QUENCHING OF FLAMES BY FLUID ADVECTION

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ABSTRACT. We consider a simple scalar reaction-advection-diffusion equation with ignition-type nonlinearity and discuss the following question: What kinds of velocity profiles are capable of quenching any given flame, provided the velocity's amplitude is adequately large? Even for shear flows, the answer turns out to be surprisingly subtle.

If the velocity profile changes in space so that it is nowhere identically constant, (or if it is identically constant only in a region of small measure) then the flow can quench any initial data. But if the velocity profile is identically constant in a sizable region, then the ensuing flow is incapable of quenching large enough flames, no matter how much larger is the amplitude of this velocity. The constancy region must be wider across than a couple of laminar propagating front-widths.

The proof uses a linear PDE associated to the nonlinear problem and quenching follows when the PDE is hypoelliptic. The techniques used allow the derivation of new, nearly optimal bounds on the speed of traveling wave solutions.

1. INTRODUCTION

We consider a mixture of reactants interacting in a region that may have a rather complicated spatial structure but is thin across. A mathematical model that describes a chemical reaction in a fluid is a system of two equations for concentration n and temperature T of the form

$$
\begin{aligned}
T_t + u \cdot \nabla T &= \kappa \Delta T + \frac{v_0^2}{\kappa} g(T) n \qquad (1)\\
n_t + u \cdot \nabla n &= \frac{\kappa}{\mathrm{Le}} \Delta n - \frac{v_0^2}{\kappa} g(T) n.
\end{aligned}
$$

The equations (1) are coupled to the reactive Euler equations for the advection velocity $u(x, y, t)$. Two assumptions are usually made to simplify the problem: the first is a constant density approximation [8] that allows to decouple the Euler equations from the system (1) and to consider $u(x, y, t)$ as a prescribed quantity that does not depend on T and n. The second assumption is that $\mathrm{Le} = 1$ (equal thermal and material diffusivities). These two assumptions reduce the above system to a single scalar equation for the temperature T. We assume in addition that the advecting flow is unidirectional. Then the system (1) becomes

$$
\begin{aligned}
T_t + Au(y) T_x &= \kappa \Delta T + \frac{v_0^2}{\kappa} f(T) \qquad (2)\\
T(0, x, y) &= T_0(x, y)
\end{aligned}
$$

with $f(T) = g(T)(1-T)$. We are interested in strong advection, and accordingly have written the velocity as a product of the amplitude A and the profile $u(y)$. In this paper we consider

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nonlinearity of the ignition type

$$\begin{aligned} &\text{(i) } f(T) \text{ is Lipschitz continuous on } 0 \le T \le 1, \\ &\text{(ii) } f(1)=0,\ \exists\theta_0 \text{ such that } f(T)=0 \text{ for } 0\le T\le\theta_0,\ f(T)>0 \text{ for } T>\theta_0, \\ &\text{(iii) } f(T)\le T. \end{aligned} \tag{3}$$

The last condition in (3) is just a normalization. We consider the reaction-diffusion equation (2) in a strip $D = \{x \in \mathbb{R},\ y \in [0, H]\}$. Equation (2) may be considered as a simple model of flame propagation in a fluid [3], advected by a shear (unidirectional) flow. The physical literature on the subject is vast, and we refer to the recent review [26] for an extensive bibliography. The main physical effect of advection for front-like solutions is the speed-up of the flame propagation due to the large scale distortion of the front. The role of the advection term in (2) for the front-like initial data was also a subject of intensive mathematical scrutiny recently. Existence of unique front-like traveling waves has been established in [3, 6], and their stability has been studied in [18, 19, 25]. A traveling front is a solution of (2) of the form

$$T(t,x,y) = U(x - c_A t, y) \tag{4}$$

such that

$$\lim_{s\to-\infty} U(s,y) = 1,\ \lim_{s\to+\infty} U(s,y) = 0,\ \ U_s(s,y) < 0. \tag{5}$$

The monotonicity property is not required for traveling wave solutions, but it is always present in the situation we consider. The speed-up of the fronts by advection mentioned above may be quantified as the dependence of the traveling front speed c_A on the amplitude A. Variational formulas for c_A were derived in [10, 11], based on the methods of [21, 22, 23, 24] where related results were proved in a slightly different context. The latter work also contains results on the asymptotic behavior of c_A when A is small for some classes of shear flows, as well as upper bounds on c_A linear in A. An alternative approach to quantifying advection effects was introduced by the present authors in [7, 15]. It is based on the notion of the bulk burning rate,

$$V(t) = \int_D T_t \frac{dxdy}{H} = \frac{v_0^2}{\kappa}\int_D f(T)\,\frac{dxdy}{H}$$

which extends the notion of front speed. We derived lower bounds for long time averages of $V(t)$ which behave like CA for large A, with constant C depending on the geometry of the flow. These bounds are valid for a class of flows that we call percolating. They are characterized by infinite tubes of streamlines connecting $\pm\infty$ and include shear flows as a particular case. Our bounds imply the estimate $c_A \ge CA$ for traveling waves. Audoly, Berestycki and Pomeau gave a formal argument [1] suggesting that in the case when the shear flow varies on the scale much larger than the laminar flame width, one should have $c_A \sim A$. One of the by-products of this paper is a rigorous proof of this conjecture.

Our main goal in the present paper is to consider advection effects for a different physically interesting situation, where initial data are compactly supported. In this case, two generic scenarios are possible. If the support of the initial data is large enough, then two fronts form and propagate in opposite directions. Fluid advection speeds up the propagation, accelerating the burning. However, if the support of the initial data is small, then the advection exposes the initial hot region to diffusion which cools it below the ignition temperature, ultimately extinguishing the flame.

We consider for simplicity periodic boundary conditions

$$T(t, x, y) = T(t, x, y + H) \tag{6}$$

in y and decay in x:

$$T(t, x, y) \to 0 \text{ as } |x| \to \infty. \tag{7}$$

We take $u(y)$ to be periodic with period H and with mean equal to zero:

$$\int_0^H u(y)dy = 0. \tag{8}$$

A constant non-zero mean can be easily taken into account by translation. We consider the case when the width of the domain is larger than the laminar front width length scale: $H \gg l = \frac{\kappa}{v_0}$. We will always assume that initial data $T_0(x, y)$ is such that $0 \le T_0(x, y) \le 1$. Then we have $0 \le T \le 1$ for all $t > 0$ and $(x, y) \in D$. Moreover, we assume that for some L and $\eta > 0$ we have

$$\begin{aligned} &T_0(x, y) > \theta_0 + \eta \text{ for } |x| \le L/2, \\ &T_0(x, y) = 0 \text{ for } |x| \ge L. \end{aligned} \tag{9}$$

