Level sets in turbulent front propagation

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1 Introduction

We study the area of level sets of the temperature in two models of combustion: a full reaction-diffusion-convection equation and a well-known Hamilton-Jacobi G-equation. Both models are used to describe the flame propagation in a turbulent flow. The G-equation

$$G_t + u \cdot \nabla G = v_0 |\nabla G| \tag{1}$$

is derived [13] assuming a vanishingly small thickness of the burning region. The parameter v_0 is the laminar velocity of $\{G(x,t) = 0\}$, the burning front. The full reaction-diffusion-convection equation is given by

$$T_t + u \cdot \nabla T - \kappa \Delta T = \frac{v_0^2}{4\kappa} f(T), \tag{2}$$

The velocity $u(x,t)$ is divergence-free in both the reaction-diffusion-convection equation and in the G-equation:

$$\mathrm{div} u = 0.$$

Here $f(T)$ is the reaction term, for instance of KPP type, such as $f(T) = T(1-T)$ (but our results extend to the case of more general chemistries), v_0 is a laminar front velocity of the burning front associated with the chemical reaction, T is a dimensionless relative temperature and κ is the molecular diffusivity that is small but not zero. Our goal in this note is to estimate the average area of the burning temperature level sets registered in a ball co-moving with the front of the propagating flame. It is believed that this area is directly related to the speed of the front propagation in the flamelet regime [9, 10, 16]. We use methods of [5], [18], [6] to obtain upper bounds on the size of these average areas as functions of the non-dimensional parameters and of the size of the co-moving ball. It turns out that the equations (1) and (2) lead to different results. Moreover, in the case of reaction-diffusion-convection equation, the turbulent velocity u influences the dimension of the level sets only if the turbulence is sufficiently rough. The area and the fractal dimension of the burning front were computed by Kerstein in [10] using physical considerations not based on either (1) or (2). Our estimate for the dimension of the front for the G-equation coincides with $d = 7/3$ obtained in [10] for the Kolmogorov turbulence, but the analysis of the reaction-diffusion-convection equation (2) yields a different result.

We also discuss the possible estimation of the turbulent burning velocity in the flamelet regime with strong turbulence based on the structure of the level sets in the case of the full diffusion-reaction-convection equation. Many studies were devoted to the estimation of the turbulent burning velocity u_T. The asymptotic theory of Yakhot [19] based on the G-equation and the heuristic argument of Kerstein [9] based on the exponential growth of the front area in time predict that

$$U_T = \exp\left(\frac{U^2}{U_T^2}\right), \tag{3}$$

where $U_T = u_T/v_0$ and $U = u_0/v_0$. Here v_0 is the laminar burning velocity and u_0 is a typical turbulent velocity. This dependence is almost linear, up to a logarithmic correction. Various other models for the relation between U_T and U were proposed in the literature: linear, or nearly linear relation by [1, 3, 17], the Clavin-Williams relation $U_T \approx 1 + U^2$ [4] for $U \ll 1$, and the power law $U_T = (1 + \beta U^\alpha)^{1/\alpha}$ [15] with a detailed review of various results presented in [16]. Expression (3) agrees qualitatively with the results of the experiments in liquids but not in gaseous mixtures, where the graph $U_T = U_T(U)$ bends and seems to tend to a finite limit as U increases at a fixed Reynolds number [2, 16]. We derive an upper bound for the turbulent burning velocity u_T for the full reaction-diffusion equation (2) using our estimate for the area of the level sets per unit volume and the physical relation between u_T and the area suggested in [9]. Our estimate predicts the following dependence of U_T on U:

$$U_T \leq CU\mathrm{Ka}^{-\frac{1}{2}}\mathrm{Re}^{\frac{1}{4}} = C(\mathrm{ReSc})^{\frac{1}{2}}.$$

Here Re is the Reynolds number, Ka is the Karlovitz number measuring the strength of the chemical reaction (Ka $\ll 1$ corresponds to the strong reaction), and Sc is the Schmidt number. We recall the definitions of these numbers in the end of Section 3.1. This expression predicts square root dependence of U_T on u_0 when all other parameters in the system are fixed, and provides an upper bound for U_T when the Reynolds and Schmidt numbers are fixed. It therefore gives a qualitative explanation of the saturation of U_T seen in the experiments in gaseous mixtures [2, 16]. The square root dependence of U_T on u_0 was previously proposed by Kerstein in [11] based on the pair exchange model for the front propagation. Our results suggest that the dependence of U_T on u_0 may be better approximated by the square root law than the linear law in the systems that are well described by the reaction-diffusion equation.

The models (1) and (2) were also recently compared in [12] in the limit $\kappa \to 0$. The velocities $u(x)$ considered in [12] were periodic with a variety of periods. Among other results it was shown that in the limit $\kappa \to 0$ the G-equation sometimes under-predicts the front speed given by the limit (homogenization) of (2). Recently, this issue was studied numerically in [14]; while confirming the results of [12] for vanishing front thickness, an interesting effect of a small but finite thickness of the burning region was demonstrated: it was shown that the G-equation may then over-predict the effective speed of the front as opposed to the limit $\kappa \to 0$.

