Series Logo
Volume 00, 19xx

,

Evolution of trajectory correlations in steady random flows

Albert Fannjiang, Leonid Ryzhik, and George Papanicolaou

To P.Lax and L. Nirenberg.

Abstract. We analyze the behavior of the correlation for two nearby trajectories of motion in a random incompressible flow with nonzero mean and small fluctuations. We show that the Fourier transform of the Richardson function of a passive scalar advected by the flow satisfies, under certain conditions, a radiative transport equation. We also study the stretching of curves advected by the flow and show that their length grows algebraically in time, and not exponentially as it does for time dependent, zero mean random flows.

Contents

1. Introduction

We consider motion in a steady, random, incompressible flow with constant drift $\mathbf{u}$. The random part of the flow $\mathbf{v}(\mathbf{x})$, with $\nabla \cdot \mathbf{v} = 0$, is a smooth spatially homogeneous random field with zero mean and rapidly decaying correlations. The trajectory $\mathbf{X}^\varepsilon(t, \mathbf{x})$ advected by the flow satisfies the scaled equation

$$\frac{d\mathbf{X}^\varepsilon}{dt} = \mathbf{u} + \sqrt{\varepsilon}\mathbf{v}\left(\frac{\mathbf{X}^\varepsilon}{\varepsilon}\right) \qquad (1.1)$$
$$\mathbf{X}^\varepsilon(0, \mathbf{x}) = \mathbf{x},$$

1991 *Mathematics Subject Classification.* Primary 60F05, 76F05, 76R50; Secondary 58F25.

©0000 American Mathematical Society
0160-7634/00 \$1.00 + \$.25 per page

where $\mathbf{x} \in R^d$, $d = 2, 3$, is the starting position. The small scaling parameter ε determines the size of the random fluctuations and their correlation length. We are interested in the behavior of the rescaled two-point difference function

$$\mathbf{Y}^\varepsilon(t, \mathbf{y}, \mathbf{x}) = \frac{1}{\varepsilon}\{\mathbf{X}^\varepsilon(t, \mathbf{x}) - \mathbf{X}^\varepsilon(t, \mathbf{x} - \varepsilon\mathbf{y})\} \tag{1.2}$$

in the limit of $\varepsilon \to 0$. The individual trajectories $\mathbf{X}^\varepsilon(t)$ are close to the trajectories of the deterministic flow $\mathbf{X} = \mathbf{x} + \mathbf{u}t$ for finite times t, when ε is small, and thus random perturbations do not play an essential role in their behavior. The rescaled trajectory differences $\mathbf{Y}^\varepsilon$ that start nearby are, however, affected and that is what we study in this paper. We note that this is different from the analysis of the long time correlations of trajectories that are initially a fixed distance apart, independently of ε [**18, 21**].

The paper is organized as follows. In Section 2 we review briefly the known results for the long time (of order ε^{-1}) behavior of single trajectories, or groups of trajectories that are not close initially. In Section 3 we show that the asymptotic behavior of $\mathbf{Y}^\varepsilon$ as $\varepsilon \to 0$ can be deduced from the limit theorems of [**18, 21, 16**], stated here in Appendix A. This allows us to derive in Sections 4 and 5 a singular diffusion equation for the rescaled Richardson function. Higher moments of the Richardson function are studied in Section 6. In Section 7 we show how the diffusion equation for the Richardson function can be transformed into a radiative transport equation for the Wigner distribution associated with the trajectory difference function $\mathbf{Y}^\varepsilon$. We study the duality between the limit two-point diffusion process and its analog in Fourier space in Section 8. A scaling more general than 1.1 is treated in Section 9. In Section 10 we show that the Jacobian matrix of the flow (1.1) also converges to a diffusion process in the limit $\varepsilon \to 0$, and we analyze in detail its law in two-dimensional potential flows in Section 11. In Section 12 we apply these results to study the stretching of curves transformed by (1.1). We show that their length grows algebraically in this case and not exponentially, as in the case of zero mean randomly time dependent flows [**2, 8, 31**].

2. Long time diffusive behavior

The long time diffusive behavior of trajectories of mean zero random velocity fields was first studied by G.I.Taylor [**25, 32**]. He considered the trajectories of the system

$$\begin{aligned} d\mathbf{X}(t) &= \mathbf{v}(t, \mathbf{X}(t)) + \kappa d\mathbf{W}(t) \\ \mathbf{X}(0) &= 0, \end{aligned} \tag{2.1}$$

where $\mathbf{v}(t, \mathbf{x})$ is a mean zero random velocity field with rapidly decaying correlations, and $\mathbf{W}(t)$ is the standard d-dimensional Brownian motion. He argued that in the long time limit the trajectories behave like those of d-dimensional Brownian motion with covariance matrix

$$\int_0^\infty E\left\{v_i(t, \mathbf{X}(t))v_j(0,0) + v_j(t, \mathbf{X}(t))v_i(0,0)\right\} dt + \kappa^2\delta_{ij}. \tag{2.2}$$

The first term in expression (2.2), which corresponds to the turbulent enhancement of the diffusion coefficient, depends also on the molecular diffusivity κ through the path $\mathbf{X}(t)$. It is convenient to scale equations (2.1) with a small parameter ε

$$\begin{aligned} d\mathbf{X}^\varepsilon(t) &= \frac{1}{\varepsilon}\mathbf{v}(\frac{t}{\varepsilon^2}, \frac{\mathbf{X}^\varepsilon}{\varepsilon}) + \kappa d\mathbf{W}(t) \\ \mathbf{X}^\varepsilon(0) &= 0 \end{aligned} \tag{2.3}$$

and analyze the behavior of the trajectories as $\varepsilon \to 0$. When the molecular diffusivity κ is positive and the flow is incompressible, $\nabla \cdot \mathbf{v} = 0$, it is shown by homogenization methods [**27, 26, 1, 23, 13**] that as $\varepsilon \to 0$ the trajectories $X^\varepsilon \to 0$ converge weakly to Brownian motion with the covariance matrix given by the homogenization formula (2.2) with $\mathbf{X}(t)$ the solution of (2.1) corresponding to $\varepsilon = 1$ in (2.3). When the molecular diffusivity κ is positive, diffusive behavior holds for random velocity fields that, in the time independent case, are spatially homogeneous, incompressible and have a square integrable stream matrix [**13**]. Time dependent flows when $\kappa > 0$ are analyzed in [**12**] and when $\kappa = 0$ and there is rapid time decorrelation of $\mathbf{v}$ in [**17**].

The weak fluctuation scaling (described in detail in Appendix A) was intensively studied (with $\kappa = 0$ and $\kappa > 0$ [**16, 18, 21**]). A typical scaling in this case, with $\kappa = 0$, is

$$\frac{d\mathbf{X}^\varepsilon}{dt} = \frac{1}{\varepsilon}\mathbf{v}\left(\frac{t}{\varepsilon^2}, \frac{\mathbf{X}^\varepsilon(t)}{\varepsilon^\gamma}\right), \qquad \mathbf{X}^\varepsilon(0) = \mathbf{x}, \tag{2.4}$$

with $0 \le \gamma < 1$, so that now the random velocity field is varying on spatial scales large compared to ε, but possibly small on the overall scale if $\gamma \neq 0$. Then, as before, the process $\mathbf{X}^\varepsilon(t)$ converges weakly as $\varepsilon \to 0$ to a diffusion process. Its generator is given by

$$L_{\mathbf{x}} f(\mathbf{x}) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(\mathbf{x}) \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

This operator is self-adjoint because the velocity field $\mathbf{v}(\mathbf{x})$ is incompressible. The corresponding generator for a compressible flow has an extra drift term [**18, 16, 21**]. The diffusion matrix is given by

$$a_{ij}(\mathbf{x}) = \int_0^\infty E\left\{v_i(0,\mathbf{x})v_j(t,\mathbf{x}) + v_j(0,\mathbf{x})v_i(t,\mathbf{x})\right\}. \tag{2.5}$$

Expression (2.5) is called the Kubo formula. Of particular interest here is the system

$$\frac{d\mathbf{X}^\varepsilon}{dt} = \frac{1}{\varepsilon}\mathbf{v}\left(\mathbf{X}^\varepsilon(t) + \frac{t}{\varepsilon^2}\mathbf{u}\right), \qquad \mathbf{X}^\varepsilon(0) = \mathbf{x}, \tag{2.6}$$

where $\mathbf{v}(\mathbf{x})$ is a space homogeneous, mean zero divergence free random field. Then $\mathbf{X}^\varepsilon(t, \mathbf{x})$ converges to Brownian motion with the covariance matrix given by the Kubo formula

$$c_{ij}(0) = \int_{-\infty}^{\infty} E\left\{v_i(t\mathbf{u})v_j(0)\right\} dt, \tag{2.7}$$

This corresponds to the Taylor prediction (2.2) with the path $\mathbf{X} = \mathbf{u}t$ being frozen and independent of the random medium. We shall see in Section 3 why the path in (2.7) is deterministic. The joint law for two trajectories of (2.6) starting at two points $\mathbf{x}, \mathbf{x}'$ was also obtained in [**18, 21**]. The joint process $(\mathbf{X}^\varepsilon, \mathbf{X}'^\varepsilon)$ converges weakly to a diffusion process $(\mathbf{X}, \mathbf{X}')$ with generator

$$\tilde{L} f(\mathbf{x}, \mathbf{x}') = L_{\mathbf{x}} f + L_{\mathbf{x}'} f + L' f,$$

where the cross-term L' is

$$L' f = \frac{1}{2} c_{ij}(\mathbf{x}, \mathbf{x}') \frac{\partial^2 f}{\partial x_i \partial x'_j}.$$

Here

$$c_{ij}(\mathbf{x} - \mathbf{x}') = \sum_{i,j=1}^{d} \int_{-\infty}^{\infty} E\left\{v_i(0) v_j(\mathbf{x}' - \mathbf{x} + t\mathbf{u})\right\} dt.$$

Thus the difference function $\mathbf{Y}^\varepsilon = \mathbf{X}^\varepsilon - \mathbf{X}'^\varepsilon$ converges weakly to a diffusion process $\mathbf{Y}$ with generator

$$L_{\mathbf{Y}} f = (c_{ij}(0) - c_{ij}(\mathbf{y})) \frac{\partial^2 f}{\partial y_i \partial y_j}. \tag{2.8}$$

The behavior of the difference function for a stochastic flow on a compact manifold was considered in [**3**], where the properties of the two-point motion for various Lyapunov exponents were investigated.

The scaling in (1.1) is different from both that of (2.3) and (2.4). As we noted above, the trajectories of (1.1) stay close to the deterministic trajectories $\mathbf{X}=\mathbf{x}+\mathbf{u}t$. We shall show, however, that after a rescaling the dynamics of many quantities of interest can be reduced to a form similar to (2.4). For example, the rescaled difference function (1.2) for the trajectories of (1.1) converges to a diffusion process with the same generator (2.8).

3. Flow correlations

Let $\mathbf{X}^\varepsilon(t,\mathbf{x})$ and $\mathbf{X}^\varepsilon(t,\mathbf{x}-\varepsilon\mathbf{y})$ be two trajectories of (1.1) that are $\varepsilon\mathbf{y}$ apart initially. We consider the rescaled difference variable

$$\mathbf{Y}^\varepsilon(t,\mathbf{y},\mathbf{x})=\frac{\mathbf{X}^\varepsilon(t,\mathbf{x})-\mathbf{X}^\varepsilon(t,\mathbf{x}-\varepsilon\mathbf{y})}{\varepsilon},$$

which is the scaled separation of the particles at time t. Then $\mathbf{X}^\varepsilon(t,\mathbf{x})$, $\mathbf{Y}^\varepsilon(t,\mathbf{y},\mathbf{x})$ satisfy

$$\begin{aligned}\frac{d\mathbf{X}^\varepsilon}{dt}&=\mathbf{u}+\sqrt{\varepsilon}\mathbf{v}\left(\frac{\mathbf{X}^\varepsilon}{\varepsilon}\right),\ \mathbf{X}^\varepsilon(0)=\mathbf{x}\\ \frac{d\mathbf{Y}^\varepsilon}{dt}&=\frac{1}{\sqrt{\varepsilon}}\left\{\mathbf{v}\left(\frac{\mathbf{X}^\varepsilon}{\varepsilon}\right)-\mathbf{v}\left(\frac{\mathbf{X}^\varepsilon}{\varepsilon}-\mathbf{Y}^\varepsilon\right)\right\},\ \mathbf{Y}^\varepsilon(0)=\mathbf{y}.\end{aligned}\tag{3.1}$$

As we noted above the trajectory $\mathbf{X}^\varepsilon(t,\mathbf{x})$ is close to the deterministic trajectory $\mathbf{X}=\mathbf{x}+\mathbf{u}t$ for finite t, and so we introduce its rescaled fluctuations

$$\mathbf{Z}^\varepsilon(t,x)=\frac{\mathbf{X}^\varepsilon(t,\mathbf{x})-\mathbf{u}t-\mathbf{x}}{\varepsilon}.\tag{3.2}$$

Then the system (3.1) becomes

$$\begin{aligned}\frac{d\mathbf{Z}^\varepsilon}{dt}&=\frac{1}{\sqrt{\varepsilon}}\mathbf{v}\left(\frac{\mathbf{x}+\mathbf{u}t}{\varepsilon}+\mathbf{Z}^\varepsilon\right),\ \mathbf{Z}^\varepsilon(0)=0\\ \frac{d\mathbf{Y}^\varepsilon}{dt}&=\frac{1}{\sqrt{\varepsilon}}\left\{\mathbf{v}\left(\frac{\mathbf{x}+\mathbf{u}t}{\varepsilon}+\mathbf{Z}^\varepsilon\right)-\mathbf{v}\left(\frac{\mathbf{x}+\mathbf{u}t}{\varepsilon}+\mathbf{Z}^\varepsilon-\mathbf{Y}^\varepsilon\right)\right\},\ \mathbf{Y}^\varepsilon(0)=\mathbf{y}.\end{aligned}\tag{3.3}$$

This system has the general form (14.1) to which the limit theorem of Appendix A applies. The full statement of this theorem and the necessary assumptions on the random velocity field in the general case are recalled in Appendix A. We shall assume in particular that $\mathbf{v}(\mathbf{y})$ is a mean zero, divergence-free, space homogeneous and strongly mixing velocity field bounded in $C^3(R^d)$. Then the limit theorem implies the following.

