

Exponential decay for the fragmentation or cell-division equation

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Abstract

We consider a classical integro-differential equation that arises in various applications as a model for cell-division or fragmentation. In biology, it describes the evolution of the density of cells that grow and divide. We prove the existence of a stable steady dynamics (first positive eigenvector) under general assumptions in the variable coefficients case. We also prove the exponential convergence, for large times, of solutions toward such a steady state.

1 Introduction

This paper is concerned with the equation

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} n(t, x) + \frac{\partial}{\partial x} n(t, x) + b(x) n(t, x) = 4b(2x) n(t, 2x), \quad t > 0, x \geq 0, \\ n(t, x = 0) = 0, \quad t > 0, \\ n(0, x) = n^0(x) \in L^1(\mathbb{R}^+). \end{array} \right. \quad (1)$$

It arises in many applications, in particular as a basic model for size-structured populations: $n(t, x)$ denotes the population density of cells of size x at time t . The cells grow at a constant rate but also divide into two cells of equal size at a rate $b(x)$ when mitosis occurs (see [13] for a classical reference on the subject and [3] for a more recent application to cell division). This model also appears in physics to describe a fragmentation (degradation) phenomenon in polymers, droplets ([12], [9], [1] and references therein) and in telecommunications systems to describe some internet protocols [2]. Several variants of the model are possible and we refer to [4] for a probabilistic study of a model with the same feature (competition between a 'growth process' and a 'fragmentation process').

Our first purpose is to study existence of the first eigenvector $N(x)$

$$\left\{ \begin{array}{l} \frac{\partial}{\partial x} N(x) + (\lambda + b(x)) N(x) = 4b(2x) N(2x), \quad x \geq 0, \\ N(0) = 0, \quad N(x) > 0 \text{ for } x > 0, \quad \int_0^\infty N(x) dx = 1, \end{array} \right. \quad (2)$$

with λ the first eigenvalue (sometimes called the Malthus parameter in biology). Then, and our second purpose is to make this rigorous, one can expect that this density plays the role of a so-called stable

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steady dynamics, that is, after a time renormalization, all the solutions to (1) converge to a multiple of N . To be more precise, we need to introduce the dual operator

$$\begin{cases} \frac{\partial}{\partial x}\psi(x) - (\lambda + b(x))\psi(x) = -2b(x)\psi(\frac{x}{2}), & x \geq 0, \\ \psi(x) > 0 \quad \text{for } x \geq 0, & \int_0^\infty N(x)\psi(x)dx = 1. \end{cases} \quad (3)$$

It plays a fundamental role in the dynamics of (1) because its solution allows to define a conservation law for the evolution equation for $n(t, x)$:

$$\int_0^\infty n(t, x)e^{-\lambda t}\psi(x)dx = \int_0^\infty n^0(x)\psi(x)dx := \langle n^0 \rangle. \quad (4)$$

The construction of the eigenvalue λ and eigenfunctions N, ψ as well as the long time asymptotics face some technical difficulties although the former is a variant of the Krein-Rutman theorem (see [7] for instance for a recent presentation and different versions). First, we work on the half line that lacks compactness, second, the regularizing effect (positivity) of the division term is very indirect and the fact that all the derivatives $N^{(p)}(x=0)$ vanish is a specific difficulty. Therefore, we base our results on (i) original a priori estimates for the steady states N, ψ and (ii) a perturbation method around the constant coefficient case which is simpler for the exponential convergence. Indeed, we have the following

Theorem 1.1 *Assume $b(x) = B$, then $\lambda = B$, $\psi = 1$ and there is a unique solution $N \in \mathcal{S}(\mathbb{R}^+)$ to (2). Furthermore, all solutions to (1) satisfy*

$$\|n(t, x)e^{-Bt} - \langle n^0 \rangle N(x)\|_{L^1(\mathbb{R}^+)} \leq e^{-Bt} [\|n^0(x) - \langle n^0 \rangle N(x)\|_{L^1(\mathbb{R}^+)} + 6B\|H^0\|_{L^1(\mathbb{R}^+)}], \quad (5)$$

where $\langle n^0 \rangle = \int_0^\infty n^0 dx$ and

$$H^0(x) = \int_0^x [n^0(y) - \langle n^0 \rangle N(y)] dy \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

It remains unknown whether the L^1 bound on the initial data H^0 is necessary for the exponential decay (it is not needed for the mere decay without rate, in any L^p space, see [14]).

Our purpose in this paper is to prove a similar result (Theorem 4.1) for variable functions $b(x)$. We would like to point out that proving exponential decay to steady states in transport equations is still an active subject because no unifying tool can be used because of lack of symmetry and lack of general spectral theory due to the hyperbolic nature of (2). We refer to [8, 17, 6, 16] for other examples of recent studies on such problems. In a discrete setting and when the growth operator is nonlinear (and then it involves quite specific method) a convergence result to steady state for the fragmentation equation is available in [10]. We finally note that the existence of a unique solution $n \in C(\mathbb{R}^+; L^1(\mathbb{R}^+))$ to (1) follows from classical analysis of transport equations, while additional integrability and L^∞ bounds follow from the entropy structure of such models (see [14]).

This paper is organized as follows. In the second section we consider the case of a constant coefficient $b(x) = B > 0$ and prove Theorem 1.1. Next, we study the existence of eigenfunctions for the problems (2), (3) with a general bounded function $b(x)$. These are used in Section 4, where we prove an explicit exponential decay rate. It relies on the knowledge of the large x behavior of ψ , a result we establish for functions $b(x)$ that are equal to a constant outside of a compact set (Theorem 4.2).

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2 The constant coefficient case

In this section we prove Theorem 1.1 for the case of a constant division rate $b(x) = B$. Then the equation reads

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} n(t, x) + \frac{\partial}{\partial x} n(t, x) + Bn(t, x) = 4Bn(t, 2x), \quad t > 0, x \geq 0, \\ n(t, x = 0) = 0, \quad t > 0, \\ n(0, x) = n^0(x) \in L^1(\mathbb{R}^+). \end{array} \right. \quad (6)$$

and its eigenelements are known explicitly, but still the large time asymptotics and decay rates are not known. This problem serves also as a model for the general case we treat later.

