The Harnack inequality for second-order elliptic equations with divergence-free drifts

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Abstract

We consider an elliptic equation with a divergence-free drift b. We prove that an inequality of Harnack type holds under the assumption $b \in L^{n/2+\delta} \cap L^2$ where $\delta > 0$. As an application we provide a one sided Liouville's theorem provided that $b \in L^{n/2+\delta}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

1 Introduction

In this paper, we consider elliptic equations of the form

$$-\Delta u + b \cdot \nabla u + au = 0 \tag{1.1}$$

in a domain $\Omega \subset \mathbb{R}^n$. Here $a(x)$ is a given function and $b(x)$ is a prescribed divergence free vector field, that is, $\operatorname{div} b = 0$. The qualitative properties of solutions to elliptic and parabolic equations in divergence form with low regularity of the coefficients have been studied extensively, starting with the classical papers of De Giorgi [DG], Nash [N], and Moser [M]. We are mostly interested in the improved regularity for divergence free drifts b, which arise in fluid dynamics models (c.f. [BKNR, CV1, FV, K, SSSZ, KNSS, Z]).

As can be easily seen from a simple scaling argument, the natural Lebesgue spaces for the coefficients in the equation for the local regularity theory to hold are $a \in L^{n/2}$, $b \in L^n$, and, indeed, regularity properties of solutions in this case have been known since the work of Stampaccia [S]. It is well known that a strong divergence free flow may induce better regularity and decay of solutions of elliptic and parabolic problems by means of improved mixing—see, for instance [CKRZ] and references therein. It is also known that a divergence free-drift of relatively low regularity can still lead to regular solutions [CV1, CV2]. The question we study in this paper is whether the divergence free condition on b allows to relax the regularity assumptions on b given by Stampaccia.

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Let us recall some recent results in this direction. In a recent paper [NU], Nazarov and Ural'tseva significantly relaxed the classical regularity assumptions for divergence-free b by establishing the Harnack inequality and the Liouville theorem for weak solutions to (1.1) if b belongs to a Morrey space $M_q^{n/q-1}$ with $n/2 < q \leq n$, which lies between L^n and BMO^{-1}. In [FV], Friedlander and Vicol proved the Hölder continuity of weak solutions to drift-diffusion equations with a drift in BMO^{-1}. In [SSSZ], Seregin et al. established the Liouville theorem and the Harnack inequality for elliptic and parabolic equations with divergence free drifts b lying in the scale invariant space BMO^{-1}. All these spaces share the same scaling properties as L^n and are thus the natural candidates for good regularity theory.

In the present paper, we establish the Harnack inequality and the one-sided Liouville theorem for Lipschitz generalized solutions to (1.1) when $a(x)$ and $b(x)$ lie in the space $L^q(\Omega)$ with $n/2 < q \leq n$, and b is divergence free. Our results also hold for weak solutions provided that the drift b satisfies certain additional assumptions (c.f. equation (27) in [NU]). More precisely, we establish a Harnack-type inequality

$$\sup_{y \in B_R(x)} u(y) \leq C \inf_{y \in B_R(x)} u(y), \tag{1.2}$$

for all $R > 0$ (see Theorem 2.1), and use this estimate to establish the one-sided Liouville theorem when $a = 0$ in Theorem 2.3. The constant C in (1.2) depends on the L^q-norms of a and b, where $q > n/2$, but not on the solution u. Note that the $L^{n/2}$-norm is not scale invariant: if we set $b_l(x) = (1/l)b(x/l)$ then $\|b_l\|_{L^{n/2}} = l^{n/2}\|b\|_{L^{n/2}}$. Because of that, one can not expect the constant C to be independent of R, and, indeed, the constant given explicitly in Theorem 2.1 blows up as $R \to 0$.

The paper is organized as follows. In Section 2, we state our main results, Theorems 2.1 and 2.3. The proof is based on two auxiliary results, Lemmas 2.4 and 2.5. We first show (see Lemma 2.4) that weak solutions of (1.1) are locally bounded by employing the classical Moser iteration technique. Then, in Lemma 2.5, we derive a weak Harnack inequality, the proof of which is inspired by the proof of Han and Lin [HL, Theorem 4.15] for elliptic equations without lower-order coefficients. Our main results, Theorems 2.1 and 2.3, are direct consequences of Lemma 2.4 and 2.5.

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2 The main results

Our first result is the Harnack inequality.

Theorem 2.1. *(Harnack inequality) Let u be a nonnegative Lipschitz solution to the elliptic equation* (1.1). *Assume that $a \in L^q(\Omega)$, $b \in L^{\bar{q}}(\Omega)$ for $n/2 < q, \bar{q} \leq n$ and $\bar{q} \geq 2$, and that* $\operatorname{div} b = 0$ *in the sense of distributions. Then for any $B_R \subset \Omega$ we have*

$$\sup_{B_R} u \leq C \inf_{B_R} u. \tag{2.1}$$

Here C is a constant depending on n, q, $\bar{q}$, R, and $M_1 = 1 + \|a\|_{L^q} + \|b\|_{L^2}^2 + \|b\|_{L^{\bar{q}}}$.

Remark 2.2. From the proof we can deduce that

$$C = C(n,q,\bar{q}) \left(R^{-1} + (R^{-1}\|a\|_{L^q})^{1/(1-n/2q)} + (R^{-1}\|b\|_{L^{\bar{q}}})^{1/(1-n/2\bar{q})} \right)^{C(n)R^{-1}M_1}, \tag{2.2}$$

where M_1 is as in the statement of Theorem 2.1. □

Theorem 2.1 has the following consequence when $\Omega = \mathbb{R}^n$.

