Time Reversal for Classical Waves in Random Media

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Abstract. We propose a mathematical theory for the refocusing properties observed in time-reversal experiments, where classical waves propagate through a medium, are recorded in time, then time-reversed and sent back into the medium. The salient feature of such experiments is that the refocusing quality of the time-reversed reemitted signals is greatly enhanced when the underlying medium is heterogeneous. Based on the Wigner transform formalism, we show that random media indeed greatly improve refocusing. We analyze two different types of random media, where in the limit of high frequencies, the Wigner transform satisfies a random Liouville equation or a linear transport equation.

Renversement du temps pour la propagation d'ondes classiques en milieu aléatoire

Résumé. Dans des expériences de renversement temporel, où un signal acoustique qui se propage dans un milieu sous-jacent, est enregistré en temps, puis subit un renversement temporel et est réémis dans le milieu, il a été observé que le signal refocalise d'autant mieux à l'emplacement de la source initiale que le milieu est hétérogène. Nous donnons une explication mathématique de ce phénomène fondée sur le formalisme de la transformée de Wigner. Nous obtenons deux exemples de refocalisation, en fonction du type de milieu aléatoire considéré, selon que la transformée de Wigner, à la limite des hautes fréquences, satisfait soit une équation de Liouville avec coefficients aléatoires, ou une équation de transport.

Version française abrégée

Dans cette note, nous nous intéressons à la propagation d'ondes classiques en milieu aléatoire et à leurs propriétés de refocalisation après renversement temporel. Dans les expériences conduites par M. Fink [6] et ses collaborateurs, une source localisée émet des ondes acoustiques à travers un milieu hétérogène. Le signal est enregistré en temps par des capteurs localisés en espace et positionnés loin de la source, puis renversé en temps et enfin retransmis dans le milieu (ce qui est arrivé en dernier repart en premier).

Dans ces expériences, il a été observé le phénomène suivant, qui à première vue pourrait paraître contre-intuitif : le signal réémis refocalise à l'emplacement de la source initiale d'autant mieux que le milieu est hétérogène, alors que la qualité de la refocalisation est très mauvaise en milieu homogène. L'explication réside dans la multiplicité des "chemins" menant de la source aux capteurs. Plus le milieu est aléatoire, plus il donne d'informations sur la source aux capteurs, pourvu que les enregistrements soient suffisamment longs.

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Plusieurs travaux sur ce phénomène existent dans la littérature physique [4, 6, 7]. Les résultats mathématiques existants portent sur le cas mono-dimensionnel [3] et le régime de faisceaux étroits [2]. Dans cette note, nous proposons une théorie qui explique le phénomène de refocalisation pour des ondes classiques à hautes fréquences en milieu multi-dimensionnel. Notre approche, s'inspirant de [2], se fonde sur la transformée de Wigner, définie en (4), qui permet l'analyse des corrélations de noyaux de Green en des points voisins. Nous pouvons écrire le signal retransmis après renversement du temps en fonction de cette transformée de Wigner (5).

Selon les caractéristiques du milieu aléatoire, la transformée de Wigner satisfait différentes équations limites quand la fréquence caractéristique des ondes tend vers l'infini. Nous analysons les deux cas où nous obtenons à la limite soit une équation de Liouville (limite semiclassique), soit une équation de transport de type Boltzmann (limite de couplage faible). Dans le premier cas, nous sommes capables d'obtenir une dérivation rigoureuse mathématiquement. La formule de refocalisation est donnée en (8)-(12). Dans le second cas, la dérivation de la formule de reconstruction (8)-(14) est formelle en certains endroits. La théorie permettant d'obtenir l'équation de transport limite n'est complète que pour l'équation (scalaire) de Schrödinger [5, 14]. De plus elle nécessite de moyenner sur les réalisations du milieu aléatoire, bien que les expériences physiques montrent que ce soit superflu.