The main purpose of this paper is to study the possibility of quenching of flames by strong fluid advection in a model (2). The phenomena associated with flame quenching are of great interest for physical, astrophysical and engineering applications. The problem of extinction and flame propagation in the mathematical model (2) under conditions (3), (7), (9), was first studied by Kanel [14] in one dimension and with no advection. He showed that, in the absence of fluid motion, there exist two length scales $L_0 < L_1$ such that the flame becomes extinct for $L < L_0$, and propagates for $L > L_1$. More precisely, he has shown that there exist L_0 and L_1 such that

$$\begin{aligned} &T(t, x, y) \to 0 \text{ as } t \to \infty \text{ uniformly in } D \text{ if } L < L_0 \\ &T(t, x, y) \to 1 \text{ as } t \to \infty \text{ for all } (x, y) \in D \text{ if } L > L_1. \end{aligned} \tag{10}$$

In the absence of advection, the flame extinction is achieved by diffusion alone, given that the support of initial data is small compared to the scale of the laminar flame width $l = \kappa/v_0$. However, in many applications the quenching is the result of strong wind, intense fluid motion and operates on larger scales. There are few results available for such situations in the framework of the reaction-diffusion model. Kanel's result was extended to non-zero advection by shear flows by Roquejoffre [20] who has shown that (10) holds also for $u \neq 0$ with L_0 and L_1 depending, in particular, on A and $u(y)$. The second (propagation) part in (10) was also proved in [25] for general periodic flows. However the interesting question about more explicit quantitative dependence of L_0, L_1 on A and $u(y)$ remained open. Is it possible to quench the initial data that previously lead to an expanding solution by increasing A, but not changing the profile? How does this possibility depend on geometry of the profile $u(y)$? Anyone who has tried to light a match in the wind has some intuition about this phenomenon. Yet, the mathematical answer turns out to be surprisingly subtle.

In this paper, we also limit ourselves to shear flows. We are interested in the understanding the behavior of L_0 and L_1 for large A. The answer depends strongly on the geometry of the flow. In some cases the maximal extinction size grows $L_0 \sim A$, and in others even the propagation size L_1 remains finite as A goes to infinity. In the first case, we will say that $u(y)$ is quenching.

Definition 1. *We say that the profile $u(y)$ is quenching if for any L and any initial data $T_0(x, y)$ supported inside the interval $[-L, L] \times [0, H]$, there exists A_0 such that solution of (2) becomes extinct:*

$$T(t, x, y) \to 0 \text{ as } t \to \infty \text{ uniformly in } D$$

for all $A \geq A_0$. We call the profile $u(y)$ strongly quenching if the critical amplitude of advection A_0 satisfies $A_0 \leq CL$ for some constant $C(u, \kappa, v_0, H)$ (which has the dimension of inverse time).

The key feature that distinguishes quenching from non-quenching velocities is the absence or presence of large enough flat parts in the profile $u(y)$.

Definition 2. *We say that the profile $u(y) \in C^\infty[0, H]$ satisfies the H-condition if*

$$\text{there is no point } y \in [0, H], \text{ where all derivatives of } u(y) \text{ vanish.} \tag{11}$$

The H-condition guarantees that the operator

$$\frac{\partial}{\partial t} + u(y)\frac{\partial}{\partial x} - \frac{\partial^2}{\partial y^2} \tag{12}$$

is hypoelliptic [12]. The study of existence of smooth fundamental solutions for such operators was initiated by Kolmogorov [16]. Kolmogorov's work with $u(y) = y$ served in part as a motivation for the fundamental result on characterization of hypoelliptic operators of Hörmander [12]. The hypoellipticity of the operator (12) plays a key role in some of our considerations. Our first result is that the H-condition implies strong quenching.

Theorem 1. *Let $f(T)$ be an ignition type nonlinearity, and let $u \in C^\infty[0, H]$ satisfy the H-condition. Then $u(y)$ is strongly quenching. That means that there exists a constant $C(u, \kappa, v_0, H) > 0$ that may depend on H, κ, v_0 and $u(y)$ but is independent of A such that*

$$T(t, x, y) \to 0 \text{ as } t \to \infty \text{ uniformly in } D \tag{13}$$

whenever the initial temperature $T_0(x, y)$ is supported in an interval $[-L, L] \times [0, H]$, with $L < C(u, \kappa, v_0, H)A$.

The next result shows that a plateau on the order of the laminar front width

$$l = \kappa / v_0 \tag{14}$$

in the profile $u(y)$ prohibits quenching. (And therefore the conditions in Theorem 1 are natural.)

Theorem 2. *There exist universal constants $C_0, C_1 > 0$, such that, if $u(y) = \bar{u} = const$ for $y \in [a - h, a + h]$ for some $a \in [0, H]$ and $h \geq h_0 = C_0 l$, then*

$$T(t, x, y) \to 1 \text{ as } t \to \infty \tag{15}$$

uniformly on compact sets, for all $A \in \mathbb{R}$ whenever the initial temperature $T_0(x, y)$ satisfying (9) is supported in an interval $[-L, L] \times [0, H]$ with $L \geq C_1 h_0$.

An interesting by-product of the proof of Theorem 2 is an estimate for the speed of traveling front solutions of (2) when the shear flow varies slowly on the scale of the laminar flame width l. Let us define

$$|u_+|_{h_0} = \max_{0 \le y_0 \le H} \left\{ \min_{y \in [y_0 - h_0, y_0 + h_0]} u_+ \right\}$$

with $h_0 = Cl$ given by Theorem 2, and $u_+ = \max(u(y), 0)$.

Theorem 3. *The speed of the traveling front c satisfies the upper and lower bounds*

$$A|u_+|_{h_0} \le c \le A\|u_+\|_\infty + v_0.$$

The upper bound of Theorem 3 is contained in [7] (it is shown there for KPP type reaction, but this immediately implies the corresponding bound for ignition nonlinearity by a simple application of maximum principle). The left hand side is close to $\|u_+\|_\infty$ if $u(y)$ is slowly varying on the scale h_0. This agrees with the formal prediction of Audoly, Berestycki and Pomeau [1], and also (up to the addition of v_0) with the results of Majda and Souganidis [17] in the homogenization regime $\kappa \to 0$. The scale h_0 in the lower bound in Theorem 3 is natural. It is known that rapid oscillations in $u(y)$ reduce the enhancement of the travelling front speed [7, 15, 11], so $\|u\|_\infty$ itself is not the right measure for a lower bound for c.

Unlike hypoellipticity, the quenching property is stable to small L^∞ perturbations: a small enough plateau (on the scale of the laminar front width l) does not stop quenching.

Theorem 4. *For every $\theta_0 > 0$ in (3) there exists a constant $B > 0$ such that, if a profile $u(y)$ satisfies the H condition outside an interval $y \in [a-h, a+h]$ with $h \le h_1 = Bl$, then it is strongly quenching.*

Moreover, the strongly quenching profiles are generic in the following sense:

Theorem 5. *The set of all strongly quenching shear flows $u(y)$ contains a dense open set in $C[0,H]$ (here $C[0,H]$ is the space of continuous functions on $[0,H]$).*

In Section 3 we show that all results on quenching, namely, Theorems 1, 4, 5 extend to the case of full system (1) with Le $\neq 1$.

Finally, in the last section we prove that initial data of sufficiently small size (of the same order as in the case $u(y) = 0$) will be quenched by any shear flow $Au(y)$.

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2. Quenching by a shear flow

We prove Theorem 1 in this section.