Theorem 2.3. *(One-sided Liouville's theorem) Let $a(x) \equiv 0$ and $b(x)$ as in Theorem 2.1. Then any nonnegative Lipschitz solution u to the elliptic equation (1.1) in $\mathbb{R}^n$ is equal to a constant.*

We note that [NU] provides a two-sided Liouville's theorem under the same assumptions, that is, the only solutions of (1.1) that are bounded both from above and from below are constants. However, the one-sided Liouville's theorem in [NU] requires b to belong to a Morrey space which is in the same scaling class as L^n.

Proof of Theorem 2.3. Without loss of generality, we may assume that u is a nonnegative Lipschitz solution to (1.1) with $\inf_{\mathbb{R}^n} u = 0$. Then for every $\epsilon > 0$, we have $\inf_{B_R} u \le \epsilon$ for any sufficiently large ball B_R. By Theorem 2.1, $\sup_{B_R} u \le C \inf_{B_R} u \le C\epsilon$ for all sufficiently large $R > 0$. Observe that the constant C given explicitly by (2.2) depends on R but remains bounded as $R \to \infty$. Therefore, the assertion is established. □

Theorem 2.1 is an immediate consequence of the following two lemmas that compare $\sup_{B_{\theta R}} u$ and $\inf_{B_{\theta R}} u$ to $\|u\|_{L^p(B_{\tau R})}$ with some small $p > 0$ and $0 < \theta < \tau < 1$.

Lemma 2.4. *Assume that u is a nonnegative Lipschitz subsolution to the equation*

$$-\Delta u + b \cdot \nabla u + au = 0 \tag{2.3}$$

with $a \in L^q(\Omega)$, $b \in L^{\bar{q}}(\Omega)$ for $n/2 < q, \bar{q} \le n$ and $\operatorname{div} b \le 0$ in the sense of distributions. Then for any $B_R \subset \Omega$, $p > 0$, and $0 < \theta < \tau < 1$

$$\sup_{B_{\theta R}} u \le C \left(R^{-n/p} + \left(R^{-1/(2-n/q)} \|a\|_{L^q(\Omega)}^{1/(2-n/q)} \right)^{n/p} + \left(R^{-1/(2-n/\bar{q})} \|b\|_{L^{\bar{q}}(\Omega)}^{1/(2-n/\bar{q})} \right)^{n/p} \right) \|u\|_{L^p(B_{\tau R})}, \tag{2.4}$$

where $C = C(n, p, \bar{q}, \theta, \tau)$ is a positive constant.

Lemma 2.5. *Assume that u is a nonnegative Lipschitz supersolution to (1.1) satisfying the assumptions of Theorem 2.1. Then for any $B_R \subset \Omega$ and $0 < \theta < \tau < 1$ there exists a small positive number $p_0 = p_0(n, q, \bar{q}, \theta, \tau, R, M_1)$ such that*

$$\inf_{B_{\theta R}} u \ge C \left(\int_{B_{\tau R}} u^{p_0} \right)^{1/p_0} \tag{2.5}$$

where $C = C(n, q, \bar{q}, \theta, \tau, R, M_1)$ is a positive constant and $M_1 = 1 + \|a\|_{L^q} + \|b\|_{L^2}^2 + \|b\|_{L^{\bar{q}}}$.

The rest of the paper contains the proofs of Lemmas 2.4 and 2.5. Both lemmas are proved using the Moser iteration, with the general strategy based on the proof of the Harnack inequality in [HL].

3 The proof of Lemma 2.4

Let u be a nonnegative Lipschitz subsolution of (2.3) in Ω, that is,

$$\int_\Omega (\partial_j u)(\partial_j \varphi) + \int_\Omega b_j(\partial_j u)\varphi + \int_\Omega au\varphi \le 0 \tag{3.1}$$

for any Lipschitz function $\varphi \ge 0$ in Ω such that $\varphi = 0$ in Ω^c.

For simplicity of presentation, we assume $a = 0$. The proof consists of a priori estimates which can be made rigorous as in [G, HL]. First, we obtain an a priori bound on the L^{p_1}-norm of u on a smaller ball B_{r_1}, in terms of an L^{p_2}-norm of u on a larger ball B_{R_2} with $r_1 < r_2$ but $p_1 > p_2$. Then an iterative procedure is used to bring the gap between r_1 and r_2 to zero and simultaneously send p_1 to infinity.

