# Radiative Transfer of Sound Waves in a Random Flow: Turbulent Scattering and Mode-Coupling

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June 29, 2000

**Abstract.** We study the sound wave propagation in a random flow, whose mean flow is large compared with its fluctuation, in the infinite three-dimensional space. We consider the intermediate regime, where the range of acoustic wave numbers overlaps with the range of turbulent wave numbers.

We use the multi-scale expansions systematically to derive the radiative transport equations that describe the evolution of acoustic correlation and the turbulent scattering and mode-coupling of sound waves. We show that, because of the flow-straining term, the turbulent scattering becomes nonconservative in general and, depending on the propagation direction, a sound wave can gain as well as lose energy through mode coupling. We calculate the attenuation/amplification coefficients due to mode coupling and turbulent scattering with flow-straining. These coefficients depict interesting dependence on the propagating direction and the wave length of sound wave.

We also obtain the diffusion equations on the physical space and, thus, further reduced the dimension of the flow-acoustic equations.

**Keywords.** Radiative transfer, Wigner function, acoustics.

**AMS subject classification.** Primary 76Q05; Secondary 82D30.

**Abbreviated title.** Acoustic transfer in random flows.

# Contents

# 1 Introduction

Flow acoustics concerns three different processes: propagation of sound through flow, generation of sound by flow and generation of flow by sound. Correspondingly, there are three main effects: scattering (and refraction), acoustic radiation and absorption. Refraction alters the direction of a beam of sound and is well studied in geometrical acoustics. Scattering is the redistribution of acoustic energy among different wave numbers or different components of the same wave number and results in spectral or directional broadening of the sound wave. Both refraction and scattering processes are commonly assumed to preserve the acoustic energy. This is not the case, however, when the effect of flow-straining is taken into account. As a sound wave propagates through an extensive body of the turbulence it may be attenuated or amplified through two mechanisms: wave mode-coupling and scattering. These effects are usually small, but can be important in an intense turbulent flow, like a jet or a grid turbulence in a wind tunnel [4], [5], [26].

Turbulent scattering of sound waves has been much studied both theoretically and experimentally since the single-scattering theory of Blokhintzev [3], Lighthill [22] and Kraichnan [19] (see [27] and [16] and the references therein). The experimental studies have been mainly of sound propagation in the atmosphere or oceans, but relatively few experiments have been conducted under laboratory conditions. This is more so when it comes to the study of turbulent *absorption* of sound waves, with [17], [24] and [13] being notable exceptions. These experiments suggest loss of acoustic energy which can not be accounted for by turbulent scattering or refraction caused by the variation of the mean flow. These experiments typically have turbulent intensity of 20%, turbulent Mach number on the order of $10^{-3}$, the integral scale roughly 10 cm and the Kolmogorov dissipation scale roughly 1 mm. The acoustic wave lengths used (e.g., 600 to 5000 Hz in [17]) in the experiments are typically larger than the integral scale. This is a *long* wave regime. The work [17] suggests a turbulent absorption mechanism which is frequency independent in the above frequency range in contrary to the frequency-square dependence of absorption by molecular dissipation. Theories have been proposed to explained this phenomena qualitatively ([26], [15]). The work [26] invokes the semi-empirical viscoelasticity theory of turbulence (see also [7]); while [15] extends Lighthill's approach. Both assume certain temporal structure of the turbulence and ignore the effect of the mean flow, but acknowledging that the absorption can become significantly greater in the presence of a mean flow (see also [14]).

We have also seen significant advances in the study of the short wave regime where the geometrical acoustics or the parabolic approximation is valid [27], [30]. In the present study we consider the intermediate regime where the sound wave lengths are comparable to the sizes of the turbulent eddies under the influence of a mean flow. This regime would correspond to the high-end audible range or the low-end ultrasonic frequencies in the experimental setting of [17]. Under such a circumstance turbulent scattering and absorption of sound waves are expected to be more pronounced.

Our main goal is to derive the transport equation for the phase-space distribution of the sound field with the explicit formulas for the scattering cross-section and the attenuation/amplification coefficient. The radiative transport equations describe naturally the aforementioned mechanisms of turbulent scattering and mode-coupling. We show that, because of the flow-straining term, the turbulent scattering becomes nonconservative in general and, depending on the propagation direction, a sound wave can gain as well as lose energy through mode coupling (i.e. the flow can emit acoustic radiation as a result of flow-straining).

## 2 Phase-space formulation

### 2.1 Flow-acoustic equations

Flow field and sound field are different modes of the total fluid motion described by the compressible fluid equations; they are distinguished by dispersion relation (see Section 3.2). Although flow acoustics is nonlinear in general, the fraction of the total fluid energy contained in the acoustic field is very small in a subsonic flow (the ratio of sound-to-flow-amplitude is of order $10^{-3}$ for the loudest sounds of interest.) The mode coupling/conversion is more important in this case.

We think of the sound (generated by the flow or other sources) as a small perturbation of a background flow field and study the sound propagation via the linearized Euler equations for weakly compressible, homentropic fluids:

$$\frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p + \frac{1}{\kappa_0} \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p_1 = 0 \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho_0} + \mathbf{u} \cdot \nabla \mathbf{V} = 0 \tag{2}$$

where $p_1, \mathbf{V}$ are the pressure and the velocity fields of the underlying random, incompressible flow, $\rho_0 \approx$ const. is the density, $\kappa_0 = 1/(c_0^2 \rho_0)$ the compressibility of the fluid, and $p, \mathbf{u}$ are the pressure and velocity fluctuations due to the perturbation by sound waves. Here $c_0$ is the sound speed of the (still) fluid and assumed to be a constant.

When $\kappa_0$ is relatively small, we can make the following approximation

$$\frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p + \frac{1}{\kappa_0} \nabla \cdot \mathbf{u} = 0 \tag{3}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho_0} + \mathbf{u} \cdot \nabla \mathbf{V} = 0. \tag{4}$$

That is, the turbulent pressure field is neglected. Further simplification can be made under suitable conditions by neglecting altogether the lowest order terms responsible for flow-straining effect:

$$\frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p + \frac{1}{\kappa_0} \nabla \cdot \mathbf{u} = 0 \tag{5}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho_0} = 0. \tag{6}$$

This approximation is valid for very high frequency waves.

We do not consider the effect of solid structure (such as walls, edges, corners), so Eq. (1)-(2), (3)-(4) or (5)-(6) are studied in an infinite medium (see [16]).

The effect of molecular viscosity is neglected. Although viscosity is ultimately responsible for the conversion of mechanical energy into heat, it has a negligible influence on the transfer of energy between the sound and the turbulence. The incompressibility of the flow is not essential to our approach; we assume it to simplify the presentation of the method and the results.

Specifically, we assume the background flow field has a small turbulent intensity with a mean flow, $\bar{\mathbf{u}}$, and write the velocity field as

$$\mathbf{V}(\mathbf{x}) = \bar{\mathbf{u}} + \sqrt{\varepsilon} \mathbf{v}(\mathbf{x}) \tag{7}$$

where $\sqrt{\varepsilon}$ is the turbulent intensity ($\ll 1$, typical of wind tunnels), $\sqrt{\varepsilon}\mathbf{v}(\mathbf{x})$ the turbulent velocity fluctuation assumed to be a mean zero, time independent, space homogeneous, divergence free random field with covariance matrix $\mathbf{R} = [R_{ij}(\mathbf{x})]$, $R_{ij}(\mathbf{x}) = \langle v_i(\cdot) v_j(\cdot + \mathbf{x})\rangle$. The homogeneity of the turbulence is a reasonable assumption for, e.g., the core region of fully developed turbulent (pipe, jet or grid) flows, or the outer region of a turbulent boundary layer. The incompressibility of the flow requires that the (turbulent) Mach number measured in the reference frame of the mean flow is infinitesimally small ($M_f \equiv \sqrt{\varepsilon\langle|\mathbf{v}|^2\rangle}/c_0 \ll 1$); while the mean-flow Mach number ($M_a \equiv |\bar{\mathbf{u}}|/c_0$) is assumed to be small to moderate. We assume for simplicity a *uniform* mean flow, in which case the mean-flow Mach number can be arbitrary. Stratified wind velocity profile is an important factor in sound propagation in the atmosphere; refraction due to variation of the mean flow can be accounted for by geometrical acoustics or parabolic approximation. Such an effect is decoupled from the turbulent scattering because of separation of scales.

When the turbulent intensity is small, time dependence of the turbulent fluctuation is negligible compared to the pulsation seen from the moving reference frame of the mean flow. This is closely related to the Taylor hypothesis of frozen turbulence (see [11]). Also, when the turbulent Mach number is small and the incident sound wave lengths are comparable to the eddy sizes, the frequencies of the turbulent pulsations are much smaller than those of the incident sound. Thus only the spatial correlations of the turbulence is pertinent to sound propagation and the turbulence is effectively frozen for our purpose.

The covariance function $R_{ij}$ can be determined from its Fourier transform, which take the following form in the case of locally isotropic turbulence

$$\widehat{R}_{ij}(\mathbf{k}) := \mathcal{R}(|\mathbf{k}|)\left(\delta_{ij} - k_i k_j |\mathbf{k}|^{-2}\right)|\mathbf{k}|^{1-d}, \quad \forall i, j.$$

The factor $(\delta_{ij} - k_i k_j |\mathbf{k}|^{-2})$, resulting from the incompressibility assumption, is the projection unto the plane orthogonal to $\mathbf{k}$. A typical example is given by the power-law spectrum

$$\mathcal{R}(|\mathbf{k}|) = R_0 |\mathbf{k}|^{-\nu}, \quad \ell_0^{-1} \ll |\mathbf{k}| \ll \ell_1^{-1}, \quad \text{some } \nu \in (-\infty, \infty),\ R_0 > 0, \tag{8}$$

and decaying rapidly elsewhere. Kolmogorov's spectrum corresponds to $\nu = 5/3$ with $\ell_1, \ell_0$ being the dissipation and the integral lengths, respectively ([11]).