Theorem 3.1. *The processes $\mathbf{Z}^\varepsilon$, and $\mathbf{Y}^\varepsilon$ converge weakly to the correlated diffusion processes $\mathbf{Z}$ and $\mathbf{Y}$ whose joint generator is*

$$\begin{aligned}Lf&=\frac{1}{2}\sum_{i,j=1}^{d}\left[\int_{-\infty}^{\infty}R_{ij}(\mathbf{u}t)dt\frac{\partial^2 f}{\partial z_i\partial z_j}+\int_{-\infty}^{\infty}\left(2R_{ij}(\mathbf{u}t)-R_{ij}(\mathbf{u}t+\mathbf{y})-R_{ij}(\mathbf{u}t-\mathbf{y})\right)dt\frac{\partial^2 f}{\partial y_i\partial y_j}\right.\\ &\left.+\int_{-\infty}^{\infty}\left(2R_{ij}(\mathbf{u}t)-2R_{ij}(\mathbf{u}t-\mathbf{y})\right)dt\frac{\partial^2 f}{\partial z_i\partial y_j}\right].\end{aligned}\tag{3.4}$$

Here the covariance tensor R_{ij} is defined by

$$E\left\{v_i(y)v_j(y+h)\right\}=R_{ij}(h),\tag{3.5}$$

The individual generators for Z and Y are:

$$L_Z f(z)=\frac{1}{2}\sum_{i,j=1}^{d}\int_{-\infty}^{\infty}R_{ij}(\mathbf{u}t)dt\frac{\partial^2 f}{\partial z_i\partial z_j},\tag{3.6}$$

and

$$L_Y f(y) = \sum_{i,j=1}^{d} \int_{-\infty}^{\infty} \left[R^s_{ij}(\mathbf{u}t) - R^s_{ij}(\mathbf{y} + \mathbf{u}t) \right] dt \frac{\partial^2 f}{\partial y_i \partial y_j}, \tag{3.7}$$

respectively, where

$$R^s_{ij} = \frac{R_{ij} + R_{ji}}{2}.$$

Thus the fluctuations $\mathbf{Z}^\varepsilon(t)$ converge to Brownian motion with covariance matrix given by the Kubo formula (2.7). It is clear in this case that the Kubo formula (2.7) is the expression (2.2) predicted by Taylor. The path $\mathbf{X}(t) = \mathbf{x} + \mathbf{u}t$ is now the deterministic trajectory of (1.1) around which $\mathbf{X}^\varepsilon(t, \mathbf{x})$ is fluctuating because these two paths are close to each other. The generator (3.7) for the separation process $\mathbf{Y}(t, \mathbf{y})$ coincides with the one given by (2.8), as expected. We note that the diffusion coefficient in (3.7) vanishes for $\mathbf{y} = c\mathbf{u}$, so that if two particles start at two nearby positions on the same deterministic trajectory then their separation is not changed by the flow in the limit. The generator L_Y is asymptotically close to L_Z for all $\mathbf{y}$ that have large component in the direction perpendicular to the mean flow $\mathbf{u}$. The reason for this is that when the two starting points are separated by a large distance in the direction normal to $\mathbf{u}$, their trajectories are almost independent, and the rescaled difference trajectory behaves like the fluctuations of each individual trajectory. The other end of the asymptotics, the small $\mathbf{y}$ behavior of L_Y, is related to the Jacobian of the map $\mathbf{x} \to \mathbf{X}^\varepsilon(t, \mathbf{x})$, and is described in detail in Section 10.

4. The Richardson function and its evolution

The Richardson function of a scalar $\phi(t, \mathbf{x})$ advected by a random flow

$$\frac{\partial \phi}{\partial t} + \mathbf{v}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} \phi = 0$$
$$\phi(0, \mathbf{x}) = \phi_0(\mathbf{x})$$

is defined [**25**] by

$$Q(t, \mathbf{x}, \mathbf{y}) = \phi(\mathbf{x})\phi(\mathbf{x} + \mathbf{y}).$$

It was predicted by Richardson [**29**] that the expectation of this function satisfies the usual diffusion equation in $\mathbf{y}$, in the long time limit. This problem was studied extensively by physicists (see [**19**] for an extensive review and references), especially for δ-correlated in time velocity fields, but to the best of our knowledge the only mathematical results are those in [**18**] and [**24, 28**]. Molchanov and Piterbarg [**24**] considered in particular a convection-diffusion equation of the form

$$\frac{\partial T^\varepsilon}{\partial t} + \frac{1}{\varepsilon} \mathbf{v}(\frac{t}{\varepsilon^2}, \mathbf{x}) \cdot \nabla_{\mathbf{x}} T^\varepsilon = \kappa \Delta T^\varepsilon \tag{4.1}$$
$$T^\varepsilon(0, \mathbf{x}) = T_0(\mathbf{x}).$$

They argued that the expectation $E\{Q^\varepsilon(\mathbf{x}, \mathbf{y})\}$ of the Richardson function has a limit $Q(\mathbf{x}, \mathbf{y})$ as $\varepsilon \to 0$. If the initial density $T_0(\mathbf{x})$ is a space homogeneous isotropic random process with correlation function $Q_0(r)$, the limit Richardson function satisfies the diffusion equation

$$\frac{\partial Q}{\partial t} = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} r^{d-1} (2\kappa + F(r)) \frac{\partial Q}{\partial r} \tag{4.2}$$
$$Q(0, r) = Q_0(r).$$

Here

$$F(r) = \int_{-\infty}^{\infty} (R_L(t, 0) - R_L(t, r))\, dt,$$

and $R_L(t,r)$ comes from the correlation matrix $R(t,\mathbf{x})$ of the isotropic velocity field $\mathbf{v}(\mathbf{x})$:

$$R_{ij}=\left(R_L(t,r)+\frac{r}{d-1}\frac{\partial R_L}{\partial r}\right)\delta_{ij}-\frac{x_ix_j}{r(d-1)}\frac{\partial R_L}{\partial r}.$$

Equation (4.2) is the generalization of (2.8) for $\kappa\neq 0$. We note that the additional term due to non-zero molecular diffusivity is additive. This is different from the homogenization formulas of the type (2.2) which come from the homogenization scaling of (2.3).

We study here the rescaled Richardson function of oscillatory initial densities advected by the random flow. Let $\phi^\varepsilon(\mathbf{x})$ be the solution of

$$\begin{aligned}&\frac{\partial\phi^\varepsilon}{\partial t}+\left(\mathbf{u}+\sqrt{\varepsilon}\mathbf{v}\left(\frac{\mathbf{x}}{\varepsilon}\right)\right)\cdot\nabla\phi^\varepsilon=0\\&\phi^\varepsilon(0,\mathbf{x})=\phi_0^\varepsilon(\mathbf{x}),\end{aligned}\tag{4.3}$$

where the initial density $\phi_0^\varepsilon(\mathbf{x})$ is ε-oscillatory (see Appendix B for the precise definition of this notion). It may be either deterministic, or random but independent of $\mathbf{v}(\mathbf{x})$. An example of such random initial data is

$$\phi_0^\varepsilon(\mathbf{x})=\phi_0(\frac{\mathbf{x}}{\varepsilon}),\tag{4.4}$$

where $\phi_0(\mathbf{x})$ is a spatially homogeneous ergodic random process. Interesting deterministic initial data of oscillatory form are the localized initial densities

$$\phi_0^\varepsilon(\mathbf{x})=\frac{1}{\varepsilon^{d/2}}\phi_0(\frac{\mathbf{x}}{\varepsilon}),\tag{4.5}$$

where $\phi_0(\mathbf{x})$ is an $L^1(R^d)\cap L^2(R^d)$ function, and the inhomogeneous wave family

$$\phi^\varepsilon(\mathbf{x})=A(\mathbf{x})e^{iS(\mathbf{x})/\varepsilon}\tag{4.6}$$

where $A(\mathbf{x})$ and $S(\mathbf{x})$ are smooth functions, and $S(\mathbf{x})$ is real valued. The latter family describes the distribution of tracers which have the form of high frequency waves propagating in the direction $\nabla S(\mathbf{x})$ with amplitude $A(\mathbf{x})$. A particular case of (4.6) is the high frequency plane wave

$$\phi^\varepsilon(\mathbf{x})=Ae^{i\mathbf{x}\cdot\mathbf{p}/\varepsilon}.\tag{4.7}$$

The rescaled Richardson function of the family $\phi^\varepsilon(\mathbf{x})$ is defined by

$$\hat{W}^\varepsilon(t,\mathbf{x},\mathbf{y})=Q(t,\mathbf{x}-\varepsilon\mathbf{y},\varepsilon\mathbf{y})=\phi^\varepsilon(t,\mathbf{x}-\varepsilon\mathbf{y})\phi^{\varepsilon*}(t,\mathbf{x}),\tag{4.8}$$

so that its expectation is the correlation function of the filed $\phi^\varepsilon(\mathbf{x})$ at two nearby points separated by $\varepsilon\mathbf{y}$. It has a weak limit $\hat{W}(t,\mathbf{x},\mathbf{y})$ as $\varepsilon\to 0$ in the space $\mathcal{A}'$ dual to (see Appendix B)

$$\mathcal{A}=\left\{f(\mathbf{x},\mathbf{y}):\int d\mathbf{y}\sup_{\mathbf{x}}|f(\mathbf{x},\mathbf{y})|<\infty\right\},$$

introduced in **[22]**. The Richardson function of the family (4.4) is

$$\hat{W}(\mathbf{x},\mathbf{y})=R_0(\mathbf{y}),\tag{4.9}$$

with probability one, by the ergodic theorem, and is independent of $\mathbf{x}$. Here $R_0(\mathbf{x})$ is the covariance function of the random process $\phi_0(\mathbf{x})$. The Richardson functions of the localized deterministic family (4.5) is

$$\hat{W}(\mathbf{x},\mathbf{y})=\delta(\mathbf{x})\int\hat{Q}_0(\mathbf{z},\mathbf{y})d\mathbf{z},\tag{4.10}$$

where

$$\hat{Q}_0(\mathbf{x},\mathbf{y})=\phi_0(\mathbf{x}-\mathbf{y})\phi_0^*(\mathbf{x}).\tag{4.11}$$

The limit Richardson function of the WKB family (4.6) is

$$\hat{W}(\mathbf{x},\mathbf{y})=|A(\mathbf{x})|^2e^{-i\nabla S(\mathbf{x})\cdot\mathbf{y}},\tag{4.12}$$

which reduces to

$$\hat{W}(\mathbf{x},\mathbf{y}) = |A|^2 e^{-i\mathbf{p}\cdot\mathbf{y}} \tag{4.13}$$

for the plane waves (4.7). The limits (4.10) and (4.12) correspond to localization in space and direction, respectively.

The rescaled Richardson function $\hat{W}^\varepsilon(t,\mathbf{x},\mathbf{y})$ of the oscillatory field $\phi^\varepsilon(t,\mathbf{x})$ may be reduced to the form studied in [**18, 24**], when the initial data has the form (4.4) and $\phi_0(\mathbf{x})$ is Gaussian. In that case $\phi^\varepsilon(t,\mathbf{x}) = \psi^\varepsilon(\dfrac{\mathbf{x}-\mathbf{u}t}{\varepsilon}, t)$, where $\psi^\varepsilon(t,\mathbf{x})$ satisfies an equation of the form (4.1) with $\kappa = 0$

$$\begin{aligned} &\frac{\partial\psi^\varepsilon}{\partial t} + \frac{1}{\varepsilon}\mathbf{v}\left(\mathbf{x} + \mathbf{u}\frac{t}{\varepsilon^2}\right)\cdot\nabla_{\mathbf{x}}\psi^\varepsilon = 0 \\ &\psi^\varepsilon(0,\mathbf{x}) = \phi_0(\mathbf{x}). \end{aligned} \tag{4.14}$$

The rescaled Richardson function $W^\varepsilon(t,\mathbf{x},\mathbf{y})$ of $\phi^\varepsilon(\mathbf{x})$ and the unscaled Richardson function $Q^\varepsilon(t,\mathbf{x},\mathbf{y})$ of $\psi^\varepsilon(\mathbf{x})$ are related by

$$W^\varepsilon(t,\mathbf{x},\mathbf{y}) = Q^\varepsilon\left(t, \frac{\mathbf{x}-\mathbf{u}t}{\varepsilon}, \mathbf{y}\right).$$

When the initial data for (4.3) is random, space homogeneous of the form (4.4), both the scaled and unscaled Richardson functions are independent of $\mathbf{x}$ in the limit $\varepsilon \to 0$, hence they coincide in this limit, and the analysis for (4.3) can be reduced to the one in [**24**] and [**18**]. We consider here a more general class of ε-oscillatory initial data, not necessarily random and Gaussian, and also a more general scaling in Section 9. The following theorem shows that a version of (2.8) and (4.2) still holds.