1. Refocusing in time-reversal

Time-reversal experiments [6] can be briefly described as follows. A point source signal is sent through an inhomogeneous medium and is recorded by a spatially localized array of receivers-transducers. The signal is subsequently reversed in time and reemitted back into the medium; that is, the part of the signal that was recorded last is sent back first. The refocusing of the reemitted signal is then observed on the spot of the original source.

Refocusing in a homogeneous medium is poor when only a few receivers are used to record the signal. The most striking and somewhat counter-intuitive observation is that inhomogeneities in the medium *enhance* the refocusing effects. This is because waves recorded at the time-reversal array have sampled a larger part of the medium than in the homogeneous case and carry more information about the *source* location. Furthermore, the refocused signal in a random medium is self-averaging, that is, independent of the realization of the random medium.

Many time-reversal experiments have been conducted in the recent past and studied theoretically [4, 6, 7]. Mathematical studies have concentrated so far on the one space dimension case [3], the narrow beam approximation [2, 12] and time-reversal in a stochastic cavity [1]. In this note we present an analysis of the enhanced refocusing property in genuinely three dimensional heterogeneous media in the regimes of random geometrical acoustics and radiative transport. Our main tool is the asymptotic analysis of the equation for the Wigner transform [8, 11]. It turns out that in the high frequency limit the refocused signal may be written as a convolution with a kernel F expressible in terms of the limit Wigner distribution. More precisely, this kernel is expressed (9) in terms of the Fourier transform in $\mathbf{k}$ of the energy densities $a(t, \mathbf{x}, \mathbf{k})$ in the phase space (6). The latter are smooth in $\mathbf{k}$ under evolution in random media but not in a homogeneous domain. Therefore the spatial localization of the kernel F is improved in a random medium compared to the homogeneous case. In the *geometric acoustics* limit, the Wigner transform asymptotically satisfies a Liouville or a Fokker-Plank equation similar to the one used to describe the refocusing of narrow beam signals [2, 12]. In the *radiative transfer* regime, the Wigner distribution satisfies a linear Boltzmann equation. The rigorous passage to this limit is not complete for classical waves [5, 9, 13, 14] and our results in this part remain formal.

2. Reemitted signal and the Wigner transform

The propagation of many classical waves is described by systems of m first-order differential equations of the form (repeated indices are summed over throughout the paper)

$$A(\mathbf{x})\frac{\partial \mathbf{u}(t,\mathbf{x})}{\partial t} + D^j \frac{\partial \mathbf{u}(t,\mathbf{x})}{\partial x_j} = 0, \quad \mathbf{x} \in \mathbb{R}^d, d = 2 \text{ or } 3 \tag{1}$$

with initial conditions $\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x})$. In this paper, we address time-reversal for acoustic waves with $d = 3$ to fix notation. The corresponding theory for other types of waves will be presented elsewhere. In this context, $\mathbf{u}$ is the 4-vector composed of the three components of the velocity field $\mathbf{v}$ and the scalar pressure field p. The matrix $A(\mathbf{x}) = \mathrm{Diag}(\rho(\mathbf{x}), \rho(\mathbf{x}), \rho(\mathbf{x}), \kappa(\mathbf{x}))$, where $\rho(\mathbf{x})$ is the density of the underlying medium and $\kappa(\mathbf{x})$ its compressibility. The 4×4 matrices D^j have entries $D^j_{ml} = \delta_{m4}\delta_{lj} + \delta_{l4}\delta_{mj}$. The solution of (1) is $\mathbf{u}(t,\mathbf{x}) = \int_{\mathbb{R}^d} G(t,\mathbf{x};\mathbf{y})\mathbf{u}_0(\mathbf{y})d\mathbf{y}$. Here $G(t,\mathbf{x};\mathbf{y})$ is the matrix-valued Green's propagator from $\mathbf{y}$ to $\mathbf{x}$. It solves (1) with the initial condition $G(0,\mathbf{x};\mathbf{y}) = I\delta(\mathbf{x}-\mathbf{y})$ where I is the 4×4 identity matrix. Our time reversal setting is as follows. At time t the wave field is truncated in a region Ω, where the transducers-receivers are located. Then the direction of the acoustic velocity field is reversed and the signal is re-emitted into the medium. The back-propagated signal emanating from the source term $\mathbf{u}_0$ at time 0 can be written using the Green's propagator as