Let $\beta \ge 0$ and $\eta(x)$ be a Lipschitz cut-off in the ball $B_{\tau R}$ such that $0 \le \eta(x) \le 1$. We use $(\beta/2+1)u^{\beta+1}\eta^{2\gamma}$ as a test function in (3.1) to obtain

$$\left(\frac{\beta}{2}+1\right)\int (\partial_j u)\partial_j(u^{\beta+1})\eta^{2\gamma} + \left(\frac{\beta}{2}+1\right)\int u^{\beta+1}(\partial_j u)\partial_j(\eta^{2\gamma}) + \left(\frac{\beta}{2}+1\right)\int b_j u^{\beta+1}(\partial_j u)\eta^{2\gamma} \le 0. \tag{3.2}$$

Let $w = u^{\beta/2+1}$ so that $\partial_j w = (\beta/2+1)u^{\beta/2}\partial_j u$. By (3.2), we get

$$\frac{\beta+1}{\beta/2+1}\int |\partial_j w|^2\eta^{2\gamma} \le -2\gamma\int w(\partial_j w)\eta^{2\gamma-1}(\partial_j\eta) - \int b_j w(\partial_j w)\eta^{2\gamma}. \tag{3.3}$$

For the first term in the right side we have

$$-2\gamma\int w(\partial_j w)\eta^{2\gamma-1}\partial_j\eta = \gamma\int w^2\left(\eta^{2\gamma-1}\Delta\eta + (2\gamma-1)\eta^{2\gamma-2}|\partial_j\eta|^2\right), \tag{3.4}$$

while for the second

$$-\int b_j w(\partial_j w)\eta^{2\gamma} = \frac{1}{2}\int(\partial_j b_j)w^2\eta^{2\gamma} + \gamma\int b_j w^2\eta^{2\gamma-1}\partial_j\eta \le \gamma\int b_j w^2\eta^{2\gamma-1}\partial_j\eta, \tag{3.5}$$

as $\operatorname{div} b \le 0$.

Next, set $\gamma_0 = n/\bar q$. Then, as $\bar q > n/2$, we have $\gamma_0 \in (0,2)$ and, in addition

$$\frac{1}{\bar q} + \frac{\gamma_0}{2^*} + \frac{2-\gamma_0}{2} = 1 \tag{3.6}$$

for $n \ge 3$. Note that if $n = 2$ then γ_0 can be also chosen so that (3.6) is satisfied.

Assume also that γ is sufficiently large so that $\gamma\gamma_0 \le 2\gamma - 1$. Then, by Hölder's inequality we have, using (3.6)

$$\int b_j w^2\eta^{2\gamma-1}\partial_j\eta \le \int |b_j||w\eta^\gamma|^{\gamma_0}|w|^{2-\gamma_0}|\partial_j\eta| \le \|b\|_{L^{\bar q}}\|w\eta^\gamma\|_{L^{2^*}}^{\gamma_0}\|w|\nabla\eta|^{1/(2-\gamma_0)}\|_{L^2}^{2-\gamma_0}, \tag{3.7}$$

as $0 \le \eta \le 1$. By Young's and the Gagliardo-Nirenberg inequalities, this leads to

$$\int b_j w^2\eta^{2\gamma-1}\partial_j\eta \le \frac{1}{2}\|\nabla(w\eta^\gamma)\|_{L^2}^2 + C\|b\|_{L^{\bar q}}^{1/(1-n/2\bar q)}\|w|\nabla\eta|^{1/(2-n/\bar q)}\|_{L^2}^2. \tag{3.8}$$

By (3.3), (3.4), and (3.8), we obtain

$$\int |\nabla(u^{\beta/2+1}\eta^\gamma)|^2 \le C\int u^{\beta+2}\eta^{2\gamma-1}|\Delta\eta| + C\int u^{\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2 + C\|b\|_{L^{\bar q}}^{1/(1-n/2\bar q)}\|u^{\beta/2+1}(\nabla\eta)^{1/(2-n/\bar q)}\|_{L^2}^2. \tag{3.9}$$

By Sobolev embedding used in the left side of (3.9), we get

$$\|u^{\beta/2+1}\eta^\gamma\|_{L^{2\chi}} \le C\left(\int u^{\beta+2}\eta^{2\gamma-1}|\Delta\eta|\right)^{1/2} + C\left(\int u^{\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2\right)^{1/2} + C\|b\|_{L^{\bar q}}^{1/(2-n/\bar q)}\|u^{\beta/2+1}(\nabla\eta)^{1/(2-n/\bar q)}\|_{L^2} \tag{3.10}$$

where $\chi = n/(n-2)$ if $n \ge 3$ and $\chi > 2$ is arbitrary if $n = 2$. Now, let $\eta \in C_0^\infty(\Omega)$ be such that $\eta \equiv 1$ in $B_{\theta R}$, $\eta \equiv 0$ in $B_{\tau R}^c$, $|\nabla\eta| \le C/[R(\tau-\theta)]$ and $|\Delta\eta| \le C/[R^2(\tau-\theta)^2]$. Then, we have

$$\|u^{\beta/2+1}\|_{L^{2\chi}(B_{\theta R})} \le \frac{C}{R(\tau-\theta)}\left(\int_{B_{\tau R}} u^{\beta+2}\right)^{1/2} + \frac{C}{(R(\tau-\theta))^{1/(2-n/\bar q)}}\|b\|_{L^{\bar q}(B_{\tau R})}^{1/(2-n/\bar q)}\|u^{\beta/2+1}\|_{L^2(B_{\tau R})}. \tag{3.11}$$

The main point of (3.11) is that, since $\chi > 1$, we have a bound on a higher norm of u on a smaller ball in terms of the lower norm of u on a larger ball. We now apply the estimate (3.11) iteratively on pairs of balls $B_{r_{i+1}} \subset B_{r_i}$, and also let $\beta_i \to +\infty$. More precisely, we choose $\beta_i = 2(\chi^i - 1)$ and $r_i = \theta R + (\tau-\theta)R2^{-i}$ for $i = 0, 1, 2, \ldots$, so that $r_i - r_{i+1} = (\tau-\theta)R2^{-(i+1)}$. We obtain

$$\|u\|_{L^{2\chi^{i+1}}(B_{r_{i+1}})} \le C^{1/\chi^i}2^{i/\chi^i}(R(\tau-\theta))^{-1/\chi^i}\|u\|_{L^{2\chi^i}(B_{r_i})} + \left(C2^{i/(2-n/\bar q)}(R(\tau-\theta))^{-1/(2-n/\bar q)}\|b\|_{L^{\bar q}(B_{r_i})}^{1/(2-n/\bar q)}\right)^{1/\chi^i}\|u\|_{L^{2\chi^i}(B_{r_i})}. \tag{3.12}$$

By iteration, letting $i \to +\infty$, we conclude that the estimate (2.4) holds for $p \ge 2$.