THEOREM 4.1. *Let $\phi^\varepsilon(t,\mathbf{x})$ satisfy (4.3) with the initial data $\phi_0^\varepsilon(\mathbf{x})$ being either*

(i) *random of the form (4.4) with $\phi_0(\mathbf{x})$ spatially homogeneous and independent of $\mathbf{v}(\mathbf{x})$, or*
(ii) *deterministic of the form (4.5) with $\phi_0(\mathbf{x}) \in L^2(R^d)$, or*
(iii) *the plane wave (4.7).*

Let $\hat{W}(t,\mathbf{x},\mathbf{y})$ be the weak limit of the expectation of the rescaled Richardson function $\hat{W}^\varepsilon(t,\mathbf{x},\mathbf{y})$ in $\mathcal{A}'$ as $\varepsilon \to 0$, and $\hat{W}_0(\mathbf{x},\mathbf{y})$ be the limit of the expectation of the scaled Richardson function for $\phi_0^\varepsilon(\mathbf{x})$. Then $\hat{W}(t,\mathbf{x},\mathbf{y})$ satisfies the diffusion equation with drift:

$$\begin{aligned} &\frac{\partial\hat{W}}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\hat{W} = L_Y\hat{W} \\ &\hat{W}(0,\mathbf{x},\mathbf{y}) = \hat{W}_0(\mathbf{x},\mathbf{y}), \end{aligned} \tag{4.15}$$

where the operator L_Y is given by (3.7). The initial data $\hat{W}_0(\mathbf{x},\mathbf{y})$ in (4.15) are given by (4.9) in the random case, and by (4.10) and (4.13) in the two deterministic cases.

We shall give the proof of Theorem 4.1 for the deterministic case (ii) of a localized pulse, with the modification for random initial data (4.4) and the plane waves (4.7) being routine. Our proof given in Section 5 is in several steps. First, we obtain an equation for $\hat{W}^\varepsilon(t,\mathbf{x},\mathbf{y})$ in the form (14.1) but with initial data $\hat{W}^\varepsilon(0,\mathbf{x},\mathbf{y})$ depending on ε. In the second step we show that these initial data may be replaced by the limit rescaled Richardson function $\hat{W}_0(\mathbf{x},\mathbf{y})$. Finally we apply the limit theorem of Appendix A to obtain equation (4.15) for $\hat{W}(t,\mathbf{x},\mathbf{y}) = \lim\limits_{\varepsilon\to 0} E\left\{W^\varepsilon(t,\mathbf{x},\mathbf{y})\right\}$.

We believe that this theorem is true for a much wider class of initial data than (4.4), (4.5) and (4.7). The technical difficulty in generalizing this result arises in the second step of the proof, that is, replacing the initial data $\hat{W}_0^\varepsilon$ by the limit $\hat{W}_0$. The limit theorem of Appendix A does not provide any information on the uniformity of the convergence to the limit process with respect to the starting point in all of R^n. In particular, we do not have uniform bounds for the probability to visit a fixed set, which are needed in the general case when we cannot control the convergence of $\hat W_0^\varepsilon(\mathbf{x},\mathbf{y})$ to $\hat W_0(\mathbf{x},\mathbf{y})$. This does not allow us to include the physically important case of general wave data (4.6) in Theorem 4.1.

5. The expectation of the Richardson function

Let the initial data for (4.3) be of the form (4.5). Then $\hat W_0^\varepsilon(\mathbf{x},\mathbf{y}) = \frac{1}{\varepsilon^d}\hat Q_0(\frac{\mathbf{x}}{\varepsilon},\mathbf{y})$, and equation (4.3) implies that $\hat W^\varepsilon$ satisfies the initial value problem

$$\frac{\partial \hat W^\varepsilon}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\hat W^\varepsilon + \sqrt{\varepsilon}\mathbf{v}\left(\frac{\mathbf{x}}{\varepsilon}\right)\cdot\nabla_{\mathbf{x}}\hat W^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\left\{\mathbf{v}\left(\frac{\mathbf{x}}{\varepsilon}\right) - \mathbf{v}\left(\frac{\mathbf{x}}{\varepsilon}-\frac{\mathbf{y}}{\varepsilon}\right)\right\}\cdot\nabla_{\mathbf{y}}\hat W^\varepsilon = 0 \tag{5.1}$$

with

$$\hat W^\varepsilon(0,\mathbf{x},\mathbf{y}) = \frac{1}{\varepsilon^d}\hat Q_0(\frac{\mathbf{x}}{\varepsilon},\mathbf{y}), \tag{5.2}$$

where $\hat Q_0$ is given by (4.11). The solution of (5.1) is given explicitly by

$$\hat W^\varepsilon(t,\mathbf{x},\mathbf{y}) = \hat W_0^\varepsilon(\mathbf{x}-\mathbf{u}t+\varepsilon\mathbf{Z}^\varepsilon(-t,\mathbf{x}),\mathbf{Y}^\varepsilon(-t,\mathbf{y},\mathbf{x})).$$

Let

$$\hat W'^\varepsilon(t,\mathbf{x},\mathbf{y}) = \hat W_0(\mathbf{x}-\mathbf{u}t+\varepsilon\mathbf{Z}^\varepsilon(-t,\mathbf{x}),\mathbf{Y}^\varepsilon(-t,\mathbf{y},\mathbf{x}))$$

be the solution of (5.2) with the initial data $\hat W_0^\varepsilon(\mathbf{x},\mathbf{y})$ replaced by $\hat W_0(\mathbf{x},\mathbf{y})$, which is given by (4.10). We claim that

$$E\{\hat W^\varepsilon(t,\mathbf{x},\mathbf{y}) - \hat W'^\varepsilon(t,\mathbf{x},\mathbf{y})\} \to 0 \tag{5.3}$$

weakly in $\mathcal{A}'$ as $\varepsilon\to 0$. Let $f(\mathbf{x},\mathbf{y})\in\mathcal{A}$ be a continuous non-random test function of compact support. Then, since the flow is incompressible

$$\begin{aligned}
&E\left\{\iint d\mathbf{x}d\mathbf{y}(\hat W^\varepsilon(t,\mathbf{x},\mathbf{y}) - \hat W'^\varepsilon(t,\mathbf{x},\mathbf{y}))f(\mathbf{x},\mathbf{y})\right\}\\
&= \iint d\mathbf{x}d\mathbf{y}(\hat W_0^\varepsilon(\mathbf{x},\mathbf{y}) - \hat W_0(\mathbf{x},\mathbf{y}))E\left\{f(\mathbf{x}+\mathbf{u}t+\varepsilon\mathbf{Z}^\varepsilon(t,\mathbf{x}),\mathbf{Y}^\varepsilon(t,\mathbf{y},\mathbf{x}))\right\}\\
&= \iint d\mathbf{x}d\mathbf{y}(\frac{1}{\varepsilon^d}\hat Q_0(\frac{\mathbf{x}}{\varepsilon},\mathbf{y}) - \delta(\mathbf{x})\int d\mathbf{p}\hat Q_0(\mathbf{p},\mathbf{y}))E\left\{f(\mathbf{x}+\mathbf{u}t+\varepsilon\mathbf{Z}^\varepsilon(t,\mathbf{x}),\mathbf{Y}^\varepsilon(t,\mathbf{y},\mathbf{x}))\right\}\\
&= \iint d\mathbf{x}d\mathbf{y}\hat Q_0(\mathbf{x},\mathbf{y})E\left\{f(\varepsilon\mathbf{x}+\mathbf{u}t+\varepsilon\mathbf{Z}^\varepsilon(t,\varepsilon\mathbf{x}),\mathbf{Y}^\varepsilon(t,\mathbf{y},\varepsilon\mathbf{x})) - f(\mathbf{u}t+\varepsilon\mathbf{Z}^\varepsilon(t,0),\mathbf{Y}^\varepsilon(t,\mathbf{y},0))\right\}.
\end{aligned} \tag{5.4}$$

Theorem 3.1 implies that

$$E\left\{f(\mathbf{u}t+\varepsilon\mathbf{Z}^\varepsilon(t,0),\mathbf{Y}^\varepsilon(t,\mathbf{y},0))\right\} \to \bar f(0,\mathbf{y}), \tag{5.5}$$

strongly in the uniform norm on compact sets in $\mathbf{y}$. Here the function $\bar f(\mathbf{x},\mathbf{y})$ satisfies the diffusion equation with a drift

$$\begin{aligned}
\frac{\partial\bar f}{\partial t} &= \mathbf{u}\cdot\nabla_{\mathbf{x}}\bar f + L_Y\bar f\\
\bar f(0,\mathbf{x},\mathbf{y}) &= f(\mathbf{x},\mathbf{y}),
\end{aligned}$$

with the operator L_Y given by (3.7). The limit theorem of Appendix A applies to the system

$$\begin{aligned}
\frac{d\tilde{\mathbf{Z}}^\varepsilon}{dt} &= \frac{1}{\sqrt{\varepsilon}}\mathbf{v}\left(\frac{\mathbf{u}t}{\varepsilon}+\tilde{\mathbf{Z}}^\varepsilon\right),\ \tilde{\mathbf{Z}}^\varepsilon(0)=\mathbf{x}\\
\frac{d\tilde{\mathbf{Y}}^eps}{dt} &= \frac{1}{\sqrt{\varepsilon}}\left[\mathbf{v}\left(\frac{\mathbf{u}t}{\varepsilon}+\tilde{\mathbf{Z}}^\varepsilon\right) - \mathbf{v}\left(\frac{\mathbf{u}t}{\varepsilon}+\tilde{\mathbf{Z}}^\varepsilon-\tilde{\mathbf{Y}}^\varepsilon\right)\right],
\end{aligned}$$

where $\tilde{\mathbf{Z}}^\varepsilon(t,\mathbf{x}) = \mathbf{x} + \mathbf{Z}^\varepsilon(t,\varepsilon\mathbf{x})$, and $\tilde{\mathbf{Y}}^\varepsilon(t,\mathbf{y},\mathbf{x}) = \mathbf{Y}^\varepsilon(t,\mathbf{y},\varepsilon\mathbf{x})$. It implies that the expectation

$$E\left\{f(\varepsilon\mathbf{x} + \mathbf{u}t + \varepsilon\mathbf{Z}^\varepsilon(t,\varepsilon\mathbf{x}), \mathbf{Y}^\varepsilon(t,\mathbf{y},\varepsilon\mathbf{x}))\right\}$$

has the same limit as in (5.5). Then by the dominated convergence theorem (5.5) vanishes in the limit $\varepsilon \to 0$ since $\hat{Q}_0(\mathbf{x},\mathbf{y}) \in L^1(R^d \times R^d)$. Thus (5.3) holds, and we may replace the initial data $\hat{W}_0^\varepsilon$ by $\hat{W}_0$ in (5.1), and

$$\hat{W}(t,\mathbf{x},\mathbf{y}) = \lim_{\varepsilon\to 0} E\left\{\hat{W}^\varepsilon(t,\mathbf{x},\mathbf{y})\right\} = \lim_{\varepsilon\to 0} E\left\{\hat{W}'^\varepsilon(t,\mathbf{x},\mathbf{y})\right\}.$$

The limit theorem applies to the weak form of the resulting system, that is, for any test function $f(\mathbf{x},\mathbf{y})$ we have

$$\begin{aligned}\iint E\left\{\hat{W}'^\varepsilon(t,\mathbf{x},\mathbf{y})\right\} f(\mathbf{x},\mathbf{y}) d\mathbf{x} d\mathbf{y} &= \iint \hat{W}_0(\mathbf{x},\mathbf{y}) E\left\{f(\mathbf{x} + \mathbf{u}t + \varepsilon\mathbf{Z}^\varepsilon(t,\mathbf{x}), \mathbf{Y}^\varepsilon(t,\mathbf{x},\mathbf{y}))\right\} d\mathbf{x} d\mathbf{y} \\ = \iint d\mathbf{x} d\mathbf{y} \hat{Q}_0(\mathbf{x},\mathbf{y}) E\left\{f(\mathbf{u}t + \varepsilon\mathbf{Z}^\varepsilon(t,0), \mathbf{Y}^\varepsilon(t,0,\mathbf{y}))\right\} d\mathbf{x} d\mathbf{y} &\to \iint d\mathbf{x} d\mathbf{y} \hat{Q}_0(\mathbf{x},\mathbf{y}) \bar{f}(\mathbf{u}t,\mathbf{y}) d\mathbf{x} d\mathbf{y}\end{aligned}$$

by the dominated convergence theorem. Thus the function $\hat{W}(t,\mathbf{x},\mathbf{y})$ is the weak solution of

$$\begin{aligned}&\frac{\partial \hat{W}}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\hat{W} = L_Y\hat{W} \\ &\hat{W}(0,\mathbf{x},\mathbf{y}) = \hat{W}_0(\mathbf{x},\mathbf{y}).\end{aligned} \tag{5.6}$$

Here the operator L_Y is given by (3.7). This finishes the proof of Theorem 4.1.