The form of the velocity field (7) is more of mathematical convenience than necessity. $\bar{\mathbf{u}}$ can be seen as representing the flow component on the integral scale and $\sqrt{\varepsilon}\mathbf{v}$ as that in the inertial subrange. Small $\varepsilon$ corresponds, in this connection, to the fact that the integral scale is the energy-containing scale. Indeed, the kinetic energy contained in the shell $[k, 2k]$ diverges as $k \to 0$ for any $\nu \geq 1$, including the Kolmogorov spectrum. In this view, (7) is a simple, if not simplistic, model of real turbulence which may not exhibit separation of scales between the integral scale and the inertial subrange as implied by (7).

## 2.2 The Wigner distribution, the Wigner equation and transport scaling

Radiative transfer theory is well established ( [6], [1], [18], [28]) for *strictly hyperbolic* waves, such as acoustic, electromagnetic and elastic waves propagation in inhomogeneous media *at rest*, governed by equations of the form

$$\mathbf{C}\frac{\partial \mathbf{w}}{\partial t} + \mathbf{D}^j \frac{\partial \mathbf{w}}{\partial x_j} = 0, \quad \mathbf{x} = (x_1, x_2, x_3) \tag{9}$$

where the positive definite matrix $\mathbf{C} = \mathbf{C}(\mathbf{x})$ represents nonuniform materials properties pertaining to the speed of wave propagation and $\mathbf{D}^j$ are *constant*, symmetric matrices. The wave filed $\mathbf{w}$ may be a scalar, a vector or a tensor. In the case of underwater sound propagation in a density stratified fluid, the medium fluctuation occurs in materials properties, such as density and compressibility, as opposed to nonuniform movements of otherwise uniform medium, and is relatively small

$$\mathbf{C}(\mathbf{x}) = \mathbf{C}_0 + \sqrt{\varepsilon}\mathbf{C}_1(\mathbf{x}). \tag{10}$$

Assuming the medium perturbation $\mathbf{C}_1$ is smooth, wave propagation on the scale of the inhomogeneities is only slightly perturbed. On larger scales, however, multiple scattering results in significant effects. If interactions between the wave and the medium inhomogeneities are incoherent, then we expect a Markovian-type of transport to take place on larger scales. This motivates the transport scaling

$$t \to t/\varepsilon, \quad \mathbf{x} \to \mathbf{x}/\varepsilon, \quad \mathbf{k} \to \mathbf{k} \tag{11}$$

where $\mathbf{k}$ is the Fourier variable associated with the scale of the inhomogeneities. Incoherent scattering is a result of the randomness of medium fluctuation and the dimensionality *three*. For periodic medium fluctuation, Block waves arise. In lower dimensions, coherent backscattering may be strong and results in localization, instead of transport, of waves.

Waves usually do not have a well-defined phase space energy density, a fundamental distinction between waves and particles. However, a *pseudo-density* function, called the Wigner distribution, can be defined as

$$\mathbf{W}(t,\mathbf{x},\mathbf{k}) = \int \frac{d\mathbf{y}}{(2\pi)^d} e^{\boldsymbol{i}\mathbf{k}\cdot\mathbf{y}} \mathbf{w}(t,\mathbf{x}-\frac{\mathbf{y}}{2}) \otimes \mathbf{w}^*(t,\mathbf{x}+\frac{\mathbf{y}}{2})$$

before the scaling factor $\varepsilon$ of the medium fluctuation is considered. As the Wigner distribution preserves all the wave properties, it usually *loses* positive semi-definiteness as it evolves in time. Here and below $*$ denotes the complex conjugate for scalars and the conjugate transpose for vectors and matrices, and $\boldsymbol{i}$ denotes the imaginary number $\sqrt{-1}$. In [28], the Wigner distribution was used to derive radiative transfer equations for hyperbolic waves.

In contrast, for the flow-acoustic problem at hand, the materials properties are uniform and represented by a constant matrix

$$\mathbf{C} = \mathrm{diag}(\rho_0, \rho_0, \rho_0, \kappa_0).$$

(The case of variable $\mathbf{C}$ can be treated also, but we do not pursue it here for the sake of presentation). Instead of materials inhomogeneities, we have a non-uniform fluid flow $\mathbf{u}(\mathbf{x})$. We rewrite the linearized Euler equation as

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{u}(\mathbf{x}) \cdot \nabla \mathbf{w} + \mathbf{C}^{-1}\mathbf{D}^j \frac{\partial \mathbf{w}}{\partial x^j} + \mathbf{G}\mathbf{w} = 0, \quad \mathbf{w} = (\mathbf{u}, p) \tag{12}$$

with symmetric, constant matrices $\mathbf{D}^j$ and

$$\mathbf{G} = \begin{pmatrix} \tilde{\mathbf{G}} & 0 \\ (\nabla P)^* & 0 \end{pmatrix}, \quad \tilde{\mathbf{G}} = [\frac{\partial v_i}{\partial x_j}].$$

The lower order term (not displayed) corresponds to the flow-straining effect. It should be noted that Eq. (12) is *not hyperbolic* because of the presence of the non-propagating vortical mode (see Section 3.1).

By the transport scaling, the flow field (7) becomes

$$\bar{\mathbf{u}} + \sqrt{\varepsilon}\mathbf{v}(\frac{\mathbf{x}}{\varepsilon}). \tag{13}$$

We also assume

$$p_1 = \sqrt{\varepsilon} P(\frac{\mathbf{x}}{\varepsilon})$$

for some space-homogeneous random function $P$. The rescaled flow-sound field $\mathbf{w}_\varepsilon$ now satisfies the equation

$$\frac{\partial \mathbf{w}_\varepsilon}{\partial t} + (\bar{\mathbf{u}} + \sqrt{\varepsilon}\mathbf{v}(\frac{\mathbf{x}}{\varepsilon})) \cdot \nabla \mathbf{w}_\varepsilon + \mathbf{C}^{-1}\mathbf{D}^j \frac{\partial \mathbf{w}_\varepsilon}{\partial x_j} + \frac{1}{\sqrt{\varepsilon}} \mathbf{G}(\frac{\mathbf{x}}{\varepsilon})\mathbf{w}_\varepsilon = 0$$
$$\mathbf{w}_\varepsilon(0, \mathbf{x}) = \mathbf{w}_\varepsilon^0(\mathbf{x}).$$

For the rescaled field $\mathbf{w}_\varepsilon$ the Wigner distribution should rescale accordingly as

$$\mathbf{W}_\varepsilon(t, \mathbf{x}, \mathbf{k}) = \int \frac{d\mathbf{y}}{(2\pi)^{d/2}} e^{\boldsymbol{i}\mathbf{k}\cdot\mathbf{y}} \mathbf{w}_\varepsilon(t, \mathbf{x} - \frac{\varepsilon \mathbf{y}}{2}) \otimes \mathbf{w}_\varepsilon^*(t, \mathbf{x} + \frac{\varepsilon \mathbf{y}}{2}) \tag{14}$$

which satisfies the Wigner equation

$$\begin{aligned}
&\frac{\partial \mathbf{W}_\varepsilon}{\partial t} + \bar{\mathbf{u}} \cdot \nabla_{\mathbf{x}} \mathbf{W}_\varepsilon + \frac{1}{2}\mathbf{C}^{-1}\mathbf{D}^j \frac{\partial \mathbf{W}_\varepsilon}{\partial x_j} + \frac{1}{2}\frac{\partial \mathbf{W}_\varepsilon}{\partial x_j}\mathbf{D}^j\mathbf{C}^{-1} + \frac{\boldsymbol{i}}{\varepsilon} k_j \mathbf{C}^{-1}\mathbf{D}^j \mathbf{W}_\varepsilon - \frac{\boldsymbol{i}}{\varepsilon}\mathbf{W}_\varepsilon k_j \mathbf{D}^j \mathbf{C}^{-1} \\
&= \frac{\boldsymbol{i}}{\sqrt{\varepsilon}} \int \frac{d\mathbf{p}}{(2\pi)^d} e^{\boldsymbol{i}\mathbf{p}\cdot\mathbf{x}/\varepsilon} \mathbf{k} \cdot \hat{\mathbf{v}}(\mathbf{p}) \left[ \mathbf{W}_\varepsilon(\mathbf{k} + \frac{\mathbf{p}}{2}) - \mathbf{W}_\varepsilon(\mathbf{k} - \frac{\mathbf{p}}{2}) \right] \\
&- \frac{\sqrt{\varepsilon}}{2} \int \frac{d\mathbf{p}}{(2\pi)^d} e^{\boldsymbol{i}\mathbf{p}\cdot\mathbf{x}/\varepsilon} \hat{v}_j(\mathbf{p}) \left[ \frac{\partial \mathbf{W}_\varepsilon(\mathbf{k} - \frac{\mathbf{p}}{2})}{\partial x_j} + \frac{\partial \mathbf{W}_\varepsilon(\mathbf{k} + \frac{\mathbf{p}}{2})}{\partial x_j} \right] \\
&- \frac{1}{\sqrt{\varepsilon}} \int \frac{d\mathbf{p}}{(2\pi)^d} e^{\boldsymbol{i}\mathbf{p}\cdot\mathbf{x}/\varepsilon} \left[ \hat{\mathbf{G}}(\mathbf{p}) \mathbf{W}_\varepsilon(\mathbf{k} - \frac{\mathbf{p}}{2}) + \mathbf{W}_\varepsilon(\mathbf{k} + \frac{\mathbf{p}}{2}) \hat{\mathbf{G}}^*(\mathbf{p}) \right]
\end{aligned} \tag{15}$$

with $\hat{\mathbf{G}}^*(\mathbf{p})$ the Fourier transform of $\mathbf{G}$:

$$\hat{\mathbf{G}}(\mathbf{p}) = \boldsymbol{i}(\hat{\mathbf{v}}(\mathbf{p}), \hat{P}(\mathbf{p})) \otimes \tilde{\mathbf{p}} \tag{16}$$

where $\tilde{\mathbf{p}} = (p_1, p_2, p_3, 0)$. For ease of notation here and below, we omit writing the independent variables of physical quantities as much as possible. For example, $\mathbf{W}_\varepsilon(\mathbf{k}+\frac{\mathbf{p}}{2})$ denotes $\mathbf{W}_\varepsilon(t, \mathbf{x}, \mathbf{k}+\frac{\mathbf{p}}{2})$ here. We also adopt the summation convention for repeated indices except for Greek letters in the superscript that distinguish different modes. Only the summation over *superscripts in Greek letters* will be displayed explicitly. The derivation of the Wigner equation from the wave equation is lengthy but straightforward, so we leave it to the reader.

