Transport Theory for Acoustic Waves with Reflection and Transmission at Interfaces

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Abstract

Transport theoretic boundary conditions are derived for acoustic wave reflection and transmission at a rough interface with small random fluctuations. The Wigner distribution is used to go from waves to energy transport in the high frequency limit, and the Born expansion is used to calculate the effect of the random rough surface. The smoothing method is also used to remove the grazing angle singularity due to the Born approximation. The results are presented in a form that is convenient both for theoretical analysis and for numerical computations.

Contents

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1 Introduction

Wave propagation in weakly fluctuating random media over distances long compared to the wave length can be described by the energy transport equation. This is the radiative transport regime described in the next section. Near boundaries and interfaces the waves undergo coherent or partially coherent reflection and transmission, according as the interfaces are smooth or randomly rough. Starting from the acoustic equations, we derive systematically, boundary conditions for the radiative transport equation. This equation is asymptotically valid away from the boundaries and interfaces. The asymptotic limit that leads to radiative transport corresponds to weak fluctuations and high frequency waves, with the correlation length of the inhomogeneities comparable to the wave length. We use the Wigner distribution to analyze this limit, as we did in [23]. The main result of this work is the derivation of the transport boundary conditions (1.3) and (1.4) for reflection and transmission at a rough interface. The boundary conditions are intuitively clear. They combine coherent or specular reflection and transmission, and incoherent or diffuse reflection and transmission, in the form of a linear input-output relation. If there is no dissipation, as we assume, energy flux is conserved as (1.5) shows. We calculate the explicit form of the coherent and incoherent energy flux reflection and transmission coefficients and cross-sections when the fluctuations of the interface are weak. Special cases of these coefficients have been obtained before in the Born approximation [5, 10, 29]. Our results are summarized at the end of section 1.2.

The reflection and transmission coefficients, computed by a Born expansion, are singular at grazing angles. To eliminate the singularity, we recalculate them in Appendix B by using the smoothing method, as was done in [31].

1.1 The Radiative Transport Equation

Radiative transport is a theory that was introduced phenomenologically to describe the propagation of light intensity through the Earth's atmosphere. It has been applied successfully to many other problems of wave propagation in complex media. In its simplest form, $a(t, \mathbf{x}, \mathbf{k})$ denotes the angularly resolved energy density defined for wave vector $\mathbf{k}$, position $\mathbf{x}$ and time t. Because of interaction with the inhomogeneous medium through which it propagates, a wave with wave vector $\mathbf{k}$ may be scattered into any other direction $\hat{\mathbf{k}}'$, where $\hat{\mathbf{k}} = \dfrac{\mathbf{k}}{|\mathbf{k}|}$. Energy balance leads to the transport equation

$$\begin{aligned}\frac{\partial a(t,\mathbf{x},\mathbf{k})}{\partial t} &+ \nabla_{\mathbf{k}}\omega(\mathbf{x},\mathbf{k})\cdot\nabla_{\mathbf{x}}a(t,\mathbf{x},\mathbf{k}) - \nabla_{\mathbf{x}}\omega(\mathbf{x},\mathbf{k})\cdot\nabla_{\mathbf{k}}a(t,\mathbf{x},\mathbf{k}) \qquad (1.1)\\ &= \int_{R^n}\sigma(\mathbf{x},\mathbf{k},\mathbf{k}')a(t,\mathbf{x},\mathbf{k}')d\mathbf{k}' - \Sigma(\mathbf{x},\mathbf{k})a(t,\mathbf{x},\mathbf{k}).\end{aligned}$$

Here n is the dimension of space ($n = 2$ or 3), $\omega(\mathbf{x}, \mathbf{k})$ is the local frequency at position $\mathbf{x}$ of a wave with wave vector $\mathbf{k}$, the differential scattering cross-section $\sigma(\mathbf{x}, \mathbf{k}, \mathbf{k}')$ is the rate at which energy with wave vector $\mathbf{k}'$ is converted to wave energy with wave vector $\mathbf{k}$ at position $\mathbf{x}$, and

$$\Sigma(\mathbf{x},\mathbf{k}) = \int \sigma(\mathbf{x},\mathbf{k}',\mathbf{k})d\mathbf{k}' \qquad (1.2)$$

is the total scattering cross-section. The function $\sigma(\mathbf{x}, \mathbf{k}, \mathbf{k}')$ is nonnegative and usually symmetric in $\mathbf{k}$ and $\mathbf{k}'$. The left side of (1.1) is the total time derivative of $a(t, \mathbf{x}, \mathbf{k})$ at a point moving along a trajectory in phase space $(\mathbf{x}, \mathbf{k})$ and the right side describes the effects of scattering.

The transport equation (1.1) is conservative when (1.2) holds because then the total energy is independent of time:

$$\iint a(t,\mathbf{x},\mathbf{k})d\mathbf{x}d\mathbf{k} = \text{const.}$$

Absorption may be accounted for easily by letting the total scattering cross-section be the sum of two terms

$$\Sigma(\mathbf{x},\mathbf{k}) = \Sigma_{sc}(\mathbf{x},\mathbf{k}) + \Sigma_{ab}(\mathbf{x},\mathbf{k})$$

where $\Sigma_{sc}(\mathbf{x},\mathbf{k})$ is the total scattering cross-section given by the right side of (1.2) and $\Sigma_{ab}(\mathbf{x},\mathbf{k})$ is the absorption cross section.

Derivations of (1.1) from equations for wave propagation by many authors are nicely presented in a recent review [22]. (See [23] for references). In our work [23], radiative transport theory for scalar and vector waves, including mode conversion and polarization, was derived in the following regime:

- Distances of propagation L are much larger than the wavelength λ,
- The medium parameters vary on a scale comparable to the wavelength,
- The mismatch between the inhomogeneities and the background medium is small,
- Absorption is small.

This regime arises in many physically important situations such as seismic wave propagation, where teleseismic events can be modeled by radiative transport equations [1, 13].

1.2 Half Space Problems and Boundary Conditions

The radiative transport equation (1.1) has been derived in an unbounded domain. Boundary conditions at a flat boundary or interface were derived for time harmonic acoustic waves for the monochromatic transport equation without scattering. They can be used with (1.1) in a domain with curved boundaries and interfaces varying on a scale large compared to the wave length because of the separation of scales and weak scattering.

The problem of wave scattering from rough boundaries has been studied extensively [5, 10, 29]. We show that scattering from a rough interface can be incorporated in radiative transport theory under the following conditions:

- Wavelength small compared to the curvature of the mean interface
- Roughness height smaller than or comparable to the wavelength
- Correlation length of the surface fluctuations comparable to the wavelength.

Then the boundary conditions for (1.1) are as follows:

Let $a^1_{in}(\mathbf{x}',\mathbf{k}')$ and $a^2_{in}(\mathbf{x}',\mathbf{k}')$ be the energy densities of the waves coming to the interface from above and below, respectively, with $\mathbf{x}'$ being a horizontal coordinate on the rough surface, and $\mathbf{k}'$ the horizontal wave vector. Then the outgoing energy densities are given by

$$\begin{aligned} v^1\hat{k}^1_n a^1_{out}(\mathbf{x}',\mathbf{k}') &= v^1\hat{k}^1_n|R_{11}(\mathbf{k}')|^2 a^1_{in}(\mathbf{x}',\mathbf{k}') + v^2\hat{k}^2_n|T_{12}(\mathbf{k}')|^2 a^2_{in}(\mathbf{x}',\mathbf{k}') \\ &+ \int d\mathbf{p}' \left(v^1\hat{k}^1_n(\mathbf{p}')\tau_{11}(\mathbf{x}',\mathbf{k}',\mathbf{p}')a^1_{in}(\mathbf{x}',\mathbf{p}') + v^2\hat{k}^2_n(\mathbf{p}')\tau_{12}(\mathbf{x}',\mathbf{k}',\mathbf{p}')a^2_{in}(\mathbf{x}',\mathbf{p}')\right) \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} v^2\hat{k}^2_n a^2_{out}(\mathbf{x}',\mathbf{k}') &= v^2\hat{k}^2_n|R_{22}(\mathbf{k}')|^2 a^2_{in}(\mathbf{x}',\mathbf{k}') + v^1\hat{k}^1_n|T_{21}(\mathbf{k}')|^2 a^1_{in}(\mathbf{x}',\mathbf{k}') \\ &+ \int d\mathbf{p}' \left(v^2\hat{k}^2_n(\mathbf{p}')\tau_{22}(\mathbf{x}',\mathbf{k}',\mathbf{p}')a^2_{in}(\mathbf{x}',\mathbf{p}') + v^1\hat{k}^1_n(\mathbf{p}')\tau_{21}(\mathbf{x}',\mathbf{k}',\mathbf{p}')a^1_{in}(\mathbf{x}',\mathbf{p}')\right). \end{aligned} \tag{1.4}$$

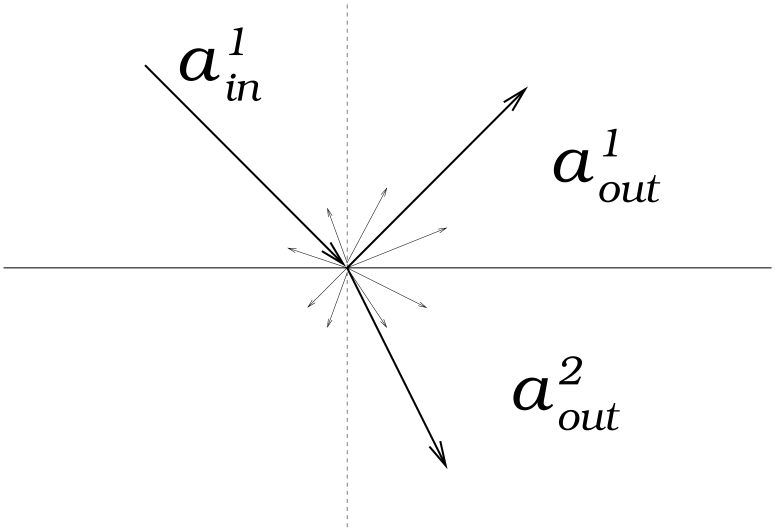

Figure 1.1: Incident, reflected, transmitted, and diffusely scattered waves at a flat interface.

	Flat surface	Rough surface, reflection coefficient correction	Rough surface, scattering cross-section
Dirichlet	1	(4.27) entry $2,2$	(4.28) entry $2,2$
Neumann	1	(4.27) entry $1,1$	(4.28) entry $1,1$
Interface	(3.11)	(4.23)	(4.22)

Table 1.1: References to the explicit formulas for the reflection and transmission coefficients and scattering cross-sections

The first and second terms on the right in (1.3) and (1.4) represent coherent reflection and transmission respectively, and the integrals represent diffuse or incoherent scattering. The coefficients in these equations satisfy the energy flux conservation law

$$1 \;=\; |R_{11}(\mathbf{k}')|^2 \;+\; |T_{21}(\mathbf{k}')|^2 \;+\; \int d\mathbf{p}' \left[\tau_{11}(\mathbf{p}',\mathbf{k}') + \tau_{21}(\mathbf{p}',\mathbf{k}')\right], \tag{1.5}$$

with a similar relation holding for R_{22}, T_{12}, τ_{12} and τ_{22}. This is the analog of (1.2) for surface scattering. Here v^j are the acoustic wave speeds in medium j, and $\hat{k}^j_n$ is the normal component of the normalized wave vector $\hat{\mathbf{k}}^j = \mathbf{k}^j/|\mathbf{k}^j|$ where $\mathbf{k}^j$ has tangential component $\mathbf{k}'$, $j = 1,2$.

We derive explicit formulas for the surface differential cross sections τ_{ij} and the reflection and transmission coefficients. For simplicity, we consider a homogeneous background medium. A generalization to an inhomogeneous medium is presented in [2]. The results remain essentially unchanged but their derivation is substantially more involved. The equation numbers for the explicit formulas are given in Table 1.1.

Interface conditions (1.3) and (1.4) are derived when the adjacent media are homogeneous. We expect that they remain valid when the media are random. This is because waves are not trapped near the boundary when the fluctuations of the interface are small. A mathematical justification of this assumption is not available at present.

2 Acoustic Wave Transport

We recall how transport equations for the phase space energy densities are constructed [23, 14, 24]. For simplicity we assume here that the space domain is $\mathbb{R}^3$ ($n = 3$). The acoustic equations for the velocity $\mathbf{v}$ and pressure p are

$$\begin{aligned} \rho \frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0 \\ \kappa \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{v} = 0. \end{aligned} \tag{2.1}$$

This system may be written as a symmetric hyperbolic system (with summation over repeated indices):

$$A(\mathbf{x}) \frac{\partial \mathbf{u}}{\partial t} + D^j \frac{\partial \mathbf{u}}{\partial x^j} = 0, \tag{2.2}$$

where $\mathbf{u} = (\mathbf{v}, p)$, and $\mathbf{x} \in \mathbb{R}^n$. The matrix $A(\mathbf{x}) = \mathrm{diag}(\rho, \rho, \rho, \kappa)$ is symmetric and positive definite and the matrices D^j are symmetric and independent of $\mathbf{x}$ and t. We consider high frequency solutions of (2.2). Physically this means that the typical wave length λ of the initial data is much smaller than the overall propagation distance L, so that $\varepsilon = \lambda/L \ll 1$. The spatial energy density for the solutions of (2.2) is given by

$$\mathcal{E}(t, \mathbf{x}) = \frac{\rho \mathbf{v}^2}{2} + \frac{\kappa p^2}{2} = \frac{1}{2}(A(\mathbf{x})\mathbf{u}(t, \mathbf{x}) \cdot \mathbf{u}(t, \mathbf{x})) = \frac{1}{2} A_{ij}(\mathbf{x}) u_i(t, \mathbf{x}) \bar{u}_j(t, \mathbf{x}) \tag{2.3}$$

and the flux $\boldsymbol{\mathcal{F}}(\mathbf{x})$ by

$$\boldsymbol{\mathcal{F}}_i(t, \mathbf{x}) = p\mathbf{v} = \frac{1}{2}(D^i \mathbf{u}(t, \mathbf{x}) \cdot \mathbf{u}(t, \mathbf{x})). \tag{2.4}$$