The construction of the first eigenvalue and adjoint eigenfunction for (2)-(3) is rather easy and we readily check that

$$\lambda = B, \quad \psi = 1.$$

This implies that we have

$$\int_0^\infty n(t, x) e^{-Bt} dx = \int_0^\infty n^0(x) dx, \quad \int_0^\infty |n(t, x)| e^{-Bt} dx \leq \int_0^\infty |n^0(x)| dx. \quad (7)$$

The first eigenvector, that is, the (smooth) solution $N(x)$ to

$$\left\{ \begin{array}{l} \frac{\partial}{\partial x} N(x) + 2B N(x) = 4B N(2x), \quad x \geq 0, \\ N(0) = 0, \quad N(x) > 0 \text{ for } x > 0, \quad \int_0^\infty N(x) dx = 1, \end{array} \right. \quad (8)$$

is less explicit but is given by a series (see [2]) that converges absolutely and uniformly.

Lemma 2.1 *Let $\alpha_0 = 1$, $\alpha_n = \frac{2}{2^n - 1} \alpha_{n-1}$, then the function*

$$N(x) = \bar{N} \sum_{n=0}^{\infty} (-1)^n \alpha_n e^{-2^{n+1} Bx}, \quad (9)$$

belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^+)$ and is the unique solution to (8) with an appropriate normalization constant $\bar{N} > 0$.

Proof. We first prove that the equation is satisfied. We have

$$\begin{aligned} N'(x) &= \bar{N} \sum_{n=0}^{\infty} (-1)^n \alpha_n 2^{n+1} B e^{-2^{n+1} Bx} = -2BN + \bar{N} \sum_{n=0}^{\infty} (-1)^n \alpha_n 2(2^n - 1) B e^{-2^{n+1} Bx} \\ &= -2BN + 2\bar{N} \sum_{n=1}^{\infty} (-1)^n 2\alpha_{n-1} B e^{-2^n B2x} = -2BN + 4BN(2x). \end{aligned}$$

Next, we prove that $N(0) = 0$. We have

$$\alpha_n = \frac{2^n}{(2^n - 1) \dots (2^1 - 1)}$$

so that

$$\alpha_0 - \alpha_1 = 1 - \frac{2}{2-1} = \frac{-1}{2-1},$$

and

$$\alpha_0 - \alpha_1 + \alpha_2 = \frac{-1}{2-1} + \frac{2^2}{(2^2-1)(2^1-1)} = \frac{-1}{(2^2-1)(2^1-1)}.$$

One readily checks by induction that

$$\sum_{n=0}^k (-1)^n \alpha_n = \frac{(-1)^k}{(2^k - 1) \dots (2^1 - 1)},$$

which proves the result as $k \rightarrow \infty$.

The positivity of $N(x)$ is a consequence of the equation itself. Multiplying (8) by $\text{sgn}(N(x))$ we obtain

$$\frac{\partial}{\partial x} |N(x)| + 2B|N(x)| = 4BN(2x)\text{sgn}(N(x)).$$

After integration over the half line $x \geq 0$, we find

$$2B \int_0^\infty |N(x)| dx = 4B \int_0^\infty N(2x)\text{sgn}(N(x)) dx.$$

Thus, dividing by $2B$ and changing variable $y = 2x$ in the second integral, we obtain

$$\int_0^\infty |N(x)| dx = \int_0^\infty N(y)\text{sgn}(N(\frac{y}{2})) dy.$$

This proves that $\text{sgn}(N(y)) = \text{sgn}(N(\frac{y}{2}))$ for all $y > 0$. A bounded function with such a property being constant we have obtained the result.

Notice that the above argument also proves the uniqueness of N . Indeed, if we had two solutions with same integral, the difference would have a vanishing integral while the above argument implies that it would have a constant sign. \square

We are now ready for the

Proof of Theorem 1.1. We set

$$h(t, x) = n(t, x)e^{-Bt} - \langle n^0 \rangle N(x), \quad H(t, x) = \int_0^x h(t, y) dy.$$

These functions satisfy

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} h(t, x) + \frac{\partial}{\partial x} h(t, x) + 2Bh(t, x) = 4Bh(t, 2x), \quad t > 0, x \geq 0, \\ h(t, x=0) = 0, \quad \int_0^\infty h(t, x) dx = 0, \quad \forall t > 0, \end{array} \right. \quad (10)$$

and

$$\begin{cases} \frac{\partial}{\partial t}H(t, x) + \frac{\partial}{\partial x}H(t, x) + 2BH(t, x) = 2BH(t, 2x), & t > 0, x \geq 0, \\ H(t, x = 0) = 0, \quad H(t, \infty) = 0, & \forall t > 0. \end{cases} \quad (11)$$

As a *first step*, we begin with a study of H . We have

$$\frac{\partial}{\partial t}[H(t, x)e^{Bt}] + \frac{\partial}{\partial x}[H(t, x)e^{Bt}] + B[H(t, x)e^{Bt}] = 2B[H(t, 2x)e^{Bt}],$$

and thus

$$\frac{\partial}{\partial t}|H(t, x)e^{Bt}| + \frac{\partial}{\partial x}|H(t, x)e^{Bt}| + B|H(t, x)e^{Bt}| \leq 2B|H(t, 2x)e^{Bt}|.$$

We find after integration in x , using that H vanishes at infinity that

$$\frac{d}{dt} \int_0^\infty |H(t, x)e^{Bt}| dx \leq 0, \quad \int_0^\infty |H(t, x)| dx \leq e^{-Bt} \int_0^\infty |H^0(x)| dx. \quad (12)$$