Proof of Theorem 1. It suffices to show that there exists some time t_0 such that

$$T(t_0, x, y) \le \theta_0 \text{ for all } (x,y) \in D. \tag{16}$$

Then it follows from the maximum principle that $T(t,x,y) \le \theta_0$ for all $t \ge t_0$, and hence T satisfies the linear advection-diffusion equation

$$T_t + Au(y)T_x = \kappa \Delta T$$

for $t \geq t_0$, which implies (13). We actually show that (16) holds at $t_0 = \kappa/v_0^2$ for sufficiently large A. Recall that $f(T) \leq T$, and hence $T(t,x,y)$ can be bounded from above using the maximum principle as follows:

$$T(t,x,y) \leq \Phi(t,x,y)e^{v_0^2 t/\kappa}. \tag{17}$$

Here the function $\Phi(t,x,y)$ satisfies the linear problem

$$\begin{aligned} &\Phi_t + Au(y)\Phi_x = \kappa\Delta\Phi \\ &\Phi(0,x,y) = T_0(x,y) \\ &\Phi(t,x,y) = \Phi(t,x,y+H). \end{aligned} \tag{18}$$

Furthermore, we have

$$\Phi(t,x,y) = \int_{-\infty}^{\infty} dz G(t,x-z)\Psi(t,z,y)$$

with the function $\Psi(t,x,y)$ satisfying the degenerate parabolic equation

$$\begin{aligned} &\Psi_t + Au(y)\Psi_x = \kappa\Psi_{yy} \\ &\Psi(0,x,y) = T_0(x,y) \\ &\Psi(t,x,y) = \Psi(t,x,y+H) \end{aligned} \tag{19}$$

and

$$G(t,x) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right).$$

We note that if $u(y)$ satisfies the H-condition (11) then the diffusion process defined by (19) has a unique smooth transition probability density. Indeed, the Lie algebra generated by the operators ∂_y and $\partial_t + u(y)\partial_x$ consists of vector fields of the form

$$\frac{\partial}{\partial y}, \frac{\partial}{\partial t} + u(y)\frac{\partial}{\partial x}, u'(y)\frac{\partial}{\partial x}, u''(y)\frac{\partial}{\partial x}, \ldots, u^{(n)}(y)\frac{\partial}{\partial x}, \ldots$$

which span $\mathbb{R}^2$ if $u(y)$ satisfies (11). Then the theory of Hörmander [12], and the results of Ichihara and Kunita [13] imply that there exists a smooth transition probability density $p_A(t,x,y,y')$ such that

$$\Psi(t,x,y) = \int\limits_{\mathbb{R}} dx' \int\limits_0^H dy' p_A(t, x-x', y, y')\, T_0(x',y').$$

In particular, the function $p_A(t)$ is uniformly bounded from above for any $t > 0$ [13]. Then we have

$$T(t,x,y) \leq e^{v_0^2 t/\kappa}\|\Phi(t)\|_{L^\infty_{x,y}} \leq e^{v_0^2 t/\kappa}\|\Psi(t)\|_{L^\infty_{x,y}} \leq e^{v_0^2 t/\kappa}\|p_A(t)\|_{L^\infty_{x,y}}\|T_0\|_{L^1_{x,y}}.$$

It is straightforward to observe that

$$p_A(t,x,y,y') = \frac{v_0}{A} p_0(t, \frac{v_0}{A}x, y, y')$$

with p_0 being the transition probability density for (19) with $A = v_0$. That is, p_0 satisfies

$$\frac{\partial p_0}{\partial t} + v_0 u(y)\frac{\partial p_0}{\partial x} = \kappa\frac{\partial^2 p_0}{\partial y^2}$$
$$p_0(0,x,y,y') = \delta(x)\delta(y-y')$$
$$p_0(t,x,y) = p_0(t,x,y+H).$$

Therefore we obtain

$$T(t,x,y) \le e^{v_0^2 t/\kappa}\frac{v_0}{A}\|p_0(t)\|_{L^\infty_{x,y}}\|T_0\|_{L^1_{x,y}} \le 2e^{v_0^2 t/\kappa}\frac{v_0}{A}\|p_0(t)\|_{L^\infty_{x,y}}LH$$

and in particular at time $t_0 = \frac{\kappa}{v_0^2}$ we have

$$T(t_0,x,y) \le C\frac{v_0}{A}\|p_0(t_0)\|_{L^\infty_{x,y}}LH \le \theta_0 \tag{20}$$

as long as

$$\frac{A}{v_0} \ge C\frac{\|p_0(t_0)\|_{L^\infty_{x,y}}LH}{\theta_0}.$$

Theorem 1 follows from (20) as explained in the beginning of this Section. □

We prove now Theorem 4 that shows that a sufficiently small plateau in the profile $u(y)$ is not an obstruction to quenching.

Proof of Theorem 4. Let us define the set

$$D_r = \mathbb{R}\times[a-r,a+r].$$

As before, it suffices to show that solution of (18) satisfies

$$\Phi(t_0,x,y) \le \frac{\theta_0}{e},\ t_0 = \frac{\kappa}{v_0^2} \tag{21}$$

and that is what we will do. First, we split the initial data for (19) into two parts: one supported on a strip D_{h_1}, containing the flat part of $u(y)$, and another supported outside it. We will choose $h_1 = C_1 l > h$ such that any solution of (19) that is independent of x and with initial data supported inside D_{h_1} will be small at time $t_0 = \kappa/v_0^2$. The second part is supported away from the strip D_h, where $u(y)$ is flat. Therefore for a sufficiently small time it behaves like a solution of (19) with advection satisfying the H-condition.

We choose h_1 as follows. Let $\phi(t,y)$ be a periodic solution of

$$\phi_t = \phi_{yy},\ \phi(0,y) = \phi_0(y),\ 0 \le \phi_0(y) \le 1$$

given by

$$\phi(t,y) = \sum_{j\in Z}\phi_j(0)e^{2ij\pi y/H - 4\kappa t\pi^2 j^2/H^2}.$$

Then we have

$$\left|\phi(t,y) - \frac{\|\phi(0,y)\|_{L^1_y}}{H}\right| \le 2\frac{\|\phi(0,y)\|_{L^1_y}}{H}\sum_{j\ge 1}e^{-4\kappa t\pi^2 j^2/H^2}.$$

A simple estimate shows that

$$\sum_{j\ge 1}e^{-\frac{4\kappa^2\pi^2 j^2}{v_0^2 H^2}} \le C\frac{Hv_0}{\kappa},$$

and hence

$$\left|\phi(t_0,y) - \frac{\|\phi(0,y)\|_{L^1_y}}{H}\right| \le C\frac{\|\phi(0,y)\|_{L^1_y}}{l},\ t_0 = \frac{\kappa}{v_0^2}. \tag{22}$$