Now, let $p \in (0, 2)$. We have just shown that

$$\begin{aligned}\sup_{B_{\theta R}} u &\le C\left((R(\tau-\theta))^{-n/2} + \left((R(\tau-\theta))^{-1/(2-n/\bar q)}\|b\|_{L^{\bar q}(B_{r_i})}^{1/(2-n/\bar q)}\right)^{n/2}\right)\|u\|_{L^2(B_{\tau R})} \\ &\le C\left((R(\tau-\theta))^{-n/2} + \left((R(\tau-\theta))^{-1/(2-n/\bar q)}\|b\|_{L^{\bar q}(B_{r_i})}^{1/(2-n/\bar q)}\right)^{n/2}\right)\|u\|_{L^\infty(B_{\tau R})}^{1-p/2}\|u\|_{L^p(B_{\tau R})}^{p/2}\end{aligned} \tag{3.13}$$

which implies

$$\sup_{B_{\theta R}} u \le \frac{1}{2}\|u\|_{L^\infty(B_{\tau R})} + C\left((R(\tau-\theta))^{-n/p} + \left((R(\tau-\theta))^{-1/(2-n/\bar q)}\|b\|_{L^{\bar q}(B_{r_i})}^{1/(2-n/\bar q)}\right)^{n/p}\right)\|u\|_{L^p(B_{\tau R})}.$$

A standard iteration argument (c.f. [HL, Lemma 4.3]) then implies

$$\sup_{B_{\theta R}} u \le C\left((R(\tau-\theta))^{-n/p} + \left((R(\tau-\theta))^{-1/(2-n/\bar q)}\|b\|_{L^{\bar q}(B_{r_i})}^{1/(2-n/\bar q)}\right)^{n/p}\right)\|u\|_{L^p(B_{\tau R})} \tag{3.14}$$

and the proof of Lemma 2.4 is complete. □

4 Proof of Lemma 2.5

We assume without loss of generality that $R = 1$. The proof is similar in spirit to that of Lemma 2.4: we obtain an a priori bound and use it iteratively.

Assume that u is a nonnegative Lipschitz supersolution to (1.1), and consider $v = 1/u$. The function v satisfies

$$-\Delta v + b \cdot \nabla v - av \leq 0 \quad \text{in} \quad \Omega \tag{4.1}$$

or equivalently

$$\int (\partial_j v)(\partial_j \varphi) + \int b_j(\partial_j v)\varphi - \int av\varphi \leq 0 \tag{4.2}$$

for any function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \geq 0$ in Ω. By Lemma 2.4, it follows that for any $0 < \theta < \tau < 1$ and $p > 0$, we have

$$\sup_{B_\theta} v \leq C\|v\|_{L^p(B_\tau)} \tag{4.3}$$

with $C = C(n, p, q, \bar{q}, \tau, \theta, M_1)$. Therefore, we have

$$\inf_{B_\theta} u \geq \frac{1}{C}\left(\int_{B_\tau} u^{-p} \int_{B_\tau} u^p\right)^{-1/p} \left(\int_{B_\tau} u^p\right)^{1/p}. \tag{4.4}$$

We claim that there exists $p_0 > 0$ such that

$$\int_{B_\tau} u^{-p_0} \int_{B_\tau} u^{p_0} \leq C \tag{4.5}$$

with a constant $C = C(n, q, \bar{q}, \tau, M_1)$, which would finish the proof of Lemma 2.5.

Reduction to an exponential bound

In order to prove (4.5) for some sufficiently small $p_0 > 0$, denote

$$(\log u)_{B_\tau} = \frac{1}{|B_\tau|}\int_{B_\tau} \log u,$$

and set

$$w = \log u - (\log u)_{B_\tau}. \tag{4.6}$$

We shall show that there exists $p_0 > 0$ such that

$$\int_{B_\tau} e^{p_0|w|} \leq C \tag{4.7}$$

where $C = C(\tau)$, which in turn implies (4.5). Indeed, if we assume that (4.7) holds, then

$$\int_{B_\tau} e^{p_0(\log u - (\log u)_{B_\tau})} \leq C \tag{4.8}$$

and

$$\int_{B_\tau} e^{-p_0(\log u - (\log u)_{B_\tau})} \leq C. \tag{4.9}$$

Therefore, we have $e^{-p_0(\log u)_{B_\tau}} \int_{B_\tau} e^{p_0 \log u} \leq C$ and $e^{p_0(\log u)_{B_\tau}} \int_{B_\tau} e^{-p_0 \log u} \leq C$. Multiplying these two inequalities then leads to (4.5).