In a *second step* we work on $K(t, x) = \frac{\partial}{\partial t}H(t, x)$. We have

$$\begin{cases} \frac{\partial}{\partial t}K(t, x) + \frac{\partial}{\partial x}K(t, x) + 2BK(t, x) = 2BK(t, 2x), & t > 0, x \geq 0, \\ K(t, x = 0) = 0, \quad K(t, \infty) = 0, & \forall t > 0. \end{cases} \quad (13)$$

Therefore, as in the first step, we deduce, since

$$K^0(x) = -h^0(x) - 2BH^0(x) + 2BH^0(2x),$$

that

$$\begin{aligned} \int_0^\infty |K(t, x)| dx &\leq e^{-Bt} \int_0^\infty |K^0(x)| dx \leq e^{-Bt} \int_0^\infty [|h^0(x)| + 2B|H^0(x)| + 2B|H^0(2x)|] dx \\ &= e^{-Bt} \int_0^\infty [|h^0(x)| + 3B|H^0(x)|] dx. \end{aligned} \quad (14)$$

In the *third step*, we deduce the time decay of h from this time decay property of H . Indeed, we compute from (11)

$$h(t, x) = \frac{\partial}{\partial x}H = -\frac{\partial}{\partial t}H(t, x) - 2BH(t, x) + 2BH(t, 2x),$$

and thus

$$\int_0^\infty |h(t, x)| dx \leq \int_0^\infty |K(t, x)| dx + 3B \int_0^\infty |H(t, x)| dx \leq e^{-Bt} \left\{ \int_0^\infty [|h^0(x)| + 6B \int_0^\infty |H^0(x)| dx] \right\}.$$

From this, we directly deduce the estimate of Theorem 1.1. \square

3 The variable coefficient case (eigenfunctions)

The purpose of this section is to prove the existence of a first eigenvalue and positive eigenvectors for the problem (2) and (3). These are fundamental for studying the exponential decay. We prove the

Theorem 3.1 *Assume that*

$$b \in C(\mathbb{R}^+), \quad 0 < b_m := \min b(x), \quad B_M := \max b(x) < \infty, \quad (15)$$

then there is a unique solution (λ, N, ψ) to equations (2), (3) with $\psi, N \in C^1(\mathbb{R}^+)$. Moreover,

$$\int_0^\infty x^p N(x) dx < \infty \quad \forall p > 0,$$

and we have the bounds

$$\begin{aligned} b_m &\leq \lambda \leq B_M, \\ \frac{c}{(1+x)^k} &\leq \psi(x) \leq C(1+x^k), \end{aligned} \quad (16)$$

with two positive constants c, C and k such that $2^k b_m > B_M$.

In order to prove this Theorem (in Section 3.2), we first consider the case of a bounded domain $x \in [0, L]$ and then pass to the limit. A related problem is also mentioned in Section 3.3, for the sake of completeness.

3.1 Bounded domain

The problem on a bounded interval $[0, L]$ is to find the first eigenvalue λ_L and (N_L, ψ_L) such that

$$\begin{cases} \frac{\partial}{\partial x} N_L(x) + (\lambda_L + b(x)) N_L(x) = 4b(2x) N_L(2x) \mathbf{1}_{\{2x \leq L\}}, & 0 \leq x \leq L, \\ N_L(0) = 0, \quad N_L(x) > 0 \quad \text{for } x > 0, & \int_0^L N_L(x) dx = 1, \end{cases} \quad (17)$$

and

$$\begin{cases} \frac{\partial}{\partial x} \psi_L(x) - (\lambda_L + b(x)) \psi_L(x) = -2b(x) \psi_L\left(\frac{x}{2}\right), & x \geq 0, \\ \psi_L(L) = 0, \\ \psi_L(x) > 0 \quad \text{for } x \geq 0, & \int_0^L N_L(x) \psi_L(x) dx = 1. \end{cases} \quad (18)$$

Lemma 3.2 *Assume (15) and let $L > 0$. Then there is a unique solution (λ_L, N_L, ψ_L) to (17), (18) such that N_L is Lipschitz continuous and $\psi_L \in C^1$.*

Proof of Lemma 3.2. This Lemma can be derived from the Perron-Frobenius theorem after discretization on the equation with source, and λ large enough, as

$$\begin{cases} \frac{n_i - n_{i-1}}{h} + (\lambda + b_i) n_i = 4b_{2i} n_{2i} \mathbf{1}_{\{2i \leq L/h\}} + f_i, & 1 \leq i \leq L/h, \\ n_0 = 0. \end{cases}$$

The underlying matrix is an M-matrix (positive dominant diagonal for λ large enough, other coefficients non-positive). Therefore there is a first positive eigenvalue associated with a positive eigenvector $(\lambda_{\text{disc}}, N_{\text{disc}})$. The solution of this discrete model converges to a solution to (17) just by positivity and uniform a priori Lipschitz bounds using the Ascoli theorem. Similar arguments are detailed below and we do not insist on this standard limit here. Similar arguments also apply to the dual problem and allow us to construct the dual eigenfunction ψ_L . \square

3.2 Limit $L \rightarrow \infty$

We begin with the *existence of a limit* as $L \rightarrow 0$ of the eigenfunctions and eigenvalues constructed above. We first establish some uniform bounds, then pass to the limit and prove uniqueness.

The first step. The limit of N_L . We first prove the uniform bounds for N_L with $L \geq 1$.

Lemma 3.3 *The solution to (17) satisfies*

$$b_m - \frac{1}{L} \leq \lambda_L \leq B_M. \quad (19)$$

Moreover, the family N_L is compact in $L^1(\mathbb{R}^+)$, bounded in $L^\infty(\mathbb{R}^+)$ and $\int_0^\infty x^p N_L(x) dx \leq C(p)$ independent of L .