Therefore we have

$$\phi(t_0,x,y) \le \frac{\theta_0}{10}$$

as long as

$$\|\phi(0,y)\|_{L^1_y} \le C_1 l, \tag{23}$$

where $C_1 = \theta_0 Hl/(10(l+CH))$. Let us choose h_1 so that (23) is automatically verified for initial data supported on D_{2h_1} :

$$h_1 \le C_1 l/4$$

with C_1 as in (23). Let us assume that the width of the interval $[a-h, a+h]$, on which $u(y)$ is constant satisfies

$$h \le \frac{h_1}{4}.$$

This condition determines the constant B in the statement of Theorem 4. We may now split the initial data for (19) as follows:

$$T_0(x,y) \le \chi_0(y) + \psi_0(x,y).$$

Here the smooth function $\chi_0(y)$ is supported in the interval $[a-2h_1, a+2h_1]$ while the function $\psi_0(x,y)$ is supported outside the set $[a-h_1, a+h_1]$. Both of these functions satisfy in addition $0 \le \chi_0(y), \psi_0(x,y) \le 1$. Then the function $\Phi(t,x,y)$ satisfies the inequality

$$\Phi(t,x,y) \le \chi(t,y) + \psi(t,x,y)$$

with the functions χ and ψ satisfying (19) with the initial data χ_0 and ψ_0, respectively. It follows from our choice of h_1 that

$$\chi(t_0,y) \le \frac{\theta_0}{10},$$

so it remains only to estimate $\psi(t_0,x,y)$. We will do it separately for (x,y) inside and outside of the strip $D_{h_1/2}$. For the points $(x,y) \in D_{h_1/2}$ we have:

$$\psi(t,x,y) = \int_{\mathbf{R}} dx' \int_0^H dy' p_A(t, x-x', y, y')\psi_0(x',y') \le \int_{\mathbf{R}} dx' \int_{|y'-a|\ge h_1} dy' p_A(t, x-x', y, y')$$
$$\le \mathcal{P}\left\{|W(t)| \ge \frac{h_1}{2}\right\} \le \frac{\theta_0}{10}$$

for sufficiently small t. Here $W(t)$ is the one-dimensional Brownian motion with diffusivity κ, and $\mathcal{P}$ denotes probability with respect to it. Thus (21) holds inside $D_{h_1/2}$.

In order to estimate the function $\psi(t,x,y)$ outside $D_{h_1/2}$ we introduce a profile $\tilde{u}(y)$ that coincides with $u(y)$ outside of the interval $[a-3h/2, a+3h/2]$ and satisfies the H-condition on the whole interval $[0,H]$. We define the process $(X(t), Y(t))$ by

$$dX(t) = u(Y(t))dt,\ dY(t) = \sqrt{2\kappa}dW(t),\ X(0) = x,\ Y(0) = y. \tag{24}$$

Consider the stopping time τ which is the first time when $Y(t)$ enters the interval $[a-3h/2, a+3h/2]$. Then we have

$$\begin{aligned}
&\psi(t,x,y) \le P_{x,y}\left\{(X(t),Y(t)) \in \text{supp }\psi_0\right\} = P_{x,y}\left\{(X(t),Y(t)) \in \text{supp }\psi_0 | \tau > t\right\} P_y(\tau > t) \\
&+P_{x,y}\left\{(X(t),Y(t)) \in \text{supp }\psi_0 | \tau < t\right\} P_y(\tau < t) \\
&\le P_{x,y}\left\{(\tilde X(t),\tilde Y(t)) \in \text{supp }\psi_0 | \tau > t\right\} P_y(\tau > t) + P_y(\tau < t).
\end{aligned}$$

Here $P_{x,y}$ denotes probability with respect to the process $(X(t), Y(t))$ starting at (x, y), while P_y denotes probability with respect to $Y(t)$ starting at y (recall that $Y(t)$ is independent of x). The process $X(t)$ for $t < \tau$ is identical to the process $(\tilde X(t), \tilde Y(t))$ defined by (24) with $u(Y)$ replaced by $\tilde u(Y)$. Therefore we have

$$\begin{aligned}
\psi(t,x,y) &\le P_{x,y}\left\{(\tilde X(t),\tilde Y(t)) \in \text{supp }\psi_0 | \tau > t\right\} P_y(\tau > t) + P_y(\tau < t) \\
&\le P_{x,y}\left\{(\tilde X(t),\tilde Y(t)) \in \text{supp }\psi_0\right\} + P_y(\tau < t).
\end{aligned}$$

Recall that $(x, y) \notin D_{h_1/2}$ and $h \le h_1/4$. Therefore the point y is a fixed distance away from the interval $[a - 3h/2, a + 3h/2]$. Hence we may choose $t_1 < t_0$ sufficiently small so that

$$P_y(\tau < t_1) \le \frac{\theta_0}{10}.$$

Furthermore, the function $\tilde\psi(t, x, y) = P_{x,y}\left\{(\tilde X(t),\tilde Y(t)) \in \text{supp }\psi_0\right\}$ satisfies (19) with the initial data

$$\tilde\phi(0,x,y) = \begin{cases} 1, & (x,y) \in \text{supp }\psi_0 \\ 0, & (x,y) \notin \text{supp }\psi_0 \end{cases}$$

However, $\tilde u(y)$ is quenching and thus we may choose A so large that

$$\tilde\psi(t_1,x,y) \le \frac{\theta_0}{10}.$$

Therefore we have at $t = t_1$:

$$\psi(t_1,x,y) \le \frac{\theta_0}{5}$$

and hence the same upper bound holds at $t = t_0 > t_1$. Therefore (21) holds also outside $D_{h_1/2}$, and Theorem 4 follows. The fact that $u(y)$ is strongly quenching follows from this property of $\tilde u(y)$. □

Theorem 5 is a simple corollary of Theorem 1.

Proof of Theorem 5. The set of all profiles $u(y)$ satisfying H-condition is dense in $C[0, H]$, so by Theorem 1 the set of strongly quenching profiles is dense. To complete the proof, we will show that if $\tilde u(y)$ satisfies H-condition, then there exists $\delta(\tilde u)$ such that if

$$\|u(y) - \tilde u(y)\|_{C[0,H]} < \delta(\tilde u)$$

then $u(y)$ is strongly quenching. From the proof of Theorem 1, we know that there exists a constant $C(\tilde u)$ such that the solution $\tilde\Psi(x, y, t)$ of the equation (19) with advection $A\tilde u(y)$ satisfies

$$\tilde\Psi(x,y,t_0) \le \frac{\theta_0}{e}$$

if the initial data $\tilde\Psi(x,y,0)$ is supported on the interval $[-L,L]$ with $L < C(\tilde u)A$ (recall that such inequality implies quenching of T with the same initial data). Let $\Psi(x,y,t)$ be a solution of (19) with advection $Au(y)$, and initial data supported in $[-L',L']$. Then by the Feynman-Kac formula (see, e.g. [9])

$$\Psi(x,y,t) \le P_{x,y}(x + A\int_0^t u(y+W(s))\,ds \in [-L',L']) \le$$

$$P_{x,y}(x + A\int_0^t \tilde u(y+W(s))\,ds \in [-L'-A\delta t, L'+A\delta t]) \le \tilde\Psi(x,y,t),$$

where $\tilde\Psi(x,y,t)$ is the solution of (19) with advection $Au(y)$ and initial data equal to the characteristic function of the interval $[-L'-A\delta t_0, L'+A\delta t_0]$ for $t \le t_0$. Now choose $\delta < C(\tilde u)/t_0$. Then

$$\Psi(x,y,t_0) \le \tilde\Psi(x,y,t_0) \le \frac{\theta_0}{e}$$

provided that $L' \le (C(\tilde u) - \delta t_0)A$, and hence $u(y)$ is strongly quenching. □