We have the energy conservation law

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}} = 0. \tag{2.5}$$

and thus the total energy is conserved:

$$\frac{d}{dt} \int \mathcal{E}(t, \mathbf{x}) d\mathbf{x} = 0. \tag{2.6}$$

The high frequency limit $\varepsilon \to 0$ of the energy density $\mathcal{E}(t, \mathbf{x})$ is described in terms of the Wigner transform, which is defined by

$$W_\varepsilon(t, \mathbf{x}, \mathbf{k}) = \left(\frac{1}{2\pi}\right)^n \int e^{i\mathbf{k}\cdot\mathbf{y}} \mathbf{u}_\varepsilon(t, \mathbf{x} - \varepsilon\mathbf{y}/2) \mathbf{u}^*_\varepsilon(\mathbf{x} + \varepsilon\mathbf{y}/2) d\mathbf{y}, \tag{2.7}$$

where $\mathbf{u}_\varepsilon(t, \mathbf{x})$ is the solution of (2.2). The Wigner transform W_ε is a 4×4 Hermitian matrix. Its limit as $\varepsilon \to 0$ is called the Wigner distribution and is denoted by $W(t, \mathbf{x}, \mathbf{k})$. The limit Wigner matrix is not only Hermitian but also positive definite. The limit energy density and flux are expressed in terms of $W(t, \mathbf{x}, \mathbf{k})$ by

$$\mathcal{E}(t, \mathbf{x}) = \frac{1}{2} \int \mathrm{Tr}(A(\mathbf{x}) W(t, \mathbf{x}, \mathbf{k})) d\mathbf{k}$$

and

$$\mathcal{F}_i(t,\mathbf{x},\mathbf{k}) = \frac{1}{2}\int \mathrm{Tr}(D^i W(t,\mathbf{x},\mathbf{k}))d\mathbf{k}.$$

The limit Wigner distribution may be decomposed into different wave modes in a way that generalizes the plane wave decomposition in a homogeneous medium. The dispersion matrix of the system (2.2) is defined by

$$L(\mathbf{x},\mathbf{k}) = A^{-1}(\mathbf{x})k_i D^i = \begin{pmatrix} 0 & 0 & 0 & k_1/\rho \\ 0 & 0 & 0 & k_2/\rho \\ 0 & 0 & 0 & k_3/\rho \\ k_1/\kappa & k_2/\kappa & k_3/\kappa & 0 \end{pmatrix}. \tag{2.8}$$

It has one double eigenvalue $\omega_1 = \omega_2 = 0$ and two simple eigenvalues

$$\omega_f = v|\mathbf{k}| \ , \ \ \omega_b = -v|\mathbf{k}| \ , \tag{2.9}$$

where $|\mathbf{k}| = \sqrt{k_1^2 + k_2^2 + k_3^2}$ and v is the sound speed

$$v = \frac{1}{\sqrt{\kappa\rho}}. \tag{2.10}$$

The corresponding basis of eigenvectors is

$$\begin{aligned} \mathbf{b}_1 &= \frac{1}{\sqrt{\rho}}(\mathbf{z}^{(1)}(\mathbf{k}),0)^t, \quad \mathbf{b}_2 = \frac{1}{\sqrt{\rho}}(\mathbf{z}^{(2)}(\mathbf{k}),0)^t, \\ \mathbf{b}_f &= (\frac{\hat{\mathbf{k}}}{\sqrt{2\rho}}, \frac{1}{\sqrt{2\kappa}})^t, \quad \mathbf{b}_b = (\frac{\hat{\mathbf{k}}}{\sqrt{2\rho}}, -\frac{1}{\sqrt{2\kappa}})^t, \end{aligned} \tag{2.11}$$

where the vectors $\hat{\mathbf{k}}$, $\mathbf{z}^{(1)}(\mathbf{k})$ and $\mathbf{z}^{(2)}(\mathbf{k})$ form an orthonormal triplet. The eigenvectors $\mathbf{b}_1(\mathbf{k})$ and $\mathbf{b}_2(\mathbf{k})$ correspond to transverse advection modes, orthogonal to the direction of propagation. These modes do not propagate because $\omega_1 = \omega_2 = 0$. The eigenvectors $\mathbf{b}_f(\mathbf{k})$ and $\mathbf{b}_b(\mathbf{k})$ represent forward and backward acoustic waves, which are longitudinal , and which propagate with the sound speed v given by (2.10).

The limit Wigner distribution matrix $W(t,\mathbf{x},\mathbf{k})$ has the form [23]:

$$W(t,\mathbf{x},\mathbf{k}) = \sum_{\tau=1}^{2} W_{ij}^{\tau}(t,\mathbf{x},\mathbf{k})\mathbf{b}^i(\mathbf{k})\mathbf{b}^{j*}(\mathbf{k}) + a_f(t,\mathbf{x},\mathbf{k})\mathbf{b}_f(\mathbf{k})\mathbf{b}_f^*(\mathbf{k}) + a_b(t,\mathbf{x},\mathbf{k})\mathbf{b}_b(\mathbf{k})\mathbf{b}_b^*(\mathbf{k}). \tag{2.12}$$

The first term corresponds to non-propagating modes and may be set to zero without loss of generality. The last two terms correspond to forward and backward propagating sound waves. The scalar functions a_f and a_b are related by $a_f(t,\mathbf{x},\mathbf{k}) = a_b(t,\mathbf{x},-\mathbf{k})$, and a_f satisfies the Liouville equation

$$\frac{\partial a}{\partial t} + \nabla_{\mathbf{k}}\omega\cdot\nabla_{\mathbf{x}}a - \nabla_{\mathbf{x}}\omega\cdot\nabla_{\mathbf{k}}a = 0. \tag{2.13}$$

The functions a_f and a_b can be interpreted as phase space energy densities since they are non-negative (because the matrix $W(t,\mathbf{x},\mathbf{k})$ is non-negative) and

$$\mathcal{E}(\mathbf{x}) = \frac{1}{2}\int d\mathbf{k}[a_f(t,\mathbf{x},\mathbf{k}) + a_b(t,\mathbf{x},\mathbf{k})] = \int d\mathbf{k} a_f(t,\mathbf{x},\mathbf{k}).$$

The flux is given by

$$\mathcal{F} = \frac{v}{2}\int d\mathbf{k}[\hat{\mathbf{k}}a_f(t,\mathbf{x},\mathbf{k}) - \hat{\mathbf{k}}a_b(t,\mathbf{x},\mathbf{k})] = v\int d\mathbf{k}\hat{\mathbf{k}}a_f(t,\mathbf{x},\mathbf{k}). \tag{2.14}$$

The radiative transport equation (1.1) arises when the density ρ and compressibility κ are random and varying on the scale of the wave length, so we assume they have the forms

$$\rho \to \rho(1+\sqrt{\varepsilon}\rho_1(\frac{\mathbf{x}}{\varepsilon})), \quad \kappa \to \kappa(1+\sqrt{\varepsilon}\kappa_1(\frac{\mathbf{x}}{\varepsilon})).$$

The random processes ρ_1 and κ_1 are mean zero space homogeneous with power spectral densities $\hat{R}_{\rho\rho}$, $\hat{R}_{\kappa\kappa}$, and cross spectral density $\hat{R}_{\kappa\rho}$. The radiative transport equation for $a(t,\mathbf{x},\mathbf{k}) = a_f(t,\mathbf{x},\mathbf{k})$ has the form

$$\begin{aligned}\frac{\partial a}{\partial t} + v\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}a - |\mathbf{k}|\nabla_{\mathbf{x}}v\cdot\nabla_{\mathbf{k}}a &= \frac{\pi v^2|\mathbf{k}|^2}{2}\int \delta(v|\mathbf{k}| - v|\mathbf{k}'|)[a(\mathbf{k}') - a(\mathbf{k})] \\ &\cdot\left\{(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')^2\hat{R}_{\rho\rho}(\mathbf{k}-\mathbf{k}') + 2(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')\hat{R}_{\rho\kappa}(\mathbf{k}-\mathbf{k}') + \hat{R}_{\kappa\kappa}(\mathbf{k}-\mathbf{k}')\right\}d\mathbf{k}'.\end{aligned} \tag{2.15}$$

This equation is of the form (1.1). A systematic derivation using the Wigner distribution is given in [23].

Equation (1.1) has been derived from equations governing particular wave motions by various authors, such as Stott [26], Watson et.al. [32, 33, 19], Barabanenkov et.al. [3], Besieris and Tappert [6], Howe [16], Ishimaru [17] and Besieris et. al. [7], with a recent survey presented in [4]. See also [12] for similar results. These derivations also determine the functions $\omega(\mathbf{x},\mathbf{k})$ and $\sigma(\mathbf{x},\mathbf{k},\mathbf{k}')$ and show how a is related to the wave field. In [23], (1.1) and these functions are derived as a special case of a more general theory.

3 Flat Boundary

3.1 Reflection and Transmission of Waves in Homogeneous Media

We will derive transport theoretic boundary conditions assuming that the media adjacent to the interface are homogeneous, with the interface flat in this section, and random, in later sections.

In this section we recall briefly the transmission and reflection of time harmonic, acoustic plane waves at a plane interface. The time harmonic solutions of (2.1) with frequency ω satisfy

$$\begin{aligned}\nabla p - i\omega\rho\mathbf{v} &= 0 \\ \nabla\cdot\mathbf{v} - i\kappa p &= 0.\end{aligned} \tag{3.1}$$

We consider the high frequency regime, and replace accordingly $\omega \to \omega/\varepsilon$ with $\varepsilon \ll 1$. Then (3.1) becomes

$$\begin{aligned}\varepsilon\nabla p - i\omega\rho\mathbf{v} &= 0 \\ \varepsilon\nabla\cdot\mathbf{v} - i\kappa p &= 0.\end{aligned} \tag{3.2}$$

Consider a medium composed of two homogeneous half spaces in $\mathbb{R}^n$, characterized by constant densities and compressibilities ρ^i and κ^i, $i = 1,2$. Here $i = 1$ in the upper half space $x_n > 0$ and $i = 2$ in the lower one $x_n < 0$. The velocity of propagation is $v^i = 1/\sqrt{\rho^i\kappa^i}$ in each half space.

The wavenumber $|\mathbf{k}^i|$ of a plane wave is related to the frequency ω by the dispersion relation

$$\omega \;=\; v^i|\mathbf{k}^i| \quad \text{for } i = 1,2. \tag{3.3}$$

Any solution of (3.2) in $x_n > 0$ can be decomposed into a sum of incoming and outgoing waves,

$$\mathbf{u}_\varepsilon^i(\mathbf{x}) \;=\; \mathbf{u}_{I,\varepsilon}^i(\mathbf{x}) + \mathbf{u}_{R,\varepsilon}^i(\mathbf{x}). \tag{3.4}$$

When the medium is homogeneous

$$\mathbf{u}_{I,\varepsilon}^i(\mathbf{x}) = \int \frac{d\mathbf{k}'}{(2\pi\varepsilon)^{(n-1)/2}} \exp\,[i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}]\exp\,[-i\frac{\epsilon^i k_n^i x_n}{\varepsilon}]\;\alpha_\varepsilon^i(\mathbf{k}')\;\mathbf{b}_f^i(\mathbf{k}_-^i) \tag{3.5}$$

$$\mathbf{u}_{R,\varepsilon}^i(\mathbf{x}) = \int \frac{d\mathbf{k}'}{(2\pi\varepsilon)^{(n-1)/2}} \exp\,[i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}]\exp\,[i\frac{\epsilon^i k_n^i x_n}{\varepsilon}]\;\beta_\varepsilon^i(\mathbf{k}')\;\mathbf{b}_f^i(\mathbf{k}_+^i). \tag{3.6}$$

The wave vector $\mathbf{k}'$ is the horizontal component of $\mathbf{k}$ and k_n^i, $i = 1,2$ is its normal component:

$$\mathbf{k}_\pm^i \;=\; (\mathbf{k}', \pm\epsilon^i k_n^i), \qquad k_n^i(\mathbf{k}') = \sqrt{|\mathbf{k}^i|^2 - |\mathbf{k}'|^2}, \qquad \epsilon^1 = 1, \qquad \epsilon^2 = -1 \tag{3.7}$$

We consider only propagating waves because the energy in evanescent waves is exponentially small in the high frequency limit, and vanishes as $\varepsilon \to 0$. Therefore we assume that the normal components of the wave vectors k_n^1 and k_n^2 are real. Then the support of the amplitudes α_ε^i and β_ε^i is uniformly inside the ball $\{|\mathbf{k}'| < \omega/v^i\}$. Then $\hat{\mathbf{u}}_{I,\varepsilon}^i$, the Fourier transform of $\mathbf{u}_{I,\varepsilon}^i$, and the corresponding amplitude $\alpha_\varepsilon^i(\mathbf{k}')$ are related by

$$\hat{\mathbf{u}}_{I,\varepsilon}^i(\frac{\mathbf{k}}{\varepsilon}) \;=\; (2\pi\varepsilon)^{(n+1)/2}\;\alpha_\varepsilon^i(\mathbf{k}')\;\mathbf{b}^i(\mathbf{k}_-^i)\;\delta(k_n + \epsilon^i k_n^i). \tag{3.8}$$

An analogous relation holds for the reflected waves.