Proof. The upper bound for λ_L in (19) follows from the positivity of $N_L \geq 0$ because, after integrating (17) in x we have

$$\lambda_L \leq N_L(L) + \lambda_L \int_0^L N_L(y) dy = \int_0^L b(y) N_L(y) dy \leq B_M \int_0^L N_L(y) dy = B_M, \quad (20)$$

Next, we multiply (17) by x and integrate in x . This gives

$$LN_L(L) - \int_0^L N_L(y) dy + \lambda_L \int_0^L y N_L(y) dy = 0, \quad (21)$$

so that using expression (20) for $N_L(L)$, we deduce

$$(L \lambda_L + 1) \int_0^L N_L(y) dy = \lambda_L \int_0^L y N_L(y) dy + L \int_0^L b(y) N_L(y) dy \geq L b_m \int_0^L N_L(y) dy. \quad (22)$$

Then the lower bound in (19) follows. The boundedness of N_L in $L^\infty(\mathbb{R}^+)$ also follows from these bounds on λ_L because $\frac{\partial}{\partial x} N_L$ is a bounded measure.

Next, we show the compactness of the family N_L in L^1 . It follows (see for instance [5]) from the two bounds already proved

$$\int_0^L y N_L(y) dy \leq \frac{1}{\lambda_L} \int_0^L N_L(y) dy \leq \frac{1}{b_m - \frac{1}{L}},$$

as implied by (21), and

$$\frac{\partial}{\partial x} N_L = 4b(2x)N_L(2x) - (\lambda_L + b(x))N_L(x) \text{ is bounded in } L^1(\mathbb{R}^+).$$

Such estimates, pushed further in fact prove boundedness of $\int_0^\infty x^p N_L(x) dx$ for all $p > 0$. \square

The second step. Positivity of N . Now, extracting convergent subsequences of λ_L and N_L we obtain an eigenvalue λ and an eigenfunction $N \geq 0$ for (2) in $\mathcal{C}^1(\mathbb{R}^+)$. The x -moment control shows that $\int_0^\infty N dx = 1$. It remains to prove the positivity of N . We use the method of characteristics. Let us denote $a = \inf\{x \text{ s.t. } N(x) > 0\}$. Then, using the method of characteristics, we have

$$N(x) = 4e^{-\mathcal{J}(x)} \int_0^x b(2y)N(2y)e^{\mathcal{J}(y)} dy, \quad \mathcal{J}'(x) = \lambda + b(x).$$

Therefore, for $x > a/2$, we deduce that $N(x) > 0$ since there is an open subset (N is continuous) where $N(2y) > 0$ in this integral. Therefore $a = 0$ and $N(x) > 0$ for $x > 0$.

The third step. The limit of ψ_L . The proof of the existence of a positive limit of ψ_L (the dual problem) also requires a specific analysis in order to obtain uniform in L upper and lower bounds (for x close to 0 and to ∞). These turn out to be longer to establish. We use the notation

$$1 \leq J(x) = \exp\left\{\int_0^x [\lambda_L + b(z)] dz\right\} \leq e^{[\lambda_L + B_M]x}.$$

Lemma 3.4 *The solution to (18) satisfies*

$$N_L(x)\psi_L(x) \leq \frac{2}{x}, \quad \psi_L(x) \leq C(1 + x^k), \quad (23)$$

with a constant C independent of L , and k large enough so that $2^k > \frac{B_M}{b_m}$.

Proof. We prove the first inequality. Using (17) and (18), we observe that the product $N_L \psi_L$ satisfies

$$\begin{cases} \frac{\partial}{\partial x}[N_L \psi_L] = 4b(2x)N_L(2x)\psi(x) \mathbf{1}_{\{2x \leq L\}} - 2b(x)N_L(x)\psi_L\left(\frac{x}{2}\right), \\ N_L \psi_L(0) = N_L \psi_L(L) = 0. \end{cases}$$

Therefore we have

$$N_L(x)\psi_L(x) = 2 \int_x^{\inf(L, 2x)} b(z)N_L(z)\psi_L\left(\frac{z}{2}\right) dz, \quad (24)$$

and we obtain after integration in x , using the Fubini theorem

$$1 = \int_0^L N_L(x)\psi_L(x) dx = \int_{z=0}^L zb(z)N_L(z)\psi_L\left(\frac{z}{2}\right) dz. \quad (25)$$

We can use this bound in the right side of (24) again (since $z \geq x$ here) and we obtain the first inequality in (23).

Secondly, we prove an intermediate upper bound, namely

$$\sup_{0 \leq x \leq A} \psi_L(x) \leq C(A). \quad (26)$$

For this inequality we use again (18) and first derive, for $y < z$

$$\psi_L\left(\frac{z}{2}\right)J^{-1}\left(\frac{z}{2}\right) \leq \psi_L\left(\frac{y}{2}\right)J^{-1}\left(\frac{y}{2}\right) \leq \psi_L\left(\frac{y}{2}\right). \quad (27)$$

We now insert this bound into the inequality (again deduced from (18))

$$\psi_L(y) \leq \psi_L(x_L) + 2 \int_y^{x_L} b(z)\psi_L\left(\frac{z}{2}\right)dz \leq \psi_L(x_L) + 2B_M\psi_L\left(\frac{y}{2}\right) \int_y^{x_L} J\left(\frac{z}{2}\right)dz,$$

with $y \leq x_L$ and x_L chosen such that

$$2B_Mx_LJ\left(\frac{x_L}{2}\right) = 1/2.$$

Note that x_L is uniformly bounded: $c \leq x_L \leq C$, as implied by the uniform bounds (19) on λ_L . We obtain thus for all $x < x_L$,

$$\sup_{0 \leq y \leq x} \psi_L(y) \leq \psi_L(x_L) + 2B_M \left[\sup_{0 \leq y \leq x} \psi_L(y) \right] x_L J\left(\frac{x_L}{2}\right) \leq \psi_L(x_L) + \frac{1}{2} \sup_{0 \leq y \leq x} \psi_L(y),$$

and hence the inequality (26) follows for $A < x_L$. The bound (26) for A large follows from above and the first inequality in (23).