3. Quenching for a system

All results on quenching for equation (2) proved in the previous section extend directly to the case of the system (1). In this respect, the situation is similar to the case $u(y) = 0$, where all results on quenching proved by Kanel [14] for a single equation extend to the system case. We make an assumption $g(T) \le T$, which is just a normalization and corresponds to the condition $f(T) \le T$ in the case of a single equation. We take compactly supported initial data for the temperature, while for the concentration we assume $0 \le n(x,y,0) \le 1$. Notice that by the maximum principle, $n(x,y,t) \le 1$ for every t. Therefore, T satisfies

$$T_t + u(y)T_x - \kappa\Delta T \le T,$$

and from this point the analysis proceeds in the same way as for scalar equation (starting from (17)). We summarize the results in

Theorem 6. *Assume that $g(T) = 0$ for $T \le \theta_0$, and $g(T) \le T$. Suppose that the velocity profile $u(y) \in C_0^\infty[0,H]$ satisfies H-condition, then it is strongly quenching. The same conclusion holds if the profile $u(y)$ has a plateau of the size $h \le h_1 = Bl$ (where B is a universal constant). Moreover, the set of all quenching shear flows contains a dense open set in $C[0,H]$.*

4. Flame propagation

We prove Theorem 2 in this section. Let $T(t,x,y)$ be solution of (2) with the initial data as in (7). We will use the following result of Xin [25] that holds for more general types of advection (its version for the shear flow was also proved by Roquejoffre in [20]). Consider

$$T_t + u(x,y)\cdot\nabla T = \kappa\Delta T + \frac{v_0^2}{\kappa}f(T) \tag{25}$$
$$T(0,x,y) = T_0(x,y) \in L^2(D)$$

with $u(x,y)$ being periodic in both variables, and $0 \le T_0(x,y) \le 1$.

Proposition 1 (Xin). *Assume that the initial data in (25) are such that*

$$\lim_{|x|\to\infty} T_0(x,y) = 0 \text{ uniformly in } D$$

and

$$T_0(x,y) > \theta_0 + \eta \text{ for } |x| \le L.$$

Then there exists $L_1(\eta,u)$ that depends on η and $u(x,y)$ such that if $L \ge L_1$ then

$$\lim_{t\to\infty} T(x-st,y,t) = \begin{cases} 1, & \text{if } c_l < s < c_r \\ 0, & \text{if } c_l > s \text{ or } s > c_r \end{cases}$$

Here c_l and c_r are the speeds of left and right traveling waves, respectively ($c_l < 0$, $c_r > 0$).

The right traveling wave is described by (4), (5), and the left traveling wave satisfies (4) and

$$\lim_{s\to-\infty} U(s,y) = 0,\ \lim_{s\to+\infty} U(s,y) = 1,\ U_s(s,y) > 0.$$

Notice that Theorems 1 and 4 imply an estimate $L_1 \ge CA$ if u has no flat parts larger than certain critical size. On the other hand, Theorem 2 shows that $L_1 = h_0$ is independent of A if u has a sufficiently large flat part.

Proof of Theorem 2. The proof of Theorem 2 proceeds in several steps. We consider the initial data satisfying (9). First we find h_0 such that there exists a C^2 function $\phi(x,y)$ such that $0 \le \phi < \theta_0 + \eta$, and

$$\kappa\Delta\phi + \frac{v_0^2}{\kappa} f(\phi) \ge 0 \tag{26}$$

and ϕ vanishes on the boundary of the disc of radius h_0 centered at the point $(0,a)$:

$$\phi|_{\partial\Omega_0} = 0,\ \Omega_0 = B((0,a);h_0).$$

Then in the system of coordinates that moves with the speed $\bar{u}$ the function $\phi(x,y)$ is a sub-solution of (2) in Ω_0. Therefore, initial data that start above ϕ will not decay to zero. Next we consider the special solution $\Phi(t,x,y)$ of (2) with the initial data given by

$$\Phi_0(x,y) = \begin{cases} \phi(x,y) & (x,y) \in \Omega_0 \\ 0 & (x,y) \notin \Omega_0 \end{cases} \tag{27}$$

We show that $\Phi(t,x,y)$ satisfies (15) and that implies that (15) holds for arbitrary initial data $T_0 \ge \Phi_0$, in particular, such as described in Theorem 2.

Step 1. Construction of a stationary sub-solution. Choose θ_1, θ_2 so that $\theta_0 + \eta > \theta_2 > \theta_1 > \theta_0$ and define $f_1(T)$ by

$$f_1(T) = \begin{cases} 0, & T \le \theta_1 \\ \dfrac{f(\theta_2)(T-\theta_1)}{\theta_2-\theta_1}, & \theta_1 \le T \le \theta_2 \\ f(T), & \theta_2 \le T \le 1 \end{cases}$$

The function $f(T)$ is Lipschitz continuous, and hence we may choose θ_1 and θ_2 so that $f_1(T) \le f(T)$. Therefore if ϕ satisfies

$$\kappa\Delta\phi + \frac{v_0^2}{\kappa} f_1(\phi) = 0 \tag{28}$$

then ϕ satisfies (26). We are going to construct an explicit radial solution $\phi(r)$ of (28) with the "initial" conditions

$$\phi(0) = \theta_2, \ \frac{\partial \phi}{\partial r}(0) = 0.$$

Indeed, $\phi(r)$ is given explicitly by

$$\phi(r) = \theta_1 + (\theta_2 - \theta_1) J_0\left(\frac{r v_0 \sqrt{\alpha}}{\kappa}\right), \ \alpha = \frac{f(\theta_2)}{\theta_2 - \theta_1} \text{ for } r \le R_1 = \frac{\kappa \xi_1}{v_0 \sqrt{\alpha}}. \tag{29}$$

Here $J_0(\xi)$ is the Bessel function of order zero, and ξ_1 is its first zero. Furthermore, we have

$$\phi(r) = B \ln \frac{r}{R}, \text{ for } R_1 \le r \tag{30}$$

with B and R determined by matching (29) and (30) at $r = R_1$. Then we get

$$R = l \xi_1 \sqrt{\frac{\theta_2 - \theta_1}{f(\theta_2)}} \exp\left[\frac{\theta_1}{(\theta_2 - \theta_1)\xi_1 |J_0'(\xi_1)|}\right] = Cl, \ l = \frac{\kappa}{v_0}.$$

Then $\phi(r)$ satisfies

$$\begin{aligned} &\kappa \Delta \phi + \frac{v_0^2}{\kappa} f(\phi) \ge 0, \ \phi(r) > 0, 0 \le r < R \\ &\phi(R) = 0. \end{aligned}$$

Thus we will take the critical size of plateau in the velocity profile to be $2R$ so that the disc of radius R will fit in.