2 The level sets of the G-equation

We first consider the case of the G-equation defined in a domain D. Our analysis is local following some point x_0 that is initially located on the front away from the boundary of domain D. We pass to co-moving variables

$$y = x - X_0(t), \tag{4}$$

where $X_0(t)$ is the trajectory of the point x_0:

$$\frac{dX_0}{dt} = u(X_0, t), \quad X_0(0) = x_0, \tag{5}$$

in order to keep track of the front. Then the G-equation takes the form

$$\tilde{G}_t + \tilde{u}(y,t) \cdot \nabla_y \tilde{G} = v_0 |\nabla \tilde{G}|, \tag{6}$$

where

$$\tilde{u}(y,t) = u(y + X_0(t), t) - u(X_0(t), t) \tag{7}$$

and $\tilde{G}(y,t) = G(y + X_0(t), t)$.

In [5], [18] and a series of related works (references in [6]) the co-area formula [8] of geometric measure theory was used to estimate the area of level sets in turbulent flows. The co-area formula relates the area of the level set $\{y|\tilde{G}(y) = g\}$ inside a ball to a spatial integral. To be specific, let $\psi(y)$ be a compactly supported function, denote by $dH^{(2)}(y)$ the two-dimensional Hausdorff measure [7] and set

$$a_\psi(g) = \int_{\{y|\tilde{G}(y)=g\}} \psi(y) dH^{(2)}(y).$$

$a_\psi(g)$ represents the area of the level set $\{y \in \mathrm{spt}\psi | \tilde{G}(y) = g\}$ weighed by ψ. The co-area formula is then

$$\int_{-\infty}^{\infty} \phi(g) a_\psi(g) dg = \int_{\mathbf{R}^3} \phi(\tilde{G}(y))\psi(y)|\nabla \tilde{G}(y)| dy \tag{8}$$

and holds for any test function $\phi(g)$. In order to make contact with the equation of motion (1) we define an auxiliary function

$$H(g) = \int_0^g \phi(\lambda) d\lambda,$$

where $\phi(\lambda)$ is an arbitrary integrable function with

$$\int_0^\infty |\phi(\lambda)| d\lambda = 1.$$

Note that $H(g)$ is uniformly bounded:

$$|H(g)| \leq 1.$$

Let η be the minimal distance between $X_0(s)$ and ∂D, $0 \leq s \leq t$. Let $\psi(y)$ be some C_0^∞ function supported inside the ball of radius η. Define $h_\psi(y,t) = H(\tilde{G})\psi(y)$. A direct computation based on (6) shows that h_ψ satisfies

$$\frac{\partial h_\psi}{\partial t} + \tilde{u} \cdot \nabla_y h_\psi - v_0 \psi(y) H'(\tilde{G}) |\nabla_y \tilde{G}| = (\tilde{u} \cdot \nabla_y \psi) H(\tilde{G}). \tag{9}$$

Integrating (9) in space and time we obtain

$$v_0 t^{-1} \int_0^t \int_{B_\eta} \psi(y)\phi(\tilde{G})|\nabla \tilde{G}|\, dy dt = t^{-1} \int_{B_\eta} (h_\psi(y,t) - h_\psi(y,0))\, dy - t^{-1} \int_0^t \int_{B_\eta} (\tilde{u} \cdot \nabla \psi) H(\tilde{G})\, dy dt. \tag{10}$$

We apply the co-area formula (8) in the expression on the left-hand side of (10):

$$t^{-1} \int_0^t \int_{B_\eta} \psi(y)\phi(G)|\nabla G|\, dy dt = \int_{-\infty}^{\infty} \phi(g) \left(t^{-1} \int_0^t a_\psi(g,s)\, ds \right) dg,$$

where

$$a_\psi(g,s) = \int_{\{y;\tilde{G}(y,s)=g\}} \psi(y) dH^{(2)}(y).$$

Therefore, from (10) we obtain

$$\int_{-\infty}^{\infty} \phi(g) \left(t^{-1} \int_0^t a_\psi(g,s)\, ds \right) dg = \frac{1}{v_0 t} \int_{B_\eta} (h_\psi(y,t) - h_\psi(y,0))\, dy - \frac{1}{v_0 t} \int_0^t \int_{B_\eta} (\tilde{u} \cdot \nabla\psi) H(\tilde{G})\, dydt. \quad (11)$$

Now let $\psi_0(y)$ be a non-negative smooth function supported in the ball of radius two, which is identically equal to 1 in $\{y; |y| \le 1\}$, and vanishes for $|y| \ge 2$, and let

$$\psi(y) = \psi_0\left(\frac{y}{r}\right), \quad (12)$$

for $r \le \frac{\eta}{2}$. Such choice of ψ allows us to study the level sets of $\tilde{G}(y,t)$ near $y=0$, which is the same as the level sets of $G(x,t)$ in the neighborhood of the Lagrangian trajectory $X_0(t)$ in the original variables. Note that we have with $C > 0$

$$0 \le \psi(y) \le 1, \quad |\nabla\psi(y)| \le Cr^{-1}, \quad |\Delta\psi(y)| \le Cr^{-2}. \quad (13)$$