Why do we use the Wigner distribution? Among other things, the energy density

$$\mathcal{E}(t, \mathbf{x}) = (\mathbf{C}\mathbf{w}_\varepsilon \cdot \mathbf{w}_\varepsilon)$$

and the energy flux

$$\mathcal{F}_j(t, \mathbf{x}) = u_j(\mathbf{x})(\mathbf{C}\mathbf{w}_\varepsilon \cdot \mathbf{w}_\varepsilon) + (\mathbf{D}^j \mathbf{w}_\varepsilon \cdot \mathbf{w}_\varepsilon)$$

can be recast in terms of the Wigner distribution as

$$\mathcal{E}(t, \mathbf{x}) = \mathrm{Tr} \int d\mathbf{k} \mathbf{C} \mathbf{W}_\varepsilon(t, \mathbf{x}, \mathbf{k}) \tag{17}$$

$$\mathcal{F}_j(t, \mathbf{x}) = \mathrm{Tr} \int d\mathbf{k} \left[ u_j(\mathbf{x}) \mathbf{C} \mathbf{W}_\varepsilon(t, \mathbf{x}, \mathbf{k}) + \mathbf{D}^j \mathbf{W}_\varepsilon(t, \mathbf{x}, \mathbf{k}) \right]. \tag{18}$$

The advantage of the phase-space formulation is that while a transport equation of the form

$$\frac{\partial \mathcal{E}}{\partial t} + \sum_j \frac{\partial \mathcal{F}_j}{\partial x_j} = \mathcal{S} \tag{19}$$

where $\mathcal{S}$ is the source term accounting for the generation or absorption of sound ([16]) is not always valid, the Wigner equation is. Moreover, in the transport limit $\varepsilon \to 0$, the non-positiveness of the Wigner distribution disappears and the weak limit of the averaged Wigner distribution $\langle \mathbf{W}_\varepsilon \rangle$ satisfies the radiative transfer equation which corresponds to a *Markov* process in the phase space and preserves the positive semi-definiteness. In the following sections, we derive and analyze such transport equations.

Before ending the section, We pause to note that the transport limit of the Wigner equation when $\mathbf{D}^j = \mathbf{G} = 0$ (i.e. the transport of a passive scalar) has been rigorously obtained by an entirely probabilistic method in [10]. Another case where the transport limit has also been rigorosuly obtained is the Schrödinger wave in a random potential of the form (10) [9] (see also [29] and [2]). In both cases, the unknowns are scalar. In addition, the inhomogeneity of the Schrödinger equation is in the lower-order term.

# 3 Radiative transfer

## 3.1 Linear dispersion: acoustical and vortical modes

To analyze (15) we consider a multi-scale expansion

$$\mathbf{W}_\varepsilon(t,\mathbf{x},\mathbf{k}) = \mathbf{W}_0(t,\mathbf{x},\mathbf{k}) + \sqrt{\varepsilon}\mathbf{W}_1(t,\mathbf{x},\frac{\mathbf{x}}{\varepsilon},\mathbf{k}) + \varepsilon \mathbf{W}_2(t,\mathbf{x},\frac{\mathbf{x}}{\varepsilon},\mathbf{k}) + \ldots \tag{20}$$

We introduce the fast scale variable $\mathbf{z} = \frac{\mathbf{x}}{\varepsilon}$ and make the substitution

$$\nabla_{\mathbf{x}} \to \nabla_{\mathbf{x}} + \frac{1}{\varepsilon}\nabla_{\mathbf{z}}$$

in the Wigner equation (15). Inserting (20) we obtain in the leading order $O(\varepsilon^{-1})$:

$$\mathbf{L}(\mathbf{k})\mathbf{W}_0 - \mathbf{W}_0\mathbf{L}^*(\mathbf{k}) = 0, \tag{21}$$

where $\mathbf{L}(\mathbf{k})$ is the dispersion matrix given by

$$\mathbf{L}(\mathbf{k}) = \mathbf{C}^{-1}k_j\mathbf{D}^j.$$

As $\mathbf{L}$ is symmetric w.r.t. the scalar product $\mathbf{Cx}\cdot\mathbf{y}$, $\mathbf{L}$ has a complete set of eigenvectors and associated real eigenvalues. Let $\omega_\alpha(\mathbf{k})$ be the eigenvalues of $\mathbf{L}(\mathbf{k})$ with the multiplicities $r_\alpha$, and let $\mathbf{b}^{\alpha,j}(\mathbf{k})$, $j = 1, \ldots r_\alpha$ be the corresponding eigenvectors

$$\mathbf{L}(\mathbf{k})\mathbf{b}^{\alpha,i} = \omega_\alpha \mathbf{b}^{\alpha,i}, \tag{22}$$

normalized so that

$$\mathbf{C}\mathbf{b}^{\alpha,i}(\mathbf{k})\cdot\mathbf{b}^{\beta,j}(\mathbf{k}) = \delta_{ij}\delta_{\alpha\beta}. \tag{23}$$

For our system (5)-(6), the matrix $\mathbf{C} = \mathrm{diag}(\rho_0,\rho_0,\rho_0,\kappa_0)$. The dispersion matrix $\mathbf{L}(\mathbf{k})$ is

$$\mathbf{L}(\mathbf{k}) = \begin{pmatrix} 0 & 0 & 0 & k_1/\rho_0 \\ 0 & 0 & 0 & k_2/\rho_0 \\ 0 & 0 & 0 & k_3/\rho_0 \\ k_1/\kappa_0 & k_2/\kappa_0 & k_3/\kappa_0 & 0 \end{pmatrix}. \tag{24}$$

It has three eigenvalues:

$$\omega_1 = c_0|\mathbf{k}|, \quad \omega_2 = -c_0|\mathbf{k}|, \quad \omega_3 = 0$$

where $c_0 = 1/\sqrt{\kappa_0\rho_0}$ is the speed of sound. The eigenvalues $\omega_1(\mathbf{k}), \omega_2(\mathbf{k})$ are simple whereas the zero eigenvalue has multiplicity two. The eigenvalues $\omega_1(\mathbf{k})$ and $\omega_2(\mathbf{k})$ correspond to the (acoustical) longitudinal modes (forward and backward, respectively) while the eigenvalue $\omega_3$ corresponds to the (vortical) transverse mode. The corresponding eigenvectors are

$$\mathbf{b}^1 = (\frac{\hat{\mathbf{k}}}{\sqrt{2\rho_0}}, \frac{1}{\sqrt{2\kappa_0}}), \quad \mathbf{b}^2 = (\frac{\hat{\mathbf{k}}}{\sqrt{2\rho_0}}, -\frac{1}{\sqrt{2\kappa_0}}), \quad \mathbf{b}^{3,i} = (\frac{\hat{\mathbf{k}}_\perp^{(i)}}{\sqrt{\rho_0}}, 0), \quad i = 1, 2.$$

To fix the idea, we take the first coordinate axis to be in the direction of $\bar{\mathbf{u}}$, the second coordinate axis to be on the $\bar{\mathbf{u}} - \mathbf{k}$ plane and the third to be orthogonal to the $\bar{\mathbf{u}} - \mathbf{k}$ plane. Let $\theta$ be the angle between $\bar{\mathbf{u}}$ and $\mathbf{k}$. Then, we have

$$\hat{\mathbf{k}} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad \hat{\mathbf{k}}_\perp^{(1)} = \begin{pmatrix} -\cos\phi\sin\theta \\ \cos\phi\cos\theta \\ \sin\phi \end{pmatrix}, \quad \hat{\mathbf{k}}_\perp^{(2)} = \begin{pmatrix} \sin\phi\sin\theta \\ -\sin\phi\cos\theta \\ \cos\phi \end{pmatrix}, \tag{25}$$

in terms of the angular coordinates $\theta, \phi$. Note that, when $\kappa_0 \ll 1$, the acoustic wave modes point predominantly in the direction of the pressue variable.

The solution $\mathbf{W}_0$ of (21) is given by

$$\begin{aligned} \mathbf{W}_0(t, \mathbf{x}, \mathbf{k}) &= \sum_{\alpha,i,j} W^\alpha_{ij}(t, \mathbf{x}, \mathbf{k})\mathbf{b}^{\alpha,i}(\mathbf{k}) \otimes \mathbf{b}^{\alpha,j}(\mathbf{k}) \\ &= W^1\mathbf{b}^1 \otimes \mathbf{b}^1 + W^2\mathbf{b}^2 \otimes \mathbf{b}^2 + \sum_{i,j=1}^{2} W^3_{ij}\mathbf{b}^{3,i} \otimes \mathbf{b}^{3,j} \end{aligned} \tag{26}$$

where $\mathbf{W}^\alpha = [W^\alpha_{ij}]$ are the $r_\alpha \times r_\alpha$ Wigner distribution matrices associated with $\omega_\alpha, \alpha = 1, 2, 3$.