The amplitudes α_ε^i and β_ε^i are not independent of one another. They are related through the interface conditions at $x_n = 0$, expressed by the continuity of the normal velocity and pressure

$$\begin{aligned} u_n^1(\mathbf{x}',0) &= u_n^2(\mathbf{x}',0) \\ p^1(\mathbf{x}',0) &= p^2(\mathbf{x}',0). \end{aligned} \tag{3.9}$$

Equations (2.11) and (3.5-3.6) give the following relations for the wave amplitudes (for $i \neq j$)

$$(k_n^i)^{1/2}(k^i)^{-1}\beta_\varepsilon^i(\mathbf{k}') \;=\; \mathcal{R}^i(\mathbf{k}')\;(k_n^i)^{1/2}(k^i)^{-1}\alpha_\varepsilon^i(\mathbf{k}') \;+\; \mathcal{T}^j(\mathbf{k}')\;(k_n^j)^{1/2}(k^j)^{-1}\alpha_\varepsilon^j(\mathbf{k}'). \tag{3.10}$$

Here $\mathcal{R}^i$ is the Fresnel reflection coefficient for medium i, and $\mathcal{T}^j$ is the Fresnel transmission coefficient from medium j into medium i. They are given by (see e.g. [18, 8])

$$\mathcal{R}^i(\mathbf{k}') \;=\; \frac{\rho^j k_n^i - \rho^i k_n^j}{\rho^j k_n^i + \rho^i k_n^j}, \qquad \mathcal{T}^i(\mathbf{k}') \;=\; \frac{2(\rho^i\rho^j k_n^i k_n^j)^{1/2}}{\rho^j k_n^i + \rho^i k_n^j}\;, \quad i \neq j. \tag{3.11}$$

Thus $\mathcal{R}^2(\mathbf{k}') = -\mathcal{R}^1(\mathbf{k}')$ and $\mathcal{T}^1(\mathbf{k}') = \mathcal{T}^2(\mathbf{k}')$. Moreover, the conservation of energy flux implies that

$$(\mathcal{R}^i(\mathbf{k}'))^2 + (\mathcal{T}^i(\mathbf{k}'))^2 = 1\;, \quad i = 1,2. \tag{3.12}$$

The angles of incidence and refraction θ^i and θ^j are related by Snell's law

$$\frac{\sin\theta^i}{v^i} \;=\; \frac{\sin\theta^j}{v^j}. \tag{3.13}$$

3.2 High Frequency Limit: Reflection and Transmission of Energy

To derive interface conditions on a at a flat boundary we consider the Wigner transform of the incoming and outgoing waves. We denote by $W[\mathbf{u}]$ the limit as $\varepsilon \to 0$ of $W_\varepsilon[\mathbf{u}_\varepsilon]$, the Wigner transform of $\mathbf{u}_\varepsilon$ defined by (2.7). The directions of the incident and reflected waves are not overlapping. Then by the general theory [14, 24], they decouple in the high frequency limit $\varepsilon \to 0$ so that

$$W[\mathbf{u}^i](\mathbf{x},\mathbf{k}) \;=\; W[\mathbf{u}^i_I](\mathbf{x},\mathbf{k}) \;+\; W[\mathbf{u}^i_R](\mathbf{x},\mathbf{k}). \tag{3.14}$$

This also may be checked directly using (3.5) and (3.6). We will now show that

$$\begin{aligned} W[\mathbf{u}^i_I](\mathbf{x},\mathbf{k}) &= W^{n-1}[\alpha^i]\;(\mathbf{k}', -[\mathbf{x}' - \frac{x_n}{k_n}\mathbf{k}'])\;\delta(k_n + \epsilon^i k^i_n)\;\mathbf{b}^i(\mathbf{k}^i_-)\mathbf{b}^{i*}(\mathbf{k}^i_-) \\ W[\mathbf{u}^i_R](\mathbf{x},\mathbf{k}) &= W^{n-1}[\beta^i]\;(\mathbf{k}', -[\mathbf{x}' - \frac{x_n}{k_n}\mathbf{k}'])\;\delta(k_n - \epsilon^i k^i_n)\;\mathbf{b}^i(\mathbf{k}^i_+)\mathbf{b}^{i*}(\mathbf{k}^i_+), \end{aligned} \tag{3.15}$$

where $W^{n-1}[\alpha^i_\varepsilon]$ is the limit as $\varepsilon \to 0$ of the $(n-1)$-dimensional Wigner transform of α^i_ε.

Let us compute $W[\mathbf{u}^i_I](\mathbf{x},\mathbf{k})$. By virtue of the relation [14, 24]

$$W_\varepsilon[f_\varepsilon(.)](\mathbf{x},\mathbf{k}) \;=\; W_\varepsilon[\frac{1}{(2\pi\varepsilon)^{n/2}}\hat{f}_\varepsilon(\frac{\cdot}{\varepsilon})](\mathbf{k},-\mathbf{x}),$$

and (3.8), we have

$$\begin{aligned} W_\varepsilon[\mathbf{u}^i_{I,\varepsilon}](\mathbf{x},\mathbf{k}) = \frac{2\pi\varepsilon}{(2\pi)^n}\int d\mathbf{p}\,\alpha^i_\varepsilon(\mathbf{k}' + \frac{\varepsilon\mathbf{p}'}{2})\alpha^{i*}_\varepsilon(\mathbf{k}' - \frac{\varepsilon\mathbf{p}'}{2})\delta(k_n + \varepsilon\frac{p_n}{2} + \epsilon^i k^i_n(\mathbf{k}' + \varepsilon\frac{\mathbf{p}'}{2})) \\ \times\delta(k_n - \varepsilon\frac{p_n}{2} + \epsilon^i k^i_n(\mathbf{k}' - \varepsilon\frac{\mathbf{p}'}{2}))\exp{(i\mathbf{p}\cdot\mathbf{x})}\mathbf{b}^i((\mathbf{k}+\varepsilon\frac{\mathbf{p}}{2})^i_-)\mathbf{b}^{i*}((\mathbf{k}-\varepsilon\frac{\mathbf{p}}{2})^i_-). \end{aligned} \tag{3.16}$$

From the dispersion relation (3.3) we deduce

$$\begin{aligned} (k_n + \varepsilon\frac{p_n}{2})^2 &= \frac{\omega^2}{(v^i)^2} - (\mathbf{k}' + \varepsilon\frac{\mathbf{p}'}{2})^2 \\ (k_n - \varepsilon\frac{p_n}{2})^2 &= \frac{\omega^2}{(v^i)^2} - (\mathbf{k}' - \varepsilon\frac{\mathbf{p}'}{2})^2. \end{aligned}$$

Then

$$|\mathbf{k}|^2 + |\frac{\varepsilon\mathbf{p}}{2}|^2 \;=\; \frac{\omega^2}{(v^i)^2}, \qquad k_n p_n \;=\; \mathbf{k}'\cdot\mathbf{p}'.$$

On the other hand one readily checks that for constants c_1 and c_2 we have

$$\delta(k_n + \frac{\varepsilon p_n}{2} + c_1)\;\delta(k_n - \frac{\varepsilon p_n}{2} + c_2) \;=\; \delta(k_n + \frac{c_1 + c_2}{2})\;\delta(p_n - \frac{c_2 - c_1}{\varepsilon}).$$

By using this relation and the change of variables $p_n \to \varepsilon p_n$ in (3.16), we get for ε small,

$$\begin{aligned} W[\mathbf{u}^i_I](\mathbf{x},\mathbf{k}) \;\sim\; & \frac{1}{(2\pi)^{n-1}}\int d\mathbf{p}'\alpha^i_\varepsilon(\mathbf{k}' + \frac{\varepsilon\mathbf{p}'}{2})\alpha^{i*}_\varepsilon(\mathbf{k}' - \frac{\varepsilon\mathbf{p}'}{2})\exp i[\mathbf{p}'\cdot(\mathbf{x}' - \frac{x_n}{k_n}\mathbf{k}')] \\ & \times\mathbf{b}^i(\mathbf{k}^i_-)\mathbf{b}^{i*}(\mathbf{k}^i_-)\;\delta(k_n + \epsilon^i k^i_n). \end{aligned} \tag{3.17}$$

When $\varepsilon \to 0$ in (3.17) it yields the first equation in (3.15), and the second equation in (3.15) can be derived similarly.

The explicit representation (3.15) implies that the Wigner matrix of the incoming waves has the form

$$W_I^i = a_I^i(\mathbf{x}, \mathbf{k})\mathbf{b}^i(\mathbf{k})\mathbf{b}^{i*}(\mathbf{k})$$

with a_I^i supported on the wave vectors $\mathbf{k}$ with $k_n = -\epsilon^i k_n^i$. In the interior, the function a_I^i satisfies the transport equation

$$\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} a_I^i = 0. \tag{3.18}$$

Similarly

$$W_R^i = a_R^i(\mathbf{x}, \mathbf{k})\mathbf{b}^i(\mathbf{k})\mathbf{b}^{i*}(\mathbf{k}).$$

The function a_R^i is supported on the hemisphere of outgoing wave vectors with $k_n = \epsilon^i k_n^i$ and it satisfies the same transport equation (3.18).

Since the coefficients $\mathcal{R}^i$ and $\mathcal{T}^i$ do not depend on ε, we deduce from (3.10) that

$$\begin{aligned} \frac{k_n^i}{(k^i)^2} W^{n-1}[\beta^i] \; &= \; \frac{k_n^i}{(k^i)^2} |\mathcal{R}^i(\mathbf{k}')|^2 \; W^{n-1}[\alpha^i] \; + \; \frac{k_n^j}{(k^j)^2} |\mathcal{T}^j(\mathbf{k}')|^2 \; W^{n-1}[\alpha^j] \\ &\quad + \frac{\sqrt{k_n^i}}{k^i} \frac{\sqrt{k_n^j}}{k^j} [\mathcal{R}^i(\mathbf{k}') \; \overline{\mathcal{T}^j}(\mathbf{k}') + \overline{\mathcal{R}^i}(\mathbf{k}') \; \mathcal{T}^j(\mathbf{k}')] \; W^{n-1}[\alpha^i, \alpha^j]. \end{aligned} \tag{3.19}$$

By virtue of (3.15), the first two terms on the right side of (3.19) are related to the incident energy $W[\mathbf{u}_I^i]$. The third term, involving $W^{n-1}[\alpha^i, \alpha^j]$, corresponds to the correlation between amplitudes of waves coming to the surface from opposite sides. We assume that they are uncorrelated, so that

$$W^{n-1}[\alpha^i, \alpha^j] \; = \; 0. \tag{H1}$$

Condition (H1) is not satisfied in general; in particular not for two incoming plane waves. However it is satisfied when the wave packets arriving at the interface from different sides have uncorrelated random phases. This is the case in the radiative transport regime when waves arriving at the interface have undergone independent multiple scattering.

When (H1) holds, (3.19) shows that the a^i satisfy the interface conditions

$$v^i \hat{k}_n^i a^i(\mathbf{x}', 0, \mathbf{k}', \epsilon^i k_n^i) = |\mathcal{R}^i(\mathbf{k}')|^2 \; v^i \hat{k}_n^i a^i(\mathbf{x}', 0, \mathbf{k}', -\epsilon^i k_n^i) \; + \; |\mathcal{T}^j(\mathbf{k}')|^2 \; v^j \hat{k}_n^j a^j(\mathbf{x}', 0, \mathbf{k}', -\epsilon^j k_n^j). \tag{3.20}$$

Note that evanescent waves disappear in the high frequency regime. Therefore $a^j(\mathbf{x}', 0, \mathbf{k}', k_n^j) = 0$ when k_n^j is imaginary. For incident angles below the critical angle (obtained through Snell's law (3.13)), the reflection coefficient $|\mathcal{R}^i(\mathbf{k}')|^2 = 1$ and no energy is transmitted. These relations also hold if the two adjacent media are deterministic but inhomogeneous, as is shown in [24]. We expect that they hold when the adjacent media are random with weak fluctuations.

4 Rough Boundaries

4.1 Surface Reflection Operator

Let us assume that the boundary of the domain or the interface between two subdomains is a surface given by $(\mathbf{x}', h(\mathbf{x}'))$. First we describe the reflected and transmitted waves generated by an incident wave, and then derive boundary conditions for the phase space energy density in the high frequency limit.

The first part of this problem has been studied at length in the past. (See [5, 11, 29].) For simplicity we deal first with reflected waves without transmission, so the superscript corresponding to the medium can be dropped. Consider an incident plane wave, with $\mathbf{k}'_0$ the horizontal component of the wave vector and with amplitude $\alpha_\varepsilon = 1$:

$$\psi_{in}(\mathbf{x}) \;=\; \exp\left[i\mathbf{k}'_0 \cdot \mathbf{x}' - ik_n(\mathbf{k}'_0)x_n\right]\, \mathbf{b}(\mathbf{k}_-).$$

In the region $x_n > h_{max}$, the scattered wave can be decomposed into a sum of outgoing plane waves:

$$\psi_{sc}(\mathbf{x}) \;=\; \int d\mathbf{k}'\; S(\mathbf{k}', \mathbf{k}'_0) \exp\left[i\mathbf{k}' \cdot \mathbf{x}' + ik_n(\mathbf{k}')x_n\right]\, \mathbf{b}(\mathbf{k}_+).$$

The kernel $S(\mathbf{k}', \mathbf{k}'_0)$ is the *scattering amplitude* for waves of incident horizontal wave number $\mathbf{k}'_0$ scattered to waves of horizontal wave number $\mathbf{k}'$. It is determined by the height $h(\mathbf{x}')$ of the surface. S can be obtained by solving an integral equation that follows from the boundary conditions.

Let the boundary vary on the scale of the wavelength, let ε be proportional to the wave length, and let the scaled boundary h_ε be given by

$$h_\varepsilon(\mathbf{x}') \;=\; \varepsilon h(\frac{\mathbf{x}'}{\varepsilon}). \tag{4.1}$$

Here h is a bounded function. This scaling implies that the roughness height, the scale length of the boundary, and the wavelength are all of the same order. Then S is independent of ε. In the high frequency limit an incident wave has scaled horizontal wave vector $\mathbf{k}_0/\varepsilon$. Then the change of variables $\mathbf{x} \to \mathbf{x}/\varepsilon$ yields the same equation for S as when $\varepsilon = 1$. However, the incoming wave still depends on ε through the amplitude α_ε.

In order to express conservation of energy flux at the boundary in a natural way we introduce the amplitudes for energy fluxes:

$$\tilde{\alpha}_\varepsilon \;=\; \sqrt{v\hat{k}_n}\; \alpha_\varepsilon, \qquad \tilde{\beta}_\varepsilon \;=\; \sqrt{v\hat{k}_n}\; \beta_\varepsilon.$$

These are energy flux densities, as can be seen from the expression (2.14) for the total flux.