Finally, we prove the polynomial growth. To do that we simply argue that the function $v = C(1+x^k)$ is an upper solution for the equation on ψ_L when C and k large enough. Because this equation satisfies the maximum principle as a backward equation, we give the argument when working on the variable $y = L - x$ and denoting by $\underline{\psi}_L(y) = \psi_L(x)$, $\underline{b}(y) = b(x)$, $\underline{v}(y) = v(x)$. We have

$$\frac{\partial}{\partial y} \underline{\psi}_L(y) + (\lambda_L + \underline{b})\underline{\psi}_L(y) = 2 \underline{b} \underline{\psi}_L\left(\frac{L+y}{2}\right).$$

On the other hand, we have $\underline{v}(y) = C(L-y)^k$ satisfies

$$\frac{\partial}{\partial y} \underline{v} + (\lambda_L + \underline{b})\underline{v} \geq -Ck(L-y)^{k-1} + Cb_m(L-y)^k \geq 2CB_M\left(\frac{L-y}{2}\right)^k$$

for $0 \leq y \leq L - A$ with the choice of k and for A large enough. For $x \leq A$ the second upper bound in (23) is implied by (26). Hence we have $\underline{\psi}_L(y) \leq \underline{v}(y)$ for C large enough and thus $\psi_L(x) \leq v(x)$ for all $0 \leq x \leq L$. This concludes the proof of Lemma 3.4. \square

We conclude from Lemma 3.4 that ψ_L converges to a solution ψ to (3) (this is local strong convergence) satisfying the same bounds as in the Lemma. Thanks to the uniform decay of N_L at infinity faster than any polynomial, we deduce that

$$\int_0^\infty \psi(x)N(x) = 1.$$

Thus the eigenfunction ψ does not vanish and from (27), we deduce that ψ does not vanish on some interval $[0, x_0]$ with $x_0 > 0$. Then we can deduce the lower bound on ψ in (16) because, arguing as above in the variable $y = L - x$, $w = \frac{c}{(1+x)^k} - \frac{c}{(1+L)^k}$ is a sub-solution to (3).

The fourth step. Uniqueness of the limit. Given another solution $\tilde{\lambda}, \tilde{N}$, we have

$$\frac{\partial}{\partial x}(\psi\tilde{N}) + (\tilde{\lambda} - \lambda)(\psi\tilde{N}) = 4b(2x)\psi\tilde{N}(2x) - 2b(x)\psi\left(\frac{x}{2}\right)\tilde{N}(x),$$

after integration, we find

$$(\tilde{\lambda} - \lambda) \int_0^\infty \psi(x)\tilde{N}(x) dx = 0,$$

which implies that $\tilde{\lambda} = \lambda$. Next, we prove that $N = \tilde{N}$. To do that we subtract the equations and multiply by $\text{sgn}(N - \tilde{N})$, we find

$$\frac{\partial}{\partial x}|N - \tilde{N}| + (\lambda + b(x))|N - \tilde{N}| = 4b(2x)(N(2x) - \tilde{N}(2x))\text{sgn}(N(x) - \tilde{N}(x))$$

and after multiplication by ψ and integration, we obtain

$$\int_0^\infty b(x)|N(x) - \tilde{N}(x)|\psi\left(\frac{x}{2}\right)dx = \int_0^\infty b(x)(N(x) - \tilde{N}(x))\text{sgn}\left(N\left(\frac{x}{2}\right) - \tilde{N}\left(\frac{x}{2}\right)\right)\psi\left(\frac{x}{2}\right)dx.$$

This means that $\text{sgn}(N(x) - \tilde{N}(x)) = \text{sgn}\left(N\left(\frac{x}{2}\right) - \tilde{N}\left(\frac{x}{2}\right)\right)$, which means that this sign is constant, that is, say, $N \geq \tilde{N}$. However, this contradicts the fact that both are probability measures. The same argument proves that $\psi = \tilde{\psi}$. The proof of Theorem 3.1 is now complete. \square

3.3 A related problem

A related problem, that exhibits a similar structure and also arises in the context of cell dynamics, is

$$\begin{cases} \frac{\partial}{\partial x}M(x) + 2b(x)M(x) = 4b(2x)M(2x), & x \geq 0, \\ M(0) = 0, \quad M(x) > 0 \text{ for } x > 0, & \int_0^\infty M(x)dx = 1. \end{cases} \quad (28)$$

We present its study as another application of the method developed above.

Note that the dual solution is trivial since $\Psi = 1$ solves

$$\begin{cases} \frac{\partial}{\partial x}\Psi(x) + 2b(x)\Psi(x) = 2b(x)\Psi\left(\frac{x}{2}\right), & x \geq 0, \\ \Psi(x) > 0 \text{ for } x \geq 0, & \int_0^\infty \Psi(x)M(x)dx = 1. \end{cases} \quad (29)$$

We claim that the existence of a solution (in Schwartz space) to (28) may be obtained by the same method and estimates. Again, this follows from the study of the problem set on a bounded domain.

$$\begin{cases} \frac{\partial}{\partial x}M_L(x) + (2b(x) - \mu_L)M_L(x) = 4b(2x)M_L(2x) \mathbf{1}_{\{2x \leq L\}}, & 0 \leq x \leq L, \\ M_L(0) = 0, \quad M_L(x) > 0 \text{ for } 0 < x \leq L, & \int_0^L M_L(x)dx = 1. \end{cases}$$

We observe that this eigenvalue problem has a unique solution which satisfies the uniform estimates (as $L \rightarrow \infty$):

$$0 \leq \mu_L \leq \frac{b_m}{Lb_m - 1}, \quad \int_0^L xb(x)M_L(x)dx \leq 1.$$

Indeed, integration against the weights 1 and x gives now

$$M_L(L) = \mu_L,$$

and

$$LM_L(L) - \int_0^L M_L(x)dx + \int_0^L xb(x)M_L(x)dx = \mu_L \int_0^L xM_L(x)dx.$$

The first equation above implies that $\mu_L \geq 0$. Combining these two equalities, we also deduce

$$L\mu_L + \int_0^L xb(x)M_L(x)dx = 1 + \mu_L \int_0^L xM_L(x)dx \leq 1 + L\mu_L,$$

which proves the inequality $\int_0^L xb(x)M_L(x)dx \leq 1$. Then, we also write

$$L\mu_L \leq 1 + \mu_L \int_0^L xM_L(x)dx \leq 1 + \frac{\mu_L}{b_m} \int_0^L xb(x)M_L(x)dx \leq 1 + \frac{\mu_L}{b_m},$$

from which we derive the upper bound on μ_L : $\mu_L \leq (L - b_m^{-1})^{-1}$ so that $\mu_L \rightarrow 0$ as $L \rightarrow \infty$ and $\mu = 0$. The uniform bounds on M_L and uniqueness of the limit are shown as before.