The matrices $\mathbf{W}^\alpha$ provide a decomposition of the total fluid energy density on the phase space among different modes and polarizations so that

$$\mathcal{E}(t, \mathbf{x}) = \sum_\alpha \int d\mathbf{k} \mathrm{Tr}\mathbf{W}^\alpha(t, \mathbf{x}, \mathbf{k}).$$

Likewise, the flux $\boldsymbol{\mathcal{F}}$ can be decomposed as

$$\mathcal{F}_j = \sum_\alpha \int d\mathbf{k} \left[\bar{u}_j + \frac{\partial\omega_\alpha}{\partial k_j}\right] \mathrm{Tr}\mathbf{W}^\alpha(t, \mathbf{x}, \mathbf{k}).$$

Note that the first term corresponds to the mean flow, and the second to the phase speed.

### 3.2 The radiative transport equations

Throughout this section we use $\tilde{\mathbf{a}}$ to denote the 4-vector

$$\tilde{\mathbf{a}} = (a_1, a_2, a_3, 0)$$

for any given 3-vector $\mathbf{a} = (a_1, a_2, a_3)$. We denote by $\mathbf{R} = [R_{ij}]$ the $4 \times 4$ correlation tensor of the vector $(\mathbf{v}, P)$. Note that the divergence free property of the flow implies the symmetry

$$\hat{\mathbf{R}}(\mathbf{p} - \mathbf{k})\tilde{\mathbf{k}} \cdot \tilde{\mathbf{k}} = \hat{\mathbf{R}}(\mathbf{p} - \mathbf{k})\tilde{\mathbf{k}} \cdot \tilde{\mathbf{p}} = \hat{\mathbf{R}}(\mathbf{p} - \mathbf{k})\tilde{\mathbf{p}} \cdot \tilde{\mathbf{p}}. \tag{27}$$

Assuming the expansion (20) and (26) we derive, in Appendix A, the transport equations

$$\frac{\partial\langle W^\tau_{nm}\rangle}{\partial t}+(\bar{\mathbf{u}}+\nabla_{\mathbf{k}}\omega_\tau)\cdot\nabla_{\mathbf{x}}\langle W^\tau_{nm}\rangle+(\langle\mathbf{W}^\tau\rangle(\mathbf{k})\Sigma^\tau(\mathbf{k}))_{nm}+(\Sigma^{\tau*}(\mathbf{k})\langle\mathbf{W}^\tau\rangle(\mathbf{k}))_{nm} \tag{28}$$
$$=\sum_\beta\int\frac{d\mathbf{p}}{(2\pi)^{d-1}}\hat{R}_{jl}(\mathbf{p}-\mathbf{k})S^{\tau\beta}_{jms}(\mathbf{k},\mathbf{p})S^{\tau\beta}_{lnr}(\mathbf{k},\mathbf{p})\langle W^\beta_{rs}\rangle(\mathbf{p})\delta(\omega_\tau(\mathbf{k})-\omega_\beta(\mathbf{p})+\bar{\mathbf{u}}\cdot(\mathbf{k}-\mathbf{p}))$$

where the total scattering cross-section matrices $\Sigma^\tau(\mathbf{k})$ are given by

$$\Sigma^\tau_{sm}(\mathbf{k})=-\sum_\beta \boldsymbol{i}\int\frac{d\mathbf{p}}{(2\pi)^d}\frac{\hat{R}_{jl}(\mathbf{p}-\mathbf{k})S^{\tau\beta}_{jmr}(\mathbf{k},\mathbf{p})S^{\beta\tau}_{lrs}(\mathbf{p},\mathbf{k})}{\omega_\tau(\mathbf{k})-\omega_\beta(\mathbf{p})+\bar{\mathbf{u}}\cdot(\mathbf{k}-\mathbf{p})-\boldsymbol{i}0}. \tag{29}$$

Note that $[S^{\tau\beta}_{lnr}]$ is rank-1 for $\tau,\beta=1,2$, rank-2 if exactly one of two superscripts is 3 and rank-3 if both superscripts are 3. They can be rewritten more concretely as follows.

$$\begin{aligned}
\mathbf{S}^{11}(\mathbf{k},\mathbf{p}) &= \frac{1}{2}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+1)\tilde{\mathbf{k}}-\frac{1}{2}(|\mathbf{p}|-\mathbf{k}\cdot\hat{\mathbf{p}})(\hat{\mathbf{k}},\sqrt{\frac{\kappa_0}{\rho_0}})\\
\mathbf{S}^{22}(\mathbf{k},\mathbf{p}) &= \frac{1}{2}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+1)\tilde{\mathbf{k}}-\frac{1}{2}(|\mathbf{p}|-\mathbf{k}\cdot\hat{\mathbf{p}})(\hat{\mathbf{k}},-\sqrt{\frac{\kappa_0}{\rho_0}})\\
\mathbf{S}^{12}(\mathbf{k},\mathbf{p}) &= \frac{1}{2}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}-1)\tilde{\mathbf{k}}-\frac{1}{2}(|\mathbf{p}|-\mathbf{k}\cdot\hat{\mathbf{p}})(\hat{\mathbf{k}},\sqrt{\frac{\kappa_0}{\rho_0}})\\
\mathbf{S}^{21}(\mathbf{k},\mathbf{p}) &= \frac{1}{2}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}-1)\tilde{\mathbf{k}}-\frac{1}{2}(|\mathbf{p}|-\mathbf{k}\cdot\hat{\mathbf{p}})(\hat{\mathbf{k}},-\sqrt{\frac{\kappa_0}{\rho_0}})\\
\mathbf{S}^{13}_{\cdot m}(\mathbf{k},\mathbf{p}) &= \frac{1}{\sqrt{2}}\tilde{\mathbf{k}}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}^{(m)}_\perp)+\frac{1}{\sqrt{2}}\mathbf{k}\cdot\hat{\mathbf{p}}^{(m)}_\perp(\hat{\mathbf{k}},\sqrt{\frac{\kappa_0}{\rho_0}}),\quad m=1,2\\
\mathbf{S}^{23}_{\cdot m}(\mathbf{k},\mathbf{p}) &= \frac{1}{\sqrt{2}}\tilde{\mathbf{k}}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}^{(m)}_\perp)+\frac{1}{\sqrt{2}}\mathbf{k}\cdot\hat{\mathbf{p}}^{(m)}_\perp(\hat{\mathbf{k}},-\sqrt{\frac{\kappa_0}{\rho_0}}),\quad m=1,2\\
\mathbf{S}^{31}_{\cdot m}(\mathbf{k},\mathbf{p}) &= \frac{1}{\sqrt{2}}\tilde{\mathbf{k}}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}^{(m)}_\perp)+\frac{1}{\sqrt{2}}(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{p}}(\hat{\mathbf{k}}^{(m)}_\perp,0)=\mathbf{S}^{32}_{\cdot m}(\mathbf{k},\mathbf{p}),\quad m=1,2\\
\mathbf{S}^{33}_{\cdot ml} &= \tilde{\mathbf{k}}(\hat{\mathbf{k}}^{(m)}_\perp\cdot\hat{\mathbf{p}}^{(l)}_\perp)+\mathbf{k}\cdot\hat{\mathbf{p}}^{(l)}_\perp(\hat{\mathbf{k}}^{(m)}_\perp,0),\quad m,l=1,2,
\end{aligned}$$

with $\hat{\mathbf{k}}^{(m)}_\perp,\hat{\mathbf{p}}^{(l)}_\perp$ defined as in (25). $\mathbf{S}^{11},\mathbf{S}^{22},\mathbf{S}^{33}$ account for the self-coupling of acoustic and vortical modes, $\mathbf{S}^{12},\mathbf{S}^{21}$ the coupling between the forward and backward acoustic modes and $\mathbf{S}^{13},\mathbf{S}^{23}$ the coupling between the acoustic and the vortical modes. For weak compressibility $\kappa_0\ll 1$, we have the following approximations:

$$\mathbf{S}^{11}(\mathbf{k},\mathbf{p})\approx\mathbf{S}^{22}(\mathbf{k},\mathbf{p})\approx(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+1/2-|\mathbf{p}|/(2|\mathbf{k}|))\tilde{\mathbf{k}} \tag{30}$$
$$\mathbf{S}^{12}(\mathbf{k},\mathbf{p})\approx\mathbf{S}^{21}(\mathbf{k},\mathbf{p})\approx(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}-1/2-|\mathbf{p}|/(2|\mathbf{k}|))\tilde{\mathbf{k}} \tag{31}$$
$$\mathbf{S}^{13}_{\cdot m}(\mathbf{k},\mathbf{p})\approx\mathbf{S}^{23}_{\cdot m}(\mathbf{k},\mathbf{p})\approx\sqrt{2}\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}^{(m)}_\perp\tilde{\mathbf{k}},\quad m=1,2 \tag{32}$$

Eq. (28) is a coupled system of equations for the limiting distributions $\langle W^1\rangle(t,\mathbf{x},\mathbf{k}),\langle W^2\rangle(t,\mathbf{x},\mathbf{k})$ and $\langle\mathbf{W}^3\rangle(t,\mathbf{x},\mathbf{k})$ for the acoustic and the vortical modes, respectively. The coupling occurs between the level surfaces (ellipsoids) of the (Doppler) acoustic frequency and the level surfaces (planes) of the vortical frequency (see the next section).

Equation (28) preserves the positive-definiteness of the initial data so that if initially the matrices $\langle\mathbf{W}^\tau\rangle(0,\mathbf{x},\mathbf{k})$ are positive definite, the solution $\langle\mathbf{W}^\tau\rangle(t,\mathbf{x},\mathbf{k})$ will remain positive definite. However, it does not in general conserve the energy

$$E(t)=\int\mathcal{E}(t,\mathbf{x})d\mathbf{x}=\text{const}$$

(of the perturbation) due to the flow-straining.