An L^2-bound for w

We now prove (4.7). First, we establish bounds on the L^2-norm of w. The function w satisfies

$$|\nabla w|^2 \le -\Delta w + b \cdot \nabla w + a \quad \text{ in } B_1. \tag{4.10}$$

Fix $\tau \in (0,1)$, and let $\eta \in C_0^1(\Omega)$ with $0 \le \eta \le 1$ be a cutoff such that $\eta \equiv 1$ on $B_{(1+\tau)/2}$, $\eta \equiv 0$ on B_1^c, and $|\nabla \eta| \le C/(1-\tau)$. Multiplying (4.10) by η^2 and integrating over B_1, we obtain

$$\begin{aligned}\int_{B_1} |\nabla w|^2\eta^2 &\le 2\int_{B_1}(\partial_j w)\eta(\partial_j\eta) + \int_{B_1} b_j(\partial_j w)\eta^2 + \int_{B_1} a\eta^2 \\ &\le 2\|\eta\nabla w\|_{L^2}\|\nabla\eta\|_{L^2} + \|b\|_{L^2}\|\eta\nabla w\|_{L^2}\|\eta\|_{L^\infty} + \|a\|_{L^q}\|\eta^2\|_{L^{q'}}\end{aligned} \tag{4.11}$$

where $1/q + 1/q' = 1$. Absorbing the factors $\|\eta\nabla w\|_{L^2}$ on the right using the term on the left, we get

$$\int_{B_{(1+\tau)/2}} |\nabla w|^2 \le C_\tau M_0 \tag{4.12}$$

where $M_0 = 1 + \|a\|_{L^q} + \|b\|_{L^2}^2$, and the constant C_τ may depend on $\tau \in (0,1)$. Also, since

$$\int_{B_\tau} w = 0,$$

and $(1+\tau)/2 \ge \tau$, we have by the Poincaré inequality

$$\int_{B_{(1+\tau)/2}} w^2 \le C\int_{B_{(1+\tau)/2}} |\nabla w|^2 \le C_\tau M_0. \tag{4.13}$$

Bounds on the higher norms of w

Next, we need to estimate $\int_{B_\tau} |w|^\beta$ for all $\beta \ge 1$. As in the proof of Lemma 2.4 the idea is to bound first the higher norms of w on smaller balls in terms of the lower norms of w on larger balls and then use the iteration process.

We multiply (4.10) by $|w|^{2\beta}\eta^{2\gamma}$ and integrate over B_1 in order to obtain

$$\begin{aligned}\int_{B_1} |w|^{2\beta}|\nabla w|^2\eta^{2\gamma} \le 2\beta\int_{B_1} |w|^{2\beta-2}w|\nabla w|^2\eta^{2\gamma} + 2\gamma\int_{B_1}|w|^{2\beta}(\partial_j w)\eta^{2\gamma-1}(\partial_j\eta) \\ - \frac{2\gamma}{2\beta+1}\int_{B_1} b_j|w|^{2\beta}w\eta^{2\gamma-1}(\partial_j\eta) + \int_{B_1} a|w|^{2\beta}\eta^{2\gamma}.\end{aligned} \tag{4.14}$$

Here we utilized $\operatorname{div} b = 0$ and $\partial_j|w| = w\partial_j w/|w|$. For the first term in the right side of (4.14) we use

$$2\beta|w|^{2\beta-1} \le \frac{1}{4}|w|^{2\beta} + (8\beta)^{2\beta}, \tag{4.15}$$

while for the second

$$2\gamma\int_{B_1}|w|^{2\beta}(\partial_j w)\eta^{2\gamma-1}(\partial_j\eta) \le \frac{1}{4}\int_{B_1}|w|^{2\beta}|\nabla w|^2\eta^{2\gamma} + C\gamma^2\int_{B_1}|w|^{2\beta}\eta^{2\gamma-2}|\nabla\eta|^2. \tag{4.16}$$

This leads to

$$\begin{aligned}\int_{B_1} |w|^{2\beta}|\nabla w|^2\eta^{2\gamma} \le C(8\beta)^{2\beta}\int_{B_1}|\nabla w|^2\eta^{2\gamma} + C\gamma^2\int_{B_1}|w|^{2\beta}\eta^{2\gamma-2}|\nabla\eta|^2 \\ + \frac{C\gamma}{\beta+1}\int_{B_1}|b||w|^{2\beta+1}\eta^{2\gamma-1}|\nabla\eta| + C\int_{B_1}|a||w|^{2\beta}\eta^{2\gamma}.\end{aligned} \tag{4.17}$$

Let $\tau \le r \le R \le (1+\tau)/2$. We now choose a cutoff $\eta \in C_0^1(\Omega)$ with $0 \le \eta \le 1$ such that $\eta \equiv 1$ on B_r, $\eta \equiv 0$ on B_R^c, and $|\nabla\eta| \le C/(R-r)$. By (4.11), for the first term in the right side of (4.17) we have

$$(8\beta)^{2\beta}\int_{B_1}|\nabla w|^2\eta^{2\gamma} \le (8\beta)^{2\beta}\int_{B_{(1+\tau)/2}}|\nabla w|^2 \le C_\tau(8\beta)^{2\beta}M_0. \tag{4.18}$$

On the other hand, for the left side of (4.17), we use

$$\left|\nabla(|w|^{\beta+1}\eta^\gamma)\right|^2 \le 2\gamma^2|w|^{2\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2 + 2(\beta+1)^2|w|^{2\beta}|\nabla w|^2\eta^{2\gamma}. \tag{4.19}$$