Consider the general incoming and outgoing plane waves given by (3.5), (3.6) with the incident and reflected amplitudes related by

$$\tilde{\beta}_\varepsilon(\mathbf{k}') \;=\; \int d\mathbf{k}'_0\; S(\mathbf{k}', \mathbf{k}'_0)\tilde{\alpha}_\varepsilon(\mathbf{k}'_0). \tag{4.2}$$

The incoming and outgoing Wigner transforms are given in (3.15) with $W^{n-1}[\tilde{\alpha}_\varepsilon]$ being the known incoming energy flux. The unknown $W^{n-1}[\tilde{\beta}]$ is given by

$$\begin{aligned} W^{n-1}[\tilde{\beta}](\mathbf{k}', -\mathbf{x}') \;&=\; \lim_{\varepsilon\to 0} \int d\mathbf{p}'\; \exp\left[i\mathbf{p}' \cdot \mathbf{x}'\right] \int d\mathbf{k}'_0 S(\mathbf{k}' - \varepsilon\frac{\mathbf{p}'}{2}, \mathbf{k}'_0)\tilde{\alpha}_\varepsilon(\mathbf{k}'_0) \int d\tilde{\mathbf{k}}'_0 \overline{S}(\mathbf{k}' + \varepsilon\frac{\mathbf{p}}{2}, \tilde{\mathbf{k}}'_0)\overline{\tilde{\alpha}}_\varepsilon(\tilde{\mathbf{k}}'_0) \\ &=\; \int d\mathbf{p}'\; \exp\left[i\mathbf{p}' \cdot \mathbf{x}'\right] \left| \int d\mathbf{k}'_0 S(\mathbf{k}', \mathbf{k}'_0)\tilde{\alpha}_\varepsilon(\mathbf{k}'_0) \right|^2 + O(\varepsilon), \end{aligned}$$

since S does not depend on ε. It is clear from this expression that we cannot write $W^{n-1}[\tilde{\beta}]$ only in terms of $W^{n-1}[\tilde{\alpha}]$ for a general scattering kernel S.

We consider random boundaries that are stationary, i.e. statistically invariant under translation. The physical problem is not changed by translation and the incident and reflected waves change phase only. Then the scattering operator for a surface shifted by a vector $\mathbf{d}'$ must satisfy the relation

$$S_{h(\mathbf{x}'-\mathbf{d}')}(\mathbf{k}', \mathbf{k}'_0) \;=\; \exp\left[i(\mathbf{k}' - \mathbf{k}'_0) \cdot \mathbf{d}'\right] S_{h(\mathbf{x}')}(\mathbf{k}', \mathbf{k}'_0). \tag{4.3}$$

Since $\langle h(\mathbf{x}')\rangle$, the mean over all realizations, is a constant, we conclude from (4.3) that

$$\langle S\rangle(\mathbf{k}',\mathbf{k}_0') \;=\; \exp\left[i(\mathbf{k}'-\mathbf{k}_0')\cdot\mathbf{d}'\right]\langle S\rangle(\mathbf{k}',\mathbf{k}_0').$$

Because this relation is true for all shifts $\mathbf{d}'$ we conclude that $\langle S\rangle$ has the form

$$\langle S\rangle(\mathbf{k}',\mathbf{k}_0') \;=\; R(\mathbf{k}')\delta(\mathbf{k}'-\mathbf{k}_0'). \tag{4.4}$$

Here $R(\mathbf{k}')$ is the coherent reflection coefficient.

Consider now the fluctuation ΔS of the scattering amplitude:

$$S(\mathbf{k}',\mathbf{k}_0) \;=\; R(\mathbf{k}')\delta(\mathbf{k}'-\mathbf{k}_0') \;+\; \Delta S(\mathbf{k}',\mathbf{k}_0'). \tag{4.5}$$

The product of two fluctuations are given by

$$\Pi(\mathbf{k}',\mathbf{k}_0';\mathbf{a}',\mathbf{a}_0') \;=\; \Delta S\left(\mathbf{k}'-\frac{\mathbf{a}'}{2},\mathbf{k}_0'-\frac{\mathbf{a}_0'}{2}\right)\overline{\Delta S}\left(\mathbf{k}'+\frac{\mathbf{a}'}{2},\mathbf{k}_0'+\frac{\mathbf{a}_0'}{2}\right).$$

Again, because of translational invariance, the average of Π is proportional to $\delta(\mathbf{a}'-\mathbf{a}_0')$. Thus there exists a kernel $\Delta\mathcal{E}$ such that:

$$\langle\Pi\rangle(\mathbf{k}',\mathbf{k}_0';\mathbf{a}',\mathbf{a}_0') \;=\; \Delta\mathcal{E}(\mathbf{k}',\mathbf{k}_0';\mathbf{a}')\delta(\mathbf{a}'-\mathbf{a}_0'). \tag{4.6}$$

Conservation of energy means that for every incident field $\tilde\alpha(\mathbf{k}_0)$,

$$\int|\tilde\beta(\mathbf{k}')|^2 d\mathbf{k}' = \int|\tilde\alpha(\mathbf{k}_0)|^2 d\mathbf{k}_0.$$

Squaring and averaging (4.2), using (4.5), yield then

$$|R(\mathbf{k}')|^2 + \int_{|\mathbf{k}_0'|<\omega/v}\tau(\mathbf{k}_0',\mathbf{k}')\,d\mathbf{k}_0' \;=\; 1, \qquad |\mathbf{k}'|<\omega/v. \tag{4.7}$$

Here

$$\tau(\mathbf{k}_0',\mathbf{k}') \;=\; \Delta\mathcal{E}(\mathbf{k}_0',\mathbf{k}';0) \tag{4.8}$$

is the differential cross section for the incoherent or diffuse surface scattering.

We are now in a position to calculate the average energy associated with the reflected waves in terms of the incident energy. We have

$$\begin{aligned}
&\langle W^{n-1}[\tilde\beta]\rangle(\mathbf{k}',-\mathbf{x}')\\
=\;& \lim_{\varepsilon\to0}\int d\mathbf{p}'\;\exp\left[i\mathbf{p}'\cdot\mathbf{x}'\right]\langle\int d\mathbf{k}_0' S(\mathbf{k}'-\varepsilon\frac{\mathbf{p}'}{2},\mathbf{k}_0')\tilde\alpha_\varepsilon(\mathbf{k}_0')\int d\tilde{\mathbf{k}}'_0\overline{S}(\mathbf{k}'+\varepsilon\frac{\mathbf{p}'}{2},\tilde{\mathbf{k}}'_0)\overline{\tilde\alpha}_\varepsilon(\tilde{\mathbf{k}}'_0)\rangle\\
=\;& \lim_{\varepsilon\to0}\int d\mathbf{p}'\;\exp\left[i\mathbf{p}'\cdot\mathbf{x}'\right]\int d\mathbf{k}_0'\int d\tilde{\mathbf{k}}'_0\langle S(\mathbf{k}'-\varepsilon\frac{\mathbf{p}'}{2},\mathbf{k}_0')\overline{S}(\mathbf{k}'+\varepsilon\frac{\mathbf{p}'}{2},\tilde{\mathbf{k}}'_0)\rangle\tilde\alpha_\varepsilon(\mathbf{k}_0')\overline{\tilde\alpha}_\varepsilon(\tilde{\mathbf{k}}'_0).
\end{aligned}$$

Because of (4.4)–(4.6) we have

$$\begin{aligned}
< S(\mathbf{k}'-\varepsilon\tfrac{\mathbf{p}'}{2},\mathbf{k}_0')\overline{S}(\mathbf{k}'+\varepsilon\tfrac{\mathbf{p}'}{2},\tilde{\mathbf{k}}'_0)\rangle \;&=\; |R(\mathbf{k}')|^2\delta(\mathbf{k}'-\varepsilon\tfrac{\mathbf{p}'}{2}-\mathbf{k}_0')\delta(\mathbf{k}'+\varepsilon\tfrac{\mathbf{p}'}{2}-\tilde{\mathbf{k}}'_0)\\
&+\;\; \Delta\mathcal{E}(\mathbf{k}',\tfrac{\mathbf{k}_0'+\tilde{\mathbf{k}}'_0}{2};\varepsilon\mathbf{p}')\;\delta(\mathbf{k}_0'-\tilde{\mathbf{k}}'_0-\varepsilon\mathbf{p}')\;+\;O(\varepsilon)
\end{aligned}$$

since R is independent of ε. Since $\Delta\mathcal{E}$ is also independent of ε, we get

$$
\begin{aligned}
\langle W^{n-1}[\tilde{\beta}]\rangle(\mathbf{k}',-\mathbf{x}') &= |R(\mathbf{k}')|^2\langle W^{n-1}[\tilde{\alpha}]\rangle(\mathbf{k}',-\mathbf{x}') \\
&+ \int d\mathbf{p}' \; \exp\left[i\mathbf{p}'\cdot\mathbf{x}'\right] \int d\mathbf{k}_0'\Delta\mathcal{E}(\mathbf{k}',\mathbf{k}_0'+\varepsilon\mathbf{p}';0)\tilde{\alpha}_\varepsilon(\mathbf{k}_0')\overline{\tilde{\alpha}}_\varepsilon(\mathbf{k}_0'-\varepsilon\mathbf{p}') \;+\; O(\varepsilon) \\
&= |R(\mathbf{k}')|^2\langle W^{n-1}[\tilde{\alpha}]\rangle(\mathbf{k}',-\mathbf{x}') + \int d\mathbf{k}_0' \; \tau(\mathbf{k}',\mathbf{k}_0')\langle W^{n-1}[\tilde{\alpha}]\rangle(\mathbf{k}_0',-\mathbf{x}').
\end{aligned}
$$

This implies the following boundary condition for the average scalar energy $a(\mathbf{x},\mathbf{k})$ at $x_n = 0$:

$$
\begin{aligned}
v\hat{k}_n(\mathbf{k}') \; a(\mathbf{x}',0,\mathbf{k}',k_n) &= |R(\mathbf{k}')|^2 v\hat{k}_n(\mathbf{k}') \; a(\mathbf{x}',0,\mathbf{k}',-k_n) \\
&+ \int d\mathbf{k}_0' \; \tau(\mathbf{k}',\mathbf{k}_0') \; v\hat{k}_n(\mathbf{k}_0') \; a(\mathbf{x}',0,\mathbf{k}_0',-k_n(\mathbf{k}_0')), \quad \mathbf{k}_n > 0.
\end{aligned}
\tag{4.9}
$$

Here $R(\mathbf{k})$ defined by (4.4) is the reflection coefficient and $\tau(\mathbf{k},\mathbf{k}_0)$ defined by (4.8) is the differential scattering cross-section of the boundary.

4.2 Reflection and Transmission at a Rough Interface

Let us now consider two homogeneous half spaces separated by a rough interface h. The scalar scattering operator S must be replaced by a two by two matrix-valued operator $(S^{ij})_{1\le i,j\le 2}$. When a wave of unit amplitude is incident in medium j, S^{ij} determines the amplitude of the wave scattered into medium i. As before, we introduce the flux amplitudes $\tilde{\alpha}^i_\varepsilon$ and $\tilde{\beta}^i_\varepsilon$:

$$
\tilde{\alpha}^i_\varepsilon \;=\; \sqrt{v^i\hat{k}^i_n}\;\alpha^i_\varepsilon, \qquad \tilde{\beta}^i_\varepsilon \;=\; \sqrt{v^i\hat{k}^i_n}\;\beta^i_\varepsilon \;=\; \frac{\sqrt{\omega}}{k^i}\sqrt{k^i_n}\;\beta^i_\varepsilon. \tag{4.10}
$$

We consider general incoming and outgoing plane waves given by (3.5) and (3.6), with the incident and reflected amplitudes related by

$$
\tilde{\beta}^i_\varepsilon(\mathbf{k}') \;=\; \int d\mathbf{k}_0' \; S^{ii}(\mathbf{k}',\mathbf{k}_0')\tilde{\alpha}^i_\varepsilon(\mathbf{k}_0') \;+\; \int d\mathbf{k}_0' \; S^{ij}(\mathbf{k}',\mathbf{k}_0')\tilde{\alpha}^j_\varepsilon(\mathbf{k}_0').
$$

Again we deduce from translational invariance that

$$
S^{ij}(\mathbf{k}',\mathbf{k}_0') \;=\; R^{ij}(\mathbf{k}')\delta(\mathbf{k}'-\mathbf{k}_0') \;+\; \Delta S^{ij}(\mathbf{k}',\mathbf{k}_0'). \tag{4.11}
$$

Moreover,

$$
\langle \Pi^{i;kl}\rangle(\mathbf{k}',\mathbf{k}_0';\mathbf{a}',\mathbf{a}_0') \;=\; \Delta\mathcal{E}^{i;kl}(\mathbf{k}',\mathbf{k}_0';\mathbf{a}')\delta(\mathbf{a}'-\mathbf{a}_0'), \tag{4.12}
$$

where

$$
\Pi^{i;kl}(\mathbf{k}',\mathbf{k}_0';\mathbf{a},\mathbf{a}_0) \;=\; \Delta S^{ik}\left(\mathbf{k}'-\frac{\mathbf{a}'}{2},\mathbf{k}_0'-\frac{\mathbf{a}_0'}{2}\right)\overline{\Delta S^{il}}\left(\mathbf{k}'+\frac{\mathbf{a}'}{2},\mathbf{k}_0'+\frac{\mathbf{a}_0'}{2}\right).
$$

Therefore we have

$$
\begin{aligned}
\langle S^{ik}(\mathbf{k}'-\varepsilon\tfrac{\mathbf{p}'}{2},\mathbf{k}_0')\overline{S^{il}}(\mathbf{k}'+\varepsilon\tfrac{\mathbf{p}'}{2},\tilde{\mathbf{k}}'_0)\rangle &= R^{ik}(\mathbf{k}')\overline{R}^{il}(\mathbf{k}')\delta(\mathbf{k}'-\varepsilon\tfrac{\mathbf{p}'}{2}-\mathbf{k}_0')\delta(\mathbf{k}'+\varepsilon\tfrac{\mathbf{p}'}{2}-\tilde{\mathbf{k}}'_0) \\
&+ \Delta\mathcal{E}^{i;kl}(\mathbf{k}',\tfrac{\mathbf{k}_0'+\tilde{\mathbf{k}}'_0}{2};\varepsilon\mathbf{p}') \;\delta(\mathbf{k}_0'-\tilde{\mathbf{k}}'_0-\varepsilon\mathbf{p}') \;+\; O(\varepsilon)
\end{aligned}
$$

Here $R^{ik}(\mathbf{k}')$ are the reflection coefficients.