Step 2. A sub-solution. Let us now assume that $h \ge h_0 = R$. We make a coordinate change

$$\xi = x - \bar{u} t$$

In new coordinates we have a function $T(t, \xi, y)$ that solves

$$T_t + A(-\bar{u} + u(y)) T_\xi = \kappa \Delta_{\xi, y} T + \frac{v_0^2}{\kappa} f(T) \tag{31}$$

with the initial data given by (27):

$$T(0, \xi, y) = \begin{cases} \phi(\xi, y), & (\xi, y) \in \Omega_0 = B((0, a); h_0) \\ 0, & (\xi, y) \notin \Omega_0 \end{cases}.$$

Observe that $\phi(\xi, y)$ satisfies (26) inside Ω_0 since $u(y) = \bar{u}$ in Ω_0. Moreover, $T(t, \xi, y) \ge \phi(\xi, y)$ on $\partial \Omega_0$, where ϕ vanishes. Therefore the maximum principle implies that inside Ω_0 we have

$$T(t, \xi, y) \ge \phi(\xi, y) \text{ for all } t > 0 \text{ and } (\xi, y) \in \Omega_0. \tag{32}$$

Note that $T_h(t, \xi, y) = T(t + h, \xi, y)$ solves (31) with the initial data

$$T_h(0, \xi, y) = T(h, \xi, y) \ge T(0, \xi, y). \tag{33}$$

The inequality in (33) follows from (32) inside Ω_0 and the fact that $T(t, x, y) \ge 0$ outside Ω_0. Therefore we have

$$T(t + h, \xi, y) - T(t, \xi, y) \ge 0 \text{ for all } h > 0, \, t > 0 \text{ and } (\xi, y) \in D$$

and thus the limit

$$\bar{T}(\xi, y) = \lim_{t \to \infty} T(t, \xi, y)$$

exists since $T \le 1$. Moreover, the standard parabolic regularity implies that $T(t,\xi,y)$ converges to $\bar T(\xi,y)$ uniformly on compact sets together with its derivatives up to the second order. Therefore $\bar T$ satisfies the stationary problem

$$A(-\bar u+u(y))\bar T_\xi=\kappa\Delta_{\xi,y}\bar T+\frac{v_0^2}{\kappa}f(\bar T). \tag{34}$$

We also have

$$\bar T(\xi,y)>\phi(\xi,y) \text{ for } (\xi,y)\in\Omega_0.$$

It is easy to show using the sliding method of Berestycki and Nirenberg [5] that for any $(\xi,y)\in D_h$

$$\bar T(\xi,y)>\phi(\xi-r,y) \tag{35}$$

where the right side is any translation of ϕ along the ξ axis. Indeed, assume that there exists the smallest (say, positive) r such that $\bar T(\xi_0,y_0)=\phi(\xi_0-r,y_0)$ at some point $(\xi_0,y_0)\in D_h$. Then the strong maximum principle implies that $\bar T(\xi,y)=\phi(\xi-r,y)$ for all (ξ,y) inside the translate $\Omega_r=\Omega_0-re_1$ of the disc Ω_0. But that contradicts the fact that $\bar T(\xi,y)>\phi(\xi-r,y)=0$ on the boundary $\partial\Omega_r$. Then (35) implies that

$$\bar T(\xi,a)>\theta_2=\sup_{(\xi,y)\in\Omega_0}\phi(\xi,y),\ \text{for all } \xi\in\mathbb{R}. \tag{36}$$

The next two lemmas show that (34) and (36) imply that $\bar T(\xi,y)\equiv 1$.

Lemma 1. *Let $\bar T$ be a solution of (31) such that $0\le\bar T\le 1$. Then we have*

$$\int_D d\xi dy f(\bar T(x,y))<\infty,\ \int_D d\xi dy|\nabla\bar T|^2<\infty. \tag{37}$$

Proof. In order to show that integral of $f(\bar T)$ is finite we integrate (34) over the set $(-L+\zeta,L+\zeta)\times[0,H]$ with L large and $\zeta\in[0,l]$. We get

$$A\int_0^H dy[-\bar u+u(y)][\bar T(L+\zeta,y)-\bar T(-L+\zeta,y)]=\kappa\int_0^H dy[\bar T_\xi(L+\zeta,y)-\bar T_\xi(-L+\zeta,y)]$$
$$+\frac{v_0^2}{\kappa}\int_0^H dy\int_{-L+\zeta}^{L+\zeta}d\xi f(\bar T(\xi,y)$$

and average this equation in $\zeta\in[0,l]$:

$$\frac{A}{l}\int_0^l d\zeta\int_0^H dy[-\bar u+u(y)][\bar T(L+\zeta,y)-\bar T(-L+\zeta,y)]$$
$$=\frac{\kappa}{l}\int_0^H dy[\bar T(L+l,y)-\bar T(L,y)-\bar T(-L+l,y)+\bar T(-L,y)]$$
$$+\frac{v_0^2}{l\kappa}\int_0^l d\zeta\int_0^H dy\int_{-L+\zeta}^{L+\zeta}d\xi f(\bar T(\xi,y).$$

Therefore we obtain

$$\frac{v_0^2}{\kappa}\int\limits_0^H dy \int\limits_{-L+l}^{L} d\xi f(\bar T(\xi,y)) \le \frac{4\kappa H}{l} + 2HA[\bar u + \|u\|_\infty]$$

for all L, and hence the first inequality in (37) holds. In order to obtain the second inequality we multiply (34) by $\bar T$ and perform the same integration and averaging as above. This leads to

$$\begin{aligned}
&\frac{A}{2l}\int\limits_0^l d\zeta \int\limits_0^H dy[-\bar u + u(y)][\bar T^2(L+\zeta,y) - \bar T^2(-L+\zeta,y)]\\
&= \frac{\kappa}{2l}\int\limits_0^H dy[\bar T^2(L+l,y) - \bar T^2(L,y) - \bar T^2(-L+l,y) + \bar T^2(-L,y)]\\
&+\frac{1}{l}\int\limits_0^l d\zeta \int\limits_0^H dy \int\limits_{-L+\zeta}^{L+\zeta} d\xi\left[\frac{v_0^2}{\kappa}\bar T f(\bar T) - \kappa|\nabla \bar T|^2\right] \qquad (38)
\end{aligned}$$

and then the second inequality in (37) follows from (38) and the first inequality in (37). □

Lemma 2. *The limit function $\bar T(\xi,y)\equiv 1$*

Proof. Notice that $\bar T$ can not achieve local minima in D as follows from the maximum principle. Therefore if we define $\mu_{\alpha,\beta} = \min\limits_{\alpha\le\xi\le\beta, 0\le y\le H} \bar T(\xi,y)$ and $\mu(\alpha) = \min\limits_{0\le y\le H} \bar T(\alpha,y)$ then $\mu_{\alpha,\beta} = \mu(\alpha)$ or $\mu_{\alpha,\beta} = \mu(\beta)$. Furthermore, if $\bar T(\xi,y) = 1$ at some point, then $\bar T(\xi,y)\equiv 1$ everywhere by the strong maximum principle. In particular, if $\mu(0) = 1$, then $\bar T(\xi,y)\equiv 1$. Therefore we consider only the case that $\mu(0) < 1$ and argue by contradiction. We have either $\mu(\xi) < \mu(0)$ for any $\xi > 0$, or for any $\xi < 0$. Otherwise the minimum of $\bar T$ over the set $[-\xi,\xi]\times[0,H]$ would be achieved inside. Let us assume without loss of generality that $\mu(\xi) < \mu(0)$ for any $\xi > 0$. Consider

$$\delta = \min\left(\frac{1-\mu(0)}{2}, \frac{\theta_2-\theta_0}{2}\right).$$

For any $\xi > 0$, we have three options.