6. Higher moment equations

We show how the joint behavior of trajectories starting at several points may be studied using the limit theorem of Appendix A. Consider first two pairs of trajectories, $\mathbf{X}^\varepsilon(t,\mathbf{x}_1)$, $\mathbf{X}^\varepsilon(t,\mathbf{x}_1 - \varepsilon\mathbf{y}_1)$, and $\mathbf{X}^\varepsilon(t,\mathbf{x}_2)$, $\mathbf{X}^\varepsilon(t,\mathbf{x}_2 - \varepsilon\mathbf{y}_2)$. Then the corresponding processes $\mathbf{Z}_j^\varepsilon$ and $\mathbf{Y}_j^\varepsilon$ satisfy the following system:

$$\begin{aligned}\frac{d\mathbf{Z}_1^\varepsilon}{dt} &= \frac{1}{\sqrt{\varepsilon}}\mathbf{v}\left(\frac{\mathbf{x}_1 + \mathbf{u}t}{\varepsilon} + \mathbf{Z}_1^\varepsilon\right),\ \mathbf{Z}_1^\varepsilon(0) = 0 \\ \frac{d\mathbf{Y}_1^\varepsilon}{dt} &= \frac{1}{\sqrt{\varepsilon}}\left\{\mathbf{v}\left(\frac{\mathbf{x}_1 + \mathbf{u}t}{\varepsilon} + \mathbf{Z}_1^\varepsilon\right) - \mathbf{v}\left(\frac{\mathbf{x}_1 + \mathbf{u}t}{\varepsilon} + \mathbf{Z}_1^\varepsilon - \mathbf{Y}_1^\varepsilon\right)\right\},\ \mathbf{Y}_1^\varepsilon(0) = \mathbf{y}_1 \\ \frac{d\mathbf{Z}_2^\varepsilon}{dt} &= \frac{1}{\sqrt{\varepsilon}}\mathbf{v}\left(\frac{\mathbf{x}_2 + \mathbf{u}t}{\varepsilon} + \mathbf{Z}_2^\varepsilon\right),\ \mathbf{Z}_2^\varepsilon(0) = 0 \\ \frac{d\mathbf{Y}_2^\varepsilon}{dt} &= \frac{1}{\sqrt{\varepsilon}}\left\{\mathbf{v}\left(\frac{\mathbf{x}_2 + \mathbf{u}t}{\varepsilon} + \mathbf{Z}_2^\varepsilon\right) - \mathbf{v}\left(\frac{\mathbf{x}_2 + \mathbf{u}t}{\varepsilon} + \mathbf{Z}_2^\varepsilon - \mathbf{Y}_2^\varepsilon\right)\right\},\ \mathbf{Y}_2^\varepsilon(0) = \mathbf{y}_2.\end{aligned} \tag{6.1}$$

The joint process $(\mathbf{Z}_1^\varepsilon, \mathbf{Y}_1^\varepsilon, \mathbf{Z}_2^\varepsilon, \mathbf{Y}_2^\varepsilon)$ converge to the process $(\mathbf{Z}_1, \mathbf{Y}_1, \mathbf{Z}_2, \mathbf{Y}_2)$ with generator of the form

$$L^{(2)} = L_1 + L_2 + L_{12},$$

where the operators L_1 and L_2 are given by (3.4) in the variables $(\mathbf{z}_1,\mathbf{y}_1)$ and $(\mathbf{z}_2,\mathbf{y}_2)$, respectively. The cross term L_{12} involves terms of the type

$$c_{ij}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{y}_1,\mathbf{y}_2)\frac{\partial^2}{\partial z_1^i \partial y_2^j},\ d_{ij}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{y}_1,\mathbf{y}_2)\frac{\partial^2}{\partial y_1^i \partial y_2^j},$$

and other similar ones. The coefficients c_{ij}, d_{ij}, ... are non-zero only for points $\mathbf{x}_1$, $\mathbf{x}_2$ which lie on the same deterministic trajectory, that is, $\mathbf{x}_1 - \mathbf{x}_2 = \lambda\mathbf{u}$ for some $\lambda \in R$. Thus when the points $\mathbf{x}_1$ and $\mathbf{x}_2$ do not lie on the same deterministic trajectory, the pairs of the limit processes $\mathbf{Z}_1$, $\mathbf{Y}_1$ and $\mathbf{Z}_2$, $\mathbf{Y}_2$ are independent and their joint generator is the sum of the corresponding generators:

$$L^{(2)} = L_1 + L_2.$$

This phenomenon occurs also for all higher moments.

Let us consider now the case when the two trajectories start at the same point $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}$, so that $\mathbf{Z}_1^\varepsilon = \mathbf{Z}_2^\varepsilon$. Then the system (6.1) reduces to

$$\begin{aligned} \frac{d\mathbf{Z}^\varepsilon}{dt} &= \frac{1}{\sqrt{\varepsilon}}\mathbf{v}\left(\frac{\mathbf{x}+\mathbf{u}t}{\varepsilon}+\mathbf{Z}^\varepsilon\right), \ \mathbf{Z}^\varepsilon(0)=0 \\ \frac{d\mathbf{Y}_1^\varepsilon}{dt} &= \frac{1}{\sqrt{\varepsilon}}\left\{\mathbf{v}\left(\frac{\mathbf{x}+\mathbf{u}t}{\varepsilon}+\mathbf{Z}^\varepsilon\right)-\mathbf{v}\left(\frac{\mathbf{x}+\mathbf{u}t}{\varepsilon}+\mathbf{Z}^\varepsilon-\mathbf{Y}_1^\varepsilon\right)\right\}, \ \mathbf{Y}_1^\varepsilon(0)=\mathbf{y}_1 \\ \frac{d\mathbf{Y}_2^\varepsilon}{dt} &= \frac{1}{\sqrt{\varepsilon}}\left\{\mathbf{v}\left(\frac{\mathbf{x}+\mathbf{u}t}{\varepsilon}+\mathbf{Z}^\varepsilon\right)-\mathbf{v}\left(\frac{\mathbf{x}+\mathbf{u}t}{\varepsilon}+\mathbf{Z}^\varepsilon-\mathbf{Y}_2^\varepsilon\right)\right\}, \ \mathbf{Y}_2^\varepsilon(0)=\mathbf{y}_2. \end{aligned} \tag{6.2}$$

The joint generator for the limit processes $\mathbf{Y}_1$, $\mathbf{Y}_2$ has the form:

$$L^{(2)}f(\mathbf{y}_1,\mathbf{y}_2) = \frac{1}{2}a_{ij}^{\alpha\beta}(\mathbf{y}_1,\mathbf{y}_2)\frac{\partial^2 f}{\partial y_i^\alpha \partial y_j^\beta},$$

where the Greek indices label the points and the Latin indices label the coordinates. The coefficients $a_{ij}^{\alpha\beta}(\mathbf{y}_1,\mathbf{y}_2)$ are given by

$$a_{ij}^{\alpha\beta}(\mathbf{y}_1,\mathbf{y}_2) = \int_{-\infty}^{\infty}\left[R_{ij}(\mathbf{u}t) - R_{ij}(\mathbf{u}t-\mathbf{y}_\alpha) - R_{ji}(\mathbf{u}t-\mathbf{y}_\beta) + R_{ij}(\mathbf{u}t-\mathbf{y}_\beta+\mathbf{y}_\alpha)\right]dt.$$

Let $\hat{G}(t,\mathbf{x},\mathbf{y}_1,\mathbf{y}_2)$ be the second moment of the Richardson function for the solution of (4.3)

$$\begin{aligned} \hat{G}(t,\mathbf{x},\mathbf{y}_1,\mathbf{y}_2) &= \lim_{\varepsilon\to 0} E\left\{\hat{W}^\varepsilon(t,\mathbf{x},\mathbf{y}_1)\hat{W}^\varepsilon(t,\mathbf{x},\mathbf{y}_2)\right\} \\ &= \lim_{\varepsilon\to 0} E\left\{\phi^\varepsilon(t,\mathbf{x}-\varepsilon\mathbf{y}_1)\phi^\varepsilon(t,\mathbf{x})\phi^\varepsilon(t,\mathbf{x}-\varepsilon\mathbf{y}_2)\phi^\varepsilon(t,\mathbf{x})\right\}. \end{aligned}$$

Assume that the initial data is of the form (4.4) with $\phi_0(\mathbf{x})$ being Gaussian. Then $\hat{G}$ satisfies the initial value problem:

$$\begin{aligned} &\frac{\partial \hat{G}}{\partial t} = L^{(2)}\hat{G}, \\ &\hat{G}(0,\mathbf{y}_1,\mathbf{y}_2) = 2R_0(\mathbf{y}_1)R_0(\mathbf{y}_2) + R_0(\mathbf{y}_1-\mathbf{y}_2)R_0(0). \end{aligned} \tag{6.3}$$

All the higher moment equations for the Richardson function can be derived in a similar way.

7. The Wigner distribution and connections with radiative transport theory

The Wigner distribution of an oscillatory family $\phi^\varepsilon(\mathbf{x})$ is defined as the inverse Fourier transform of the rescaled Richardson function

$$W^\varepsilon(t,\mathbf{x},\mathbf{k}) = \int \frac{d\mathbf{y}}{(2\pi)^d}e^{i\mathbf{k}\cdot\mathbf{y}}\phi^\varepsilon(t,\mathbf{x}-\varepsilon\mathbf{y})\phi^{\varepsilon *}(t,\mathbf{x}). \tag{7.1}$$

The Wigner transform has a weak limit $W(t,\mathbf{x},\mathbf{k})$ as $\varepsilon \to 0$, which is a non-negative measure **[14]** (see Appendix B). The limit Wigner distribution may be interpreted as the limit energy (or particle) density of an ε-oscillatory family $\phi^\varepsilon(\mathbf{x})$, travelling in the direction $\mathbf{k}$ at position $\mathbf{x}$. In particular, we have

$$\lim_{\varepsilon\to 0}|\phi^\varepsilon(\mathbf{x})|^2 = \int W(t,\mathbf{x},\mathbf{k})d\mathbf{k} \tag{7.2}$$

in the weak sense if and only if $\phi^\varepsilon(\mathbf{x})$ is ε-oscillatory. The limit Wigner distribution also determines the limit values as $\varepsilon \to 0$ of various others quantities of interest, for example, the correlation functions of the derivatives of ϕ_ε. The limit of the correlation matrix of the gradient

$$\tilde{W}^\varepsilon_{ij}(t,\mathbf{x},\mathbf{y}) = \varepsilon\frac{\partial\phi^\varepsilon}{\partial x_i}(\mathbf{x}-\varepsilon\mathbf{y})\varepsilon\frac{\partial\phi^{*\varepsilon}}{\partial x_j}(\mathbf{x})$$

is the Hessian in $\mathbf{y}$ of $\hat{W}(t,\mathbf{x},\mathbf{k})$:

$$\tilde{W}^\varepsilon_{ij}(t,\mathbf{x},\mathbf{y}) \to \frac{\partial^2\hat{W}}{\partial y_i\partial y_j}. \tag{7.3}$$

The limits of the higher order derivatives can be similarly expressed via $W(t,\mathbf{x},\mathbf{k})$. Thus, in the high frequency limit the Wigner distribution gives a complete description both of the field $\phi^\varepsilon(t,\mathbf{x})$ itself and its derivatives. Relations of the type (7.2) and (7.3) are the main reason for the recent studies of the Wigner distribution for waves in random media [**30**].

Let us briefly recall how radiative transport equations for waves arise [**30**]. Let $\mathbf{w}^\varepsilon(t,\mathbf{x}) \in C^N$ be the solution of a symmetric hyperbolic system (this scaling corresponds to $\beta = 1$ in (4.3))

$$\begin{aligned} &\left\{A(\mathbf{x}) + \sqrt{\varepsilon}V\left(\frac{\mathbf{x}}{\varepsilon}\right)\right\}\frac{\partial\mathbf{w}^\varepsilon}{\partial t} + D^j\frac{\partial\mathbf{w}^\varepsilon}{\partial x^j} = 0\\ &\mathbf{w}^\varepsilon(0,\mathbf{x}) = \mathbf{w}^\varepsilon_0(\mathbf{x}). \end{aligned} \tag{7.4}$$

Here the matrix $A(\mathbf{x})$ is positive definite and the matrices D^j are symmetric. The initial data $\mathbf{w}^\varepsilon_0(\mathbf{x})$ is assumed to be ε-oscillatory and deterministic, and $V(\mathbf{y})$ is a matrix valued, space homogeneous random process. Then the $N\times N$ limit Wigner matrix $W(t,\mathbf{x},\mathbf{k})$ has a special form

$$W(t,\mathbf{x},\mathbf{k}) = \sum_{\alpha,i,j} W^\alpha_{ij}(t,\mathbf{x},\mathbf{k})\mathbf{b}^{\alpha,i}\mathbf{b}^{*\alpha,j}. \tag{7.5}$$

Here the vectors $\mathbf{b}^{\alpha,j} \in C^N$ form a basis for the eigenspace of the dispersion matrix

$$L(\mathbf{x},\mathbf{k}) = A^{-1}(\mathbf{x})k_jD^j$$

of the system (7.4), corresponding to the eigenvalue ω_α. The size of the square matrices W^α is equal to the degeneracy of the eigenvalue. They satisfy a system of radiative transport equations. This system is decoupled when the eigenvalues are simple and $\omega_\alpha(\mathbf{k}) \neq \omega_\beta(\mathbf{p})$ for all β, $\mathbf{k}$ and $\mathbf{p}$. Then the radiative transport equation for the scalar W^α has the form

$$\begin{aligned} &\frac{\partial W^\alpha}{\partial t} + \nabla_\mathbf{k}\omega_\alpha\cdot\nabla_\mathbf{x}W^\alpha - \nabla_\mathbf{x}\omega_\alpha\cdot\nabla_\mathbf{k}W^\alpha\\ &= \int d\mathbf{k}'\sigma(\mathbf{x},\mathbf{k},\mathbf{k}')\delta(\omega_\alpha(\mathbf{k})-\omega_\alpha(\mathbf{k}'))(W^\alpha(\mathbf{k}')-W^\alpha(\mathbf{k})). \end{aligned} \tag{7.6}$$

The function $\sigma(\mathbf{x},\mathbf{k},\mathbf{k}')$ is the differential scattering cross-section and is determined by the power spectrum of $V(\mathbf{y})$. Equation (7.6) has a long history. It was proposed phenomenologically by Rayleigh in the beginning of this century and then studied extensively by physicists [**10**]. Various derivations of this equation were given in 1960's (see [**30**] for references) when $\mathbf{w}^\varepsilon$ is a solution of the wave equation or Maxwell's equations. A general case was treated in [**30**] but the results were derived only formally. The only case when the radiative transport equation was obtained rigorously, to the best of our knowledge, is for the Schrödinger equation with $V(\mathbf{y})$ Gaussian and only for small t [**11, 15**].

For the Wigner distribution of the density of a passive scalar in a random flow we have the following theorem.