4 The variable coefficient case: stability

We are now ready to state the main result of this paper. We define the renormalized division rate

$$\tilde{b}(x) = b(x) \frac{\psi(\frac{x}{2})}{\psi(x)}, \quad (30)$$

where ψ is the eigenfunction of problem (3). We assume that

$$0 < \tilde{b}_m \leq \tilde{b}(x) \leq \tilde{B}_M < \infty. \quad (31)$$

This asymptotic behavior of \tilde{b} is stronger than the mere upper and lower bounds on b in (15) and we conjecture it does not hold true in general. Below, we prove (see Theorem 4.2) that it is satisfied under the stronger requirement

$$b(x) = b_\infty > 0 \quad \text{for } x \text{ large enough.} \quad (32)$$

For the moment and as a motivation, we just mention, following the proof of Lemma 3.4, that $\psi(x)$ should behave at infinity exactly as x^{k_0} in this case, with $2^{k_0} = \frac{2b_\infty}{\lambda + b_\infty}$ (notice that $k_0 < 1$). Indeed, a formal expansion allows to recover a solution as a power series

$$\sum_{k \geq 0} a_k x^{(k_0 - k)} \quad \text{with} \quad (k_0 - k) a_k = a_{k+1} (\lambda + b_\infty) (2^{(k+1)} - 1).$$

This is justified in Theorem 4.2.

Theorem 4.1 *Assume that (15) and (31) hold and that there exists a constant $B > 0$ such that*

$$\gamma := \|\tilde{b}(x) - B\|_{L^\infty(\mathbb{R}^+)} < \frac{B}{4(2+B)}. \quad (33)$$

Then the solution to (1) satisfies

$$\|(n(t, x)e^{-\lambda t} - \langle n^0 \rangle N(x))\psi(x)\|_{L^1(\mathbb{R}^+)} \leq e^{-\mu t} [2\|(n^0(x) - \langle n^0 \rangle N(x))\psi(x)\|_{L^1(\mathbb{R}^+)} + 4B\|H^0\|_{L^1(\mathbb{R}^+)}] \quad (34)$$

with a constant $\mu = B - 8\gamma - 4B\gamma$. Here $\langle n^0 \rangle$ is given by (4), the eigenfunctions N and ψ are the solutions to (2) and (3), respectively, constructed in Section 3 and

$$H^0(x) = \int_0^x [n^0(y) - \langle n^0 \rangle N(y)]\psi(y)dy \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

We do not know at present whether the smallness condition (33) is optimal for exponential convergence to the steady state but at least the condition $b_m > 0$ is necessary. Indeed, in [11] periodic solutions are built when b can vanish on a large enough set.

Proof. We introduce, mimicking the constant coefficient case in Section 2,

$$h(t, x) = n(t, x)e^{-\lambda t} - \langle n^0 \rangle N(x),$$

so that

$$\langle h^0 \rangle = \int_0^\infty h^0(x)\psi(x)dx = 0,$$

and

$$H(t, x) = \int_0^x h(t, y)\psi(y)dy, \quad H(t, \infty) = 0.$$

The proof of Theorem 4.1 is then divided in two steps: first we establish a decay rate for the anti-derivative of $q(t, x) = \psi(x)h(t, x)$, and in the second step we conclude by means of a representation formula based on characteristics.

The function h satisfies

$$\begin{cases} \frac{\partial}{\partial t}h(t, x) + \frac{\partial}{\partial x}h(t, x) + (b(x) + \lambda)h(t, x) = 4b(2x)h(t, 2x), \\ h(t, 0) = 0. \end{cases}$$

Multiplying this equation by $\psi(x)$ and using the equation for ψ we obtain

$$\begin{cases} \frac{\partial}{\partial t}(\psi(x)h(t, x)) + \frac{\partial}{\partial x}(\psi(x)h(t, x)) + 2b(x)\psi\left(\frac{x}{2}\right)h(t, x) = 4b(2x)h(t, 2x)\psi(x), \\ h(t, 0) = 0. \end{cases}$$

This yields two fundamental equations on which we work:

$$\begin{cases} \frac{\partial}{\partial t}q(t, x) + \frac{\partial}{\partial x}q(t, x) + 2\tilde{b}(x)q(t, x) = 4\tilde{b}(2x)q(t, 2x), \\ q(t, 0) = 0, \end{cases} \quad (35)$$

and

$$\begin{cases} \frac{\partial}{\partial t} H(t, x) + \frac{\partial}{\partial x} H(x) + 2 \int_0^x \tilde{b}(y) q(t, y) dy = 2 \int_0^{2x} \tilde{b}(y) q(t, y) dy, \\ H(t, 0) = H(t, \infty) = 0. \end{cases} \quad (36)$$

The main difference here is that we have used the factor ψ which is hidden in the constant coefficient case.