Given these observations we obtain from (11)

$$\int_{-\infty}^{\infty} \phi(g) \left(t^{-1} \int_0^t a_r(g,s)\, ds \right) dg \le \frac{1}{v_0 t} \int_{B_{2r}} |h_\psi(y,t) - h_\psi(y,0)|\, dy + \frac{C}{v_0 r} \int_{B_{2r}} |\tilde{u}(y,t)|_t\, dy, \quad (14)$$

where B_{2r} is a ball of radius $2r$ around $y=0$,

$$|\tilde{u}(y,t)|_t = t^{-1} \int_0^t |\tilde{u}(y,s)|\, ds,$$

and

$$a_r(g,s) = \int_{\{y; |y| \le r, \tilde{G}(y,s) = g\}} dH^{(2)}(y).$$

Let us denote

$$A_{r,t}(g) = t^{-1} \int_0^t a_r(g,s)\, ds \quad (15)$$

the time average of the areas of the g level recorded in the support of of the ball B_r with center at $y=0$. Since h_ψ is uniformly bounded, and the inequality (14) is true for any ϕ we conclude that

$$A_{r,t}(g) \le \pi r^2 \left\{ C_1 \frac{r}{v_0 t} + C_2 \frac{1}{v_0} \langle |\tilde{u}| \rangle_{2r,t} \right\} \quad (16)$$

holds for any g. Here $\langle |\tilde{u}| \rangle_{2r,t}$ is the average of the velocity $\tilde{u}(y,s)$ over a ball of radius $2r$ around $y=0$ and time interval $[0,t]$:

$$\langle |\tilde{u}| \rangle_{r,t} = \frac{3}{4\pi t r^3} \int_0^t \int_{B_r} |(\tilde{u}(y,s)| dx ds.$$

Assume that the original turbulent velocity $u(x,t)$ scales with scaling exponent ζ:

$$\frac{1}{|B_r|} \int_{B_r} |u(x_1 + x, s) - u(x_1, s)| dx ds \approx u_0 \left(\frac{r}{L}\right)^\zeta, \quad (17)$$

for any point x_1 with u_0 a typical turbulent fluctuation velocity and L a turbulence length. Then $\tilde{u}$ has the same scaling:

$$\langle|\tilde{u}|\rangle_{2r,t} \approx u_0 \left(\frac{r}{L}\right)^{\zeta}.$$

Given that, (16) becomes

$$A_{r,t}(g) \le \pi r^2 \left\{ C_1 \frac{r}{v_0 t} + C_2 \frac{u_0}{v_0} \left[\frac{r}{L}\right]^{\zeta} \right\}. \tag{18}$$

The Gibson scale L_G is defined by

$$L_G = L \left(\frac{v_0}{u_0}\right)^{\frac{1}{\zeta}}$$

that is, by

$$1 = \left(\frac{L_G}{L}\right)^{\zeta} \frac{u_0}{v_0}.$$

The inequality (18) can be written as

$$A_{r,t}(g) \le \pi r^2 \left[C_1 \frac{r}{v_0 t} + C_2 \left(\frac{r}{L_G}\right)^{\zeta} \right].$$

This is our main result concerning the G equation. We summarize it in terms of the original variable x as follows:

Theorem 1 *Let $X_0(t)$ be the Lagrangian trajectory (5) of a point x_0. There exist constants C_1, C_2 such that, for any $\eta > r > 0$ and any $t > 0$ the time average on the time interval $[0,t]$ of the area of the portion of the level set $\{x|G(x,t) = g\}$ situated inside the Lagrangian ball $B(X_0(t), r)$ satisfies*

$$\frac{1}{t}\int_0^t dH^{(2)}\left(\{x|G(x,s) = g\} \cap B(X_0(s), r)\, ds\right) \le \pi r^2 \left[C_1 \frac{r}{v_0 t} + C_2 \left(\frac{r}{L_G}\right)^{\zeta} \right].$$

If

$$\frac{r}{v_0 t} \le 1$$

and the radius r exceeds the Gibson length

$$r \ge L_G$$

then the second term is dominant. This result may be interpreted as the box counting "dimension" of the level sets of $G(x,t)$ measured with balls of radii larger than the Gibson length on the average being (less or equal to) $d = 2+\zeta$. This result is in agreement with [10] for the Kolmogorov turbulence $\zeta = 1/3$. There the fractal dimension of the level sets $d = 7/3$ was predicted for scales above the Gibson scale from physical considerations.