$$\mathbf{u}^B(\mathbf{x}) = \int_{\mathbb{R}^{3d}} \Gamma G(t,\mathbf{x};\mathbf{y})\Gamma G(t,\mathbf{y}';\mathbf{z})\chi_\Omega(\mathbf{y})\chi_\Omega(\mathbf{y}')f(\mathbf{y}-\mathbf{y}')\mathbf{u}_0(\mathbf{z})d\mathbf{y}d\mathbf{y}'d\mathbf{z}. \tag{2}$$

Here Γ is a matrix that models the time reversal process. It is given by $\Gamma = \mathrm{Diag}(-1,-1,-1,1)$, so that the velocity field $\mathbf{v}$ is replaced by $-\mathbf{v}$ and the pressure field p remains unchanged. The function $\chi_\Omega(\mathbf{y})$ may equal 1 on Ω and 0 outside, or could be a more general function modeling amplification of the time-reversed signal by various receivers. We also allow for some blurring of the recorded signal before it is transmitted back. The convolution with a filter f describes this blurring. Introducing the adjoint Green's matrix G_*, solution of

$$\frac{\partial G_*(t,\mathbf{x};\mathbf{y})}{\partial t} + \frac{\partial}{\partial x_j}(G_*(t,\mathbf{x};\mathbf{y}))D^j A^{-1}(\mathbf{x}) = 0, \tag{3}$$

with initial condition $G_*(0,\mathbf{x};\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})A^{-1}(\mathbf{x})$, we observe that $\Gamma G_*(t,\mathbf{x};\mathbf{y})A(\mathbf{x})\Gamma = G(t,\mathbf{y};\mathbf{x})$.

We now rescale our problem. Refocusing is expected to be important in the close vicinity of the source location. We take the initial source as a localized function $\mathbf{u}_0(\mathbf{x}) = \mathbf{S}\left(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon}\right)$ of finite amplitude and rescale the filter accordingly: $\frac{1}{\varepsilon^d}f(\frac{\mathbf{y}-\mathbf{y}'}{\varepsilon})$. Here $\varepsilon \ll 1$ is a small parameter that measures the ratio of the width of the initial pulse to the propagation distance L between the source location and the array of receivers-transducers. An observation point $\mathbf{x}$ is close to $\mathbf{x}_0$ and we write it as $\mathbf{x} = \mathbf{x}_0 + \varepsilon\boldsymbol{\xi}$. After a change of variables, equation (2) becomes

$$\mathbf{u}^B_\varepsilon(\boldsymbol{\xi};\mathbf{x}_0) = \int_{\mathbb{R}^{3d}} \Gamma G(t,\mathbf{x}_0+\varepsilon\boldsymbol{\xi};\mathbf{y})G_*(t,\mathbf{x}_0+\varepsilon\mathbf{z};\mathbf{y}')A(\mathbf{x}_0+\varepsilon\mathbf{z})\Gamma\mathbf{S}(\mathbf{z})\chi_\Omega(\mathbf{y})\chi_\Omega(\mathbf{y}')f(\frac{\mathbf{y}-\mathbf{y}'}{\varepsilon})d\mathbf{y}d\mathbf{y}'d\mathbf{z}.$$

The *Wigner transform* is a natural tool in the analysis of the correlation function of wave fields at neighboring points [8, 11, 13]. In our context, we define it as

$$W_\varepsilon(t,\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^{2d}} \left[\int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{z}} G(t,\mathbf{x}-\frac{\varepsilon\mathbf{z}}{2};\mathbf{y})G_*(t,\mathbf{x}+\frac{\varepsilon\mathbf{z}}{2};\mathbf{y}')\frac{d\mathbf{z}}{(2\pi)^d}\right]\chi_\Omega(\mathbf{y})\chi_\Omega(\mathbf{y}')f(\frac{\mathbf{y}-\mathbf{y}'}{\varepsilon})d\mathbf{y}d\mathbf{y}'. \tag{4}$$

