alpha} + \frac{v_0}{\lambda L_x}\left(\frac{U}{v_0}\right)^{1-m\alpha+\alpha}\right] + \kappa\lambda. \tag{90}$$

Therefore the function ϕ that we have constructed satisfies the inequality (86) with the constant B given by the right side of (90). Using the definition (87) of B and relation (85) we obtain then the following bound on the bulk burning rate:

$$\frac{\langle V\rangle_\infty}{v_0} \le Ce^{3\lambda L_x/2}\left[\frac{l}{\lambda L_x^2}\left(\frac{U}{v_0}\right)^{2\alpha}+\frac{l}{L_x}\left(\frac{U}{v_0}\right)^{\alpha}+\left(\frac{U}{v_0}\right)^{1-m\alpha}+\frac{1}{\lambda L_x}\left(\frac{U}{v_0}\right)^{1-m\alpha+\alpha}\right]+2l\lambda+\frac{1}{4l\lambda}$$

where $l=\kappa/v_0$ is the laminar front width. Then we let $\lambda = 1/L_x$ and $\alpha = 1/(1+m)$, and get the estimate in Proposition 1.□

One can see from the proof of Proposition 1 that it may be easily generalized to include cellular flows other than those of the form (81-82). The relevant assumptions are similar geometric structure of the streamlines and the appropriate rate of decay of the normal velocity at the boundary of the period cell.

The power $\xi = 2/(1+m)$ in (83) is probably not sharp, but Proposition 1 still shows several important points. Not all advection velocities with non-trivial u_1 lead to linear growth of the bulk burning rate in the advection amplitude. The presence of the closed streamlines appears to be crucial for sub-linear enhancement. Also, the exponent ξ may be made arbitrarily close to $\xi = 0$ by taking $m\to\infty$, and thus one can construct non-trivial flows for which the bulk burning rate grows slower than any given power of U/v_0.

A Homogenization regime

Here we briefly present a simple direct application of the bound we obtained in Section 3. We will consider a homogenization regime where the reaction is very weak, and investigate an effect of the periodic advection velocity in this limit. Let us consider the reaction-diffusion-advection equation (1) with the laminar velocity v_0 being small: $v_0 \to \frac{1}{N}v_0$, $N \gg 1$. The domain is then taken to be finite but very large: $D_N = ND$, where D is some fixed region, such as a rectangle. Initial data varies on the large scale:

$$\begin{aligned}&(T_N)_t + u(\mathbf{x},t)\cdot\nabla T_N = \kappa\Delta T_N + \frac{v_0^2}{4N^2\kappa}f(T_N)\\&\frac{\partial T_N}{\partial n} = 0 \;\text{ on }\; \partial D_N\\&T_N(\mathbf{x},0) = T_0(\frac{\mathbf{x}}{N}),\;\; \mathbf{x}\in D_N\end{aligned}\tag{91}$$

Then after rescaling $\mathbf{x}, t \to N\mathbf{x}, N^2 t$ the rescaled problem is

$$\begin{aligned}&(T_N)_t + Nu(N\mathbf{x},N^2t)\cdot\nabla T_N = \kappa\Delta T_N + \frac{v_0^2}{4\kappa}f(T_N)\\&\frac{\partial T_N}{\partial n} = 0 \;\text{ on }\; \partial D\\&T_N(\mathbf{x},0) = T_0(\mathbf{x}),\;\; \mathbf{x}\in D_N\end{aligned}\tag{92}$$

We assume that $u(\mathbf{x},t)$ is periodic in x with period cell Q and in t with period τ, and vanishes on the boundary of the cell C. Moreover, it is convenient to assume that D contains an integer number of cells, so that $u(N\mathbf{x},Nt)$ vanishes on the boundary ∂D. The bulk burning rate is given as before by

$$V_N(t) = \int_D d\mathbf{x}\frac{\partial T_N}{\partial t} = \frac{v_0^2}{4\kappa}\int_D d\mathbf{x} f(T_N).$$

The following Theorem may be established using the technique of [3].

Theorem 7 *The family of solutions T_N of (92) converges strongly in $L_2([0,r]\times D)$ to the solution $\bar{T}$ of the homogenized problem*

$$\begin{aligned} &\bar{T}_t = \kappa^*_{ij}\frac{\partial^2 \bar{T}}{\partial x_i \partial x_j} + \frac{v_0^2}{4\kappa}f(\bar{T}) \\ &\frac{\partial \bar{T}}{\partial n} = 0 \quad on \quad \partial D \\ &\bar{T}(\mathbf{x},0) = T_0(\mathbf{x}). \end{aligned} \tag{93}$$

The tensor κ^* is given by

$$\kappa^*_{ij} = \kappa\delta_{ij} - \frac{1}{|Q|\tau}\int\limits_0^\tau ds \int\limits_Q d\mathbf{y}\, u_i(\mathbf{y},s)\theta_j(\mathbf{y},s)$$

with θ_i being the periodic solution of the cell problem

$$\frac{\partial \theta_i}{\partial t} + u(\mathbf{x},t)\cdot\nabla\theta_i - \kappa\Delta\theta_i = -u_i(\mathbf{x},t).$$

Moreover, there exists a constant C such that $\|T_N - \bar{T}\|_{L^2([0,r]\times D)} \le \dfrac{C}{N}$.

Since D is finite, Theorem 7 implies that the bulk burning rate $V_N(t) \to \bar{V}(t) = \int_D \bar{T}_t(s)ds$. Let us denote k_* the minimal eigenvalue of the symmetric part of the tensor κ^*, and set $v_0^* = v_0\sqrt{k_*/\kappa}$. This Theorem may be also applied to the front propagation problem in a finite rectangle with the boundary conditions (3), (8), (9). Arguments similar to Theorem 1 imply that the bulk burning rate for the homogenized problem obeys the lower bound

$$\bar{V}(t) \ge C v_0 \sqrt{\frac{\beta k_*}{4\alpha\kappa}}(1 - e^{-\alpha v_0^2 t/2\kappa})$$

for times less than $t_* \approx \frac{\mathrm{diam} D}{v_0^*}$. Therefore the bulk burning rate in the original unscaled variables is increased from $\frac{1}{N}v_0$ to $\dfrac{1}{N}\sqrt{k_*/\kappa}$, but remains of order $O(\frac{1}{N})$. The dependence of the tensor κ^* on the advection velocity u and diffusivity κ is rather complicated. Some estimates for κ^* were obtained in [2] and [12], and they may be applied to obtain the relevant bounds for $\bar{V}(t)$. This homogenization analysis is applicable only in the limit of very weak reaction at a fixed diffusivity. This is the case when the front width $l_0 = \kappa/v_0$ is much larger than the typical scale of variations of the turbulent velocity.

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