3 The turbulent reactive model

We consider the full turbulent reaction diffusion equation (2):

$$\begin{aligned} &\frac{\partial T}{\partial t} + u \cdot \nabla T - \kappa \Delta T = \frac{v_0^2}{4\kappa} f(T) \\ &T(x,0) = T_0(x). \end{aligned} \tag{19}$$

We do not specify any boundary conditions on (19). We assume instead that maximum principle applies to (19), so that $T(t,x)$ satisfies $0 \le T(t,x) \le M = \sup_{x\in D} T_0(x) \le 1$. This is true for the usual Dirichlet or periodic boundary conditions, but we would like to allow possible application to certain astrophysical problems of flame propagation in stars, where the correct boundary conditions might be different. We are interested again in the behavior of level sets for small but non-zero molecular diffusivity $\kappa > 0$. We shall consider two possible ways to track the front. First, in the next subsection we move with the point x_0 as we did in the previous section and consider an average area of a level set inside the ball $B(X_0(t), r)$. Second, in Section 3.2 we consider the frame that is moving with the uniform speed $u(x_0, 0)$ that the point x_0 had initially and consider the level sets inside the ball $B(x_0 + u(x_0)t, t)$. We obtain different estimates for these two cases.

3.1 The area of the level sets in a Lagrangian ball

Let $X_0(t)$ be the trajectory (5) and let the co-moving variable y be defined by (4) as previously. Then the function $\tilde{T}(y,t) = T(y + X_0(t), t)$ satisfies

$$\frac{\partial \tilde{T}}{\partial t} + \tilde{u} \cdot \nabla \tilde{T} - \kappa \Delta \tilde{T} = \frac{v_0^2}{4\kappa} f(\tilde{T}) \tag{20}$$

with $\tilde{u}(y,t)$ defined by (7). The analysis follows along similar lines to the one for the G equation, but the spatial second order derivatives and the reaction nonlinearity make it technically more involved. Following [5] we let the function $H(g)$ be defined now by

$$H(g) = \int_0^g (g - \lambda)\phi^2(\lambda) d\lambda \tag{21}$$

where the test function $\phi(\lambda)$ is normalized in L^2

$$\|\phi\|^2 = \int_0^\infty \phi^2(\lambda) d\lambda = 1. \tag{22}$$

We note that in this case

$$0 \le H(g) \le g$$

and

$$H'(g) = \int_0^g \phi^2(\lambda) d\lambda, \tag{23}$$

so that

$$0 \le H'(g) \le 1.$$

Moreover, we have

$$H''(g) = \phi^2(g). \tag{24}$$

We introduce the auxiliary function $h_\psi(y)$:

$$h_\psi(y,t) = \psi(y) H(\tilde{T}(y,t))$$

with the localizer $\psi(y)$ defined as in the previous section by (12): $\psi(y) = \psi_0(y/r)$. The function h_ψ satisfies the equation

$$(\partial_t + \tilde{u} \cdot \nabla_y - \kappa \Delta_y) h_\psi + \kappa \psi H''(\tilde{T}) |\nabla_y \tilde{T}|^2 = R, \tag{25}$$

where

$$R = H(\tilde{T})(\tilde{u} \cdot \nabla_y \psi - \kappa \Delta_y \psi) - 2\kappa \nabla_y \psi \cdot \nabla_y H + \frac{v_0^2}{4\kappa} H'(\tilde{T}) f(\tilde{T}).$$

We integrate both sides of (25) over the ball B_{2r} and using the incompressibility of u we obtain

$$\frac{d}{dt}\int_{B_{2r}} h_\psi(t,y)dy + \kappa\int_{B_{2r}} \psi(y)H''(\tilde{T})|\nabla\tilde{T}|^2dy = \mathcal{R}(t), \tag{26}$$

where

$$\mathcal{R}(t) = \int_{B_{2r}} H(\tilde{T})(\tilde{u}\cdot\nabla_y\psi + \kappa\Delta_y\psi)dy + \frac{v_0^2}{4\kappa}\int_{B_{2r}} \psi(y)H'(\tilde{T})f(\tilde{T})dy.$$

Integrating (26) in time and using (24) we obtain

$$\int_0^t\int_{B_{2r}} \psi(y)\phi^2(\tilde{T}(y,t))|\nabla\tilde{T}|^2dyds = \frac{1}{\kappa}E(t) \tag{27}$$

with

$$E(t) = \int_0^t \mathcal{R}(s)ds + \int_{B_{2r}} [h_\psi(y,0) - h_\psi(y,t)]dy.$$

Applying the Cauchy-Schwartz inequality to the left side of (27) we obtain after a brief calculation

$$\int_0^M A_{r,t}^2(g)dg \le \frac{1}{\kappa t}\sup_{||\phi||=1} E(t)\int_{B_{2r}} \psi(y)dy. \tag{28}$$

Here the function $A_{r,t}(g)$ is defined by (15) and represents as before the time average of the area of the g-level set of the temperature weighed by ψ. We split $E(t)$ into three parts:

$$E(t) = I + II + III. \tag{29}$$

The three terms on the right side of (29) are estimated as follows. The function $H(\beta)$ satisfies the estimate

$$|H(\alpha) - H(\beta)| \le |\alpha - \beta| \tag{30}$$

that follows immediately from the definition (21) and (22). Then we have

$$I = \int_{B_{2r}} [h_\psi(y,0) - h_\psi(y,t)]\,dy \le \int_{B_{2r}} \psi(y)|H(\tilde{T}(y,0)) - H(\tilde{T}(y,t))|dy.$$