Hence, we obtain

$$\begin{aligned}\int_{B_1}\left|\nabla(|w|^{\beta+1}\eta^\gamma)\right|^2 \le C\gamma^2\int_{B_1}|w|^{2\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2 + C(\beta+1)^2(8\beta)^{2\beta}M_0 \\ + C\gamma^2(\beta+1)^2\int_{B_1}|w|^{2\beta}\eta^{2\gamma-2}|\nabla\eta|^2 \\ + C\gamma(\beta+1)\int_{B_1}|b||w|^{2\beta+1}\eta^{2\gamma-1}|\nabla\eta| + C(\beta+1)^2\int_{B_1}|a||w|^{2\beta}\eta^{2\gamma}.\end{aligned} \tag{4.20}$$

For the third term in the right side we utilize

$$(\beta+1)^2|w|^{2\beta} \le \frac{(\beta+1)^{2\beta+2}}{\beta+1} + \frac{\left(|w|^{2\beta}\right)^{(\beta+1)/\beta}}{(\beta+1)/\beta} \le (8\beta)^{2\beta} + |w|^{2\beta+2} \tag{4.21}$$

which gives

$$\begin{aligned}C\gamma^2(\beta+1)^2\int_{B_1}|w|^{2\beta}\eta^{2\gamma-2}|\nabla\eta|^2 \le C(8\beta)^{2\beta}\gamma^2\int_{B_1}\eta^{2\gamma-2}|\nabla\eta|^2 + C\gamma^2\int_{B_1}|w|^{2\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2 \\ \le \frac{C(8\beta)^{2\beta}\gamma^2M_0}{(R-r)^2} + C\gamma^2\int_{B_1}|w|^{2\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2,\end{aligned} \tag{4.22}$$

as $M_0 \ge 1$. The last two terms in (4.20) are estimated as follows. First, we have

$$\int_{B_1}|a||w|^{2\beta}\eta^{2\gamma} = \int_{B_1}|a|(|w|^{\beta+1}\eta^\gamma)^{2\beta/(\beta+1)}\eta^{2\gamma/(\beta+1)} \le \|a\|_{L^q}\||w|^{\beta+1}\eta^\gamma\|_{L^{2\beta q'/(\beta+1)}}^{2\beta/(\beta+1)} \tag{4.23}$$

where $1/q + 1/q' = 1$. Now, we use the Gagliardo-Nirenberg inequality

$$\||w|^{\beta+1}\eta^\gamma\|_{L^{2\beta q'/(\beta+1)}} \le C\||w|^{\beta+1}\eta^\gamma\|_{L^2}^{1-\alpha}\|\nabla(|w|^{\beta+1}\eta^\gamma)\|_{L^2}^{\alpha} \tag{4.24}$$

with $\alpha = n/2 - n/(2\beta q'/(\beta+1))$ if $2\beta q'/(\beta+1) \geq 2$, and $\alpha = 0$ otherwise. By Young's inequality, we obtain

$$\begin{aligned}\int_{B_1} |a||w|^{2\beta}\eta^{2\gamma} &\leq C\|a\|_{L^q}\||w|^{\beta+1}\eta^\gamma\|_{L^2}^{2(1-\alpha)\beta/(\beta+1)}\|\nabla(|w|^{\beta+1}\eta^\gamma)\|_{L^2}^{2\alpha\beta/(\beta+1)} \\ &\leq \left(\frac{1}{(2(\beta+1))^{2\alpha\beta/(\beta+1)}}\|\nabla(|w|^{\beta+1}\eta^\gamma)\|_{L^2}^{2\alpha\beta/(\beta+1)}\right)^{(\beta+1)/\alpha\beta} \\ &\quad + C\left((2(\beta+1))^{2\alpha\beta/(\beta+1)}\|a\|_{L^q}\||w|^{\beta+1}\eta^\gamma\|_{L^2}^{2(1-\alpha)\beta/(\beta+1)}\right)^{(\beta+1)/(\beta(1-\alpha)+1)}.\end{aligned} \tag{4.25}$$

As $\alpha \in (0,1)$, this implies

$$\int_{B_1} |a||w|^{2\beta}\eta^{2\gamma} \leq \frac{1}{(2(\beta+1))^2}\|\nabla(|w|^{\beta+1}\eta^\gamma)\|_{L^2}^2 + C(\beta+1)^{2\alpha_1}\|a\|_{L^q}^{\alpha_1}\||w|^{\beta+1}\eta^\gamma\|_{L^2}^{\alpha_2}. \tag{4.26}$$

Here we denoted $\alpha_1 = (\beta+1)/(\beta(1-\alpha)+1)$ and $\alpha_2 = 2\beta(1-\alpha)/(\beta(1-\alpha)+1)$. Observe that $\alpha_1 \geq 1$ and α_1 is smaller than a constant independent of β, while $0 < \alpha_2 < 2$ with $\alpha_2 \to 2$ as $\beta \to \infty$.