THEOREM 7.1. *Let $\phi^\varepsilon(t,\mathbf{x})$ be the solution of (4.3), with the initial data of the form (4.4), (4.5) or (4.7). Let $W_0(\mathbf{x},\mathbf{k})$ be the expectation of the limit Wigner distribution of $\phi_0^\varepsilon(\mathbf{x})$. Then $E\{W^\varepsilon(t,\mathbf{x},\mathbf{k})\}$ converges to $W(t,\mathbf{x},\mathbf{k})$ weakly in $\mathcal{A}'$, where $W(t,\mathbf{x},\mathbf{k})$ satisfies the radiative transport equation*

$$\frac{\partial W}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{x}} W = \int \frac{d\mathbf{k}'}{(2\pi)^{d-1}} k_i k_j \hat{R}^s_{ij}(\mathbf{k}'-\mathbf{k})\delta((\mathbf{k}'-\mathbf{k})\cdot\mathbf{u})(W(\mathbf{k}')-W(\mathbf{k})) \tag{7.7}$$

with the initial condition

$$W(0,\mathbf{x},\mathbf{k}) = W_0(\mathbf{x},\mathbf{k}).$$

This theorem follows immediately from Theorem 4.1 by applying the inverse Fourier transform to (5.6). Equation (7.7) has the usual form of a radiative transport equation (7.6). The dispersion law of (4.3) is

$$\omega(\mathbf{k}) = \mathbf{u}\cdot\mathbf{k}.$$

The scattering operator on the right side of (7.7) is symmetric since $(\mathbf{k},\hat{R}(\mathbf{k}-\mathbf{k}')\mathbf{k}) = (\mathbf{k}',\hat{R}(\mathbf{k}'-\mathbf{k})\mathbf{k}')$ because of the incompressibility of the random field $\mathbf{v}(\mathbf{y})$. The transport equation is valid globally in time and for random velocity fields that are not necessarily Gaussian. This tells us that in more general cases the radiative transport equation should also be valid globally in time and for non-Gaussian random fluctuations. It should also be valid for general inhomogeneous high frequency waves of the form (4.6), the restriction to plane wave initial data is technical as explained in the remarks after Theorem 4.1.

8. Diffusion–transport duality

The radiative transport equations (7.6) for waves have a nice interpretation in terms of a certain Markovian jump process [**5**]. Consider the backward characteristics of (7.6)

$$\begin{aligned}\dot{\mathbf{X}} &= -\nabla_{\mathbf{k}}\omega(\mathbf{X},\mathbf{K}),\\ \dot{\mathbf{K}} &= \nabla_{\mathbf{x}}\omega(\mathbf{X},\mathbf{K}),\end{aligned} \tag{8.1}$$

starting at $\mathbf{x}(0)=\mathbf{x}$ and $\mathbf{K}(0)=\mathbf{k}$. Let a particle move along a trajectory of (8.1) for a random time τ_1 and then switch wave vector randomly from $\mathbf{K}$ to $\mathbf{K}'$. After that it follows trajectories of (8.1) for a random time τ_2, at which moment it switches it direction again. The process is continued in an obvious manner. The probability distribution of the j-th jump time τ_j is

$$P\{\tau_j > t\} = \exp\left\{-\int_0^t \Sigma(\mathbf{x}(s,\mathbf{x}_{j-1},\mathbf{k}_{j-1}),\mathbf{k}(s,\mathbf{x}_{j-1},\mathbf{k}_{j-1}))ds\right\}$$

Here $\Sigma(\mathbf{x},\mathbf{k})$ is the total scattering cross-section:

$$\Sigma(\mathbf{x},\mathbf{k}) = \int d\mathbf{k}'\sigma(\mathbf{x},\mathbf{k}',\mathbf{k})\delta(\omega_\alpha(\mathbf{k})-\omega_\alpha(\mathbf{k}')),$$

and $\mathbf{x}(s,\mathbf{x}_{j-1},\mathbf{k}_{j-1})$, $\mathbf{k}(s,\mathbf{x}_{j-1},\mathbf{k}_{j-1})$ is the trajectory of (8.1) starting at the position and wave number of the previous jump. The probability density that the wave number jumps from direction $\mathbf{k}$ into direction $\mathbf{k}'$ is given by

$$p(\mathbf{x},\mathbf{k},\mathbf{k}') = \frac{\sigma(\mathbf{x},\mathbf{k},\mathbf{k}')}{\Sigma(\mathbf{x},\mathbf{k})}.$$

The resulting process is well defined if the total scattering cross-section is bounded from above, so that only a finite number of jumps occur during any given time interval, with probability one. The Kolmogorov equation for the resulting process is (7.6), that is, given any function $f(\mathbf{x},\mathbf{k})$, the function $\bar{f}(t,\mathbf{x},\mathbf{k}) = E\{f(\mathbf{X}(t,x,\mathbf{k}),\mathbf{K}(t,\mathbf{x},\mathbf{k}))\}$ is the solution of (7.6) with the initial data $\bar{f}(0,\mathbf{x},\mathbf{k}) = f(\mathbf{x},\mathbf{k})$.

The integral operator on the right side of (7.7)

$$\mathcal{Q}f = \int \frac{d\mathbf{k}'}{(2\pi)^{d-1}} k_i k_j \hat{R}^s_{ij}(\mathbf{k}' - \mathbf{k}) \delta((\mathbf{k}' - \mathbf{k}) \cdot \mathbf{u})(f(\mathbf{k}') - f(\mathbf{k})) \tag{8.2}$$

is of the same form as the integral operator in (7.7). It is the generator of a jump process for which the particle is moving along the deterministic trajectory $\mathbf{x}(t) = \mathbf{x}(0) + \mathbf{u}t$, while the wave vector is the jump process described above with the differential scattering cross-section

$$\sigma(\mathbf{k}, \mathbf{k}') = \frac{1}{(2\pi)^{d-1}} k_i k_j \hat{R}^s_{ij}(\mathbf{k}' - \mathbf{k}) \delta((\mathbf{k}' - \mathbf{k}) \cdot \mathbf{u})$$

and the total scattering cross-section

$$\Sigma(\mathbf{k}) = \int d\mathbf{k} \frac{1}{(2\pi)^{d-1}} k_i k_j \hat{R}^s_{ij}(\mathbf{k}' - \mathbf{k}) \delta((\mathbf{k}' - \mathbf{k}) \cdot \mathbf{u}). \tag{8.3}$$

Thus the projection of the wave vector on the mean velocity direction $\mathbf{u}$ is not changed after the jump. The total scattering cross-section (8.3) is an unbounded function of $\mathbf{k}$, and so the standard argument [**5**] that $\mathbf{K}(t, \mathbf{k})$ starting at $\mathbf{k}$ does not go to infinity in a finite time does not apply. We note that the process $\mathbf{K}(t, \mathbf{k})$ is dual to the diffusion process $\mathbf{Y}(t, \mathbf{x}, \mathbf{y})$ with generator L_Y given by (3.7). In fact we have that

$$L_Y(\mathbf{y}) e^{i\mathbf{k}\cdot\mathbf{y}} = \mathcal{Q}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{y}}$$

and hence

$$E_Y\left\{e^{i\mathbf{k}\cdot\mathbf{Y}(t,\mathbf{y})}\right\} = E_K\left\{e^{i\mathbf{K}(t,\mathbf{k})\cdot\mathbf{y}}\right\}. \tag{8.4}$$

This means that the process $\mathbf{K}(t, \mathbf{k})$ is well defined as long as the diffusion process $\mathbf{Y}(t, \mathbf{y})$ is well defined, and so it exists for all time. The duality (8.4) has an interesting qualitative implication. Given any function $f(t, \mathbf{x}, \mathbf{y})$ we may view

$$\bar{f}(t, \mathbf{y}) = E\left\{f(\mathbf{Y}(t, \mathbf{y}))\right\}$$

either as a solution of

$$\begin{aligned} \frac{\partial \bar{f}}{\partial t} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \bar{f} &= L_Y \bar{f} \\ f(0, \mathbf{x}, \mathbf{y}) &= f(\mathbf{x}, \mathbf{y}) \end{aligned}$$

or as the Fourier transform of the solution of

$$\begin{aligned} \frac{\partial \hat{f}}{\partial t} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \hat{f} &= \mathcal{Q}\hat{f} \\ \hat{f}(0, \mathbf{x}, \mathbf{k}) &= \frac{\hat{f}(\mathbf{x}, -\mathbf{k})}{(2\pi)^d}. \end{aligned}$$

The limit process $\mathbf{K}(t, \mathbf{k})$ has a unique invariant measure, which is Lebesgue measure on the plane $\mathbf{k} \cdot \mathbf{u} =$ const. This means that in the long time limit solutions of (8.5) are nearly functions of $\mathbf{k} \cdot \mathbf{u}$ only, that is, their support fills out the whole plane, and the function is close to a small constant on this set. This implies that solutions of (8.5) tend to a delta function in the directions orthogonal to the mean flow $\mathbf{u}$, and so the in the long time limit $\mathbf{Y}(t)$ tends to be parallel to the mean velocity $\mathbf{u}$. In terms of the flow, this means that particles starting nearby tend to be aligned with the flow no matter what their relative position is initially.

9. General scaling of the fluctuations

The results described above may be generalized to the case when random fluctuations are oscillating on a scale finer than ε, but their strength is also scaled appropriately. The trajectories $\mathbf{X}^\varepsilon(t,\mathbf{x})$ satisfy the scaled equation

$$\begin{aligned}\frac{d\mathbf{X}^\varepsilon}{dt} &= \mathbf{u}+\varepsilon^\alpha\mathbf{v}\left(\frac{\mathbf{X}^\varepsilon}{\varepsilon^\beta}\right), \\ \mathbf{X}^\varepsilon(0,\mathbf{x}) &= \mathbf{x}.\end{aligned} \tag{9.1}$$

The trajectory fluctuation, defined as before by

$$\mathbf{Z}^\varepsilon(t,\mathbf{x})=\frac{\mathbf{X}^\varepsilon(t,\mathbf{x})-\mathbf{u}t-\mathbf{x}}{\varepsilon} \tag{9.2}$$

has a non-trivial limit if $\alpha=1-\dfrac{\beta}{2}$, and $1\le\beta<2$. The strength of the fluctuations $\alpha=1-\beta/2$, which increases as the scale of the fluctuations decreases, is chosen so as to make the effect of the fluctuations on scale ε of order one. The case $\beta=1$ corresponds to (1.1), while $\beta=2$ corresponds to the homogenization scaling (2.3), which we do not consider here. In that case $\alpha=0$ and fluctuations of the velocity field are no longer weak, which leads to entirely different results.

The fluctuations $\mathbf{Z}^\varepsilon$, defined by (9.2), and $\mathbf{Y}^\varepsilon$, defined by

$$\mathbf{Y}^\varepsilon(t,\mathbf{y},\mathbf{x})=\frac{\mathbf{X}^\varepsilon(t,\mathbf{x})-\mathbf{X}^\varepsilon(t,\mathbf{x}-\varepsilon\mathbf{y})}{\varepsilon},$$

satisfy a scaled system similar to (3.3)

$$\frac{d\mathbf{Z}^\varepsilon}{dt} = \frac{1}{\sqrt{\varepsilon^\beta}}\mathbf{v}\left(\frac{\mathbf{x}+\mathbf{u}t}{\varepsilon^\beta}+\frac{\mathbf{Z}^\varepsilon}{\varepsilon^{\beta-1}}\right),\ \mathbf{Z}^\varepsilon(0)=0 \tag{9.3}$$

$$\frac{d\mathbf{Y}^\varepsilon}{dt} = \frac{1}{\sqrt{\varepsilon^\beta}}\left\{\mathbf{v}\left(\frac{\mathbf{x}+\mathbf{u}t}{\varepsilon^\beta}+\frac{\mathbf{Z}^\varepsilon}{\varepsilon^{\beta-1}}\right)-\mathbf{v}\left(\frac{\mathbf{x}+\mathbf{u}t}{\varepsilon^\beta}+\frac{\mathbf{Z}^\varepsilon}{\varepsilon^{\beta-1}}-\frac{\mathbf{Y}^\varepsilon}{\varepsilon^{\beta-1}}\right)\right\},\ \mathbf{Y}^\varepsilon(0)=\mathbf{y}. \tag{9.4}$$

The limit theorem of Appendix A applies also to this system [**16**] as does the limit theorem for the process $\mathbf{Z}^\varepsilon$ of Section 3 in this scaling. The limit theorem for the process $\mathbf{Y}^\varepsilon$ has to modified because the two terms in (9.4) become decorrelated in the limit $\varepsilon\to0$, as opposed to (3.3). That means that the $\mathbf{y}$-dependent cross term in (3.7) vanishes in the scaling of (9.1). The two particles behave independently of each other and there is no interaction. This happens because the initial separation $\varepsilon\mathbf{y}$ is large compared to the scale of the randomness and the two particles sample nearly uncorrelated parts of the random medium. The results of Theorems 4.1 and 7.1 for the limit Richardson and Wigner functions are modified because of this. Thus, the regime of validity of the radiative transport theory in is restricted to the case of inhomogeneities that are comparable to the wavelength [**30**]. Inclusions that are smaller than the wavelength but with higher contrast than in the scaling (4.3) will cause dissipation of energy on smaller scales. The contrast still has to be small, which corresponds to $\alpha>0$ in (9.1).