To see how the flow-straining term has affected the transport equation, we derive, in Appendix A, the radiative transfer equations by neglecting the term involving $\hat{\mathbf{G}}$ in (15):

$$\frac{\partial\langle\mathbf{W}^\tau\rangle}{\partial t}+(\bar{\mathbf{u}}+\nabla_{\mathbf{k}}\omega_\tau)\cdot\nabla_{\mathbf{x}}\langle\mathbf{W}^\tau\rangle+\Sigma^\tau\langle\mathbf{W}^\tau\rangle(\mathbf{k})+\langle\mathbf{W}^\tau\rangle(\mathbf{k})\Sigma^{\tau*}(\mathbf{k}) \tag{33}$$
$$=\sum_\beta\int\frac{d\mathbf{p}}{(2\pi)^{d-1}}(\hat{\mathbf{R}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k})\mathbf{T}^{\tau\beta}(\mathbf{k},\mathbf{p})\langle\mathbf{W}^\beta\rangle(\mathbf{p})\mathbf{T}^{\beta\tau}(\mathbf{p},\mathbf{k})\delta(\omega_\tau(\mathbf{k})+\bar{\mathbf{u}}\cdot\mathbf{k}-\omega_\beta(\mathbf{p})-\bar{\mathbf{u}}\cdot\mathbf{p})$$

where the total scattering cross-section matrices $\Sigma^\tau(\mathbf{k})$ is given by

$$\Sigma^\tau(\mathbf{k})=\boldsymbol{i}\sum_\beta\int\frac{d\mathbf{p}}{(2\pi)^d}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k})\frac{\mathbf{T}^{\tau\beta}(\mathbf{k},\mathbf{p})\mathbf{T}^{\beta\tau}(\mathbf{p},\mathbf{k})}{\omega_\tau(\mathbf{k})+\bar{\mathbf{u}}\cdot\mathbf{k}-\omega_\beta(\mathbf{p})-\bar{\mathbf{u}}\cdot\mathbf{p}+\boldsymbol{i}0}$$

with

$$T^{\alpha\beta}_{mj}(\mathbf{k},\mathbf{p})=\mathbf{C}\mathbf{b}^{\beta,m}(\mathbf{k})\cdot\mathbf{b}^{\alpha,j}(\mathbf{p}). \tag{34}$$

or, more explicitly,

$$\begin{aligned}
T^{11}(\mathbf{k},\mathbf{p})= &\quad (\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+1)/2 &&= T^{22}(\mathbf{k},\mathbf{p})\\
T^{12}(\mathbf{k},\mathbf{p})= &\quad (\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}-1)/2 &&= T^{21}(\mathbf{k},\mathbf{p})\\
T^{13}_i(\mathbf{k},\mathbf{p})= &\quad \hat{\mathbf{k}}\cdot\hat{\mathbf{p}}^{(i)}_\perp/\sqrt{2} &&= T^{23}_i(\mathbf{k},\mathbf{p}),\quad i=1,2,\\
T^{33}_{ij}(\mathbf{k},\mathbf{p})= &\quad \hat{\mathbf{k}}^{(i)}_\perp\cdot\hat{\mathbf{p}}^{(j)}_\perp, &&\quad i,j=1,2.
\end{aligned} \tag{35}$$

We know from (23) that $T^{\alpha\beta}_{mj}(\mathbf{k},\mathbf{k})=\delta_{\alpha\beta}\delta_{mj}$. It is interesting to compare Eq. (35) with Eq. (30)-(32) to see the effect of flow-straining (Note that the square of $\mathbf{k}$ has been factored out of $\mathbf{T}^{ij}$).

In addition, thanks to the reciprocity relations

$$\mathbf{T}^{\beta\tau}(\mathbf{p},\mathbf{k})=\mathbf{T}^{\tau\beta*}(\mathbf{k},\mathbf{p})$$

Eq. (33) preserves positive definiteness, as well as the averaged total energy

$$\langle E(t)\rangle=\sum_\tau\mathrm{Tr}\int\langle\mathbf{W}^\tau\rangle(\mathbf{x},\mathbf{k})d\mathbf{x}d\mathbf{k}=\mathrm{const.}$$

Energy conservation is due to the absence of the straining term in equations (5)-(6).

### 3.3 Resonant interaction: mode coupling and conversion

Because of the presence of Dirac's delta-function in the kernels of the transport equations acoustic/vortical waves interaction is best described in terms of the Doppler frequency surfaces $\mathcal{S}^\pm_h$ and $\mathcal{P}_h$, for $h\in R$,

$$\begin{aligned}
\mathcal{S}^\pm_h &= \{\mathbf{k}|\omega^\pm_a(\mathbf{k})=h\}\\
\mathcal{P}_h &= \{\mathbf{k}|\omega_v(\mathbf{k})=h\}
\end{aligned}$$

with

$$\omega^\pm_a(\mathbf{p})=\pm c_0|\mathbf{p}|+\bar{\mathbf{u}}\cdot\mathbf{p},\quad \omega_v(\mathbf{p})=\bar{\mathbf{u}}\cdot\mathbf{p}$$

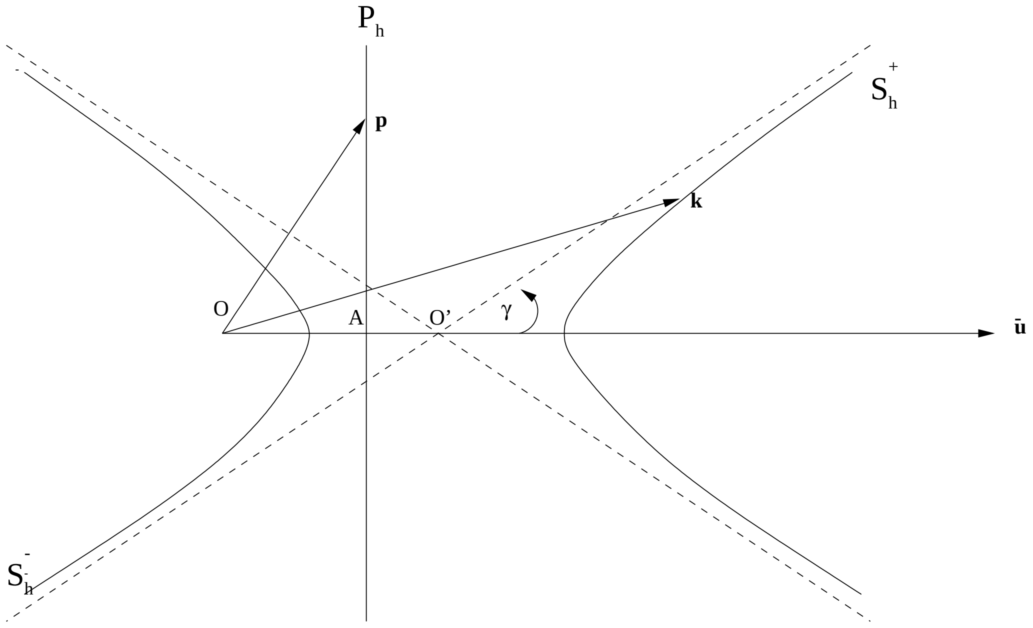



Figure 1: The supersonic diagram. The point $O$ is the origin in the wave vector space. The length of $\overline{OA}$ is $|h|/|\bar{\mathbf{u}}|$.

being the Doppler frequencies of the forward, backward acoustic modes and the vortical mode, respectively.

We have the relation

$$\mathcal{S}^-_{-h} = \{-\mathbf{k}|\mathbf{k} \in \mathcal{S}^+_h\}, \quad h \in R,$$

between the frequency surfaces $\mathcal{S}^+_h$ and $\mathcal{S}^-_h$ associated with the forward and backward acoustic modes.

In addition to self-coupling the acoustic modes can interact with the vortical mode through resonance

$$\pm c_0|\mathbf{k}| + \bar{\mathbf{u}} \cdot \mathbf{k} = \bar{\mathbf{u}} \cdot \mathbf{p}$$

and, as a result, the transport equations for the acoustic energy densities $W^i(t, \mathbf{x}, \mathbf{k}), i = 1, 2$, couple with that for the vortical Wigner distribution matrix $\mathbf{W}^3(t, \mathbf{x}, \mathbf{p})$.

The surfaces $\mathcal{P}_h$ for the vortical mode are planar. Thus the vortical energy cascades to high wave numbers regardless of coupling with the acoustic mode. The vortical energy cascade does not introduce long range correlation in velocity fluctuations and, hence, maintains the validity of the short range correlation assumption in (13).

The frequency surfaces $\mathcal{S}^\pm_h$ are hyperboloids for supersonic flows $\bar{\mathbf{u}} > c_0$ and ellipsoids for subsonic flows $\bar{\mathbf{u}} < c_0$. In the supersonic regime the acoustic energy cascades to high wave numbers since $\mathcal{S}^\pm_h, h \in R$ also have infinite surface area. The hyperboloids defined by

$$c_0^2|\mathbf{k}|^2 = |h - \bar{\mathbf{u}} \cdot \mathbf{k}|^2, \quad h > 0$$

have two branches: the forward branch $\mathcal{S}^+_h$ and the backward branch $\mathcal{S}^-_h$, corresponding to the forward acoustic mode $\omega_1(\mathbf{k}) = c_0|\mathbf{k}|$ and the backward acoustic mode $\omega_2(\mathbf{k}) = -c_0|\mathbf{k}|$ respectively,