This allows us to recast the expression for the retransmitted signal as follows

$$\mathbf{u}^B_\varepsilon(\boldsymbol{\xi};\mathbf{x}_0) = \int_{\mathbb{R}^{2d}} \Gamma W_\varepsilon(t,\mathbf{x}_0+\varepsilon\frac{\boldsymbol{\xi}+\mathbf{z}}{2},\mathbf{k})e^{-i\mathbf{k}\cdot(\mathbf{z}-\boldsymbol{\xi})}A(\mathbf{x}_0+\varepsilon\mathbf{z})\Gamma\mathbf{S}(\mathbf{z})d\mathbf{z}d\mathbf{k}. \tag{5}$$

Let us define the space $\mathcal{A}(\mathbb{R}^{2d})$ as the subset of $\mathcal{S}'(\mathbb{R}^{2d})$ of matrix-valued distributions $\eta(\mathbf{x},\mathbf{k})$ such that $\int_{\mathbb{R}^d} \sup_{\mathbf{x}} \|\hat{\eta}(\mathbf{x},\mathbf{y})\|d\mathbf{y}$ is bounded. Here $\hat{\eta}(\mathbf{x},\mathbf{y})$ is the inverse Fourier transform of $\eta(\mathbf{x},\mathbf{k})$ in the second variable only. We denote by $\mathcal{A}'$ the dual space to $\mathcal{A}$. Then we have

Lemma 1. *The Wigner transform $W_\varepsilon(t,\mathbf{x},\mathbf{k})$ is bounded in $\mathcal{C}^0(0,T;\mathcal{A}'(\mathbb{R}^{2d}))$ independent of ε provided that $\hat{f}(\mathbf{k}) \in L^1(\mathbb{R}^d)$. As a consequence, it converges weakly along a subsequence $\varepsilon_k \to 0$ to a distribution $W(t,\mathbf{x},\mathbf{k}) \in \mathcal{C}^0(0,T;\mathcal{A}'(\mathbb{R}^{2d}))$.*

The proof of this lemma is obtained by rewriting W_ε as

$$W_\varepsilon(t,\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^d} \hat{f}(\mathbf{q}) \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{z}} \tilde{G}_\varepsilon(t,\mathbf{x}-\frac{\varepsilon\mathbf{z}}{2};\mathbf{q}) \tilde{G}_{\varepsilon *}(t,\mathbf{x}+\frac{\varepsilon\mathbf{z}}{2};\mathbf{q}) \frac{d\mathbf{z}}{(2\pi)^d} d\mathbf{q},$$

where $\tilde{G}_\varepsilon(t,\mathbf{x};\mathbf{q}) = \int_{\mathbb{R}^d} G(t,\mathbf{x};\mathbf{y})\chi_\Omega(\mathbf{y}) e^{-i\mathbf{q}\cdot\mathbf{x}/\varepsilon} d\mathbf{y}$ solves (1) with initial data $\chi_\Omega(\mathbf{x})e^{-i\mathbf{q}\cdot\mathbf{x}/\varepsilon} I$, and $\tilde{G}_{\varepsilon *}(t,\mathbf{x};\mathbf{q}) = \int_{\mathbb{R}^d} G_*(t,\mathbf{x};\mathbf{y})\chi_\Omega(\mathbf{y}) e^{i\mathbf{q}\cdot\mathbf{x}/\varepsilon} d\mathbf{y}$ solves (3) with initial data $\chi_\Omega(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}/\varepsilon} A^{-1}(\mathbf{x})$. The functions $\tilde{G}_\varepsilon$ and $\tilde{G}_{\varepsilon *}$ are uniformly (in ε and $\mathbf{q}$) bounded in L^2, and hence [8, 11] $\int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{z}} \tilde{G}(t,\mathbf{x}-\frac{\varepsilon\mathbf{z}}{2};\mathbf{q}) \tilde{G}_*(t,\mathbf{x}+\frac{\varepsilon\mathbf{z}}{2};\mathbf{q}) \frac{d\mathbf{z}}{(2\pi)^d}$ is uniformly bounded in $\mathcal{C}^0(0,T;\mathcal{A}'(\mathbb{R}^{2d}))$. This implies the result of the Lemma. □