1. $\bar T(\xi,y)\in[\mu(\xi),\mu(\xi)+\delta]$ for any $y\in[0,H]$, and $\mu(\xi)\ge\theta_0+\delta$. In this case, by definition of δ, properties (3) of f, and since $\mu(\xi)\le\mu(0)<1$, we have

$$\int\limits_0^H f(\bar T(\xi,y))\,dy \ge CH.$$

2. $\mu(\xi) < \theta_0+\delta$. Inequality (36) implies that there exist y_1, y_2 such that $\bar T(\xi,y_1) = \theta_2-\delta$, $\bar T(\xi,y_2) = \theta_2$. Then

$$\left|\int\limits_{y_1}^{y_2} f(\bar T(\xi,y))\,dy\right| \ge C|y_2-y_1|,$$

$$\int_{y_1}^{y_2} |\bar{T}_y|^2\,dy \geq |y_2-y_1|^{-1}\left(\int_{y_1}^{y_2} |\bar{T}_y|\,dy\right)^2 \geq \delta^2|y_2-y_1|^{-1}.$$

Therefore,

$$\int_H^0 f(\bar{T}(\xi,y))\,dy \int_{y_1}^{y_2} |\bar{T}_y|^2\,dy \geq C\delta^2 \tag{39}$$

in this case.
3. The remaining option is that $\mu(\xi) \geq \theta_0+\delta$, and there exists $y\in[0,H]$ such that $\bar{T}(\xi,y) > \mu(\xi)+\delta$. In this case, an argument identical to the reasoning of option two leads to the same bound (39).
Overall, we see that for any $\xi>0$,

$$\int_0^H \left(f(\bar{T}(\xi,y)+|\nabla\bar{T}|^2(\xi,y)\right)\,dy \geq C,$$

where C depends only on H and δ. But this contradicts directly Lemma 1. □

Using Proposition 1, we can now complete the proof of Theorem 2. Notice that even though we showed $\bar{T}(\xi,y)\equiv 1$, we still have to show the limiting function is the same in the original coordinates. Lemma 2 implies that there exists a time t_0 so that for all $\xi \in [0, L_0(\theta_2-\theta_0, Au(y))]$ and all $y\in[0,H]$ we have

$$T(t_0,\xi,y) \geq \theta_0+\eta.$$

Here $L_0(\eta,v)$ is defined by Proposition 1. Then we may apply Proposition 1 with $s=0$ in the original coordinates (x,y), and initial data $T(t_0,x,y)$, and get (15). The fact that $c_l<0<c_r$ follows from (8) and, for instance, results of [3]. □

The sub-solution we constructed in Theorem 2 is also useful for a proof of Theorem 3. As mentioned in the Introduction, we only need to show the lower bound, the upper bound is contained in [7]. We begin the proof with two auxiliary lemmas.

Lemma 3. *Let $u(y)$ be a function that is constant on an interval:*

$$u(y)=\bar{u},\;\; y\in[a-h,a+h]$$

that may or may not satisfy the mean zero condition (8). Assume that $h\geq h_0$ with h_0 given in Theorem 2. Then we have $c_l(u)\leq\overline{u}\leq c_r(u)$ (recall that $c_l(u)$ and $c_r(u)$ are the velocities of unique left and right traveling fronts).

Proof. Let T_0 be an initial data as in Theorem 2 such that

$$T_0(x,y) \geq \begin{cases} \phi(x,y), & (x,y)\in\Omega_0 = B((0,a);h_0) \\ 0, & (\xi,y)\notin\Omega_0 \end{cases}$$

with the function $\phi(x,y)$ constructed in the proof of Theorem 2 and given explicitly by (29) and (30). Then we have

$$T(t,x-\bar{u}t,y)\geq\phi(x,y)$$

and therefore Proposition 1 implies that $c_l\leq\bar{u}\leq c_r$ because $T(t,x-\bar{u}t,y)$ may not go to zero as $t\to\infty$. □

Let y_0 be a point at which $|u_+|_{h_0}$ is achieved: $u(y) \geq |u_+|_{h_0}$ for $y \in [y_0 - h_0, y_0 + h_0]$. We define now a new velocity field

$$v(y) = \begin{cases} u(y), & y \notin [y_0 - h_0, y_0 + h_0] \\ |u_+|_{h_0}, & y \in [y_0 - h_0, y_0 + h_0] \end{cases} \tag{40}$$

so that $u(y) \geq v(y)$.

Lemma 4. *Let $u(y) \geq v(y)$ be two velocity fields. Then the corresponding right traveling front speeds satisfy $c_r(u) \geq c_r(v)$. Similarly, if $u(y) \leq v(y)$, then $c_l(u) \leq c_l(v)$.*

Proof. Let $T^{u,v}(t,x,y)$ be solutions of

$$T^u_t + Au(y)T^u_x = \kappa \Delta T^u + \frac{v_0^2}{\kappa} f(T^u)$$

and

$$T^v_t + Av(y)T^v_x = \kappa \Delta T^v + \frac{v_0^2}{\kappa} f(T^v)$$

with the same initial data

$$T^u(0,x,y) = T^v(0,x,y) = T_0(x,y).$$

The function $T_0(x,y)$ is assumed to be front-like and monotonic. That is, $T_0(x,y) = 1$ for $x \leq -L$, $T_0(x,y) = 0$ for $x \geq L$ and

$$\frac{\partial T_0}{\partial x}(x,y) \leq 0.$$

Then by maximum principle applied to the equation for $\partial T^{u,v}/\partial x$ we have

$$\frac{\partial T^{u,v}}{\partial x}(t,x,y) \leq 0 \text{ for all } t > 0$$

and hence

$$T^v_t + Au(y)T^v_x - \kappa \Delta T^v - \frac{v_0^2}{\kappa} f(T^v) = A(u(y) - v(y))T^v_x \leq 0.$$

Therefore by maximum principle we have

$$T^u(t,x,y) \geq T^v(t,x,y). \tag{41}$$

However, by results of [25, 20] similar to Proposition 1 but for the front-like initial data we have

$$\lim_{t\to\infty} T^v(x - st, y, t) = \begin{cases} 1, & \text{if } s < c_r(v) \\ 0, & \text{if } s > c_r(v) \end{cases}$$

and similarly for T^u. Then $c_r(v) > c_r(u)$ would be incompatible with (41), and hence $c_r(v) \leq c_r(u)$. Naturally, an analogous result is true for left traveling fronts. □

Proof of Theorem 3. In order to finish the proof of Theorem 3 we apply Lemma 4 to $v(y)$ given by (40), and observe that $c_r(v)$ satisfies the estimate in Theorem 3 according to Lemma 3. □

Finally, we remark that in the situation of Theorem 2 strong advection not only does not quench the flame, but has directly opposite effect, according to Theorem 3 and Proposition 1.