Let us define

$$\sigma' = \sup_t |\tilde{T}(0,0) - \tilde{T}(0,t)|,\quad \tau = \sup_{0\le s\le t}\sup_{y_1,y_2\in B(2r)} |\tilde{T}(y_1,s) - \tilde{T}(y_2,s)|, \tag{31}$$

then we get using (28), (30) and the triangle inequality

$$I \le 2Cr^3[2\tau + \sigma'].$$

The second term in (29) is given by

$$\begin{aligned} II = &\int_0^t\int_{B_{2r}} H(\tilde{T}(y,s))(\tilde{u}\cdot\nabla_y\psi + \kappa\Delta_y\psi)dyds \\ &= \int_0^t\int_{B_{2r}} (H(\tilde{T}(y,s)) - H(\tilde{T}(0,s)))\,[\tilde{u}\cdot\nabla_y\psi + \kappa\Delta_y\psi]\,dyds. \end{aligned}$$

It is estimated using the bounds (13) and (30), and (31):

$$\begin{aligned} II \quad &\le \int_0^t\int_{B_{2r}} |\tilde{T}(y,s) - \tilde{T}(0,s)|\left\{|\tilde{u}(y,s)|\frac{C}{r} + \frac{\kappa C}{r^2}\right\}dyds \\ &\le \tau Cr^2t\langle|\tilde{u}|\rangle_{2r,t} + \kappa Cr\tau t. \end{aligned}$$

The last term in (29) that is due to the chemical reaction is bounded using (23) by

$$III = \frac{v_0^2}{4\kappa}\int_0^t \int_{B_{2r}} H'(\tilde{T}(y,s))f(\tilde{T}(y,s))dyds \le \tau C\frac{v_0^2 r^3 t\sigma''}{\kappa}$$

with

$$\sigma'' = \frac{1}{\tau}\sup_{y\in B_{2r}, 0\le s\le t} |f(\tilde{T}(y,s))|.$$

Because of the fact that the gradient of T appears squared in the basic balance equation due to the presence of the Laplacian, we need to use the L^2 norm of ϕ (as opposed to the L^1 norm for the G equation). Consequently we must content ourselves with an estimate of the mean square area of $A_{r,t}(g)$

$$\langle A\rangle_{r,t}(x_0) = \left(\frac{1}{\tau}\int_0^M A_{r,t}^2(g)dg\right)^{1/2}.$$

This represents the average area of level set inside the ball $B(X_0(s),r)$. We are dividing by τ and not M here, since the variation of values of $T(x,s)$ near x_0 is bounded by τ.

We summarize our result in the following theorem.

Theorem 2 *Let $X_0(t)$ be the Lagrangian trajectory (5) of a point x_0. There exist constants C_1, C_2 such that, for any $\eta > r > 0$ and any $t > 0$ the time average on the time interval $[0,t]$ of the area of the portion of the level set $\{x|T(x,s) = g\}$ situated inside the Lagrangian ball $B(X_0(s),r)$ satisfies*

$$\langle A\rangle_{r,t}(x_0) \le \pi r^2\left[\frac{C_1 r}{\kappa^{1/2}t^{1/2}}(2+\frac{\sigma'}{\tau}) + \frac{C_2 r^{1/2}u_0^{1/2}}{\kappa^{1/2}}\left(\frac{r}{L}\right)^{\zeta/2} + C_3 + \frac{C_4 v_0\sigma'' r}{\kappa}\right]. \tag{32}$$

The first term in (32) is due to time derivative in (19). In principle, it may be made small by choosing the time t sufficiently large, so that it would not be dominant at any scale. The third term is due to the diffusive part of the turbulent reaction-diffusion equation (19). The second term is due to turbulent advection, and the last one is related to the reaction term in (19). The diffusive contribution r^2 is the largest one as $r \to 0$ with the other parameters being fixed. We are interested in those scales where the effect of the turbulent velocity on the dimension of the level sets is dominant. To find out what these scales are, we need to pick the averaging time in (32). There is a danger that if we choose this time too large, the burning front will move away, and the averaging will be done mostly over the burnt region. This would make our estimate much less precise, since the actual difference of temperatures inside the ball will be significantly smaller than our estimate for it, τ, for most of the averaging time. We choose the averaging time t to be the chemical reaction time

$$t_c = \frac{\kappa}{v_0^2},$$

since this is the smallest time scale in the problem in the regime that we are interested in. Moreover, in Section 4 we provide an estimate for the effective turbulent front velocity u_T which together with the Kerstein's [10] estimate for the size of burning region, δ_T, justifies our choice: it turns out that $t_c = \delta_T/u_T$.