The Wigner distribution at time $t=0$ is given by $W(0,\mathbf{x},\mathbf{k}) = |\chi_\Omega(\mathbf{x})|^2 \hat{f}(\mathbf{k}) A^{-1}(\mathbf{x})$. The dispersion matrix $L(\mathbf{x},\mathbf{k}) = A^{-1}(\mathbf{x}) k_j D^j$ has a double eigenvalue $\omega_0 = 0$ that corresponds to vortical modes, and simple eigenvalues $\omega_{1,2} = \pm c(\mathbf{x})|\mathbf{k}|$, $c(\mathbf{x}) = 1/\sqrt{\rho(\mathbf{x})\kappa(\mathbf{x})}$. As was shown in [8, 13] the limit Wigner distribution may be decomposed as

$$W(t,\mathbf{x},\mathbf{k}) = \sum_{j=1}^{2} a_{ij}^0(t,\mathbf{x},\mathbf{k}) \mathbf{b}_i^0 \mathbf{b}_j^{0*} + a_1(t,\mathbf{x},\mathbf{k}) \mathbf{b}^1 \mathbf{b}^{1*} + a_2(t,\mathbf{x},\mathbf{k}) \mathbf{b}^2 \mathbf{b}^{2*}. \tag{6}$$

Here $\mathbf{b}$ are eigenvectors of the matrix L. If the source $\mathbf{S}(\mathbf{z})$ has no vortical waves, that is, if $\hat{\mathbf{S}}(\mathbf{k})$ is parallel to $\mathbf{k}$, then the limit of the back-propagated signal is

$$\mathbf{u}^B(\boldsymbol{\xi};\mathbf{x}_0) = \sum_{m=1}^{2} \int_{\mathbb{R}^{2d}} a_m(t,\mathbf{x}_0,\mathbf{k}) \Gamma \mathbf{b}^m(\mathbf{x}_0,\mathbf{k}) \mathbf{b}^{m*}(\mathbf{x}_0,\mathbf{k}) e^{-i\mathbf{k}\cdot(\mathbf{z}-\boldsymbol{\xi})} A(\mathbf{x}_0) \Gamma \mathbf{S}(\mathbf{z}) d\mathbf{z} d\mathbf{k} \tag{7}$$

$$= \int_{\mathbb{R}^d} F(t,\boldsymbol{\xi}-\mathbf{z};\mathbf{x}_0) \mathbf{S}(\mathbf{z}) d\mathbf{z} = (F(t,\cdot;\mathbf{x}_0) * \mathbf{S})(\boldsymbol{\xi}), \tag{8}$$

with $a_1(0,\mathbf{x},\mathbf{k}) = a_2(0,\mathbf{x},\mathbf{k}) = |\chi_\Omega(\mathbf{x})|^2 \hat{f}(\mathbf{k})$. The quality of the refocusing of the back-propagated signal is determined by the decay properties in $\boldsymbol{\xi}$ of the kernel

$$F(t,\boldsymbol{\xi};\mathbf{x}_0) = \sum_{m=1}^{2} \int_{\mathbb{R}^d} a_m(t,\mathbf{x}_0,\mathbf{k}) \Gamma \mathbf{b}^m(\mathbf{x}_0,\mathbf{k}) \mathbf{b}^{m*}(\mathbf{x}_0,\mathbf{k}) e^{i\mathbf{k}\cdot\boldsymbol{\xi}} A(\mathbf{x}_0) \Gamma d\mathbf{k}. \tag{9}$$