We denote the differential scattering cross-sections by

$$
\tau^{i;kl}(\mathbf{k}',\mathbf{k}_0') \;=\; \Delta\mathcal{E}^{i;kl}(\mathbf{k}',\mathbf{k}_0';0). \tag{4.13}
$$

Then as above, we obtain

$$\begin{aligned}\langle W^{n-1}[\tilde{\beta}^i]\rangle(\mathbf{k}',-\mathbf{x}') &= \sum_{k,l=1}^{2} R^{ik}(\mathbf{k}')\overline{R}^{il}(\mathbf{k}')W^{n-1}[\tilde{\alpha}^k,\tilde{\alpha}^l](\mathbf{k}',-\mathbf{x}') \\ &+ \sum_{k,l=1}^{2}\int d\mathbf{k}_0'\ \tau^{i;kl}(\mathbf{k}',\mathbf{k}_0')W^{n-1}[\tilde{\alpha}^k,\tilde{\alpha}^l](\mathbf{k}_0',-\mathbf{x}').\end{aligned}$$

When the decorrelation condition (H1) stated in the previous section holds, we let $\tau^{ij} = \tau^{i;jj}$ and we get the incoming-outgoing energy flux relations

$$\begin{aligned}v^i\hat{k}_n^i(\mathbf{k}')\ a^i(\mathbf{x}',0,\mathbf{k}',\epsilon^i k_n^i) &= \sum_{j=1}^{2}|R^{ij}(\mathbf{k}')|^2\ v^j\hat{k}_n^j(\mathbf{k}')\ a^j(\mathbf{x}',0,\mathbf{k}',-\epsilon^j k_n^j) \\ &+ \sum_{j=1}^{2}\int d\mathbf{k}_0'\ \tau^{ij}(\mathbf{k}',\mathbf{k}_0')\ v^j\hat{k}_n^j(\mathbf{k}_0')\ a^j(\mathbf{x}',0,\mathbf{k}_0',-\epsilon^j k_n^j(\mathbf{k}_0'))).\end{aligned} \tag{4.14}$$

Here R^{ij} is defined by (4.11) and τ^{ij} by (4.13).

4.3 Reflection and Transmission from Small Amplitude Rough Interfaces

We now specialize to reflection and transmission at a rough interface with small random fluctuations. The interface is given by $x_n = \varepsilon\eta h(\frac{\mathbf{x}'}{\varepsilon})$, where η is another small parameter. The random process $h(\mathbf{y}')$ has mean zero and is stationary, with covariance function $Q(\mathbf{y}')$ defined by

$$\langle h(\mathbf{x}'+\mathbf{y}')h(\mathbf{x}')\rangle = Q(\mathbf{y}').$$

The power spectral density $\hat{Q}(\mathbf{p})$ of the surface fluctuations is the Fourier transform of Q:

$$Q(\mathbf{y}') = \int \frac{d\mathbf{p}'}{(2\pi)^{n-1}} e^{i\mathbf{p}'\cdot\mathbf{y}'}\hat{Q}(\mathbf{p}'). \tag{4.15}$$

To simplify the calculations, we assume that the scattering operator S, a 2×2 matrix, has an expansion of the form

$$S = S_0 + \eta\, S_1 + \eta^2 S_2 + o(\eta^2). \tag{4.16}$$

Here $S_0(\mathbf{k}',\mathbf{k}_0') = R(\mathbf{k}')\delta(\mathbf{k}'-\mathbf{k}_0')$ is the scattering operator of the unperturbed flat surface. Clearly S_1 has mean zero because it is linear in h. Then

$$\begin{aligned}R(\mathbf{k}') &= R_0(\mathbf{k}')\delta(\mathbf{k}'-\mathbf{k}_0') + \eta^2 R_2(\mathbf{k}')\delta(\mathbf{k}'-\mathbf{k}_0') + o(\eta^2) \\ \Delta S &= \eta S_1 + O(\eta^2).\end{aligned} \tag{4.17}$$

Then (4.13) implies that τ depends only on S_1 up to $o(\eta^2)$. Moreover R depends only on S_2 through its statistical average R_2.

The details of the small η computation of the reflection and transmission coefficients is given in Appendix A. The results are as follows. The leading order term is

$$R_0 = \begin{pmatrix} \mathcal{R} & \mathcal{T} \\ \mathcal{T} & -\mathcal{R} \end{pmatrix}. \tag{4.18}$$

The Fresnel coefficients $\mathcal{R}$ and $\mathcal{T}$ are given in (3.11). They can also be written in the form

$$\mathcal{R} \;=\; \frac{\delta_1^2-\delta_2^2}{\delta_1^2+\delta_2^2}\,, \qquad \mathcal{T} \;=\; \frac{2\delta_1\delta_2}{\delta_1^2+\delta_2^2}, \tag{4.19}$$

where

$$\delta_i \;=\; \sqrt{\frac{k_n^i}{\rho^i}}.$$

Now we define the auxiliary matrices P, M and N by

$$P = \begin{pmatrix} \delta_1 & \delta_2 \\ \delta_1^{-1} & -\delta_2^{-1} \end{pmatrix}, \quad M(\mathbf{p}',\mathbf{k}') = \begin{pmatrix} -\frac{(k^1)^2-\mathbf{p}'\cdot\mathbf{k}'}{k_n^1(\mathbf{k}')}\delta_1(\mathbf{k}') & \frac{(k^2)^2-\mathbf{p}'\cdot\mathbf{k}'}{k_n^2(\mathbf{k}')}\delta_2(\mathbf{k}') \\ k_n^1(\mathbf{k}')\delta_1^{-1}(\mathbf{k}') & k_n^2(\mathbf{k}')\delta_2^{-1}(\mathbf{k}') \end{pmatrix} \tag{4.20}$$

and $N(\mathbf{p}',\mathbf{k}') = P^{-1}(\mathbf{p}')V_0M(\mathbf{p}',\mathbf{k}')$, where

$$V_0 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then B_{ij} is given by $B(\mathbf{p}',\mathbf{k}') = \frac{1}{2}\left[R_0(\mathbf{k}')N(\mathbf{p}',\mathbf{k}') + N(\mathbf{p}',\mathbf{k}')R_0(\mathbf{k}')\right]$.

$$B_{ij} \;=\; \frac{\Gamma(\mathbf{p}'\cdot\mathbf{k}')\delta_i(\mathbf{p}')\delta_j(\mathbf{k}') \;+\; \epsilon^i\epsilon^j(\rho_1-\rho_2)\delta_1\delta_2(\mathbf{p}')\delta_1\delta_2(\mathbf{k}')\delta_{i'}(\mathbf{k}')\delta_{j'}(\mathbf{p}')}{(\delta_1^2+\delta_2^2)(\mathbf{p}')(\delta_1^2+\delta_2^2)(\mathbf{k}')}, \tag{4.21}$$

where $i'=2$ if $i=1$ and vice-versa, and

$$\Gamma(\mathbf{p}'\cdot\mathbf{k}') \;=\; \frac{(k^2)^2-\mathbf{k}'\cdot\mathbf{p}'}{\rho^2} - \frac{(k^1)^2-\mathbf{k}'\cdot\mathbf{p}'}{\rho^1}.$$

The interface conditions (3.9) are continuity of the normal velocity and pressure. Then from (4.13) and (4.17) we find, after a long computation given in Appendix A, that the scattering cross-section tensor is

$$\tau^{ij}(\mathbf{p}',\mathbf{k}') = 4B_{ij}^2(\mathbf{p}',\mathbf{k}')\hat{Q}(\mathbf{p}'-\mathbf{k}'). \tag{4.22}$$

The part R_2 of the coherent reflection coefficient may be split into a sum of two terms

$$R_2 = R_{2,0} + R_{2,1},$$

which are given by

$$\begin{aligned} R_{2,0} \;&=\; \frac{\mathcal{T}}{2}((k^1)^2-(k^2)^2)\;Q(0)\;\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ R_{2,1}(\mathbf{p}') \;&=\; -2\int d\mathbf{k}'N(\mathbf{p}',\mathbf{k}')\,B(\mathbf{k}',\mathbf{p}')\hat{Q}(\mathbf{p}'-\mathbf{k}'). \end{aligned} \tag{4.23}$$

Conservation of energy flux (4.7) has to be satisfied at the interface. This means that energy incident upon the interface is either reflected, transmitted, or scattered. From (4.17)

$$|R^{ij}|^2 \;=\; |R_0^{ij}+\eta^2R_2^{ij}|^2 \;=\; (R_0^{ij})^2 + 2\eta^2R_0^{ij}R_2^{ij} + o(\eta^2). \tag{4.24}$$

Therefore up to $o(\eta^2)$ conservation of energy flux for each $\mathbf{p}'$ is given by

$$\begin{aligned} 1 &= |R^{11}(\mathbf{p}')|^2 + |R^{21}(\mathbf{p}')|^2 + \eta^2 \int d\mathbf{k}'[\tau^{11}(\mathbf{k}',\mathbf{p}') + \tau^{21}(\mathbf{k}',\mathbf{p}')] \\ 1 &= |R^{22}(\mathbf{p}')|^2 + |R^{12}(\mathbf{p}')|^2 + \eta^2 \int d\mathbf{k}'[\tau^{22}(\mathbf{k}',\mathbf{p}') + \tau^{12}(\mathbf{k}',\mathbf{p}')]. \end{aligned} \tag{4.25}$$

In the limit $\eta = 0$ this simplifies to (3.12). The η^2 terms in (4.25) give the identity

$$\begin{aligned} 0 &= 2(-\mathcal{T}R^{21}_{2,0} + \mathcal{R}R^{11}_{2,1} + \mathcal{T}R^{21}_{2,1}) + \int d\mathbf{k}'[\tau^{11}(\mathbf{k}',\mathbf{p}') + \tau^{21}(\mathbf{k}',\mathbf{p}')] \\ 0 &= 2(\mathcal{T}R^{12}_{2,0} - \mathcal{R}R^{22}_{2,1} + \mathcal{T}R^{12}_{2,1}) + \int d\mathbf{k}'[\tau^{22}(\mathbf{k}',\mathbf{p}') + \tau^{12}(\mathbf{k}',\mathbf{p}')]. \end{aligned}$$

A straightforward though tedious computation shows that this is indeed true when we use the formulas given above. Note that the scattering cross section (4.22) is positive for every incident and reflected or transmitted angle. Therefore, when the incident angle is smaller than the angle of total reflection for the flat interface, only diffuse energy is transmitted.

To summarize, we have derived boundary conditions for the transport equation (3.18) in two adjacent homogeneous half spaces from the plane wave decomposition (3.5) and (3.6) for both flat and rough interfaces, with small fluctuations. For slowly varying heterogeneous media with flat or smoothly varying interfaces such conditions are derived in detail in [24]. For weakly fluctuating random media or heterogeneous media with random rough interfaces no such derivation has been given. In the high frequency limit the boundary conditions for the transport equation must remain valid, but this is difficult to justify. When the fluctuations of the interface are small, the interaction of the volume scattering with the surface fluctuations must be of higher order and therefore negligible.

4.4 Derivation of the Dirichlet and Neumann Boundary Conditions

When $\rho^1/\rho^2 \to 0$, the interface corresponds to a hard boundary with vanishing normal velocity and Neumann boundary condition for waves incident from medium 1, and to a soft boundary with vanishing pressure and Dirichlet boundary condition for waves incident from medium 2. Then $\mathcal{R} \to 1$, $\mathcal{T} \to 0$, and

$$R_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moreover

$$S_1(\mathbf{p}',\mathbf{k}') = \begin{pmatrix} -2i\dfrac{(k^1)^2 - \mathbf{p}'\cdot\mathbf{k}'}{(k^1_n(\mathbf{p}')k^1_n(\mathbf{k}'))^{1/2}}\,\hat{h}(\mathbf{p}'-\mathbf{k}') & 0 \\ 0 & -2i(k^2_n(\mathbf{p}')\;k^2_n(\mathbf{k}'))^{1/2}\,\hat{h}(\mathbf{p}'-\mathbf{k}') \end{pmatrix}. \tag{4.26}$$

As is to be expected in this limit, $R_{2,0} = 0$ and $R_2 = R_{2,1}$ is given by

$$R_2(\mathbf{p}') = \begin{pmatrix} -2\displaystyle\int d\mathbf{k}'\frac{((k^1)^2 - \mathbf{k}'\cdot\mathbf{p}')^2}{k^1_n(\mathbf{k}')k^1_n(\mathbf{p}')}\hat{Q}(\mathbf{p}'-\mathbf{k}') & 0 \\ 0 & 2\displaystyle\int d\mathbf{k}' k^2_n(\mathbf{k}')k^2_n(\mathbf{p}')\hat{Q}(\mathbf{p}'-\mathbf{k}') \end{pmatrix}. \tag{4.27}$$

The differential scattering tensor τ, deduced from the expression for S_1, is

$$\tau(\mathbf{p}',\mathbf{k}') = \begin{pmatrix} 4\dfrac{((k^1)^2 - \mathbf{k}'\cdot\mathbf{p}')^2}{k^1_n(\mathbf{k}')k^1_n(\mathbf{p}')}\hat{Q}(\mathbf{p}'-\mathbf{k}') & 0 \\ 0 & 4k^2_n(\mathbf{k}')k^2_n(\mathbf{p}')\hat{Q}(\mathbf{p}'-\mathbf{k}') \end{pmatrix}. \tag{4.28}$$

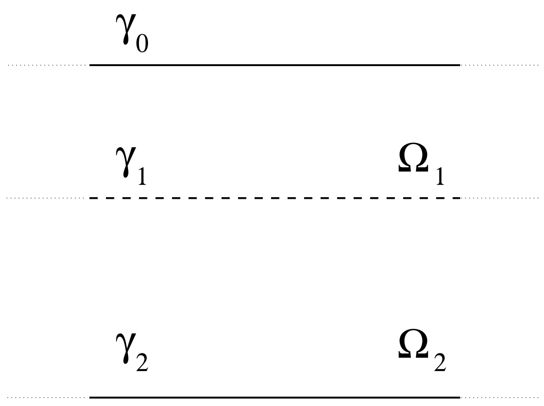

Figure 5.1: Geometry of the media

Here τ_{11} is the differential scattering cross-section for the Neumann problem in the first medium, and τ_{22} is the differential scattering cross-section for the Dirichlet problem in the second medium. Conservation of energy flux also holds:

$$0 \ = \ 2R_0^{ii}R_{2,1}^{ii}(\mathbf{p}') + \int d\mathbf{k}'\tau^{ii}(\mathbf{p}', \mathbf{k}'), \quad i = 1, 2.$$

5 The Initial Boundary Value Problem for the Radiative Transport Equation

Now we summarize the initial, boundary, and interface conditions for the radiative transport equation in two adjacent plane layers Ω_1 and Ω_2 with rough boundaries, separated by a rough interface, as is shown in Figure 5.1. Radiative transport theory is valid in the interior of each layer. The boundary and interface conditions are those for high frequency time harmonic waves derived in the previous sections. We now use them in the time domain, assuming that time variations in the transport regime are slow compared to the oscillation frequency ε^{-1}. This is so if the frequency spread of the waves about a carrier frequency is narrow. At the outer boundaries we impose a Neumann condition at γ_0 and a Dirichlet condition at γ_2.