Let $\phi^\varepsilon(t,\mathbf{x})$ be the solution of the rescaled version of (4.3):

$$\frac{\partial\phi^\varepsilon}{\partial t}+\left(\mathbf{u}+\varepsilon^{1-\beta/2}\mathbf{v}(\frac{\mathbf{x}}{\varepsilon^\beta})\right)\cdot\nabla_\mathbf{x}\phi^\varepsilon=0. \tag{9.5}$$

The limit equation for $W(t,\mathbf{x},\mathbf{k})$ in this scaling has the form

$$\frac{\partial W}{\partial t}+\mathbf{u}\cdot\nabla_\mathbf{x}W=-\int\frac{d\mathbf{k}'}{(2\pi)^{d-1}}k_ik_j\hat{R}^s_{ij}(\mathbf{k}'-\mathbf{k})\delta((\mathbf{k}'-\mathbf{k})\cdot\mathbf{u})W(\mathbf{k}), \tag{9.6}$$

and

$$\iint W(t,\mathbf{x},\mathbf{k})d\mathbf{x}d\mathbf{k}<\iint W_0(\mathbf{x},\mathbf{k})d\mathbf{x}d\mathbf{k}=\lim_{\varepsilon\to0}\int|\phi_0^\varepsilon(\mathbf{x})|^2d\mathbf{x}=\lim_{\varepsilon\to0}\int|\phi^\varepsilon(t,\mathbf{x})|^2d\mathbf{x}.$$

Thus, Proposition 1 of Appendix B implies that if the family $\phi_0^\varepsilon(\mathbf{x})$ is ε-oscillatory, then the solution $\phi^\varepsilon(t,\mathbf{x})$ does not remain ε-oscillatory for all times $t>0$, and randomness generates oscillations on its own scale when $\beta>1$.

10. Evolution of the Jacobian matrix

The behavior of the Jacobian matrix of the map induced by a mean zero time dependent random flow is the subject of recent study, both numerical [**7**] and theoretical [**6, 8, 19, 20, 31**]. One of the main quantities of interest in such a flow is the Lyapunov exponent responsible for the exponential growth in time of the norm of the Jacobian matrix. Its positivity was established in [**8**] for velocity fields that are a finite-dimensional Ornstein-Uhlenbeck process. The Jacobian and the evolution of curves in isotropic stochastic flows with zero drift is studied in [**4**]. The trajectories of (1.1) behave very differently from mean zero time dependent flows. In particular, as we shall see in Section 8, the length of the curves advected by this flow does not grow exponentially in time.

Let $J^\varepsilon(t,\mathbf{x})$ be the Jacobian matrix of the transformation $\mathbf{x}\to\mathbf{X}^\varepsilon(t,\mathbf{x})$:

$$J^\varepsilon_{ik}(t,\mathbf{x})=\frac{\partial X_i^\varepsilon(t,\mathbf{x})}{\partial x_k} \tag{10.1}$$

This map is volume preserving because the flow is incompressible, and thus, $\det J^\varepsilon=1$. We are interested in the limit of $J^\varepsilon(t,\mathbf{x})$ as $\varepsilon\to 0$. The main result of this section is the following theorem.

Theorem 10.1. *The Jacobian $J^\varepsilon(t,\mathbf{x})$ of the map $\mathbf{x}\to\mathbf{X}^\varepsilon(t,\mathbf{x})$ converges weakly to the diffusion process with generator*

$$L_J f=-\frac{1}{2}\sum_{i,k,m,n=1}^{d}\int_0^\infty\frac{\partial^2 R_{im}(\mathbf{u}t)}{\partial y_j\partial y_l}dt\,J_{jk}J_{ln}\frac{\partial^2 f}{\partial J_{ik}\partial J_{mn}}, \tag{10.2}$$

starting at $J=I$.

We derive (10.2) at the end of this section. First we note that the generator (10.2) for the limit diffusion process $J(t)$ may be interpreted naturally in terms of the generator (3.7) for the limit process $\mathbf{Y}(t)$. We recall that

$$\mathbf{Y}^\varepsilon(t,\mathbf{y},\mathbf{x})=\frac{\mathbf{X}^\varepsilon(t,\mathbf{x})-\mathbf{X}^\varepsilon(t,\mathbf{x}-\varepsilon\mathbf{y})}{\varepsilon}\approx J^\varepsilon(t,\mathbf{x})\mathbf{y}$$

for small $\mathbf{y}$. Thus the processes $\mathbf{Y}$ and $\mathbf{S}=J\mathbf{y}$ should behave in a similar way for small $\mathbf{y}$. In particular we should have that

$$E\left\{f(\mathbf{Y}(t,\mathbf{y}))\right\}\approx E\left\{f(\mathbf{S}(t,\mathbf{y}))\right\} \tag{10.3}$$

for small $\mathbf{y}$. To show that (10.3) holds we expand the generator (3.7) of $\mathbf{Y}$ for small $\mathbf{y}$ and obtain

$$\begin{aligned}
L_Y f &= \frac{1}{2}\int_{-\infty}^{\infty}\left(R^s_{im}(\mathbf{u}t)-R^s_{im}(\mathbf{u}t+\mathbf{y})\right)dt\frac{\partial^2 f}{\partial y_i\partial y_m}\\
&= \frac{1}{2}\int_{-\infty}^{\infty}\left(R_{im}(\mathbf{u}t)-\frac{1}{2}R_{im}(\mathbf{u}t+\mathbf{y})-\frac{1}{2}R_{im}(\mathbf{u}t-\mathbf{y})\right)dt\frac{\partial^2 f}{\partial y_i\partial y_m}\\
&\approx -\frac{1}{2}\int_0^\infty\frac{\partial^2 R_{im}(\mathbf{u}t)}{\partial y_j\partial y_l}y_jy_l\,dt\frac{\partial^2 f}{\partial y_i\partial y_m}=c_{imjl}y_jy_l\frac{\partial^2 f}{\partial y_i\partial y_m},
\end{aligned} \tag{10.4}$$

where

$$c_{imjl}=-\frac{1}{2}\int_0^\infty\frac{\partial^2 R_{im}(\mathbf{u}t)}{\partial y_j\partial y_l}dt. \tag{10.5}$$

Let $J(t, A, \omega)$ be the diffusion with generator L_J starting at $J(0) = A$. We claim that the generator for the process $\mathbf{S}(t, \mathbf{y}, \omega) = J(t, A, \omega)\mathbf{y}$, starting at $\mathbf{s} = A\mathbf{y}$ is given by (10.4). Theorem 10.1 implies that given any function $f(\mathbf{s})$, the expectation

$$\bar{f}(t, \mathbf{y}, A) = E\{f(\mathbf{S}(t))\} = E\{f(J(t)\mathbf{y})\}$$

satisfies the diffusion equation

$$\frac{\partial \bar{f}}{\partial t} = c_{imjl} A_{jk} A_{ln} \frac{\partial^2 \bar{f}}{\partial A_{ik} \partial A_{mn}},$$
$$\bar{f}(0, y, A) = f(Ay).$$

Then a simple calculation shows that

$$\bar{f}(t, A, \mathbf{y}) = \bar{g}(t, A\mathbf{y}),$$

where the function $\bar{g}(t, \mathbf{s})$ satisfies

$$\frac{\partial \bar{f}}{\partial t} = c_{imjl} s_j s_l \frac{\partial^2 \bar{g}}{\partial s_i \partial s_m}$$
$$\bar{g}(0, \mathbf{s}) = f(\mathbf{s}).$$

Thus, the generator for $\mathbf{S}$ coincides with the small $\mathbf{y}$ asymptotics (10.4) of the generator L_Y. We recall that the large $\mathbf{y}$ asymptotics of L_Y gives rise to the generator L_Z for the limit process $\mathbf{Z}(t)$. Thus the limit difference function $\mathbf{Y}(t)$ contains information about both limit processes $\mathbf{Z}(t)$ and $J(t)$. In physical terms the behavior of the Jacobian matrix can be recovered from the large $\mathbf{k}$ behavior of the Wigner distribution which corresponds to the small $\mathbf{y}$ behavior of the Richardson function. The behavior of the fluctuations $\mathbf{Z}(t)$ may be recovered from small $\mathbf{k}$ limit of the Wigner distribution or, equivalently, large $\mathbf{y}$ behavior of the Richardson function.

We note further that the generator (10.2) is formally of the form

$$L_J f = < cJ, J > \frac{\partial^2 f}{\partial J \partial J}. \tag{10.6}$$

Such diffusion processes have typically positive non-zero Lyapunov exponents, that is, the limit

$$\lambda = \lim_{T \to \infty} \frac{\ln ||J(t)||}{T}$$

exists with probability one, and $\lambda > 0$. The computation of λ for some time dependent flows was done in [**4, 8, 31, 19, 20**]. We shall show in Section 11 that in our case $\lambda = 0$. The reason for this is the strong degeneracy of (1.1) in the direction of the mean flow $\mathbf{u}$.

We derive now formula (10.2) for the generator of the limit process $J(t)$ and prove Theorem 10.1. The evolution equation for J^ε is obtained by differentiating (1.1) with respect to x_k:

$$\frac{dJ_{ik}^\varepsilon}{dt} = \frac{1}{\sqrt{\varepsilon}} \frac{\partial v_i}{\partial x_j} \left(\frac{\mathbf{X}^\varepsilon}{\varepsilon} \right) J_{jk}^\varepsilon \tag{10.7}$$
$$J(0, \mathbf{x}) = I.$$

We rewrite this equation using the fluctuations $\mathbf{Z}^\varepsilon(t, x)$, defined by (3.2), so as to put it in a form suitable for the limit theorem of Appendix A

$$\begin{aligned} \frac{d\mathbf{Z}^\varepsilon}{dt} &= \frac{1}{\sqrt{\varepsilon}} \mathbf{v} \left(\frac{\mathbf{x} + \mathbf{u}t}{\varepsilon} + \mathbf{Z}^\varepsilon \right), \ \mathbf{Z}^\varepsilon(0) = 0 \\ \frac{dJ^\varepsilon}{dt} &= \frac{1}{\sqrt{\varepsilon}} \nabla \mathbf{v} \left(\frac{\mathbf{x} + \mathbf{u}t}{\varepsilon} + \mathbf{Z}^\varepsilon \right) J^\varepsilon, \ J^\varepsilon(0) = I. \end{aligned} \tag{10.8}$$

From this theorem, the joint generator for the limit diffusion processes Z and J has the form

$$L = L_Z + L_J + L_{JZ} + L_{ZJ}. \tag{10.9}$$

The operator L_Z is given by (3.6) and the operator L_J is

$$\begin{aligned} L_J f &= \frac{1}{2}\sum_{i,k,m,n=1}^{d}\int_0^\infty E\left\{\frac{\partial v_i}{\partial x_j}\left(\frac{\mathbf{x}}{\varepsilon}+\mathbf{z}\right)J_{jk}\frac{\partial}{\partial J_{ik}}\left(\frac{\partial v_m}{\partial x_l}\left(\frac{\mathbf{x}}{\varepsilon}+\mathbf{u}t+\mathbf{z}\right)J_{ln}\frac{\partial f(\mathbf{z},J)}{\partial J_{mn}}\right)\right\}dt \\ &= \frac{1}{2}\sum_{i,k,m,n=1}^{d}\int_0^\infty E\left\{\frac{\partial v_i}{\partial x_j}\left(\frac{\mathbf{x}}{\varepsilon}+\mathbf{z}\right)\frac{\partial v_m}{\partial x_l}\left(\frac{\mathbf{x}}{\varepsilon}+\mathbf{u}t+\mathbf{z}\right)\right\}dt\,J_{jk}\frac{\partial}{\partial J_{ik}}\left(J_{ln}\frac{\partial f(\mathbf{z},J)}{\partial J_{mn}}\right). \end{aligned} \tag{10.10}$$

To simplify (10.10) we note that if we let

$$F_{ijml}(\mathbf{y}) = E\left\{\frac{\partial v_i}{\partial x_j}(\mathbf{x})\frac{\partial v_m}{\partial x_l}(\mathbf{x}+\mathbf{y})\right\},$$

then

$$\frac{\hat{F}_{ijml}(\mathbf{q})}{(2\pi)^d}\delta(\mathbf{p}+\mathbf{q}) = ip_j iq_l E\left\{\hat{v}_i(\mathbf{p})\hat{v}_m(\mathbf{q})\right\} = \frac{q_j q_l \hat{R}_{im}(\mathbf{q})}{(2\pi)^d}\delta(\mathbf{p}+\mathbf{q}),$$

and thus

$$F_{ijml}(\mathbf{y}) = -\frac{\partial^2 R_{im}(\mathbf{y})}{\partial y_j\partial y_l}.$$

Therefore the generator L_J has the form

$$\begin{aligned} L_J f &= -\frac{1}{2}\sum_{i,k,m,n=1}^{d}\int_0^\infty \frac{\partial^2 R_{im}(\mathbf{u}t)}{\partial y_j\partial y_l}dt\,J_{jk}\frac{\partial}{\partial J_{ik}}\left(J_{ln}\frac{\partial f}{\partial J_{mn}}\right) \\ &= -\frac{1}{2}\sum_{i,k,m,n=1}^{d}\int_0^\infty \frac{\partial^2 R_{im}(\mathbf{u}t)}{\partial y_j\partial y_l}dt\,J_{jk}J_{ln}\frac{\partial^2 f}{\partial J_{ik}\partial J_{mn}}, \end{aligned} \tag{10.11}$$

the last equality being due to incompressibility. Note that L_J is independent of $\mathbf{z}$. Next we compute L_{ZJ}:

$$\begin{aligned} L_{ZJ} f &= \frac{1}{2}\sum_{k,i,j=1}^{d}\int_0^\infty E\left\{v_k\left(\frac{\mathbf{x}}{\varepsilon}+\mathbf{z}\right)\frac{\partial}{\partial z_k}\left(\frac{\partial v_i}{\partial x_n}\left(\frac{\mathbf{x}}{\varepsilon}+\mathbf{u}t+\mathbf{z}\right)J_{nj}\frac{\partial f}{\partial J_{nj}}\right)\right\}dt \\ &= \frac{1}{2}\sum_{k,i,j=1}^{d}\int_0^\infty E\left\{v_k\left(\frac{\mathbf{x}}{\varepsilon}+\mathbf{z}\right)\frac{\partial v_i}{\partial x_n}\left(\frac{\mathbf{x}}{\varepsilon}+\mathbf{u}t+\mathbf{z}\right)\right\}dt\,J_{nj}\frac{\partial^2 f}{\partial z_k\partial J_{ij}}. \end{aligned} \tag{10.12}$$

The coefficients of L_{ZJ} are also independent of $\mathbf{z}$. The first order derivative term vanishes after taking the expectation because of the incompressibility condition. The operator L_{JZ} is

$$\begin{aligned} L_{JZ} f &= \frac{1}{2}\sum_{i,j,k=1}^{d}\int_0^\infty E\left\{\frac{\partial v_i}{\partial x_n}\left(\frac{\mathbf{x}}{\varepsilon}+\mathbf{z}\right)J_{nj}\frac{\partial}{\partial J_{ij}}\left(v_k\left(\frac{\mathbf{x}}{\varepsilon}+\mathbf{u}t+\mathbf{z}\right)\frac{\partial f}{\partial z_k}\right)\right\}dt \\ &= \frac{1}{2}\sum_{i,j,k=1}^{d}\int_0^\infty E\left\{\frac{\partial v_i}{\partial x_n}\left(\frac{\mathbf{x}}{\varepsilon}+\mathbf{z}\right)v_k\left(\frac{\mathbf{x}}{\varepsilon}+\mathbf{u}t+\mathbf{z}\right)\right\}dt\,J_{nj}\frac{\partial^2 f}{\partial z_k\partial J_{ij}}, \end{aligned} \tag{10.13}$$

it does not have first order derivative terms as well, and has coefficients independent of $\mathbf{z}$. This shows that J^ε converges by itself to a diffusion process J with generator (10.11), and thus Theorem 10.1 holds.