In the limit where $f = \delta$ and $\Omega = \mathbb{R}^d$, we verify that $\mathbf{u}^B(\boldsymbol{\xi};\mathbf{x}_0) = \mathbf{S}(\boldsymbol{\xi})$. When all the information is propagated back, the refocusing is perfect, as is physically expected. However, when the medium is homogeneous and $c(\mathbf{x}) = c_0$, the amplitudes $a_{1,2}(t,\mathbf{x}_0,\mathbf{k}) = |\chi_\Omega(\mathbf{x}_0 \mp c_0\hat{\mathbf{k}}t)|^2 \hat{f}(\mathbf{k})$ become more and more singular in $\mathbf{k}$ as time grows (their gradient in $\mathbf{k}$ grows linearly in time). The corresponding kernel $F = F_H$ given by (9) being a Fourier transform in $\mathbf{k}$ of a function with growing gradients decays more and more slowly (in $\boldsymbol{\xi}$) as time grows. This leads to poorer and poorer refocusing.

3. Refocusing via the Liouville and Fokker-Plank equations

In this section, we analyze the random geometric acoustics limit, where the correlation length δ of the random medium is small compared to the propagation distance L but large relative to the wavelength ε. The mode $a = a^1$ satisfies the Liouville equation

$$\partial_t a^\delta + \nabla_{\mathbf{k}}\omega_\delta \cdot \nabla_{\mathbf{x}} a^\delta - \nabla_{\mathbf{x}}\omega_\delta \cdot \nabla_{\mathbf{k}} a^\delta = 0, \tag{10}$$

where $\omega(\mathbf{x},\mathbf{k}) = (c_0 + \sqrt{\delta} c_1(\mathbf{x}/\delta))|\mathbf{k}|$. The Liouville equation is obtained thanks to the a priori estimate of Lemma 1. We assume that $c_1(\mathbf{x})$ is a mean-zero stationary random process with a smooth rapidly decaying correlation function $R(\mathbf{x})$. In the limit $\delta \to 0$ we have $\mathbb{E}\{a^\delta\} \to a(t, \mathbf{x}, \mathbf{k})$, where a solves the Fokker-Plank equation [10]

$$\frac{\partial a}{\partial t} = \frac{\partial}{\partial k_m}\left(k^2 D_{jm}(\hat{\mathbf{k}})\frac{\partial a}{\partial k_j}\right) - c_0 \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} a \tag{11}$$

with the initial data $a(0, \mathbf{x}, \mathbf{k}) = |\chi_\Omega(\mathbf{x})|^2 \hat{f}(\mathbf{k})$. Here, $k = |\mathbf{k}|$, and the diffusion matrix is

$$D_{jm}(\hat{\mathbf{k}}) = \frac{-1}{2}\int_{-\infty}^{\infty} \frac{\partial^2 R}{\partial z_j \partial z_m}(c_0 \hat{\mathbf{k}} s) ds,$$

with $\hat{\mathbf{k}} = \mathbf{k}/k$. The operator on the right side of (11) is degenerate because $D_{ij}(\mathbf{k})k_j = 0$. However, it is hypoelliptic on $X = \mathbb{R}^3_{\mathbf{x}} \times \mathbb{S}^2_{\mathbf{k}}$ and thus possesses a smooth Green's function $G_L(t, \mathbf{x}, \mathbf{k}; \mathbf{x}', \mathbf{k}')$ on X so that

$$a(t, \mathbf{x}, \mathbf{k}) = \int_{\mathbb{R}^{2d}} G_L(t, \mathbf{x}, \mathbf{k}; \mathbf{y}, \hat{\mathbf{k}}')|\chi_\Omega(\mathbf{y})|^2 \hat{f}(\hat{\mathbf{k}}', k') d\mathbf{y} d\mathbf{k}'.$$