The transport equation in Ω_i is

$$\frac{\partial a^i(t,\mathbf{x},\mathbf{k})}{\partial t} + v^i\hat{\mathbf{k}}^i \cdot \nabla_{\mathbf{x}} a^i(t,\mathbf{x},\mathbf{k}) = \int_{R^3} \sigma^i(\mathbf{x},\mathbf{k},\mathbf{k}') \left[a^i(t,\mathbf{x},\mathbf{k}') - a^i(t,\mathbf{x},\mathbf{k})\right] d\mathbf{k}', \quad i = 1, 2. \tag{5.1}$$

The scattering cross-sections are given by

$$\begin{aligned}\sigma^i(\mathbf{x},\mathbf{k},\mathbf{k}') = \frac{M(v^i)^2|\mathbf{k}|^2}{2}\delta(v^i|\mathbf{k}| - v^i|\mathbf{k}'|) \\ \cdot\left\{(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')^2\hat{R}^i_{\rho\rho}(\mathbf{k}-\mathbf{k}') + 2(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')\hat{R}^i_{\rho\kappa}(\mathbf{k}-\mathbf{k}') + \hat{R}^i_{\kappa\kappa}(\mathbf{k}-\mathbf{k}')\right\}.\end{aligned} \tag{5.2}$$

The mean boundaries γ_0, γ_1 and γ_2 are the flat surfaces $x_n = L_0$, $x_n = 0$ and $x_n = -L_2$ respectively. At γ_0 we require $p = 0$ and at γ_2, $v_n = 0$. Then the conditions at γ_0 and γ_2 are given respectively by

$$\begin{aligned} a^1(t,\mathbf{x}',L_0,\mathbf{k}',k_n) \ &= \ \left[1 - 2R_{2,1}^{11}(\mathbf{k}')\eta^2\right] a^1(t,\mathbf{x}',L_0,\mathbf{k}',-k_n) \\ &+ \ \eta^2\int d\mathbf{p}'\tau^{11}(\mathbf{k}',\mathbf{p}')a^1(t,\mathbf{x}',L_0,\mathbf{p}',p_n^1) \end{aligned} \tag{5.3}$$

$$
\begin{aligned}
a^2(t,\mathbf{x}',-L_2,\mathbf{k}',k_n) &= \left[1-2R^{22}_{2,1}(\mathbf{k}')\eta^2\right] a^2(t,\mathbf{x}',-L_2,\mathbf{k}',-k_n) \\
&+ \eta^2\int d\mathbf{p}'\tau^{22}(\mathbf{k}',\mathbf{p}')a^2(t,\mathbf{x}',-L_2,\mathbf{p}',p^2_n).
\end{aligned}
\tag{5.4}
$$

Here $k_n<0$, $p_n>0$ at γ_1, and $k_n>0$, $p_n<0$ at γ_2, so that the part on the left represents the outgoing or scattered wave, and the part on the right corresponds to the incoming waves. The matrices $R_{2,1}$ and τ appearing in (5.3) and (5.4) are given by (4.27) and (4.28).

The interface condition at γ_1 for the outgoing field $a^1(t,\mathbf{x}',0,\mathbf{k}',k^1_n)$ with $k_n>0$ is

$$
\begin{aligned}
a^1(t,\mathbf{x}',0,\mathbf{k}',k^1_n) =& \left(|\mathcal{R}|^2+2\mathcal{R}R^{11}_{2,1}\eta^2\right) a^1(t,\mathbf{x}',0,\mathbf{k}',-k^1_n) + \left(|\mathcal{T}|^2+2\mathcal{T}(R^{12}_{2,0}+R^{12}_{2,1})\eta^2\right) a^2(t,\mathbf{x}',0,\mathbf{k}',k^2_n) \\
&+\eta^2\int d\mathbf{p}'\tau^{11}(\mathbf{k}',\mathbf{p}')a^1(t,\mathbf{x}',0,\mathbf{p}',-p^1_n)+\eta^2\int d\mathbf{p}'\tau^{12}(\mathbf{k}',\mathbf{p}')a^2(t,\mathbf{x}',0,\mathbf{p}',p^2_n).
\end{aligned}
\tag{5.5}
$$

Here $-p^1_n<0$ and $p^2_n>0$ correspond to incoming waves. The condition on a^2 is similar to (5.5). The matrices $R_{2,0}$, $R_{2,1}$ and τ are given by (4.23) and (4.22). These interface conditions are valid under the assumption that waves incident on the interface from the two sides are uncorrelated. This assumption is valid provided that the widths of the layers L_1, L_2 are much larger than the corresponding scattering mean free paths

$$
l_j=\frac{v^j}{\Sigma^j},\quad j=1,2.
$$

The total scattering cross-section Σ^j is defined by (1.2).

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A Derivation of the Interface Conditions

We shall now calculate the reflection coefficients (4.18) and (4.23), and the scattering cross section tensor (4.22). We note first that the normal vector to the surface $[\frac{\mathbf{x}'}{\varepsilon},h(\frac{\mathbf{x}'}{\varepsilon})]$ is

$$
\nu\left[\frac{\mathbf{x}'}{\varepsilon},h(\frac{\mathbf{x}'}{\varepsilon})\right] = C\left[\mathbf{e}_z-\nabla h^\varepsilon(\frac{\mathbf{x}'}{\varepsilon})\right] = C\left[\mathbf{e}_z-\eta(\nabla h)(\frac{\mathbf{x}'}{\varepsilon})\right],
\tag{A.1}
$$

where C is a normalization constant. Using the plane wave decomposition (3.5) and (3.6), the definition (2.11) of the eigenvector $\mathbf{b}_f$, and the flux amplitudes (4.10), we obtain

$$
\begin{aligned}
&\int d\mathbf{k}'e^{i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}}e^{-i\epsilon^1k^1_n\eta h(\frac{\mathbf{x}'}{\varepsilon})}\frac{\mathbf{k}'\cdot\nu'-\epsilon^1k^1_n\nu_n}{k^1_n}\ \delta_1\tilde{\alpha}^1_\varepsilon+\int d\mathbf{k}'e^{i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}}e^{i\epsilon^1k^1_n\eta h(\frac{\mathbf{x}'}{\varepsilon})}\frac{\mathbf{k}'\cdot\nu'+\epsilon^1k^1_n\nu_n}{k^1_n}\ \delta_1\tilde{\beta}^1_\varepsilon \\
=&\int d\mathbf{k}'e^{i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}}e^{-i\epsilon^2k^2_n\eta h(\frac{\mathbf{x}'}{\varepsilon})}\frac{\mathbf{k}'\cdot\nu'-\epsilon^2k^2_n\nu_n}{k^2_n}\ \delta_2\tilde{\alpha}^2_\varepsilon+\int d\mathbf{k}'e^{i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}}e^{i\epsilon^2k^2_n\eta h(\frac{\mathbf{x}'}{\varepsilon})}\frac{\mathbf{k}'\cdot\nu'+\epsilon^2k^2_n\nu_n}{k^2_n}\ \delta_2\tilde{\beta}^2_\varepsilon
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
&\int d\mathbf{k}'e^{i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}}e^{-i\epsilon^1k^1_n\eta h(\frac{\mathbf{x}'}{\varepsilon})}\ \delta_1^{-1}\tilde{\alpha}^1_\varepsilon+\int d\mathbf{k}'e^{i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}}e^{i\epsilon^1k^1_n\eta h(\frac{\mathbf{x}'}{\varepsilon})}\ \delta_1^{-1}\tilde{\beta}^1_\varepsilon \\
=&\int d\mathbf{k}'e^{i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}}e^{-i\epsilon^2k^2_n\eta h(\frac{\mathbf{x}'}{\varepsilon})}\ \delta_2^{-1}\tilde{\alpha}^2_\varepsilon+\int d\mathbf{k}'e^{i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}}e^{i\epsilon^2k^2_n\eta h(\frac{\mathbf{x}'}{\varepsilon})}\ \delta_2^{-1}\tilde{\beta}^2_\varepsilon.
\end{aligned}
$$

Here δ_i is defined by

$$\delta_i \ = \ \sqrt{\frac{k_n^i}{\rho^i}}.$$

The incoming energy flux amplitudes $\tilde{\alpha}_\varepsilon^i$ are not random.

To calculate the reflected wave amplitudes $\tilde{\beta}_\varepsilon^i$ using a Born expansion in the height η of the rough boundary, we need the following intermediate expansions:

$$\begin{aligned}
\exp\left[ik_n\eta h(\frac{\mathbf{x}'}{\varepsilon})\right] \ &= \ 1 \ + \ ik_n h(\frac{\mathbf{x}'}{\varepsilon}) \ \eta \ - \ \frac{1}{2}(k_n)^2 h^2(\frac{\mathbf{x}'}{\varepsilon}) \ \eta^2 + o(\eta^2) \\
\exp\left[ik_n\eta h(\frac{\mathbf{x}'}{\varepsilon})\right](\mathbf{k}\cdot\mathbf{n})C^{-1} \ &= \ k_n \ + \ \left[i(k_n)^2 h(\frac{\mathbf{x}'}{\varepsilon}) - \mathbf{k}'\cdot\nabla h(\frac{\mathbf{x}'}{\varepsilon})\right] \ \eta \\
&\quad + \left[-\frac{1}{2}(k_n)^3 h^2(\frac{\mathbf{x}'}{\varepsilon}) - ik_n h(\frac{\mathbf{x}'}{\varepsilon})\mathbf{k}'\cdot\nabla h(\frac{\mathbf{x}'}{\varepsilon})\right] \ \eta^2 + o(\eta^2) \\
\tilde{\beta}_\varepsilon^i \ &= \ \tilde{\beta}_{\varepsilon,0} \ + \ \eta \ \tilde{\beta}_{\varepsilon,1} \ + \ \eta^2\tilde{\beta}_{\varepsilon,2} \ + \ o(\eta^2).
\end{aligned} \tag{A.3}$$

Using (A.3) in (A.2), we find at zeroth order in η

$$\begin{aligned}
\tilde{\beta}_{\varepsilon,0}^1\delta_1 \ - \ \tilde{\alpha}_\varepsilon^1\delta_1 \ &= \ -\tilde{\beta}_{\varepsilon,0}^2\delta_2 \ + \ \tilde{\alpha}_\varepsilon^2\delta_2 \\
\tilde{\beta}_{\varepsilon,0}^1\delta_1^{-1} \ + \ \tilde{\alpha}_\varepsilon^1\delta_1^{-1} \ &= \ \tilde{\beta}_{\varepsilon,0}^2\delta_2^{-1} \ + \ \tilde{\alpha}_\varepsilon^2\delta_2^{-1}.
\end{aligned} \tag{A.4}$$

We can now invert (A.4) to obtain

$$\tilde{\beta}_{\varepsilon,0} \ = \ R_0\tilde{\alpha}_\varepsilon \ = \ \begin{pmatrix} \mathcal{R} & \mathcal{T} \\ \mathcal{T} & -\mathcal{R} \end{pmatrix} \tilde{\alpha}_\varepsilon. \tag{A.5}$$

Here $\tilde{\alpha}_\varepsilon = (\tilde{\alpha}_\varepsilon^i)$ and $\tilde{\beta}_{\varepsilon,k} = (\tilde{\beta}_{\varepsilon,k}^i)$, with $k = 0, 1, 2$, are two-component vectors, and $\mathcal{R}$ and $\mathcal{T}$ are given by

$$\mathcal{R} \ = \ \frac{\delta_1^2 - \delta_2^2}{\delta_1^2 + \delta_2^2}\,, \qquad\qquad \mathcal{T} \ = \ \frac{2\delta_1\delta_2}{\delta_1^2 + \delta_2^2}. \tag{A.6}$$

At the first order in η in equation (A.2) we obtain

$$\begin{aligned}
&\int d\mathbf{k}' \exp\left[i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}\right] \left\{ \left(ik_n^1 h(\frac{\mathbf{x}'}{\varepsilon}) - \frac{\mathbf{k}'\cdot(\nabla h)(\frac{\mathbf{x}'}{\varepsilon})}{k_n^1} \right) \delta_1(\tilde{\alpha}_\varepsilon^1 + \tilde{\beta}_{\varepsilon,0}^1) + \delta_1\tilde{\beta}_{\varepsilon,1}^1 \right\} \\
= \ &\int d\mathbf{k}' \exp\left[i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}\right] \left\{ \left(ik_n^2 h(\frac{\mathbf{x}'}{\varepsilon}) - \frac{\mathbf{k}'\cdot(\nabla h)(\frac{\mathbf{x}'}{\varepsilon})}{k_n^2} \right) \delta_2(\tilde{\alpha}_\varepsilon^2 + \tilde{\beta}_{\varepsilon,0}^2) - \delta_2\tilde{\beta}_{\varepsilon,1}^2 \right\} \\
&\int d\mathbf{k}' \exp\left[i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}\right] \left\{ ik_n^1 h(\frac{\mathbf{x}'}{\varepsilon})\delta_1^{-1}(-\tilde{\alpha}_\varepsilon^1 + \tilde{\beta}_{\varepsilon,0}^1) + \delta_1^{-1}\tilde{\beta}_{\varepsilon,1}^1 \right\} \\
= \ &\int d\mathbf{k}' \exp\left[i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}\right] \left\{ ik_n^2 h(\frac{\mathbf{x}'}{\varepsilon})\delta_2^{-1}(\tilde{\alpha}_\varepsilon^2 - \tilde{\beta}_{\varepsilon,0}^2) + \delta_2^{-1}\tilde{\beta}_{\varepsilon,1}^2 \right\}.
\end{aligned} \tag{A.7}$$