For the last remaining term in (4.20), we have

$$C\gamma(\beta+1)\int_{B_1} |b||w|^{2\beta+1}\eta^{2\gamma-1}|\nabla\eta| = C\gamma(\beta+1)\int_{B_1} |b|\left(|w|^{\beta+1}\eta^\gamma\right)^{(2\beta+1)/(\beta+1)}\eta^{\gamma/(\beta+1)-1}|\nabla\eta|. \tag{4.27}$$

Let us choose $\gamma = \beta+1$. Then, the above expression becomes

$$C\gamma(\beta+1)\int_{B_1} |b|\left(|w|^{\beta+1}\eta^\gamma\right)^{(2\beta+1)/(\beta+1)}|\nabla\eta| \leq C(\beta+1)^2\|b\|_{L^{\bar q}}\||w|^{\beta+1}\eta^\gamma\|_{L^{(2\beta+1)\bar q'/(\beta+1)}}^{(2\beta+1)/(\beta+1)}\|\nabla\eta\|_{L^\infty} \tag{4.28}$$

where $1/\bar q + 1/\bar q' = 1$. Once again we apply the Gagliardo-Nirenberg inequality

$$\||w|^{\beta+1}\eta^\gamma\|_{L^{(2\beta+1)\bar q'/(\beta+1)}} \leq C\||w|^{\beta+1}\eta^\gamma\|_{L^2}^{1-\bar\alpha}\|\nabla(|w|^{\beta+1}\eta^\gamma)\|_{L^2}^{\bar\alpha} \tag{4.29}$$

with $\bar\alpha = n/2 - n/((2\beta+1)\bar q'/(\beta+1))$ if $(2\beta+1)\bar q'/(\beta+1) \geq 2$ and $\bar\alpha = 0$ otherwise.

Thus, by Young's inequality, we have

$$\begin{aligned}&C\gamma(\beta+1)\int_{B_1} |b|\left(|w|^{\beta+1}\eta^\gamma\right)^{(2\beta+1)/(\beta+1)}|\nabla\eta| \\ &\leq C(\beta+1)^2\|b\|_{L^{\bar q}}\||w|^{\beta+1}\eta^\gamma\|_{L^2}^{(1-\bar\alpha)(2\beta+1)/(\beta+1)}\|\nabla(|w|^{\beta+1}\eta^\gamma)\|_{L^2}^{\bar\alpha(2\beta+1)/(\beta+1)}\|\nabla\eta\|_{L^\infty} \\ &\leq \frac{1}{4}\|\nabla(|w|^{\beta+1}\eta^\gamma)\|_{L^2}^2 + \frac{C(\beta+1)^{2\bar\alpha_1}}{(R-r)^{\bar\alpha_1}}\|b\|_{L^{\bar q}}^{\bar\alpha_1}\||w|^{\beta+1}\eta^\gamma\|_{L^2}^{\bar\alpha_2}.\end{aligned} \tag{4.30}$$

Here we denoted $\bar\alpha_1 = (2\beta+2)/(2\beta(1-\bar\alpha)+2-\bar\alpha)$ and $\bar\alpha_2 = 2(2\beta+1)(1-\bar\alpha)/(2\beta(1-\bar\alpha)+2-\bar\alpha)$. Note that, as in (4.24), we have $\bar\alpha_1 \geq 1$ and $\bar\alpha_1$ is less than a constant independent of β, while $0 < \bar\alpha_2 < 2$, and $\bar\alpha_2 \to 2$ when $\beta \to \infty$.

Putting together (4.20), (4.21), (4.24), and (4.30), we obtain

$$\|\nabla(|w|^{\beta+1}\eta^\gamma)\|^2_{L^2(B_r)} \le \frac{C(\beta+1)^2}{(R-r)^2}\||w|^{\beta+1}\|^2_{L^2(B_R)} + \frac{C(\beta+1)^2(8\beta)^{2\beta}M_0}{(R-r)^2} \tag{4.31}$$
$$+ C(\beta+1)^{2\alpha_1+2}\|a\|^{\alpha_1}_{L^q(B_R)}\||w|^{\beta+1}\|^{\alpha_2}_{L^2(B_R)} + \frac{C(\beta+1)^{2\bar\alpha_1}}{(R-r)^{\bar\alpha_1}}\|b\|^{\bar\alpha_1}_{L^{\bar q}(B_R)}\||w|^{\beta+1}\|^{\bar\alpha_2}_{L^2(B_R)}.$$

Using Sobolev embedding, we may rewrite (4.31) in the form

$$\||w|^{\beta+1}\|^2_{L^{2\chi}(B_r)} \le \frac{C(\beta+1)^{2\kappa}}{(R-r)^{\bar\alpha_1+2}}\Big(\||w|^{\beta+1}\|^2_{L^2(B_R)} + (8\beta)^{2\beta}M_0 \tag{4.32}$$
$$+ \|a\|^{\alpha_1}_{L^q(B_R)}\||w|^{\beta+1}\|^{\alpha_2}_{L^2(B_R)} + \|b\|^{\bar\alpha_1}_{L^{\bar q}(B_R)}\||w|^{\beta+1}\|^{\bar\alpha_2}_{L^2(B_R)}\Big)$$

where $\kappa = \max\{\alpha_1+1, \bar\alpha_1\}$ and $\chi = n/(n-2)$ if $n \ge 3$ and $\chi > 2$ if $n = 2$. Estimate (4.32) is analogous to (3.11): a higher norm of w on a smaller ball is bounded in terms of a lower norm of w on a larger ball.