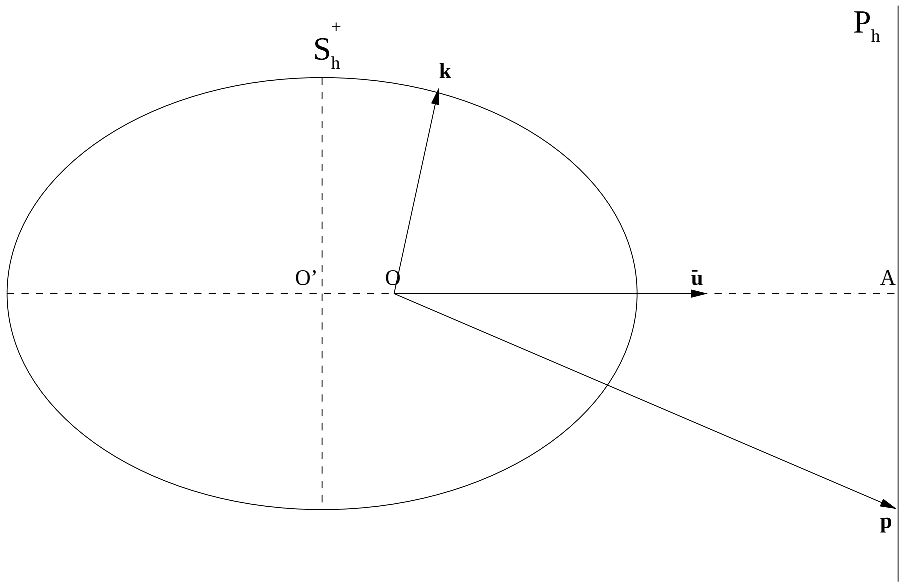



Figure 2: The subsonic diagram. The point $O$ is the origin in the wave vector space. The length of $\overline{OA}$ is $h/|\bar{\mathbf{u}}|$.

with the major axis of symmetry in the direction of $\bar{\mathbf{u}}$. The normal form of the hyperboloids is

$$(|\bar{\mathbf{u}}|^2 - c_0^2)(\xi_1 - \frac{h|\bar{\mathbf{u}}|}{|\bar{\mathbf{u}}|^2 - c_0^2})^2 - c_0^2(\xi_2^2 + \xi_3^2) = h^2 \frac{c_0^2}{|\bar{\mathbf{u}}|^2 - c_0^2}$$

where $\xi_1$ is the coordinate in the direction of $\bar{\mathbf{u}}$ and $\xi_2, \xi_3$ the coordinates in the orthogonal directions. In the long times the acoustic energy is driven in the directions of the asymptotes which form two (forward and backward) cones of semi-angle

$$\gamma = \arccos \frac{c_0}{|\bar{\mathbf{u}}|},$$

with the axis of symmetry parallel to $\bar{\mathbf{u}}$, joining their tips at

$$O' = \frac{h\bar{\mathbf{u}}}{\bar{\mathbf{u}}^2 - c_0^2}$$

(Fig. 1). Since the slope of the asymptotes is independent of $h$, the acoustic energy gradually transfers to higher and higher wave vectors parallel to the asymptotes regardless of the initial distribution and eventually dissipates into heat.

Since the subsonic regime $\bar{\mathbf{u}} < c_0$ is probably more relevant our attention is restricted to this regime in the sequel.

The frequency surfaces $\mathcal{S}_h^{\pm}$ are football-like ellipsoids with the major axis parallel to $\bar{\mathbf{u}}$. The normal form of the ellipsoids is

$$\frac{(c_0^2 - |\bar{\mathbf{u}}|^2)^2}{c_0^2 h^2}(\xi_1 + \frac{h|\bar{\mathbf{u}}|}{c_0^2 - |\bar{\mathbf{u}}|^2})^2 + \frac{c_0^2 - |\bar{\mathbf{u}}|^2}{h^2}(\xi_2^2 + \xi_3^2) = 1 \tag{36}$$

where the coordinate $\xi_1$ is in the direction of $\bar{\mathbf{u}}$ and $\xi_2, \xi_3$ are in the orthogonal directions. The major and minor radii are $hc_0/(c_0^2 - |\bar{\mathbf{u}}|^2)$ and $h/\sqrt{c_0^2 - |\bar{\mathbf{u}}|^2}$, respectively. Note that $\mathcal{S}_h^+$ and $\mathcal{S}_h^-$ exist only for positive and negative $h$, respectively. This means that the coupling of forward and backward acoustic modes disappears in the subsonic regime (Fig. 2). Furthermore, the center of $\mathcal{S}_h^+$ is shifted backward ($h > 0$) and the center of $\mathcal{S}_h^-$ forward ($h < 0$) due to the stronger backscattering of wave vectors in the direction of the mean flow. This has an important implication on the Stokes drift (see Section 4.3).

In the long times equidistribution of energy takes place between the coupled surfaces $\mathcal{S}_h^\pm$ and $\mathcal{P}_h$, and within themselves, in the wave vector space. Since the surface area of $\mathcal{S}_h^\pm$ is finite the coupled acoustic energy gradually converts into vortical energy which, in turn, is spread further and further over eddies of increasingly smaller scales and eventually dissipated as heat by molecular viscosity. This is the turbulent dissipation resulting in the absorption of sound waves. Conversely, the background turbulent flow energy can convert into acoustic energy by the mode-coupling mechanism and emit acoustic radiation. The generated sound waves are called aero- or hydro-dynamic sound [20],[21].

### 3.4 Subcritical wave numbers

The acoustic wave vector $\mathbf{k}$ is effectively decoupled from the vortical mode $\mathbf{p}$ if the sound-flow scattering cross section is much smaller than $1/\tau$ where $\tau$ is the observation time unit. In this case, coupling and conversion between acoustic and vortical modes does not occur. Such acoustic wave numbers are called *sub-critical*; otherwise, they are called *supercritical*.

For simplicity of discussion, let us assume the correlation matrix $\mathbf{R}$ is isotropic and is band-limited with support of $\hat{\mathbf{R}}$ enclosed by the ball of radius $r_c$ which roughly equals $1/\ell_1$ for the power-law spectrum (8). In the high frequency end, wave vector $\mathbf{k}$ is sub-critical if the distance from $\mathcal{S}_h^+$ to $\mathcal{P}_h$ for $h = \omega_a^+(\mathbf{k})$ is greater than $r_c$. This leads to the explicit condition

$$\omega_a^+(\mathbf{k}) \geq r_c(c_0 + |\bar{\mathbf{u}}|)|\bar{\mathbf{u}}|/c_0 \tag{37}$$

after some calculation or, equivalently,

$$|\mathbf{k}| \geq r_c(1 + M_a)M_a \tag{38}$$

where $M_a \equiv \bar{\mathbf{u}}/c_0$ is the mean flow Mach number.

In the low frequency end, we make the following observation. The ellipsoids $\mathcal{S}_h^\pm$ are nearly spheres centered at $-h\bar{\mathbf{u}}/c_0^2$ with radius $|h|/c_0$. The distance between $\mathcal{P}_h$ and $\mathcal{S}_h^\pm$ is roughly $h/|\bar{\mathbf{u}}|$. For fixed $\mathbf{p} \in \mathcal{P}_h$, $\hat{\mathbf{R}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k}$ is of the order $\mathrm{Tr}[\hat{\mathbf{R}}(|\mathbf{p}|)]h^2/c_0^2$. Sub-critical wave vectors $\mathbf{k}$ of the low frequency end can then be characterized as

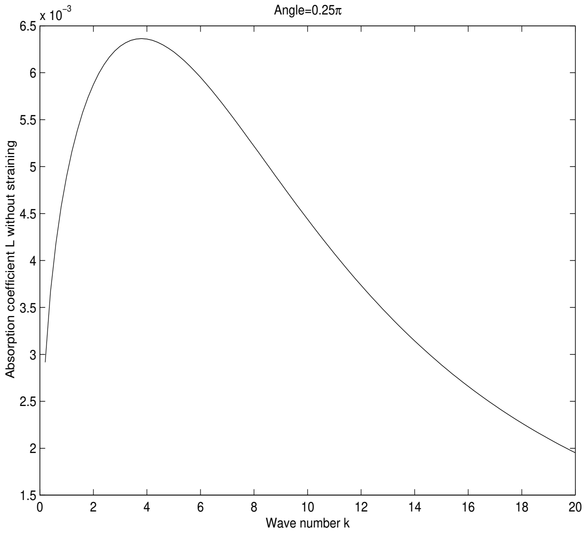
$$|\mathbf{k}| \ll \sqrt{\frac{|\bar{\mathbf{u}}|}{\tau \int \mathrm{Tr}[\hat{\mathbf{R}}(r)]r dr}}. \tag{39}$$

## 4 Subsonic regime

### 4.1 Absorption/emission due to mode coupling

In the subsonic regime the forward and backward acoustic modes are decoupled. We have the (forward) acoustic transport equation:

$$\frac{\partial \langle W^1 \rangle}{\partial t} + (\bar{\mathbf{u}} + \nabla_{\mathbf{k}}\omega_1)\cdot\nabla_{\mathbf{x}}\langle W^1\rangle + (\Sigma^1(\mathbf{k}) + \Sigma^{1*}(\mathbf{k}))\langle W^1\rangle(\mathbf{k}) \tag{40}$$



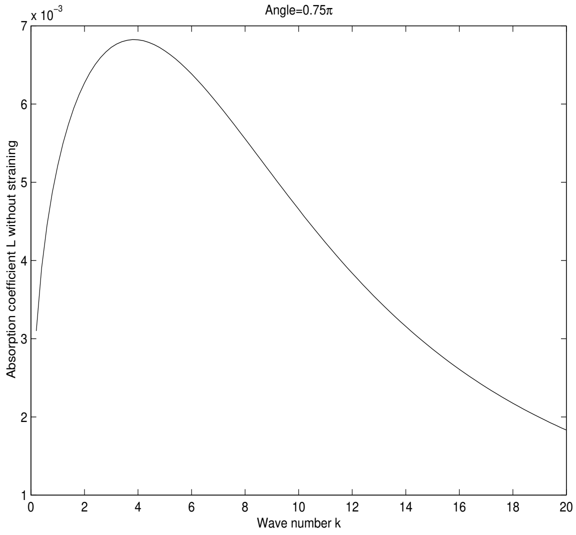




Figure 3: $L(k)$ without straining at $\theta = 0.25\pi$. Figure 4: $L(k)$ without straining at $\theta = 0.75\pi$.

$$= \sum_{\beta} \int \frac{d\mathbf{p}}{(2\pi)^{d-1}} \hat{R}_{jl}(\mathbf{p}-\mathbf{k}) S^{1\beta}_{js}(\mathbf{k},\mathbf{p}) S^{1\beta}_{lr}(\mathbf{k},\mathbf{p}) \langle W^{\beta}_{rs}\rangle(\mathbf{p}) \delta(c_0|\mathbf{k}| - \omega_\beta(\mathbf{p}) + \bar{\mathbf{u}}\cdot(\mathbf{k}-\mathbf{p}))$$

which is coupled to the vortical mode through $\mathbf{S}^{13}_{\cdot m}, \mathbf{S}^{31}_{\cdot l}$.