Denoting by $\hat{h}$ the Fourier transform of h, we have

$$h(\frac{\mathbf{x}'}{\varepsilon}) \ = \ \int \exp\left(i\frac{\xi'\cdot\mathbf{x}'}{\varepsilon}\right)\hat{h}(\xi')d\xi', \qquad (\nabla h)(\frac{\mathbf{x}'}{\varepsilon}) \ = \ \int i\xi' \exp\left(i\frac{\xi'\cdot\mathbf{x}'}{\varepsilon}\right)\hat{h}(\xi')d\xi'.$$

Now we replace $\tilde\beta_{\varepsilon,0}$ in (A.7) by its expression (A.5), and compute the Fourier transform of (A.7) with respect to $\mathbf{x}'$, denoting the conjugate variable $\mathbf{p}'$:

$$\begin{aligned} &\delta_1\tilde\beta^1_{\varepsilon,1}(\mathbf{p}') + \int d\mathbf{k}'\hat h(\mathbf{p}'-\mathbf{k}')\ \delta_1\ (ik^1_n - i\frac{\mathbf{k}'\cdot(\mathbf{p}'-\mathbf{k}')}{k^1_n})\ ((1+\mathcal{R})\tilde\alpha^1_\varepsilon + \mathcal{T}\tilde\alpha^2_\varepsilon)\\ = \quad &-\delta_2\tilde\beta^2_{\varepsilon,1}(\mathbf{p}') + \int d\mathbf{k}'\hat h(\mathbf{p}'-\mathbf{k}')\ \delta_2\ (ik^2_n - i\frac{\mathbf{k}'\cdot(\mathbf{p}'-\mathbf{k}')}{k^2_n})\ (\mathcal{T}\tilde\alpha^1_\varepsilon + (1-\mathcal{R})\tilde\alpha^2_\varepsilon)\\ &\delta_1^{-1}\tilde\beta^1_{\varepsilon,1}(\mathbf{p}') + \int d\mathbf{k}'\hat h(\mathbf{p}'-\mathbf{k}')\ \delta_1^{-1}\ ik^1_n\ ((-1+\mathcal{R})\tilde\alpha^1_\varepsilon + \mathcal{T}\tilde\alpha^2_\varepsilon)\\ = \quad &\delta_2^{-1}\tilde\beta^2_{\varepsilon,1}(\mathbf{p}') + \int d\mathbf{k}'\hat h(\mathbf{p}'-\mathbf{k}')\ \delta_2^{-1}\ ik^2_n\ (-\mathcal{T}\tilde\alpha^1_\varepsilon + (1+\mathcal{R})\tilde\alpha^2_\varepsilon). \end{aligned} \tag{A.8}$$

For $m = 1,2$ we have

$$ik^m_n - i\frac{\mathbf{k}'\cdot(\mathbf{p}'-\mathbf{k}')}{k^m_n} = i\frac{(k^m)^2 - \mathbf{k}'\cdot\mathbf{p}'}{k^m_n}.$$

Using the definition of M given in (4.20) we can rewrite (A.8) in vector form:

$$P(\mathbf{p}')\tilde\beta_{\varepsilon,1}(\mathbf{p}') = \int d\mathbf{k}'\ i\hat h(\mathbf{p}'-\mathbf{k}')(M(\mathbf{p}',\mathbf{k}')\tilde\alpha_\varepsilon(\mathbf{k}') + V_0M(\mathbf{p}',\mathbf{k}')\tilde\beta_{\varepsilon,0})(\mathbf{k}'). \tag{A.9}$$

From (A.6) we have

$$\delta_1\ (1+\mathcal{R}) \ = \ \frac{2\delta_1^3}{\delta_1^2+\delta_2^2}\ ,\quad \delta_2\ (1-\mathcal{R}) \ = \ \frac{2\delta_2^3}{\delta_1^2+\delta_2^2}$$

along with similar expressions for $\mathcal{T}$. Therefore, from (A.8) we get

$$\begin{aligned} (\delta_1\tilde\beta^1_{\varepsilon,1} + \delta_2\tilde\beta^2_{\varepsilon,1})(\mathbf{p}') &= 2i\int d\mathbf{k}'\hat h(\mathbf{p}'-\mathbf{k}')\Gamma(\mathbf{p}'\cdot\mathbf{k}')\ \frac{(\delta_1\tilde\alpha^1_\varepsilon + \delta_2\tilde\alpha^2_\varepsilon)}{(\delta_1^2+\delta_2^2)}(\mathbf{k}')\\ (\delta_1^{-1}\tilde\beta^1_{\varepsilon,1} - \delta_2^{-1}\tilde\beta^2_{\varepsilon,1})(\mathbf{p}') &= 2i\int d\mathbf{k}'\hat h(\mathbf{p}'-\mathbf{k}')(\rho_1-\rho_2)\ \frac{(\delta_1\delta_2^2\tilde\alpha^1_\varepsilon - \delta_2\delta_1^2\tilde\alpha^2_\varepsilon)}{(\delta_1^2+\delta_2^2)}(\mathbf{k}') \end{aligned} \tag{A.10}$$

where

$$\Gamma(\mathbf{p}'\cdot\mathbf{k}') \ = \ \frac{(k^2)^2 - \mathbf{k}'\cdot\mathbf{p}')}{\rho^2} - \frac{((k^1)^2 - \mathbf{k}'\cdot\mathbf{p}')}{\rho^1}.$$

Using the solution (A.6) of (A.5) we obtain in vector form

$$(\tilde\beta_{\varepsilon,1})(\mathbf{p}') \ = \ \int d\mathbf{k}' S_1(\mathbf{p}',\mathbf{k}')(\tilde\alpha_\varepsilon)(\mathbf{k}') \ = \ 2i\int d\mathbf{k}'\hat h(\mathbf{p}'-\mathbf{k}')B(\mathbf{p}',\mathbf{k}')(\tilde\alpha_\varepsilon)(\mathbf{k}') \tag{A.11}$$

where B is given by (4.21).

The terms of order η^2 in (A.2) yield

$$\int\exp(\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon})\gamma^1 d\mathbf{k}' = \int\exp(\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon})\gamma^2 d\mathbf{k}' \tag{A.12}$$

where the vectors γ^m are

$$\begin{aligned} \gamma^m_1 &= \delta_m[-(\frac{1}{2}(k^m_n)^2h^2(\frac{\mathbf{x}'}{\varepsilon}) + ih(\frac{\mathbf{x}'}{\varepsilon})\mathbf{k}'\cdot\nabla h(\frac{\mathbf{x}'}{\varepsilon}))\epsilon^m(-\tilde\alpha_\varepsilon + \tilde\beta^m_{\varepsilon,0}) + (ik^m_n h(\frac{\mathbf{x}'}{\varepsilon}) - \frac{\mathbf{k}'\cdot\nabla h(\frac{\mathbf{x}'}{\varepsilon})}{k^m_n})\tilde\beta^m_{\varepsilon,1} + \epsilon^m\tilde\beta^m_{\varepsilon,2}]\\ \gamma^m_2 &= \delta_m^{-1}[-\frac{1}{2}(k^m_n)^2h^2(\frac{\mathbf{x}'}{\varepsilon})(\tilde\alpha^m_\epsilon + \tilde\beta^m_{\varepsilon,0}) + (i\epsilon^m k^m_n h(\frac{\mathbf{x}'}{\varepsilon}))\tilde\beta^m_{\varepsilon,1} + \tilde\beta^m_{\varepsilon,2}]. \end{aligned} \tag{A.13}$$

We average the interface conditions (A.12) before inverting them.

We note first that

$$\begin{aligned}\langle h^2(\frac{\mathbf{x}'}{\varepsilon})\rangle &= \int \hat{Q}(\xi')d\xi' = Q(0)\\ \langle h(\frac{\mathbf{x}'}{\varepsilon})\ \mathbf{k}'\cdot\nabla h(\frac{\mathbf{x}'}{\varepsilon})\rangle &= \int i\mathbf{k}'\cdot\xi'\ \hat{Q}(\xi')d\xi' = \mathbf{k}'\cdot\nabla Q(0).\end{aligned}$$

We also note the following Fourier transforms for $m = 1, 2$

$$\begin{aligned}&\mathcal{F}^{-1}_{\mathbf{x}'\to\mathbf{p}'}\langle\int d\mathbf{k}'\delta_m(\mathbf{k}')\exp[i\mathbf{k}'\cdot\mathbf{x}'/\varepsilon](ik_n^m h(\frac{\mathbf{x}'}{\varepsilon}) - \frac{\mathbf{k}'\cdot\nabla h(\frac{\mathbf{x}'}{\varepsilon})}{k_n^m})\tilde{\beta}^m_{\varepsilon,1}\rangle\\ = &-2\int d\mathbf{k}'\delta_m(\mathbf{k}')\frac{(k^m)^2-\mathbf{p}'\cdot\mathbf{k}'}{k_n^m(\mathbf{k}')}\hat{Q}(\mathbf{k}'-\mathbf{p}')(B(\mathbf{k}',\mathbf{p}')\tilde{\alpha}_\varepsilon(\mathbf{p}'))_m\\ &\mathcal{F}^{-1}_{\mathbf{x}'\to\mathbf{p}'}\langle\int d\mathbf{k}'\delta_m(\mathbf{k}')\exp[i\mathbf{k}'\cdot\mathbf{x}'/\varepsilon]ik_n^m h(\frac{\mathbf{x}'}{\varepsilon})\tilde{\beta}^m_{\varepsilon,1}\rangle\\ = &-2\int d\mathbf{k}'\delta_m(\mathbf{k}')k_n^m(\mathbf{k}')\hat{Q}(\mathbf{k}'-\mathbf{p}')(B(\mathbf{k}',\mathbf{p}')\tilde{\alpha}_\varepsilon(\mathbf{p}'))_m\end{aligned}$$

Here $\mathcal{F}^{-1}_{\mathbf{x}'\to\mathbf{p}'}$ is the inverse Fourier transform with conjugate variables $\mathbf{x}'$ and $\mathbf{p}'$.

Taking the Fourier transform of the interface conditions (A.12), we get the following 2×2 system

$$\begin{aligned}&\delta_1(\mathbf{p}')[-(\frac{1}{2}(k_n^1)^2Q(0)+i\mathbf{p}'\cdot\nabla Q(0))(-\tilde{\alpha}^1_\varepsilon+\tilde{\beta}^1_{\varepsilon,0})(\mathbf{p}')]\\ &-2\int d\mathbf{k}'\delta_1(\mathbf{k}')\frac{(k^1)^2-\mathbf{p}'\cdot\mathbf{k}'}{k_n^1(\mathbf{k}')}\ \hat{Q}(\mathbf{p}'-\mathbf{k}')(B(\mathbf{k}',\mathbf{p}')\tilde{\alpha}_\varepsilon(\mathbf{p}'))_1+\delta_1(\mathbf{p}')\langle\tilde{\beta}^1_{\varepsilon,2}\rangle\\ = &\ \delta_2(\mathbf{p}')[-(\frac{1}{2}(k_n^2)^2Q(0)+i\mathbf{p}'\cdot\nabla Q(0))(\tilde{\alpha}^2_\varepsilon-\tilde{\beta}^2_{\varepsilon,0})(\mathbf{p}')]\\ &-2\int d\mathbf{k}'\delta_2(\mathbf{k}')\frac{(k^2)^2-\mathbf{p}'\cdot\mathbf{k}'}{k_n^2(\mathbf{k}')}\ \hat{Q}(\mathbf{p}'-\mathbf{k}')(B(\mathbf{k}',\mathbf{p}')\tilde{\alpha}_\varepsilon(\mathbf{p}'))_2-\delta_2(\mathbf{p}')\langle\tilde{\beta}^2_{\varepsilon,2}\rangle\end{aligned}\tag{A.14}$$

$$\begin{aligned}&\delta_1^{-1}(\mathbf{p}')[-\frac{1}{2}(k_n^1)^2Q(0)(\tilde{\alpha}^1_\varepsilon+\tilde{\beta}^1_{\varepsilon,0})]\\ &-2\int d\mathbf{k}'\delta_1^{-1}(\mathbf{k}')k_n^1(\mathbf{k}')\ \hat{Q}(\mathbf{p}'-\mathbf{k}')(B(\mathbf{k}',\mathbf{p}')\tilde{\alpha}_\varepsilon(\mathbf{p}'))_1+\delta_1^{-1}(\mathbf{p}')\langle\tilde{\beta}^1_{\varepsilon,2}\rangle\\ = &\ \delta_2^{-1}(\mathbf{p}')[-\frac{1}{2}(k_n^2)^2Q(0)(\tilde{\alpha}^2_\varepsilon+\tilde{\beta}^2_{\varepsilon,0})]\\ &+2\int d\mathbf{k}'\delta_2^{-1}(\mathbf{k}')k_n^2(\mathbf{k}')\ \hat{Q}(\mathbf{p}'-\mathbf{k}')(B(\mathbf{k}',\mathbf{p}')\tilde{\alpha}_\varepsilon(\mathbf{p}'))_2+\delta_2^{-1}(\mathbf{p}')\langle\tilde{\beta}^2_{\varepsilon,2}\rangle.\end{aligned}$$

We solve (A.14) for $\langle\tilde{\beta}_{\varepsilon,2}\rangle$ and write the result in terms of the tensor R_2. Then we split the tensor R_2 into two parts, $R_2 = R_{2,0} + R_{2,1}$, and we obtain the formulas (4.23).

B Grazing Angles and the Smoothing Method

As has been pointed out by several authors [28, 10, 30, 31] the reflection coefficient blows up as the incidence direction becomes parallel to the mean surface in the case of the Neumann boundary condition. As can be seen from (4.27), the term $R^{11}_{2,1}$ is infinite for grazing angles, since $k_n(\mathbf{p}') = 0$. This shortcoming can be avoided by using the smoothing method instead of the Born expansion. For the Neumann boundary condition, the smoothing method gives the reflection coefficient [31]

$$R(\mathbf{k}) = \frac{1+\eta^2\mathcal{Q}}{1-\eta^2\mathcal{Q}}\tag{B.1}$$

where

$$\mathcal{Q} \;=\; -\int d\mathbf{k}' \frac{((k^1)^2 - \mathbf{k}'\cdot\mathbf{p}')^2}{k_n^1(\mathbf{k}')k_n^1(\mathbf{p}')}\hat{Q}(\mathbf{p}'-\mathbf{k}'). \tag{B.2}$$

The Born expansion for the reflection coefficient gives, when (4.27) is used,

$$R(\mathbf{k}) \;=\; 1 + 2\eta^2\mathcal{Q} + O(\eta^3).$$

Thus, (B.1) agrees with the Born expansion to $O(\eta^3)$, but it is not singular at grazing angles. Note that $\mathcal{Q}$ is negative since it involves only propagating waves for which k_n^i is real. Therefore the reflection coefficient R given by (B.1) ranges between 0 and 1.