11. Application to two-dimensional flows

We apply the results of Section 10 to two dimensional flows. Any two dimensional incompressible flow in R^2 has the form

$$\mathbf{v}(x) = \left(\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1}\right), \tag{11.1}$$

where $\psi(\mathbf{x})$ is the stream function. We assume that the random field $\psi(\mathbf{x})$ is space homogeneous, zero mean and isotropic with covariance function

$$R_\psi(x) = E\left\{\psi(\mathbf{y})\psi(\mathbf{x}+\mathbf{y})\right\} = F\left(\frac{r^2}{2}\right),$$

where $r^2 = x_1^2 + x_2^2$. The Jacobian matrix of two-dimensional time dependent flows with zero mean $\mathbf{u}$ was studied in detail in [**31**], in the diffusion approximation. The results there are entirely different but may be formally recovered from our calculations as we explain below. The covariance matrix R_{ij} for the flow (11.1) is

$$R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} F' + \begin{pmatrix} -x_2^2 & x_1x_2 \\ x_1x_2 & -x_1^2 \end{pmatrix} F''. \tag{11.2}$$

The tensor c_{imjl} (10.5) in the generator (10.11) has now the form

$$\begin{aligned}
c_{1111} &= -\frac{1}{|\mathbf{u}|}\left(I_2 + I_3 + \hat{u}_1^2\hat{u}_2^2 I_4\right)\\
c_{1112} &= -\frac{1}{|\mathbf{u}|}\left(3\hat{u}_1\hat{u}_2 I_3 + \hat{u}_1\hat{u}_2^3 I_4\right)\\
c_{1122} &= -\frac{1}{|\mathbf{u}|}\left(3I_2 + 6\hat{u}_2^2 I_3 + \hat{u}_2^4 I_4\right)\\
c_{1211} &= \frac{1}{|\mathbf{u}|}\left(3\hat{u}_1\hat{u}_2 I_3 + \hat{u}_1^3\hat{u}_2 I_4\right)\\
c_{1212} &= \frac{1}{|\mathbf{u}|}\left(I_2 + I_3 + \hat{u}_1^2\hat{u}_2^2 I_4\right)\\
c_{1222} &= \frac{1}{|\mathbf{u}|}\left(3\hat{u}_1\hat{u}_2 I_3 + \hat{u}_1\hat{u}_2^3 I_4\right)\\
c_{2211} &= -\frac{1}{|\mathbf{u}|}\left(3I_2 + 6\hat{u}_1^2 I_3 + \hat{u}_1^4 I_4\right)\\
c_{2212} &= -\frac{1}{|\mathbf{u}|}\left(3\hat{u}_1\hat{u}_2 I_3 + \hat{u}_1^3\hat{u}_2 I_4\right)\\
c_{2222} &= -\frac{1}{|\mathbf{u}|}\left(I_2 + I_3 + \hat{u}_1^2\hat{u}_2^2 I_4\right).
\end{aligned} \tag{11.3}$$

Here $\mathbf{u} = (u_1, u_2)$ is the mean flow, $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2}$, $\hat{u}_j = \dfrac{u_j}{|\mathbf{u}|}$, and the constants

$$I_2 = \int_0^\infty F''\left(\frac{t^2}{2}\right)dt,\ I_3 = \int_0^\infty t^2 F'''\left(\frac{t^2}{2}\right)dt,\ I_4 = \int_0^\infty t^4 F^{(iv)}\left(\frac{t^2}{2}\right)dt$$

are related by

$$I_2 = -I_3,\ I_4 = 3I_2. \tag{11.4}$$

The other entries are determined by the symmetries $c_{imjl} = c_{mijl} = c_{imlj}$. The generator (11.5) with c_{imjl} as above is very different form the generator the limit of the Jacobian J^ε for mean zero, time dependent flows [**19, 20, 31**]. We may not set even formally $\mathbf{u} = 0$ in (11.3) because the resulting expression diverges as it always happens with the Kubo formula. However, the results of [**19, 20, 31**] can be recovered in our calculation by setting formally $I_3 = I_4 = 0$ in (11.3). Then the generator (10.2) given by Theorem 10.1 reduces formally to the one obtained in [**19, 20, 31**].

Let us choose a coordinate system so that $u_2 = 0$, and $u_1 = |\mathbf{u}|$. Then the only non-zero entry is $c_{1122} = -\dfrac{3I_2}{|\mathbf{u}|}$ and the generator L_J is

$$L_J = \frac{I_2}{2|\mathbf{u}|}\left[3J_{21}^2\frac{\partial^2}{\partial J_{11}^2} + 6J_{21}J_{22}\frac{\partial^2}{\partial J_{11}\partial J_{12}} + 3J_{22}^2\frac{\partial^2}{\partial J_{12}^2}\right] = \frac{3I_2}{2|\mathbf{u}|}\left[J_{21}\frac{\partial}{\partial J_{11}} + J_{22}\frac{\partial}{\partial J_{12}}\right]^2.$$

The matrix $J(0) = I$, and thus the only entry which is changed during the evolution in the limit is J_{12}. The generator L_J has the simple form

$$L_J f = \frac{3I_2}{2|\mathbf{u}|}\frac{\partial^2 f}{\partial J_{12}^2}. \tag{11.5}$$

The Lyapunov exponent of the matrix valued process $J(t)$ is manifestly zero. The reason that only J_{12} is changed by the dynamics can be seen from the generator L_Y. Its coefficients depend only on y_2 and are independent of the $\mathbf{y}$-coordinate along the mean velocity $\mathbf{u}$. This is reflected in (11.5).

12. Deformation of the length

Theorem 10.1 allows us to study stretching of curves by the flow (1.1) in R^2. If the initial curve has length of order one, and is parameterized by $\mathbf{x} = \mathbf{x}(s)$, $0 \le s \le 1$, its length is given by

$$l = \int_0^1 \left|\frac{d\mathbf{x}}{ds}\right| ds.$$

The length of the curve $\mathbf{X}^\varepsilon(t,s) = \mathbf{X}^\varepsilon(t,\mathbf{x}(s))$ is

$$l^\varepsilon(t) = \int_0^1 \left|\frac{d\mathbf{X}^\varepsilon}{ds}\right| ds = \int_0^1 \left|J^\varepsilon(t,\mathbf{x}(s))\frac{d\mathbf{x}}{ds}(s)\right| ds.$$

The limit theorem of Appendix A implies that for any two distinct points $\mathbf{x}$ and $\mathbf{y}$, the processes $J^\varepsilon(t,\mathbf{x})$ and $J^\varepsilon(t,\mathbf{y})$ are not only identically distributed, but also independent in the limit $\varepsilon \to 0$, unless $\mathbf{x} - \mathbf{y} = c\mathbf{u}$ for some c, which means that the points lie on the same deterministic trajectory. This is similar to the joint behavior of trajectories of (1.1) starting at points lying on different deterministic trajectories as described in Section 6. The same is true for any number of fixed initial points. Thus $l^\varepsilon(t) \to l(t)$, so that $E\{l^n(t)\} = (E\{l(t)\})^n$, and the length $l^\varepsilon(t)$ becomes deterministic in the limit.

Let us consider stretching of an interval of length one which is initially at angle θ with respect to the mean flow $\mathbf{u} = (|\mathbf{u}|, 0)$, so that $\dfrac{d\mathbf{x}}{ds} = (\cos\theta, \sin\theta)$, under the potential flow (11.1). We introduce the variables

$$\begin{aligned} J_1(\theta) &= J_{11}\cos\theta + J_{12}\sin\theta = \cos\theta + J_{12}\sin\theta \\ J_2(\theta) &= J_{21}\cos\theta + J_{22}\sin\theta = \sin\theta. \end{aligned}$$

The behavior of the joint process $(J_1(\theta), J_2(\theta))$ in our case is trivial and degenerate. The point (J_1, J_2) performs an ordinary Brownian motion along the horizontal lines $J_2 = \text{const}$ with diffusion coefficient that depends only on J_2. In the time dependent case [**31**] there is no such degeneracy, the process (J_1, J_2) is not restricted to a line, and its norm grows exponentially in time.

We have $J_1(0) = \cos\theta$, and

$$\bar{l}(t) = E\{l(t)\} = \lim_{\varepsilon\to 0} E\{l^\varepsilon(t)\} = \int_0^1 E\left\{\sqrt{J_1^2(\theta) + J_2^2(\theta)}\right\} ds = E\left\{\sqrt{J_1^2(\theta) + \sin^2\theta}\right\},$$

because the law for the limit process J is independent of the starting point. Then, using (11.5) we see that the limit length $\bar{l}(t) = g(t, \cos\theta)$, where the function $g(t,x)$ satisfies the initial value problem:

$$\begin{aligned} \frac{\partial g}{\partial t} &= \frac{3I_2}{|\mathbf{u}|}\sin^2\theta\frac{\partial^2 g}{\partial x^2} \\ g(0,x) &= \sqrt{\sin^2\theta + x^2}. \end{aligned} \tag{12.1}$$

This means that unless $\theta = 0$ the length $\bar{l}(t) \sim C(\theta)\sqrt{t}$, and does not grow exponentially. The length does not change at all when $\theta = 0$, that is, when the interval is parallel to the mean flow. The algebraic growth is similar to the algebraic growth of the length in a shear flow [**6**]. The flow (1.1) is dominated by the mean flow, and randomness is not strong enough to generate the kind of mixing needed for exponential growth of the length.

13. Summary and conclusions

We have shown that weak, time independent, incompressible fluctuations of a uniform flow produce certain non-trivial effects on time scales of order one. These are important in advection of passive scalars when the initial density varies on scales comparable to that of the inhomogeneities of the flow. This regime is similar to the one for which the radiative transport theory [**30**] holds, and we show that the Wigner distribution of the passive scalar satisfies the radiative transport equation (7.7). The limit Richardson function, or the two point correlation function, satisfies the degenerate diffusion equation (4.15). This result does not require that the velocity field or the initial tracer distribution be Gaussian.

The non-zero mean flow has a strong effect on the two-point motion introducing a degeneracy in its direction. As we explain in Section 8, using the duality between the limit diffusion process $\mathbf{Y}(t)$ and the corresponding jump process in Fourier space, the two point difference vector tends to be aligned with the mean flow in the long time limit.

We also study the evolution of the Jacobian of the flow map $\mathbf{x} \to \mathbf{X}^\varepsilon(t)$ in the limit $\varepsilon \to 0$ and show that its limit is a diffusion process. We show that because of the degeneracy caused by the mean flow the corresponding Lyapunov exponent vanishes. This implies in particular that the length of curves moving with the flow is growing only algebraically, which is quite different from strong, time dependent, mean zero flows studied in [**8, 31**]. Physically this is because of the sweeping effect of the mean flow and the fact that we are not looking at the long time limit but rather at finite time effects.

Our results can be generalized to the case of non-zero but small molecular diffusivity. The results regarding the two point motion remain essentially the same allowing for an additive term in the diffusion equation for the Richardson function, and an absorption term in the radiative transport equation.

14. Appendix A. A limit theorem for turbulent diffusion

The limit as $\varepsilon \to 0$ of the trajectories of the dynamical systems with the scaling as in (2.4) is described by a limit theorem, which was proven by Kesten and Papanicolaou [**18, 21**] for $\gamma = 1$, and later by Komorowski [**16**] for $0 \le \gamma < 1$. They considered equations of the form

$$\begin{aligned} &\frac{d\mathbf{Q}^\varepsilon}{dt} = \frac{1}{\varepsilon}\mathbf{G}\left(\frac{t}{\varepsilon^2}, \frac{\mathbf{Q}^\varepsilon(t)}{\varepsilon^\gamma}, \varepsilon\right), \\ &\mathbf{Q}^\varepsilon(0) = \mathbf{q}, \end{aligned} \tag{14.1}$$

where the function $\mathbf{G}$ satisfies the following conditions.

(A1) The function $\mathbf{G}(t, \mathbf{q}, \varepsilon, \omega)$ is jointly measurable in all its arguments and a.s. in $C^3(R^d)$ as a function of $\mathbf{q}$ for each t, ε.