We now show that the equation on H exhibits a natural decay that we dig out by a method of perturbation. We write

$$\frac{\partial}{\partial t} H(t, x) + \frac{\partial}{\partial x} H(x) + 2BH(t, x) - 2BH(t, 2x) = 2 \int_x^{2x} [\tilde{b}(y) - B] q(t, y) dy,$$

and thus

$$\frac{\partial}{\partial t} |H|(t, x) + \frac{\partial}{\partial x} |H|(x) + 2B|H|(t, x) - 2B|H|(t, 2x) \leq 2 \int_x^{2x} |(\tilde{b}(y) - B) q(t, y)| dy,$$

which gives our first fundamental inequality (we use here explicitly the condition $H(t, \infty) = 0$, that is, the specific constant $\langle n^0 \rangle$)

$$\frac{\partial}{\partial t} \|H\|_{L^1(\mathbb{R}^+)} + B\|H\|_{L^1(\mathbb{R}^+)} \leq 2 \int_{x=0}^{\infty} \int_{y=x}^{2x} |(\tilde{b}(y) - B) q(t, y)| dy \leq \gamma \|q(t, \cdot)\|_{L^1(\mathbb{R}^+)}$$

and thus

$$\|H(t)\|_{L^1(\mathbb{R}^+)} \leq e^{-Bt} \|H^0\|_{L^1(\mathbb{R}^+)} + \gamma \int_0^t e^{-B(t-s)} \|q(s, \cdot)\|_{L^1(\mathbb{R}^+)} ds \quad (37)$$

We cannot follow the constant coefficient case in the last step of the proof. Instead, we introduce $\Phi(t, x) = q(t, x)e^{2Bt}$. This function satisfies

$$\frac{\partial}{\partial t} \Phi(t, x) + \frac{\partial}{\partial x} \Phi(t, x) = 4B\Phi(t, 2x) + R(t, x) \quad (38)$$

with

$$R(t, x) = e^{2Bt} [4(\tilde{b}(2x) - B)q(t, 2x) - 2(\tilde{b}(x) - B)q(t, x)].$$

We integrate (38) along characteristics to find

$$\Phi(t, x) = \Phi^0(x - t) + 4B \int_0^t \Phi(t - s, 2x - 2s) ds + \int_0^t R(t - s, x - s) ds.$$

We use here the convention that all functions are extended by 0 for negative x arguments.

Next, we iterate the formula for Φ to find

$$\begin{aligned} \Phi(t, x) &= \Phi^0(x - t) + \int_0^t R(t - s, x - s) ds + 4B \int_0^t \Phi^0(2x - t - s) ds \\ &+ 4B \int_{s=0}^t \int_{u=0}^{t-s} R(t - s - u, 2x - 2s - u) duds + (4B)^2 \int_{s=0}^t \int_{u=0}^{t-s} \Phi(t - s - u, 4x - 4s - 2u) duds, \end{aligned}$$

and changing variables in the last two integrals ($u \rightarrow s + u = \sigma$) we get

$$\begin{aligned} \Phi(t, x) &= \Phi^0(x - t) + \int_0^t R(t - s, x - s) ds + 4B \int_0^t \Phi^0(2x - t - s) ds \\ &\quad + 4B \int_{\sigma=0}^t \int_{s=0}^{\sigma} R(t - \sigma, 2x - \sigma - s) ds d\sigma + (4B)^2 \int_{\sigma=0}^t \int_{s=0}^{\sigma} \Phi(t - \sigma, 4x - 2s - 2\sigma) ds d\sigma. \end{aligned} \quad (39)$$

In order to estimate the L^1 -norm of Φ we first observe, using the definition of H , the L^1 -norm of the last term in (39) is bounded by

$$(4B)^2 \int_0^\infty \int_{\sigma=0}^t e^{2B(t-\sigma)} |H(t - \sigma, 4x - 2\sigma) - H(t - \sigma, 4x - 4\sigma)| d\sigma dx \leq 4B^2 \int_{\sigma=0}^t e^{2B\sigma} \|H(\sigma)\|_{L^1(\mathbb{R}^+)} d\sigma.$$

Furthermore, as all functions vanish for negative arguments of x we have a bound for the second term in (39)

$$\begin{aligned} \int_0^t \int_0^\infty |R(t - s, x - s)| ds dx &= \int_{s=0}^t \int_{x=s}^\infty |R(t - s, x - s)| dx ds = \int_{s=0}^t \int_0^\infty |R(t - s, x)| dx ds \\ &= \int_{s=0}^t \int_0^\infty |R(s, x)| dx ds \leq 4\gamma \int_{s=0}^t \|\Phi(s, \cdot)\|_{L^1(\mathbb{R}^+)} ds. \end{aligned}$$

The fourth term in (39) is bounded by

$$\begin{aligned} 4B \int_0^\infty \int_{\sigma=0}^t \int_{s=0}^{\sigma} |R(t - \sigma, 2x - \sigma - s)| ds d\sigma dx &\leq 2B \int_{\sigma=0}^t \int_{x=0}^\infty \sigma |R(t - \sigma, x)| dx ds d\sigma \\ &\leq 8B\gamma \int_{\sigma=0}^t (t - \sigma) \|\Phi(\sigma)\|_{L^1(\mathbb{R}^+)} d\sigma. \end{aligned}$$

Therefore, we end up with the estimate

$$\begin{aligned} \|\Phi(t)\|_{L^1(\mathbb{R}^+)} &\leq (1 + 2Bt) \|\Phi^0\|_{L^1(\mathbb{R}^+)} + 4\gamma \int_{s=0}^t \|\Phi(s)\|_{L^1(\mathbb{R}^+)} ds + 8B\gamma \int_0^t (t - s) \|\Phi(s)\|_{L^1(\mathbb{R}^+)} ds \\ &\quad + 4B^2 \int_0^t e^{2B\sigma} \|H(\sigma)\|_{L^1(\mathbb{R}^+)} d\sigma \end{aligned}$$