As the sound waves propagate through the random flow, in addition to being scattered by the turbulent flow, the *supercritical* accoustic wave numbers can convert into the vortical mode by mode coupling, resulting in the absorption of these wave numbers and the generation of flow which can occur with or without flow-straining. Conversely, the supercritical acoustic wave numbers can cause the background turbulent fluctuation to emit acoustic radiation via mode-coupling, resulting in the amplification of these wave numbers and sound generation. This latter effect, however, requires the presence of flow-straining.

The absorption/emission coefficient $L(\mathbf{k})$ due to mode-coupling is twice the real part of $\Sigma^1$. In the absence of flow-straining, we have

$$\begin{aligned} L(\mathbf{k}) &= \frac{1}{|\bar{\mathbf{u}}|(2\pi)^{d-1}} \int_{\omega_a^+(\mathbf{k})=\omega_v(\mathbf{p})} d\Omega(\mathbf{p}) \left[\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k}\right] T^{13}_{r}(\mathbf{k},\mathbf{p}) T^{31}_{r}(\mathbf{p},\mathbf{k}) \\ &= \frac{1}{|\bar{\mathbf{u}}|(2\pi)^{d-1}} \int_{\omega_a^+(\mathbf{k})=\omega_v(\mathbf{p})} d\Omega(\mathbf{p}) \left[\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k}\right] \frac{1}{2}(1-|\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}|^2) \end{aligned} \tag{41}$$

which is always non-negative. Here $d\Omega(\mathbf{p})$ is the area element of the surface $\{\mathbf{p}|\omega_a^+(\mathbf{k}) = \omega_v(\mathbf{p})\}$.

In the presence of flow-straining, we have

$$\begin{aligned} L(\mathbf{k}) &= \int \frac{d\mathbf{p}}{(2\pi)^d} \hat{R}_{jl}(\mathbf{p}-\mathbf{k}) S^{13}_{jr}(\mathbf{k},\mathbf{p}) S^{31}_{lr}(\mathbf{p},\mathbf{k}) \delta(c_0|\mathbf{k}| + \bar{\mathbf{u}}\cdot(\mathbf{k}-\mathbf{p})) \\ &= \frac{1}{|\bar{\mathbf{u}}|(2\pi)^{d-1}} \int_{\omega_a^+(\mathbf{k})=\omega_v(\mathbf{p})} d\Omega(\mathbf{p}) \Big\{ \left[\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k}\right] (1-|\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}|^2) \\ &\qquad + \sum_{m=1}^{2} \left[\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\hat{\mathbf{p}}^{(m)}_{\perp}\right] \left[(\mathbf{p}-\mathbf{k})\cdot\hat{\mathbf{k}}\right] \left[\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}^{(m)}_{\perp}\right] \Big\}. \end{aligned} \tag{42}$$

We recall the convention that the lower index $r$ in (42) is summed over $r = 1, 2$ (see Eq. (30)-(32)). In addition to the factor 2 in the first term, the second term in (42) is entirely due to flow-straining and can be positive or negative depending on the direction of propagation and, as shown in Figures

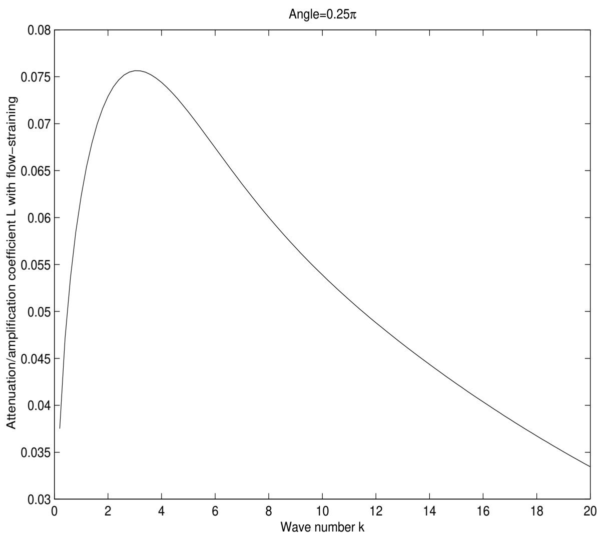



Figure 5: $L(k)$ with straining at $\theta = 0.25\pi$.

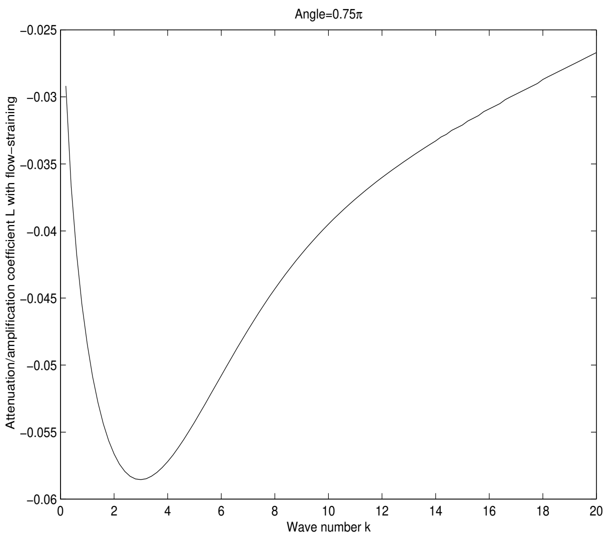



Figure 6: $L(k)$ with straining at $\theta = 0.75\pi$.

3,4, 5 and 6, dominates in magnitude over the first term at small or medium Mach numbers. The vanishing of $L(\mathbf{k})$ for small and large $k$ corresponds to the conditions for subcritical wave numbers (37)-(39).

For the numerical calculations of $L(\mathbf{k})$ (as well as $A(\mathbf{k})$ in the next section) we have used, instead of the Kolmogorov spectrum, the Von Kármán spectrum

$$\mathcal{R}(|\mathbf{k}|) = R_0|\mathbf{k}|^2(|\mathbf{k}|^2 + k_0^2)^{-17/6}, \quad |\mathbf{k}| \le k_0 = 1/\ell_1$$

and taken $c_0 = 1, \bar{\mathbf{u}} = 0.1, R_0 = 1, k_0 = 1/\ell_0 = 0.1, \ell_1 = 0.01$. The Von Kármán spectrum provides a good approximation to the turbulence spectrum but avoid the abrupt transition at $k_0$ in the Kolmogorov spectrum [27], which is more convenenient for computation. The angle $\theta$ is between the mean flow $\bar{\mathbf{u}}$ and $\mathbf{k}$ (cf. 25).

## 4.2 Attenuation/amplification due to scattering with flow-straining

In the simpler case of subcritical regime (i.e. the mode coupling is absent) we have a single closed equation for $\langle W\rangle(t, \mathbf{x}, \mathbf{k})$ (dropping the superscript 1)

$$\frac{\partial\langle W\rangle}{\partial t} + (\bar{\mathbf{u}} + c_0\hat{\mathbf{k}})\cdot\nabla_{\mathbf{x}}\langle W\rangle - A(\mathbf{k})\langle W\rangle = \mathcal{Q}\langle W\rangle \tag{43}$$

where $\mathcal{Q}$ is a conservative scattering operator with a non-symmetric kernel:

$$\mathcal{Q}\langle W\rangle = \frac{1}{(2\pi)^{d-1}}\int_{\omega_a^+(\mathbf{k})=\omega_a^+(\mathbf{p})} \frac{d\Omega(\mathbf{p})}{|\bar{\mathbf{u}} + c_0\hat{\mathbf{p}}|}\left[\sigma(\mathbf{k},\mathbf{p})\langle W\rangle(\mathbf{p}) - \sigma(\mathbf{p},\mathbf{k})\langle W\rangle(\mathbf{k})\right] \tag{44}$$

and $A(\mathbf{k})$ is the attenuation/amplification coefficient given by

$$A(\mathbf{k}) = \frac{1}{(2\pi)^{d-1}}\int_{\omega_a^+(\mathbf{k})=\omega_a^+(\mathbf{p})} \frac{d\Omega(\mathbf{p})}{|\bar{\mathbf{u}} + c_0\hat{\mathbf{p}}|}\left[\sigma'(\mathbf{k},\mathbf{p}) - \sigma(\mathbf{k},\mathbf{p}) + \sigma(\mathbf{p},\mathbf{k})\right] \tag{45}$$

with

$$\sigma(\mathbf{k},\mathbf{p}) = (\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k})\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} + \frac{1}{2} - \frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)^2 \tag{46}$$

$$+\sqrt{\frac{\kappa_0}{\rho_0}}(\hat{\mathbf{R}}_{\mathbf{v}p}(\mathbf{p}-\mathbf{k})\cdot\mathbf{k})|\mathbf{k}|(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} + \frac{1}{2} - \frac{|\mathbf{p}|}{2|\mathbf{k}|})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} - \frac{|\mathbf{p}|}{|\mathbf{k}|}) + \frac{1}{4}\frac{\kappa_0}{\rho_0}\hat{R}_p(\mathbf{p}-\mathbf{k})|\mathbf{k}|^2\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} - \frac{|\mathbf{p}|}{|\mathbf{k}|}\right)^2$$

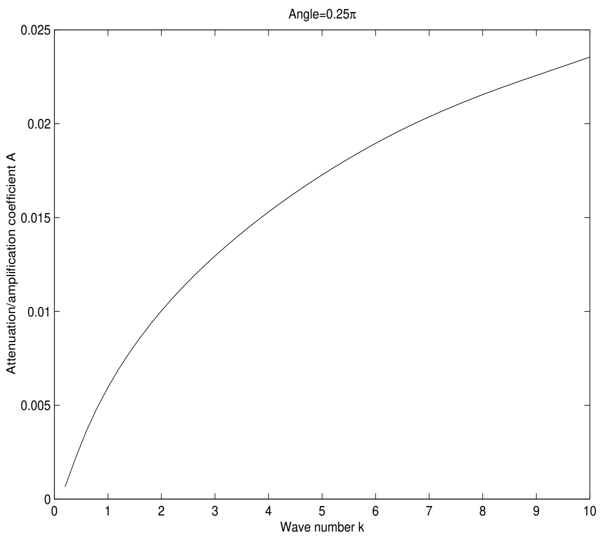



Figure 7: $A(k)$ at $\theta = 0.25\pi$.