Corollary 1. *Let $u(y) = \bar{u} = const$ for $y \in [a-h, a+h]$ for some $a \in [0, H]$, and h is larger than or equal to the critical size h_0. Then, provided that the size of initial data $L \geq h_0$, we have*

$$\lim_{t\to\infty} T(x-st, y, t) = \begin{cases} 1, & \text{if } c_l < s < c_r \\ 0, & \text{if } c_l > s \text{ or } s > c_r \end{cases}$$

Moreover, $c_r \geq |u_+|_{h_0}$ and $c_l \leq -|u_-|_{h_0}$.

Proof. The first statement is a direct corollary of Proposition 1 and Theorem 2. The second statement is the content of Theorem 3. □

5. A lower bound for the quenching size

Recall that the burning rate is defined by

$$V(t) = \int_D T_t(x, y)\frac{dxdy}{H}.$$

We also say that nonlinearity $f(T)$ is of concave KPP class if

$$f(0) = f(1) = 0,\ f(T) > 0 \text{ for } 0 < T < 1,\ f''(T) < 0.$$

We have previously shown [7] that for such nonlinearities and for front-like initial conditions the burning rate in the presence of any advection is bounded from below by Cv_0, more precisely:

$$V(t) \geq Cv_0(1 - e^{-Cv_0^2 t/\kappa}). \tag{42}$$

The physical meaning of (42) is that no advection may slow up the burning significantly. We now show that similarly there is a fixed size of initial data that is quenched by any shear flow. That means that no shear flow may help prevent quenching of initial data with a fixed small support. Let $T(t, x, y)$ be solution of

$$T_t + u(y)T_x = \kappa\Delta T + \frac{v_0^2}{\kappa}f(T) \tag{43}$$

with the initial data $T(0, x, y) = T_0(x, y)$ and periodic boundary conditions (6).

Proposition 2. *There exists a constant $C > 0$ that depends only on the non-linearity $f(T)$ such that given any initial data T_0 with*

$$\int_D T_0(x, y)dxdy \leq Cl^2,\ \ l = \frac{\kappa}{v_0},$$

we have

$$T(t, x, y) \to 0 \text{ uniformly in } D \text{ as } t \to +\infty \tag{44}$$

for all $u(y) \in C[0, H]$.

Proof. We will show that solution of (43) with $f(T)$ replaced by a KPP nonlinearity $\tilde{f}(T) = MT(1-T)$ and the same initial data drops below θ_0 before time $t_0 = \kappa/v_0^2$. That will imply (44). Choose M such that $f(T) \leq \tilde{f}(T) = MT(1-T)$ (such M exists since f is Lipschitz continuous) and let $\tilde{T}$ be the solution of

$$\tilde{T}_t + u(y)\tilde{T}_x = \kappa\Delta\tilde{T} + \frac{v_0^2}{\kappa}\tilde{f}(\tilde{T})$$

with initial data T_0 and periodic boundary conditions (6). Let us also define

$$\tilde{V}(t) = \int_D \tilde{T}_t(x,y)\frac{dxdy}{H} = \frac{Mv_0^2}{\kappa}\int_D \tilde{T}(1-\tilde{T})\frac{dxdy}{H}.$$

Note that $\tilde{T}(t,x,y) \geq T(t,x,y)$ and hence Proposition 2 follows from the following Lemma.

Lemma 5. *There exists a constant $C > 0$ that depends only on the non-linearity $f(T)$ such that given any initial data T_0 with*

$$\int_D T_0(x,y)dxdy \leq Cl^2, \;\; l = \frac{\kappa}{v_0},$$

and any $u(x,y) \in C^1(D)$, there exists a time $t_1 < t_0$ such that $\tilde{T}(t,x,y) \leq \theta_0$ for all $(x,y) \in D$.

Proof. The following inequality holds for $\tilde{V}(t)$:

$$\tilde{V}(t) + \frac{\kappa}{Mv_0^2}\frac{d\tilde{V}(t)}{dt} \geq 2\kappa \int_D |\nabla \tilde{T}|^2 \frac{dxdy}{H}.$$

We define the set

$$\mathcal{S}(t) = \left\{ y \in [0,H] : \; \exists x \in \mathbb{R} \text{ such that } T(t,x,y) \geq \theta_0/e^{2M} \right\} \subset [0,H].$$

Observe that there exists a constant $\delta > 0$ that depends only on M and θ_0 such that if the Lebesgue measure $|\mathcal{S}(\tau)| \leq \delta l$ at some time $\tau \in [0, t_0/2]$, then $\|\tilde{T}(t_0)\|_{L^\infty} \leq \theta_0$. Indeed, we have then

$$\tilde{T}(\tau,x,y) \leq \chi(y) + \psi(x,y), \;\; \|\chi(y)\| \leq \delta l, \;\; \psi(x,y) \leq \theta_0/e^{2M}.$$

Therefore we have with C as in (22)

$$T(t_0,x,y) \leq e^M \left[(C+1)\delta + \frac{\theta_0}{e^{2M}} \right] \leq \theta_0$$

for a sufficiently small δ. Hence it suffices to consider the case when $|\mathcal{S}(t)| \geq \delta l$ for all $t \in [0, t_0/2]$. We claim that then at any time $t \in [0, t_0/2]$ we have

$$\int_D |\nabla \tilde{T}|^2 dxdy \int_D \tilde{f}(\tilde{T})dxdy \geq Cl^2 \tag{45}$$

with the constant C depending only on M and θ_0. This may be proved similarly to Lemma 2 in [7]. Therefore we obtain

$$\tilde{V}(t) + \frac{\kappa}{Mv_0^2}\frac{d\tilde{V}(t)}{dt} \geq \frac{2Cv_0^2 l^2}{H^2 \tilde{V}(t)}.$$

Thus we have

$$\tilde{V}(t) \geq \frac{Cv_0 l}{H} \text{ for } t_0/2 \leq t \leq t_0.$$

Therefore

$$\int_D \tilde{T}(t_0,x,y) \geq \int_D T_0(x,y)dxdy + C\kappa t_0.$$

However, we have an a priori bound

$$\int\limits_D \tilde{T}(t_0, x, y)dxdy \le e^M \int\limits_D T_0(x, y)dxdy$$

which contradicts the previous inequality if

$$\int\limits_D T_0(x, y)dxdy \le \frac{C\kappa t_0}{e^M - 1} = C_M l^2. \tag{46}$$

Therefore $T(t, x, y)$ has to drop below θ_0 if the initial data satisfies (46) and Lemma 5 is proved. □

Proposition 2 then follows from Lemma 5. □

Remark. The uniform quenching size in Proposition 2 is optimal. Indeed, the sub-solution we have constructed in Section 4 for shear flows with a flat part has L^1 norm Cl^2, and thus one cannot expect that initial data with L^1-norm larger than Cl^2 are quenched by all shear flows.

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