The iteration process

Next, we consider the iteration process. Let $\beta_i = \chi^i - 1$ and $r_i = \tau + (1+\tau)/2^{i+1}$ for $i = 0, 1, 2, \ldots$. From (4.32), we get

$$\||w|^{\chi^i}\|^2_{L^{2\chi}(B_{r_{i+1}})} \le C\chi^{2\kappa i}2^{(\bar\alpha_1+2)(i+2)}\Big(\||w|^{\chi^i}\|^2_{L^2(B_{r_i})} + (8\chi^i)^{2\chi^i}M_0 \tag{4.33}$$
$$+ \|a\|^{\alpha_1}_{L^q(B_{r_i})}\||w|^{\chi^i}\|^{\alpha_2}_{L^2(B_{r_i})} + \|b\|^{\bar\alpha_1}_{L^{\bar q}(B_{r_i})}\||w|^{\chi^i}\|^{\bar\alpha_2}_{L^2(B_{r_i})}\Big)$$

for all $i = 0, 1, 2, \ldots$. Taking $1/(2\chi^i)$ power on both sides of (4.33) gives

$$\|w\|_{L^{2\chi^{i+1}}(B_{r_{i+1}})} \le C^{1/(2\chi^i)}\chi^{\kappa i/\chi^i}2^{(\bar\alpha_1+2)(i+2)/(2\chi^i)}\Big(\|w\|_{L^{2\chi^i}(B_{r_i})} + 8\chi^i M_0^{1/(2\chi^i)} \tag{4.34}$$
$$+ \|a\|^{\alpha_1/(2\chi^i)}_{L^q(B_{r_i})}\|w\|^{\alpha_2/2}_{L^{2\chi^i}(B_{r_i})} + \|b\|^{\bar\alpha_1/(2\chi^i)}_{L^{\bar q}(B_{r_i})}\|w\|^{\bar\alpha_2/2}_{L^{2\chi^i}(B_{r_i})}\Big).$$

This leads to the inequality

$$\|w\|_{L^{2\chi^{i+1}}(B_{r_{i+1}})} \le (CM_1)^{\widetilde\alpha/(2\chi^i)}(2\chi)^{\kappa i/\chi^i}\left(\|w\|_{L^{2\chi^i}(B_{r_i})} + 8\chi^i + \|w\|^{\alpha_2/2}_{L^{2\chi^i}(B_{r_i})} + \|w\|^{\bar\alpha_2/2}_{L^{2\chi^i}(B_{r_i})}\right)$$
$$\le (CM_1)^{\widetilde\alpha/(2\chi^i)}(2\chi)^{\kappa i/\chi^i}\left(\|w\|_{L^{2\chi^i}(B_{r_i})} + 8\chi^i\right), \tag{4.35}$$

for all $i = 0, 1, 2, \ldots$, with $\widetilde\alpha = \max\{\alpha_1, \bar\alpha_1\}$ and $M_1 = 1 + \|a\|_{L^q} + \|b\|^2_{L^2} + \|b\|_{L^{\bar q}}$. For the second inequality in (4.35) we also used $\alpha_2, \bar\alpha_2 \le 2$, so that $\|w\|^{\alpha_2/2}_{L^p} \le 1 + \|w\|^2_{L^p}$ and $\|w\|^{\bar\alpha_2/2}_{L^p} \le 1 + \|w\|^2_{L^p}$.

Note that if a sequence Y_i satisfies $Y_{i+1} \le C_i(Y_i + \chi^i)$ with $C_i \ge 1$ and $\prod_{i=1}^\infty C_i \le \bar K$, then by induction we have

$$Y_i \le C\bar K(Y_0 + \sum_{j=0}^{i}\chi^{i-1}) \le C(Y_0 + \chi^i), \tag{4.36}$$

for all $i = 0, 1, 2, \ldots$. Thus, iterating (4.35), we obtain

$$\|w\|_{L^{2\chi^{i+1}}(B_{r_{i+1}})} \leq CM_1^{C(n)} \left(CM_1 + \chi^{i+1} \right) \leq CM_1^{C(n)} \chi^{i+1}. \tag{4.37}$$

for all $i = 0, 1, 2, \ldots$, as $\sum_{j=1}^{i} j/\chi^j \leq C$ and $\sum_{j=1}^{i} \chi^j \leq \chi^{i+1}$ for $\chi > 1$.

Finally, for any $\beta \geq 1$ there exists $i = 0, 1, 2, \ldots$ such that

$$2\chi^i \leq \beta + 1 \leq 2\chi^{i+1}. \tag{4.38}$$

Thus, in particular, we have

$$\left(\int_{B_\tau} |w|^{\beta+1} \right)^{1/(\beta+1)} \leq C\|w\|_{L^{2\chi^{i+1}}(B_{r_{i+1}})} \leq CM_1^{C(n)}(\beta + 1). \tag{4.39}$$

Therefore, for all $\beta \geq 1$, we obtain

$$\int_{B_\tau} \frac{(p_0|w|)^{(\beta+1)}}{(\beta+1)!} \leq p_0^{\beta+1} \left(CM_1^{C(n)} e \right)^{(\beta+1)} \leq \frac{1}{2^{(\beta+1)}} \tag{4.40}$$

by taking

$$p_0 = \frac{1}{CM_1^{C(n)} e} \tag{4.41}$$

sufficiently small. By (4.13), we also have

$$\int_{B_\tau} |w| \leq C \int_{B_\tau} w^2 \leq CM_0 \tag{4.42}$$

which gives (4.40) for $\beta = 0$ as well. It follows from (4.40) that (4.7) holds, and therefore the proof of the lemma is complete. $\square$

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