The smoothing method gives for the scattering cross-section [31]

$$\tau_{ren}^{11}(\mathbf{p}',\mathbf{k}') \;=\; \frac{4}{(1-\eta^2\mathcal{Q}(\mathbf{k}'))^2}\,\frac{((k^1)^2 - \mathbf{k}'\cdot\mathbf{p}')^2}{k_n^1(\mathbf{k}')k_n^1(\mathbf{p}')}\hat{Q}(\mathbf{p}'-\mathbf{k}'). \tag{B.3}$$

This coefficient is nonsingular at grazing angles.

The reason why the second Born approximation (4.27) is singular at grazing incidence is that it includes only single and double scattering. At grazing incidence, all the singly scattered waves combine in phase in the direction of the reflected wave, and their sum is infinite. Double scattering alone does not reduce these waves enough to make the sum converge. The smoothing method takes into account some multiple scattering of all orders—that corresponding to the ladder diagrams of diagrammatic resummation methods [25, 27]—and this reduces the scattered waves enough to make their sum converge.

B.1 Smoothing Method for Dirichlet and Neumann Boundary Conditions

The smoothing method is an alternative to the Born expansion which includes ladder diagrams. We will present it in a form suitable for reflection from a random surface with a Dirichlet or Neumann boundary condition, as in [31].

Vanishing pressure (soft boundary condition) and vanishing normal velocity at the rough boundary (hard boundary condition) give equations of the form

$$K(-\eta)\tilde{\beta} \;=\; \pm K(\eta)\tilde{\alpha}. \tag{B.4}$$

Here $\tilde{\alpha}$ and $\tilde{\beta}$ are the incident and reflected wave amplitudes and the sign is $-$ for the Dirichlet and $+$ for the Neumann boundary condition. The operator K for the Dirichlet and Neumann conditions is

$$[K_D(\eta)\tilde{g}]\,(\mathbf{p}') = (2M)^{-n}\int d\mathbf{x}' e^{-i\mathbf{p}'\cdot\mathbf{x}'}\int d\mathbf{k}' e^{i\mathbf{k}'\cdot\mathbf{x}'}e^{-ik_n\eta h(\mathbf{x}')}\delta^{-1}(\mathbf{k}')\tilde{g}(\mathbf{k}')$$

$$[K_N(\eta)\tilde{g}]\,(\mathbf{p}') = (2M)^{-n}\int d\mathbf{x}' e^{-i\mathbf{p}'\cdot\mathbf{x}'}\int d\mathbf{k}' e^{i\mathbf{k}'\cdot\mathbf{x}'}e^{-ik_n\eta h(\mathbf{x}')}(-1+\eta\frac{\mathbf{k}'\cdot\nabla h(\mathbf{x}')}{k_n})\delta(\mathbf{k}')\tilde{g}(\mathbf{k}').$$

Expansion in powers of η gives

$$K(\eta) \;=\; 1 + \eta K_1 + \eta^2 K_2 + O(\eta^3). \tag{B.5}$$

Plugging (B.5) into (B.4) yields

$$(I - \eta K_1 + \eta^2 K_2)\tilde{\beta} = \pm(I + \eta K_1 + \eta^2 K_2)\tilde{\alpha} + O(\eta^3). \tag{B.6}$$

We denote by $< f >$ the average over realizations of a random quantity f. We denote the fluctuation by $f' = f - < f >$. We now use this notation to invert (B.6) approximately. Note that the incoming energy flux amplitude $\tilde{\alpha}$ is independent of h. We also note from (4.26) that $< K_1 >= 0$.

Averaging (B.6) over the realizations yields

$$(I + \eta^2 < K_2 >) < \tilde{\beta} > -\eta < K_1' \tilde{\beta}' > + \eta^2 < K_2' \tilde{\beta}' >= \pm (I + \eta^2 < K_2 >)\tilde{\alpha} + O(\eta^3). \tag{B.7}$$

Subtracting (B.7) from (B.6) and keeping terms through $O(\eta)$ we get an expression for the fluctuating part of the reflected wave amplitude

$$\tilde{\beta}' \;=\; \eta (K_1' < \tilde{\beta} > \pm K_1' \tilde{\alpha}) + O(\eta^2). \tag{B.8}$$

Finally we use (B.8) for $\tilde{\beta}'$ in (B.7) and obtain

$$([I + \eta^2 < K_2 >] - \eta^2 < K_1' K_1' >) < \tilde{\beta} > \;=\; \pm([I + \eta^2 < K_2 >] + \eta^2 < K_1' K_1' >)\tilde{\alpha} + O(\eta^3). \tag{B.9}$$

Dropping the error and noting that $< K_2 >$ and $< K_1' K_1' >$ are scalars, we get for the mean reflected amplitude

$$< \tilde{\beta} >= \frac{\pm(1 + \eta^2 < K_2 > + \eta^2 < K_1' K_1' >)}{1 + \eta^2 < K_2 > - \eta^2 < K_1' K_1' >}\tilde{\alpha}.$$

The factor $< K_2 >$ is not singular at grazing angles, while $< K_1' K_1' >$ is singular in the Neumann case. We can therefore divide both the numerator and denominator in (B.9) by $1 + \eta^2 < K_2 >$, use the approximation $[1 + \eta^2 < K_2 >]^{-1} \sim 1 - \eta^2 < K_2 >$ and obtain to order η^3,

$$< \tilde{\beta} >= \frac{\pm(1 + \eta^2 < K_1' K_1' >)}{1 - \eta^2 < K_1' K_1' >}\tilde{\alpha}. \tag{B.10}$$

Upon defining $\mathcal{Q} =< K_1' K_1' >$, (B.10) becomes (B.2) for the Neumann boundary condition. For the Dirichlet condition the smoothing method is not necessary since the Born approximation is bounded independently of the angle of incidence.

Using (B.8) for $\tilde{\beta}'$ and (B.10) for $< \tilde{\beta} >$ allows us to recompute the correlation function $< W^{n-1}[\tilde{\beta}] >$. By doing so we check that the recomputed scattering cross-section for the Neumann boundary condition is (B.3). We can interpret the smoothing method as a way of renormalizing the results of the Born approximation.

B.2 Extension to Interface Problems

We will now use the smoothing method to obtain nonsingular interface reflection and transmission coefficients, following unpublished calculations of John G. Watson. Continuity of pressure and normal velocity give interfacial conditions of the form

$$\begin{pmatrix} K^{11}(\eta)\delta_1 & K^{12}(-\eta)\delta_2 \\ K^{21}(\eta)\delta_1^{-1} & -K^{22}(-\eta)\delta_2^{-1} \end{pmatrix} \tilde{\beta} \;=\; \begin{pmatrix} K^{11}(-\eta)\delta_1 & K^{12}(\eta)\delta_2 \\ -K^{21}(-\eta)\delta_1^{-1} & K^{22}(\eta)\delta_2^{-1} \end{pmatrix} \tilde{\alpha}. \tag{B.11}$$

Here $\tilde{\alpha} = (\tilde{\alpha}^i)$, $\tilde{\beta} = (\tilde{\beta}^i)$, $i = 1, 2$, and we have used the following definitions:

$$\begin{aligned} K^{ij}(\eta) &= 1 + \eta K_1^{ij} + \eta^2 K_2^{ij} + O(\eta^3) \\ \left[K_1^{ij}(\eta)\gamma\right](\mathbf{p}') &= \int \hat{h}(\mathbf{p}' - \mathbf{k}') m_i^j(\mathbf{p}', \mathbf{k}')\gamma(\mathbf{p}')d\mathbf{p}' \\ \left[< K_2^{ij}(\eta) > (\gamma)\right](\mathbf{k}') &= \frac{1}{2}(m_1^j)^2 Q(0)\gamma(\mathbf{k}'). \end{aligned} \tag{B.12}$$

Only the real part of K^{1j} has been written since the imaginary part does not contribute, and

$$m_1^j = i\epsilon^j k_n^j, \qquad m_2^j(\mathbf{p}', \mathbf{k}') = i\frac{(k^j)^2 - \mathbf{p}' \cdot \mathbf{k}'}{\epsilon^j k_n^j(\mathbf{k}')}.$$

These expressions are obtained from (A.2).

We note that $R_0 = P^{-1} V_0 P$ where P is defined in (4.20). Therefore, if we multiply both sides of (B.11) by P^{-1}, we can write it in the form

$$(I + \eta R_1 + \eta^2 R_2)\, \tilde{\beta} = R_0\, (I - \eta R_1 + \eta^2 R_2)\, \tilde{\alpha}, \tag{B.13}$$

where R_1 and R_2 are defined by using (B.12) in (B.11) and adopting matrix notation.

We can now use the smoothing method, as in the previous section. With the notation of the previous section for average and fluctuation, we have

$$\tilde{\beta}' = -\eta[R_1' < \tilde{\beta} > + R_0 R_1' \tilde{\alpha}] + O(\eta^2), \tag{B.14}$$

and then up to terms of order η^3

$$< I + \eta^2 R_2 >< \tilde{\beta} > -\eta^2 < R_1' R_1' >< \tilde{\beta} > = R_0 < I + \eta^2 R_2 > \tilde{\alpha} + \eta^2 < R_1' R_0 R_1' > \tilde{\alpha}. \tag{B.15}$$

Again $< R_2 >$ is a local multiplication operator and we can multiply both sides of (B.15) by $(I + \eta^2 R_2)^{-1}$. This yields, to order η^3

$$< \tilde{\beta} > = [I - \eta^2 < R_1' R_1' >]^{-1} [R_0 + \eta^2 < R_1' R_0 R_1' > + \eta^2 (R_0 < R_2 > - < R_2 > R_0)] \tilde{\alpha} \tag{B.16}$$

uniformly in angles, including grazing angles.

The matrices appearing in (B.16) can be related to the quantities already computed in section 4.3 as follows. We deduce from (A.9) given in Appendix A that the operator $R_1 = R_1'$ has the form

$$(R_1' g)(\mathbf{p}') = \int d\mathbf{k}'\ i\hat{h}(\mathbf{p}' - \mathbf{k}') N(\mathbf{p}', \mathbf{k}') g(\mathbf{k}').$$

Therefore we have

$$\begin{aligned} (I - \eta^2 < R_1' R_1' >) &= I + \eta^2 W \\ (R_0 + \eta^2 < R_1'\ R_0\ R_1' >) &= R_0 (I - \eta^2 V), \end{aligned}$$

where

$$\begin{aligned} V(\mathbf{p}') &= \int d\mathbf{k}' P^{-1}(\mathbf{p}') M(\mathbf{p}', \mathbf{k}') P^{-1}(\mathbf{k}') M(\mathbf{k}', \mathbf{p}') \hat{Q}(\mathbf{p}' - \mathbf{k}') \\ W(\mathbf{p}') &= \int d\mathbf{k}' N(\mathbf{p}', \mathbf{k}') N(\mathbf{k}', \mathbf{p}') \hat{Q}(\mathbf{p}' - \mathbf{k}'). \end{aligned}$$

We rewrite expression (B.16) for the coherent reflection coefficient in the form

$$< \tilde{\beta} > = [\tilde{R}_0 + \eta^2 (R_0 < R_2 > - < R_2 > R_0)\,]\, \tilde{\alpha} + O(\eta^3), \tag{B.17}$$

where

$$\tilde{R}_0 = (I + \eta^2 W)^{-1} R_0 (I - \eta^2 V). \tag{B.18}$$

The differential scattering cross-sections (4.22) also need to be recomputed. From (B.14) and (B.17) we find that

$$\tilde{\beta}' = -\eta[R_1' \tilde{R}_0 + R_0 R_1'] \alpha + O(\eta^2).$$

	Born expansion	Smoothing Method
Dirichlet B.C.	(4.27) entry $2,2$ (4.28) entry $2,2$	(4.27) entry $2,2$ (4.28) entry $2,2$
Neumann B.C.	(4.27) entry $1,1$ (4.28) entry $1,1$	(B.1) (B.3)
Interface Condition	(4.23) (4.22)	(B.17) (B.20)

Table B.1: Reference to the explicit formulas for the Smoothing Method

Therefore replacing B by

$$\tilde{B}(\mathbf{p}',\mathbf{k}') \;=\; \frac{1}{2}(N(\mathbf{p}',\mathbf{k}')\tilde{R}_0(\mathbf{k}') + R_0(\mathbf{k}')N(\mathbf{p}',\mathbf{k}')), \tag{B.19}$$

we get for the cross-sections the nonsingular result

$$\tau^{ij}_{ren}(\mathbf{p}',\mathbf{k}') = = 4\tilde{B}^2_{ij}(\mathbf{p}',\mathbf{k}')\hat{Q}(\mathbf{p}'-\mathbf{k}'). \tag{B.20}$$

If we expand the inverse in (B.16) to order η^3 we get the standard Born expansion

$$< \tilde{\beta} > = \; [R_0 + \eta^2((R_0 < R_2 > - < R_2 > R_0) + < R_1' R_0 R_1' > + < R_1' R_1' > R_0)]\;\tilde{\alpha}.$$

The term $< R_1' R_0 R_1' > + < R_1' R_1' R_0 >$ corresponds to $R_{2,1}$ in (4.23). These coefficients blow up at grazing angles, as in the case of the Neumann boundary condition We note that the term $(R_0 < R_2 > - < R_2 > R_0)$, which is proportional to $Q(0)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and corresponds to the term $R_{2,0}$, does not vanish. Since it does not depend on the angle of incidence, this term is always bounded and does not need to be renormalized.

To summarize, the reflection and transmission coefficients given in Table 1.1, valid in the Born expansion, are now replaced by those in Table B.1 given by the smoothing method.

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