(A2) The process $\{\mathbf{G}(t,\cdot,\varepsilon,\omega)\}$ is stationary in t for each fixed ε.

(A3) Let

$$\mathcal{G}_s^t(\varepsilon,M)=\sigma\{\mathbf{G}(u,\mathbf{q},\varepsilon,\cdot)|s\le u\le t,|\mathbf{q}|\le M\},$$

and

$$\beta(t,M)=\sup_{s\ge 0,0<\varepsilon\le 1}\ \sup_{A\in\mathcal{G}_0^s(\varepsilon,M),B\in\mathcal{G}_{s+t}^{\infty}(\varepsilon,M)}|P(AB)-P(A)P(B)|.$$

Then

$$\int_0^\infty|\beta(t,M)|^{1/p}dt<\infty$$

for $p=6+2d$.

(A4) $E\{\mathbf{G}(t,\mathbf{q},\varepsilon)\}=0$.

(A5) For each $M<\infty$ there exists a constant $C(M)$ independent of t, $\mathbf{k}$, and ε such that

$$E\left\{\sup_{|\mathbf{q}-\mathbf{k}|\le M}|D^\beta G(t,\mathbf{q},\varepsilon)|^{\max(8,d)}\right\}\le C,\ \ 0\le|\beta|\le 3, \tag{14.2}$$

and these integrals converge uniformly in $\mathbf{k}$. All the derivatives in (14.2) are with respect to $\mathbf{q}$.

(A6) The following limits exist uniformly on compact sets and are bounded functions of $\mathbf{q}$:

$$\begin{aligned} A_{ij}(\mathbf{q}) &= \lim_{\varepsilon\to 0}\int_0^\infty E\left\{G_i(0,\mathbf{q},\varepsilon)G_j(t,\mathbf{q},\varepsilon)\right\}dt \\ c_{ij}(\mathbf{q}) &= \lim_{\varepsilon\to 0}\int_0^\infty E\left\{G_i(0,\mathbf{q},\varepsilon)\frac{\partial}{\partial q_i}G_j(t,\mathbf{q},\varepsilon)\right\}dt. \end{aligned} \tag{14.3}$$

In addition the matrix $a_{ij}=A_{ij}+A_{ji}$ is twice continuously differentiable.

(A7) The velocity field is incompressible: $\mathrm{div}\mathbf{G}^\varepsilon=0$.

By a solution of (14.1) we mean a continuous function $\mathbf{Q}^\varepsilon(t)$ which satisfies

$$\mathbf{Q}^\varepsilon(t,\omega)=\mathbf{q}+\frac{1}{\sqrt{\varepsilon}}\int_0^t\mathbf{G}\left(\frac{\sigma}{\varepsilon},\mathbf{Q}^\varepsilon(\sigma,\omega),\varepsilon,\omega\right)d\sigma. \tag{14.4}$$

THEOREM 14.1. **[18, 16]** *Let $\mathbf{G}$ be a random field that satisfies conditions (A1-A7). Then with probability one (14.4) has a unique solution for all $t\ge 0$. Let R^ε be the measure on $C([0,\infty);R^d)$ induced by this solution. For each $f\in C^2(R^d)$ put*

$$Lf(\mathbf{q})=\frac{1}{2}\sum_{i,j=1}^d a_{ij}(\mathbf{q})\frac{\partial^2 f(\mathbf{q})}{\partial q_i\partial q_j} \tag{14.5}$$

Then R^ε converges weakly to the probability measure R on $C([0,\infty);R^d)$, which corresponds to the diffusion with infinitesimal generator L and starting at the point $\mathbf{q}$, that is, $R\{\mathbf{Q}(0)=\mathbf{q}\}=1$.

The most general statement of this theorem, including more general time dependence and compressible velocity fields, may be found in **[16]**.

15. Appendix B. Oscillatory functions and the Wigner distribution

We review briefly some definitions and basic facts about oscillatory families of functions and the Wigner distribution **[14]**. Let $f^\varepsilon(\mathbf{x})$ be a bounded family of functions in $L^2_{loc}(R^d)$.

DEFINITION 1. The family $f^\varepsilon(\mathbf{x})$ is ε-oscillatory if for any smooth test function of compact support $\phi(\mathbf{x})$

$$\limsup_{\varepsilon\to 0}\int_{|\mathbf{k}|\geq R/\varepsilon}\left|\widehat{\phi f^\varepsilon}(\mathbf{k})\right|^2 d\mathbf{k}\to 0 \text{ as } R\to+\infty. \tag{15.1}$$

A sufficient condition for (15.1) to hold is

$$\exists k>0, \text{ such that } \varepsilon^k\left|\left|\frac{\partial^k f^\varepsilon}{\partial x^k}\right|\right|_{L^2_{loc}}\leq C.$$

DEFINITION 2. The family $f^\varepsilon(\mathbf{x})$ is said to be compact at infinity if

$$\limsup_{\varepsilon\to 0}\int_{|\mathbf{x}|\geq R/\varepsilon}|f^\varepsilon(\mathbf{x})|^2 d\mathbf{x}\to 0 \text{ as } R\to+\infty. \tag{15.2}$$

A sufficient condition for (15.2) to hold is that there exists a set K such that $\text{supp} f^\varepsilon(\mathbf{x})\subset K$.

DEFINITION 3. The Wigner distribution is defined by

$$W^\varepsilon(\mathbf{x},\mathbf{k})=\int\frac{d\mathbf{y}}{(2\pi)^d}e^{i\mathbf{k}\cdot\mathbf{y}}f^\varepsilon(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})f^{\varepsilon *}(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2}).$$

It has a weak limit point $W(\mathbf{x},\mathbf{k})$ in the space $\mathcal{S}'$ of Schwartz distributions provided that $f^\varepsilon(\mathbf{x})$ is bounded in L^2_{loc}. We shall assume that such a limit point is unique. The limit Wigner distribution may be interpreted as a phase space energy density. For instance, the limit Wigner distribution of the WKB family $f^\varepsilon(\mathbf{x})=A(\mathbf{x})e^{iS(\mathbf{x})/\varepsilon}$ is

$$W(\mathbf{x},\mathbf{k})=|A(\mathbf{x})|^2\,\delta(\mathbf{k}-\nabla S(\mathbf{x})).$$

The limit Wigner distribution is mostly relevant for ε-oscillatory and compact at infinity families $f^\varepsilon(\mathbf{x})$.

PROPOSITION 1. [**14**] Let f_ε be a bounded family in L^2_{loc} with limit Wigner measure $W(\mathbf{x},\mathbf{k})$. Then

(i) The limit Wigner distribution $W(\mathbf{x},\mathbf{k})\in\mathcal{S}'$ is non-negative (a measure).

(ii) If two families $f^\varepsilon(\mathbf{x})$ and $g^\varepsilon(\mathbf{x})$ coincide on an open ball $B\subset R^d$ then $W_f(\mathbf{x},\mathbf{k})=W_g(\mathbf{x},\mathbf{k})$ for $\mathbf{x}\in B$ and all $\mathbf{k}\in R^d$.

(ii) For any smooth function of compact support $\theta(\mathbf{x})$

$$\iint|\theta(\mathbf{x})|^2 W(d\mathbf{x},d\mathbf{k})\leq\limsup_{\varepsilon\to 0}\int_{R^d}|\theta(\mathbf{x})f_\varepsilon(\mathbf{x})|^2 d\mathbf{x} \tag{15.3}$$

with equality holding if and only if f_ε is ε-oscillatory. In this case lim sup can be replaced by lim on the right side of (15.3).

ACKNOWLEDGMENTS

This work was partially sponsored by a grant from the NSF, DMS-9622854, and by the Air Force Office of Scientific Research, Air Force Materials Command, USAF, under grant number F49620-95-1-0315. The US Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation thereon. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the Air Force Office of Scientific Research or the US Government.

We are grateful to Joseph B. Keller for numerous discussions on the subject.

References

[1] M. Avellaneda and A. J. Majda, An Integral Representation And Bounds On The Effective Diffusivity In Passive Advection By Laminar And Turbulent Flows, Commun. Math.Phys. **138**, 1991, 339-391.

[2] G.K. Batchelor and A.A. Townsend, Turbulent diffusion, **Surveys in Applied Mechanics**, Cambridge University Press, 1956, 352-399.
[3] P. Baxendale, Statistical equilibrium and two-point motion for a stochastic flow of diffeomorphisms, Progress in Probab., **19**, **Spatial Stochastic Processes**, eds. K. Alexander and J. Watkins, Birkhäuser, Boston, 1991, 189-218.
[4] P.Baxendale and T. Harris, Isotropic stochastic flows, Annals of Probability, **14**, 1986, 1155-1179.
[5] A. Bensoussan, J.L. Lions and G. Papanicolaou, Boundary layers and homogenization of transport processes, Publ. RIMS, **15**, 1979, 54-157.
[6] R.Carmona, Transport properties of Gaussian velocity fields, **Stochastic Analysis**, ed. M.M. Rao, 1996, CRC Press.
[7] R.Carmona, S.Grishin, and S.Molchanov, Massively parallel simulations of motions in a Gaussian velocity field, Progress in Probab. **39**, **Stochastic Modeling in Physical Oceanography**, eds R. Adler, P. Mueller and B. Rozovskii, Birkhäuser, New York, 1996, 47-68.
[8] R.Carmona, S.Grishin, L.Xu and S.Molchanov, Surface stretching for Ornstein-Ulehnbeck velocity fields, Electron. Comm. Probab., 1996.
[9] R. Carmona and L. Xu, Homogenization for time-dependent two-dimensional incompressible Gaussian flows, The Annals of Applied Probability, **7**, 1997, 265-279.
[10] S.Chandrasekhar, **Radiative transfer**, Dover, New York, 1960.
[11] G. Dell'Antonio, Large time, small coupling behavior of a quantum particle in a random field, Ann. Inst. H. Poincare Sect. A (N.S), **39**, 1983, 339-384.
[12] A. Fannjiang and T. Komorowski, An invariance principle for diffusion in time dependent turbulence, Preprint, 1997.
[13] A. Fannjiang and G. Papanicolaou, Diffusion In Turbulence, Probability Theory and Related Fields, **105**, 1996, 279-334.
[14] P. Gérard, Mesures Semi-Classiques et Ondes de Bloch, Sém. Ecole Polytechnique, exposé **XVI**, 1990-91, 1-19.
[15] T. Ho, L. Landau and A. Wilkins, On the weak coupling limit for a Fermi gas in a random potential, Rev. Math. Phys., **5**, 1993, 209-298.
[16] T. Komorowski, Diffusion approximation for the advection of particles in a strongly turbulent random environment, The Annals of Probability, **24**, 1996, 346-376.
[17] T. Komorowski and G. Papanicolaou, Motion in a Gaussian incompressible flow, The Annales of Applied Probability, **7**, 1997, 229-264.
[18] H. Kesten and G. Papanicolaou, A limit theorem in turbulent diffusion, Comm. Math. Phys., **65**, 1979, 97-128.
[19] V. Klyatskin, W. Woyczynski and D. Gurarie, Short-time correlation approximations for diffusing scalars in random velocity fields: a functional approach, in **Stochastic modeling in physical oceanography**, R. Adler, P. Müller and B. Rozovskii, eds., Birkhäuser, 1996, 221-270.
[20] V. Klyatskin, W. Woyczynski and D. Gurarie, Diffusive passive tracers in random incompressible flows: Statistical topography aspects, Journal Stat. Phys., **84**, 1996, 797-836.
[21] H. Kunita, **Stochastic flows and stochastic differential equations**, Cambridge University Press, Cambridge, 1990.
[22] P.L. Lions and T. Paul, Sur les Mesures de Wigner, Revista Mat. Iberoamericana, **9**, 1993, 553-618.
[23] S. Molchanov, Lectures on Random Media, in: P. Bernard, ed., Lectures On Probability Theory Ecole d'Eté de Probabilité de Saint-Flour XXII, 1992, 242-411, Springer Lecture Notes in Mathematics 1581.
[24] S.A. Molchanov and L.I. Piterbarg, Heat propagation in random flows, Russian Jour. Math. Phys., **1**, 1994, 353-376.
[25] A.Monin and A.Yaglom, **Statistical Fluid Mechanics: Mechanics of Turbulence**, MIT Press, Cambridge, Mass., 1975
[26] K. Oeschlager, Homogenization Of A Diffusion Process In A Divergence Free Random Field, Ann. Of Probability, **16**, 1988, 1084-1126.
[27] G.C. Papanicolaou and S.R.S. Varadhan, Boundary value problems with rapidly oscillating coefficients, **Random Fields Coll. Math. Soc Janos Bolyai**, J. Fritz, J. Lebowitz, eds., 1982, North-Holland, 835-873.
[28] L. Piterbarg, Short-correlation approximation in models of turbulent diffusion **Stochastic models in geosystems**, IMA Proc., vol 85, 1997, 313-352.
[29] L.F. Richardson, Atmospheric diffusion shown on a distance-neighbor graph, Proc. Roy. Soc., **110**, 1926, 709-727.
[30] L. Ryzhik, G. Papanicolaou and J.B. Keller, Energy transport for elastic and other waves in a random medium, Wave Motion, **24**, 1996, 327-370.
[31] A. Saichev and W. Woyczynski, Probability distributions of passive tracers in randomly moving media, **Stochastic models in geosystems**, IMA Proc., vol. 85, 1997 , 359-399.
[32] G.I. Taylor, Diffusions by continuous movements, Proc. London math. Soc. Ser. 2, **20**, 1923, 196-211.

Department of Mathematics, University of California at Davis, Davis CA 95616, fannjian@math.ucdavis.edu

Department of Mathematics, Stanford University, Stanford CA, 94305, ryzhik@math.stanford.edu, papanico@math.stanford.edu