With our choice of t, the first term in (32) is of the same order as the last term that is related to the chemical reaction. We must have then

$$r^2 < \frac{r^{5/2}u_0^{1/2}}{\kappa^{1/2}}\left(\frac{r}{L}\right)^{\zeta/2} \tag{33}$$

and

$$\frac{r^3 v_0}{\kappa} < \frac{r^{5/2} u_0^{1/2}}{\kappa^{1/2}} \left(\frac{r}{L}\right)^{\zeta/2} \tag{34}$$

for the turbulent contribution to be dominant. Let us recall that the Reynolds number is defined as $\mathrm{Re} = \dfrac{u_0 L}{\nu}$, the Schmidt number as $\mathrm{Sc} = \dfrac{\nu}{\kappa}$ and the Karlovitz number as

$$\mathrm{Ka} = \frac{t_c}{t_k} = \left(\frac{\kappa}{v_0^2}\right) \frac{u_0}{L} \mathrm{Re}^{\frac{1}{2}}.$$

Here ν is the kinematic viscosity, and $t_k = \dfrac{L}{u_0}\mathrm{Re}^{-1/2}$ is the small-eddy cutoff time. Then the lower bound (33) on the ball radius r means that

$$\left(\frac{r}{L}\right) \geq (\mathrm{ScRe})^{-\frac{1}{1+\zeta}}. \tag{35}$$

This is the Kolmogorov cut-off scale $r_1 = L\mathrm{Re}^{-3/4}$ for $\zeta = 1/3$ and the Schmidt number being of order one. The upper bound (34) leads to

$$\frac{r}{L} \leq (\mathrm{KaRe}^{-\frac{1}{2}})^{\frac{1}{1-\zeta}}. \tag{36}$$

In order for conditions (36) and (35) to be compatible, we should have

$$\mathrm{Ka} \geq \mathrm{Sc}^{-\frac{1-\zeta}{1+\zeta}} \mathrm{Re}^{\frac{3\zeta-1}{2(1+\zeta)}}. \tag{37}$$

We are interested in the regime when the turbulence is strong and the chemical reaction is fast. Then the Reynolds number is large and Karlovitz number is small. The Schmidt number may be considered of order of unity or large. The restriction (37) implies that turbulence plays an important role on a large range of scales defined by (36) and (35) only if $\zeta \leq 1/3$, so that the turbulence is sufficiently rough, with the scaling exponent not larger than the Kolmogorov exponent $\zeta = 1/3$. When $\zeta > 1/3$ so that turbulence is less rough, then (37) provides a bound on the strength of the chemical reaction so that turbulent effects may still dominate. The fractal dimension of the level sets predicted by (32) for these scales is $d = \dfrac{5}{2} + \dfrac{\zeta}{2}$. The prediction $d = 2 + \zeta$ obtained above for the G equation agrees with this estimate only in the case $\zeta = 1$, that is only for smooth velocities. Thus, the structure of the (averaged) level sets in the cases of G-equation and full reaction-diffusion-convection equation is significantly different. In addition, the full equation produces a specific interval of scales where turbulence contribution is most important.

3.2 Level sets in a ball moving with a uniform speed

Now we consider the average area of the level sets of the function $T(x,t)$ in a ball that is moving with a uniform speed $u(x_0, 0)$ and is centered at the point $x_0 + u(x_0, 0)t$. This quantity provides the snapshot of level sets in the inertial coordinate system, rather than in one following a given particle. Accordingly we make a change of variables $y = x - x_0 - u(x_0, 0)t$, so that $\tilde{T}(y,t) = T(y + x_0 + u(x_0, 0)t, t)$ satisfies equation (20) with the velocity $\tilde{u}(y,t) = u(y + x_0 + u(x_0, 0)t, t) - u(x_0, 0)$. Then the analysis of the previous section has to be modified only in the estimate for the term II_1 in (29) given by

$$II_1 = \int_0^t \int_{B_{2r}} (H(\tilde{T}(y,s)) - H(\tilde{T}(0,s)))[\tilde{u} \cdot \nabla\psi] dy ds.$$

The other terms I, III and the second part of II obey exactly the same estimates as in the previous section. We now have for II_1:

$$
\begin{aligned}
II_1 \quad & \leq \frac{\tau C}{r}\int_0^t\int_{B_{2r}} |u(y+u(x_0,0)s+x_0,s)-u(x_0,0)|dyds \\
& \leq \frac{\tau C}{r}\int_0^t\int_{B_{2r}} (|u(x_0+y+u(x_0,0)s,s)-u(x_0+u(x_0,0)s,s)| \\
& +|u(x_0+u(x_0,0)s,s)-u(x_0,s)|+|u(x_0,s)-u(x_0,0)|)dyds = II_{11}+II_{12}+II_{13}.
\end{aligned}
\tag{38}
$$

The first term may be estimated using the scaling relation (17) so that

$$
II_{11} \leq Cr^2\tau t u_0\left(\frac{r}{L}\right)^{\zeta}.
$$

The term II_{12} satisfies a similar estimate if we assume that (17) holds also with the integral over the ball B_r replaced by the interval $[x_0, x_0+r]$. We have then

$$
II_{12} \leq Cr^2\tau t u_0\left(\frac{u(x_0,0)t}{L}\right)^{\zeta} \approx Cr^2\tau t u_0\left(\frac{u_0 t}{L}\right)^{\zeta}.
$$

In order to estimate the last term in (38) we have to make an additional assumption regarding the scaling of the time correlations of the velocity $u(x,t)$:

$$
\frac{1}{t}\int |u(x,s)-u(x,0)|ds \approx u_0\left(\frac{t}{t_0}\right)^{\zeta_t}.
$$

Then we have for II_{13}:

$$
II_{13} \leq Cr^2\tau t u_0\left(\frac{t}{t_0}\right)^{\zeta_t}.
$$

Then we have the following theorem.