Hence we can introduce the new filter $F_L = F_L^1 + F_L^2$ with

$$F_L^1(t, \boldsymbol{\xi}; \mathbf{x}_0) = \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\boldsymbol{\xi}} a(t, \mathbf{x}_0, \mathbf{k}) \Gamma \mathbf{b}^1(\mathbf{k}) \mathbf{b}^{1*}(\mathbf{k}) A_0 \Gamma d\mathbf{k}, \tag{12}$$

and F_L^2 defined similarly. In the limit $\varepsilon \to 0$ and $\delta \to 0$ we obtain the reconstruction (8) for $\mathbb{E}\{u^B\}$ with $F = F_L$. Now the function $a(t, \mathbf{x}, \mathbf{k})$ is smoother in $\mathbf{k}$ than in the homogeneous case and thus the filter F_L decays faster than F_H. This produces sharper refocusing and eliminates spurious Fresnel zones. The process $a^\delta(t, \mathbf{x}, \mathbf{k})$ does not converge to the deterministic function $a(t, \mathbf{x}, \mathbf{k})$ in probability pointwise in $\mathbf{x}$ and $\mathbf{k}$. However, for any test function $\phi(\mathbf{x}, \mathbf{k}) \in \mathcal{S}(\mathbb{R}^{2d})$ the process $\phi^\delta(t) = (a^\delta, \phi)$ converges to (a, ϕ) in probability. That means that the refocused signal u^B is deterministic. This self-averaging of the refocused pulse follows from a modification of the analysis in [10].

4. Refocusing via the transport and diffusion equations

The radiative transport regime arises when the random medium has weak fluctuations at the wavelength scale. This case is more complicated than the one treated in Section 3 because the waves now fully interact with the medium. Propagation of wave correlations is described in terms of a Boltzmann-type radiative transport equation [5, 13, 14]. The random fluctuations of the density and compressibility are assumed to be of the form $\rho_\varepsilon(\mathbf{x}) = \rho_0 + \sqrt{\varepsilon}\rho_1(\frac{\mathbf{x}}{\varepsilon})$, $\kappa_\varepsilon(\mathbf{x}) = \kappa_0 + \sqrt{\varepsilon}\kappa_1(\frac{\mathbf{x}}{\varepsilon})$, where ρ_0, κ_0 are constants, and $\rho_1(\mathbf{x})$, $\kappa_1(\mathbf{x})$ are mean-zero stationary random processes. A rigorous derivation of the transport equation for the Wigner distribution is only available for Schrödinger equations [5, 14], Based on the formal analysis in [13] for hyperbolic systems, we can still conjecture that $a_{1,2}(t, \mathbf{x}, \mathbf{k})$ solve a linear transport equation, at least after ensemble averaging over realizations of the random fluctuations. It is moreover known that for long distances of propagation and large times, the transport solution becomes independent of the direction $\hat{\mathbf{k}}$ and is approximated by that of a diffusion equation:

$$\partial_t a(t, \mathbf{x}, k) - D(k)\Delta_{\mathbf{x}} a(t, \mathbf{x}, k) = 0, \tag{13}$$

where $D(k)$ is a diffusion coefficient that depends on the power spectrum of the random fluctuations (see [13] for instance). When the filter f is radially symmetrical, the solution of (13) may be written as $a_{1,2}(t, \mathbf{x}, k) = \psi(t, \mathbf{x}, k)\hat{f}(k)$. Here $\psi(t, \mathbf{x}, k)$ is the solution of (13) with initial data $\psi(0, \mathbf{x}, k) = |\chi_\Omega(\mathbf{x})|^2$. We then obtain that the new filter F_B is scalar and is given by

$$F_B(t, \boldsymbol{\xi}; \mathbf{x}_0) = \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\boldsymbol{\xi}} \psi(t, \mathbf{x}_0, k)\hat{f}(k) d\mathbf{k}. \tag{14}$$

The retransmitted field (in the weak sense and after ensemble averaging) is given now by (8) with $F = F_B I$. Note that similarly to the random geometrical optics case treated in Section 3 the amplitude $a(t, \mathbf{x}, \mathbf{k})$ is smoother in $\mathbf{k}$ than in the homogeneous case and the filter F_B decays faster than F_H, thus produces a sharper refocusing than in a homogeneous medium.

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