We may use the decay rate of H in (37) to obtain

$$\begin{aligned} \|\Phi(t)\|_{L^1(\mathbb{R}^+)} &\leq (1 + 2Bt) \|\Phi^0\|_{L^1(\mathbb{R}^+)} + 4\gamma \int_0^t (1 + 2B(t - s)) \|\Phi(s)\|_{L^1(\mathbb{R}^+)} ds \\ &\quad + 4B(e^{Bt} - 1) \|H^0\|_{L^1(\mathbb{R}^+)} + 4B\gamma \int_0^t e^{B(t-\sigma)} \|\Phi(\sigma)\|_{L^1(\mathbb{R}^+)} d\sigma \\ &\leq (1 + 2Bt) \|\Phi^0\|_{L^1(\mathbb{R}^+)} + 4B(e^{Bt} - 1) \|H^0\|_{L^1(\mathbb{R}^+)} + 4\gamma(2 + B) \int_0^t e^{B(t-\sigma)} \|\Phi(\sigma)\|_{L^1(\mathbb{R}^+)} d\sigma. \end{aligned}$$

Therefore the function $v(t) = \int_0^t e^{-Bs} \|\Phi(s)\|_{L^1(\mathbb{R}^+)} ds$ satisfies

$$\dot{v}(t) \leq 2\|\Phi^0\|_{L^1(\mathbb{R}^+)} + 4B\|H^0\|_{L^1(\mathbb{R}^+)} + 4(2 + B)\gamma Bv(t), \quad v(0) = 0.$$

The Gronwall lemma implies that

$$v(t) \leq \frac{e^{4(2+B)\gamma t}}{4(2+B)\gamma} [2\|\Phi^0\|_{L^1(\mathbb{R}^+)} + 4B\|H^0\|_{L^1(\mathbb{R}^+)}],$$

and

$$\Phi(t) = e^{Bt}\dot{v}(t) \leq [2\|\Phi^0\|_{L^1(\mathbb{R}^+)} + 4B\|H^0\|_{L^1(\mathbb{R}^+)}](1 + e^{4(2+B)\gamma t})e^{Bt}.$$

We recall the definition of the function Φ and obtain that

$$\|h(t)\psi\|_{L^1(\mathbb{R}^+)} \leq [2\|\Phi^0\|_{L^1(\mathbb{R}^+)} + 4B\|H^0\|_{L^1(\mathbb{R}^+)}](1 + e^{4(2+B)\gamma t})e^{-Bt}.$$

Thus the statement of Theorem 4.1 is proved. \square

Theorem 4.2 *Under the assumptions $b(x) > 0$ and (32) there are two constants such that*

$$c(1 + x^{k_0}) \leq \psi(x) \leq C(1 + x^{k_0}),$$

with $2^{k_0} = \frac{2b_\infty}{\lambda + b_\infty}$.

Proof. Following the argument in section 3.2, we are going to build super and sub-solutions to the problem ψ_L . Then, from the upper and lower a priori bounds for x close to 0 it is enough (and necessary) to work for $x \in (A, L)$ for a given A (large enough at least so that $b(x) = b_\infty$ for $x \geq A$) and the constants c, C serve to adjust the "boundary values" at $x \leq A$.

Super-solution. We claim that for all $k_0 < 1$, the function $x^{k_0} - \mu x^{(k_0-1)}$ is a super-solution for $\mu > k_0/[\lambda + b_\infty + \frac{k_0-1}{A}]$. More precisely, with $y = L - x$ the correct variable for comparison, the function $\bar{v}(y) = (L - y)^{k_0} - \mu(L - y)^{(k_0-1)}$ satisfies

$$\frac{\partial}{\partial y}\bar{v}(y) + (\lambda_L + b_\infty)\bar{v}(y) - 2b_\infty\bar{v}\left(\frac{L+y}{2}\right) \geq 0,$$

for $0 \leq y \leq L - A$. Indeed, the left hand side is, after dividing it by $(L - y)^{(k_0-1)}$

$$-k_0 + \mu \frac{k_0 - 1}{L - y} + \mu(\lambda + b_\infty) \geq 0,$$

under the given condition on μ (and A large enough so that the condition implies $\mu > 0$).

Sub-solution. Let us first consider the case $k_0 \geq 0$. A sub-solution has to satisfy the boundary condition $\underline{v}(y = 0) = 0$. This is why we introduce a truncation function and define

$$\underline{v}(y) = (L - y)^{k_0} \frac{y}{L}.$$

We compute

$$\begin{aligned} \frac{\partial}{\partial y}\underline{v}(y) + (\lambda_L + b_\infty)\underline{v}(y) - 2b_\infty\underline{v}\left(\frac{L+y}{2}\right) \\ = -k_0(L - y)^{k_0-1} \frac{y}{L} + (L - y)^{k_0} \frac{1}{L} + (\lambda_L + b_\infty)(L - y)^{k_0} \left[\left(\frac{y}{L}\right) - \left(\frac{L+y}{2L}\right)\right]. \end{aligned}$$

To see the sign, after dividing by $(L - y)^{k_0 - 1}$, we arrive at the condition (recall that $k_0 \geq 0$)

$$-k_0 \frac{y}{L} + \frac{L - y}{L} - (\lambda_L + b_\infty)(L - y) \frac{L - y}{2L} \leq z[1 - L(\lambda_L + b_\infty) \frac{z}{2}] \leq 0,$$

with $z = \frac{L - y}{L} \in [1 - \frac{A}{L}, 1]$. This sign condition is satisfied iff

$$(\lambda_L + b_\infty)(L - A) \geq 2.$$

This is indeed true for L large enough.

In the case $k_0 < 0$, a sub-solution is

$$\underline{v}(y) = [(L - y)^{k_0} + \mu(L - y)^{k_0 - 1}] \frac{y}{L}.$$

For μ and A large enough, and following the above computations which we do not repeat, we arrive at the conclusion that it is a sub-solution on (A, L) .

We conclude this proof by indicating why the truncation on the sub-solution is irrelevant for the final comparison. From these sub-solutions, we conclude that $\psi_L \geq cx^{k_0} \frac{L - x}{L}$ on $(0, L)$. Therefore on $(0, L/2)$ we deduce that $\psi_L \geq \frac{c}{2}x^{k_0}$, and the result is proved. \square

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