Theorem 3 *Let $B_r(t)$ be the translation of the ball $B(x_0,r)$ around a point x_0 by distance $u(x_0,0)t$. There exist constants C_1, C_2 such that, for any $\eta > r > 0$ and any $t > 0$ the time average on the time interval $[0,t]$ of the area of the portion of the level set $\{x|T(x,s)=g\}$ situated inside the ball $B_r(s)$ satisfies*

$$
\langle A\rangle_{r,t}(x_0) \leq \pi r^2\left[\frac{C_1 r}{\kappa^{1/2}t^{1/2}}(2+\frac{\sigma'}{\tau})+\frac{C_2 r^{1/2}u_0^{1/2}}{\kappa^{1/2}}\left[\left(\frac{r}{L}\right)^{\frac{\zeta}{2}}+\left(\frac{u_0 t}{L}\right)^{\frac{\zeta}{2}}+\left(\frac{t}{t_0}\right)^{\frac{\zeta_t}{2}}\right]+C_3+\frac{C_4 v_0\sigma'' r}{\kappa}\right].
\tag{39}
$$

Let us assume for simplicity that $\zeta_t = \zeta$ and, moreover, $L \approx u_0 t_0$, so that the two terms inside the bracket in (39) are of the same order. This is not necessary but is physically reasonable and simplifies the subsequent analysis. Then in order to balance the terms in (39) that have time in the numerator and denominator we choose

$$
t = \frac{L}{u_0}\left(\frac{r}{L}\right)^{1/(\zeta+1)},
$$

and (39) becomes then

$$
\langle A\rangle_{r,t}(x_0) \leq C\left[\frac{r^{5/2}u_0^{1/2}}{\kappa^{1/2}}\left[\left(\frac{r}{L}\right)^{\zeta/2}+\left(\frac{r}{L}\right)^{\zeta/2(\zeta+1)}\right]+r^2+\frac{v_0\sigma'' r^3}{\kappa}\right].
$$

Note that for $r/L \leq 1$ the second term inside the bracket is now larger than the first so we finally get

$$\langle A\rangle_{r,t}(x_0) \leq C\left[\frac{r^{5/2}u_0^{1/2}}{\kappa^{1/2}}\left(\frac{r}{L}\right)^{\zeta/2(\zeta+1)} + r^2 + \frac{v_0\sigma'' r^3}{\kappa}\right].$$

Now the turbulent term is dominant provided that

$$(\mathrm{ScRe})^{-\frac{1+\zeta}{1+2\zeta}} \leq \frac{r}{L} \leq \left(\mathrm{KaRe}^{-1/2}\right)^{\zeta+1}.$$

This range of scales is non-empty provided that

$$\mathrm{Ka} \geq \mathrm{Sc}^{-\frac{1}{2\zeta+1}}\mathrm{Re}^{\frac{2\zeta-1}{4\zeta+2}}.$$

This range includes the strong chemical reaction $\mathrm{Ka} \leq 1$ and strong turbulence $\mathrm{Re} \gg 1$ provided that $\zeta \leq 1/2$. This range of the exponents is wider than the corresponding range for the level sets inside the Lagrangian balls considered in the previous section. Observe that the range of scales obtained in the present section is larger than those for the Lagrangian balls with the lower cut-off being of order $r_1 = L(\mathrm{Re\ Sc})^{-\frac{1+\zeta}{1+2\zeta}}$ smaller than that in (35), and the upper cut-off scale $r_2 = L\left(\mathrm{KaRe}^{-1/2}\right)^{\zeta+1}$ being larger than that in (36). The cut-off scales in the two cases coincide for $\zeta = 0$, which corresponds to very rough turbulence. The exponent $\frac{\zeta}{2(\zeta+1)}$ is smaller than $\frac{\zeta}{2}$ which we obtain for Lagrangian balls. Which one is closer to the experimentally observable quantities may depend on the type of the turbulence and nature of measurements. We remark that a similar alternative estimate for area of the level sets inside uniformly moving balls may also be applied in the case of $G-$equation.

4 The structure of the level sets and the front speed

We relate here the estimate (32) for the average area of a level set in the Lagrangian ball to the average speed of the burning front. First we recall following [10] how the area of the front is related to its effective speed in a mean zero turbulent velocity field. Notice that in the strong turbulence regime the flame surface may be so convoluted, that a streamwise ray in the mean propagation direction may intersect the surface many times. The average distance between last and first such intersection, δ_T, is adopted as a measure of thickness of the turbulent flame brush. The width δ_T may be expressed as

$$\delta_T = \frac{u_0 L}{u_T}, \tag{40}$$

where u_T is the turbulent burning velocity and L is the integral turbulent scale. The effective front speed is given then by

$$u_T = v_0 A'\delta_T,$$

where A' is the surface area of the front per unit volume. This together with (40) implies that the speed of the front is given by

$$u_T = (u_0 v_0 A' L)^{1/2}. \tag{41}$$

Note that the thickness δ_T is much wider than the local width of the front, so A' corresponds to the large scale averaging. It may be estimated from (32) on the large scales when the reaction term proportional to r^3 is dominant:

$$A' \leq C\frac{v_0}{\kappa} = C\mathrm{Ka}^{-1/2}\mathrm{Re}^{3/4}\mathrm{Sc}^{1/2}L^{-1}.$$

Substituting this expression into the Kerstein's formula (41) we obtain

$$u_T \leq C u_0 \mathrm{Ka}^{-\frac{1}{2}} \mathrm{Re}^{\frac{1}{4}},$$

or

$$\frac{u_T}{v_0} \leq C \left(\frac{u_0}{v_0}\right) \mathrm{Ka}^{-\frac{1}{2}} \mathrm{Re}^{\frac{1}{4}} = C(\mathrm{ReSc})^{\frac{1}{2}}. \tag{42}$$

We notice that dependence of u_T on u_0 with other parameters fixed exhibits square root behavior, which is different from most of the existing theoretical results (see [16] for the review) which predict linear or linear up to a logarithmic correction behavior. The square root dependence was predicted in [11] from other considerations. Notice that if we keep the Reynolds number fixed, (42) provides an upper bound for u_T/v_0. It was observed in the experiments [2, 16] that u_T/v_0 stays bounded as u_0/v_0 grows at a fixed Reynolds number, which qualitatively agrees with the prediction of (42).

Finally we remark that our estimate is sufficiently robust not to depend on whether we use the bound (32) or (39) for the area of the level sets. The result would remain unchanged since the coefficient in front of r^3 that determines A' is the same in both estimates.

References

[1] M.S.Anand and S.B.Pope, Combust. Flame **67**, 1987, 127.

[2] D. Bradley, Twenty-Fourth Symposium (International) on Combustion, Combustion Institute, 1992, 247-262.

[3] K.N.C. Bray, Proc. Roy. Soc. (London) A, **431**, 1990, 315.

[4] P.Clavin and F.A.Williams, Theory of pre-mixed flame propagation in large-scale turbulence, Jour. Fluid. Mech., **90**, 1979, 589-604.

[5] P. Constantin, Navier-Stokes equations and area of interfaces, Comm. Math. Phys., **129**, 1990, 241-266.

[6] P. Constantin and I. Procaccia, The geometry of turbulent advection: sharp estimates for the dimensions of level sets, Nonlinearity, **7**, 1994, 1045-1054.

[7] L.C. Evans and R.F. Gariepy, Measure theory and fine properties of functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[8] H. Federer, Geometric measure theory, Berlin, Heidelberg, New York, Springer, 1969

[9] A. Kerstein, Simple derivation of Yakhot's turbulent premixed flamespeed formula, Combust. Sci. and Tech., **60**, 1988, 163-165.

[10] A. Kerstein, Fractal dimension of turbulent premixed flames, Combust. Sci. and Tech., **60**, 1988, 441-445.

[11] A. Kerstein, Pair-exchange model of turbulent premixed flame propagation, Twenty-first Symposium (International) on Combustion, Combustion Institute, 1986, 1281-1289.

[12] A. Majda and P. Souganidis, Large scale front dynamics for turbulent reaction-diffusion equations with separated velocity scales, Nonlinearity, **7**, 1994, 1-30.

[13] G.H. Markstein, Nonsteady flame propagation, Macmillan, 1964.

[14] R. McLaughlin and J. Zhao, The effect of finite front thickness on the enhanced speed of propagation, to appear in Combustion Sci. and Tech., 1998.

[15] A. Pocheau, Phys. Rev. E, **49**, 1994, 1109.

[16] P. Ronney, Some open issues in premixed turbulent combustion, in Modeling in Combustion Science, J.Buckmuster and T.Takeno, eds., Springer-Verlag, Berlin, 1995.

[17] G. Sivashinsky, in: *Dissipative Structures in Transport Processes and Combustion* (D. Meinköhn, ed.), Springer Series in Synergetics, Vol. 48, Springer-Verlag, Berlin, 1990, 30.

[18] P. Constantin, I. Procaccia, and K.R. Sreenivasan, Phys. Rev. Lett. **67**, 1991, 1739.

[19] V. Yakhot, Propagation velocity of premixed turbulent flames, Combust. Sci. and Tech., **60**, 191-214.