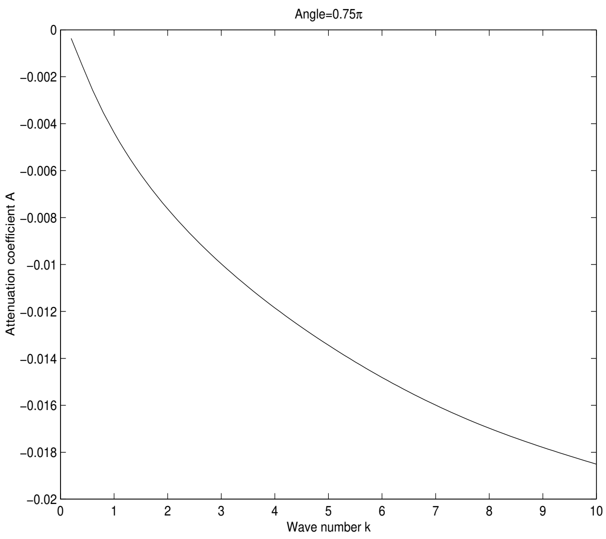



Figure 8: $A(k)$ at $\theta = 0.75\pi$.

and

$$\begin{aligned}\sigma'(\mathbf{k},\mathbf{p}) = {} & (\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k})\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+\frac{1}{2}-\frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)\left(\frac{|\mathbf{k}|}{2|\mathbf{p}|}-\frac{|\mathbf{p}|}{2|\mathbf{k}|}\right) \\ & +\frac{1}{2}\sqrt{\frac{\kappa_0}{\rho_0}}(\hat{\mathbf{R}}_{\mathbf{v}p}(\mathbf{p}-\mathbf{k})\cdot\mathbf{k})|\mathbf{k}|\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+\frac{1}{2}-\frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)\left(1+\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}\right)\left(1-\frac{|\mathbf{p}|}{|\mathbf{k}|}\right) \\ & +\frac{1}{2}\sqrt{\frac{\kappa_0}{\rho_0}}(\hat{\mathbf{R}}_{\mathbf{v}p}(\mathbf{p}-\mathbf{k})\cdot\mathbf{k})|\mathbf{k}|\left(\frac{|\mathbf{k}|}{2|\mathbf{p}|}-\frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}-\frac{|\mathbf{p}|}{|\mathbf{k}|}\right) \\ & +\frac{1}{4}\frac{\kappa_0}{\rho_0}\hat{R}_p(\mathbf{p}-\mathbf{k})|\mathbf{k}|^2\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}-\frac{|\mathbf{p}|}{|\mathbf{k}|}\right)\left(1+\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}\right)\left(1-\frac{|\mathbf{p}|}{|\mathbf{k}|}\right).\end{aligned} \tag{47}$$

Note that $\sigma(\mathbf{k},\mathbf{p})$ is a quadratic form associated with the non-negative-definite matrix $\hat{\mathbf{R}}(\mathbf{p}-\mathbf{k})$ for all $\mathbf{k},\mathbf{p}$ and, thus, is non-negative for all $\mathbf{k},\mathbf{p}$. The rate of pumping or draining energy $A(\mathbf{k})$ vanishes, however, if $\bar{\mathbf{u}} = 0$. Both the straining mechanism and the mean flow are necessary for destroying energy conservation in the radiative transfer equation when the temporal variation of the turbulent fluctuation is negligible.

For small $\kappa_0 \ll 1$, the scattering cross sections become

$$\sigma(\mathbf{k},\mathbf{p}) = (\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k})\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+\frac{1}{2}-\frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)^2 \tag{48}$$

$$\sigma'(\mathbf{k},\mathbf{p}) = (\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k})\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+\frac{1}{2}-\frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)\left(\frac{|\mathbf{k}|}{2|\mathbf{p}|}-\frac{|\mathbf{p}|}{2|\mathbf{k}|}\right). \tag{49}$$

Then, we have $\sigma'(\mathbf{k},\mathbf{p})-\sigma'(\mathbf{p},\mathbf{k}) = \sigma(\mathbf{k},\mathbf{p})-\sigma(\mathbf{p},\mathbf{k})$ and, consequently,

$$A(\mathbf{k}) = \frac{1}{(2\pi)^{d-1}}\int_{\omega_a^+(\mathbf{k})=\omega_a^+(\mathbf{p})}\frac{d\Omega(\mathbf{p})}{|\bar{\mathbf{u}}+c_0\hat{\mathbf{p}}|}\sigma'(\mathbf{p},\mathbf{k}). \tag{50}$$

Figures 7 and 8 show the dependence of $A$, as given by (50), on the wave numbers and the direction of propagation.

Setting $\bar{\mathbf{u}} = P = 0$ we recover the transport equation of Howe [14]:

$$\frac{\partial\langle W\rangle}{\partial t}+c_0\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}\langle W\rangle = \mathcal{Q}_0\langle W\rangle \tag{51}$$

with

$$\mathcal{Q}_0 f := \frac{1}{c_0(2\pi)^{d-1}} \int d\Omega(\mathbf{p})(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2\,[f(\mathbf{p})-f(\mathbf{k})]\,. \tag{52}$$

The geometric factor in (52) prohibits orthogonal scattering which would result in $\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} = 0$.

### 4.3 Diffusion approximation without flow-straining

In the absence of flow-straining the transport equation for the subcritical wave numbers becomes, after droping the superscript,

$$\frac{\partial\langle W\rangle}{\partial t} + (\bar{\mathbf{u}} + c_0\hat{\mathbf{k}})\cdot\nabla_{\mathbf{x}}\langle W\rangle = \mathcal{Q}\langle W\rangle \tag{53}$$

with

$$\mathcal{Q}f := \frac{1}{(2\pi)^{d-1}} \int_{\omega_a^+(\mathbf{k})=\omega_a^+(\mathbf{p})} \frac{d\Omega(\mathbf{p})}{|c_0\hat{\mathbf{p}}+\bar{\mathbf{u}}|}(\hat{\mathbf{R}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k})\frac{1}{4}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+1)^2\,[f(\mathbf{p})-f(\mathbf{k})]\,. \tag{54}$$

Notice that the geometric factor changes from $(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2$ in (52) to $(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+1)^2/4$ in (54) due to straining. In constrast to that in (52), the scattering kernel prohibits back scattering which would result in $\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+1=0$.

Next we consider a situation where energy transfer *per frequency* can be adequately described by a physical-space transport equation like (19): sound propagation over distances much longer than the mean free path. In this case the transport process can be decomposed into an *effective* transport and a fluctuation.

Since we expect the fluctuation to be diffusion-like, we make the diffusive scaling after subtracting the effective part

$$t = \frac{s}{\epsilon^2}, \quad \mathbf{x} - \bar{\mathbf{u}}_{\text{eff}}t = \frac{\mathbf{y}}{\epsilon}, \quad \mathbf{k}\to\mathbf{k} \tag{55}$$

with the transport velocity

$$\bar{\mathbf{u}}_{\text{eff}}(h) = \bar{\mathbf{u}} + \frac{1}{\Omega_h}\int_{\mathcal{S}_h^+} c_0\hat{\mathbf{k}}\frac{d\Omega(\mathbf{k})}{|\bar{\mathbf{u}}+c_0\hat{\mathbf{k}}|}, \quad \Omega_h = \int_{\mathcal{S}_h^+}\frac{d\Omega(\mathbf{k})}{|\bar{\mathbf{u}}+c_0\hat{\mathbf{k}}|} \tag{56}$$

where the *Stokes drift*

$$\frac{1}{\Omega_h}\int_{\mathcal{S}_h^+} c_0\hat{\mathbf{k}}\frac{d\Omega(\mathbf{k})}{|\bar{\mathbf{u}}+c_0\hat{\mathbf{k}}|}$$

accounts for the difference between the transport velocity and the Eulerian mean flow. Note that $\bar{\mathbf{u}}_{\text{eff}}$ is parallel to $\bar{\mathbf{u}}$ but is smaller in magnitude since the ellipsoid $\mathcal{S}_h^+$ is shifted to the left from the origin (Fig. 2). Thus turbulent scattering reduces the transport velocity of sound resulting in *negative* Stokes' drift. This can be explained physically as follows: the wave vectors parallel to the mean flow are scattered more intensely than those anti-parallel to the mean flow due to the difference in speed relative to the turbulent fluctuation. Notice also that the reduction in speed is linearly proportional to $\bar{\mathbf{u}}$ for low Mach number and is roughly independent of frequency.

In the multi-scale expansion $\langle W\rangle = \bar{W}_0 + \varepsilon\bar{W}_1 + \cdots$, the leading order term $\bar{W}_0$ satisfies a diffusion equation on the physical space (see Appendix B for derivation)

$$\frac{\partial\bar{W}_0}{\partial t} = \nabla_{\mathbf{y}}\cdot\mathbf{D}(\omega_a^+(\mathbf{k}))\nabla_{\mathbf{y}}\bar{W}_0 \tag{57}$$

where the anisotropic diffusion matrix $\mathbf{D} = [D_{ij}]$ is given by

$$\mathbf{D}(h) = \frac{1}{\Omega_h} \int_{\mathcal{S}_h^+} (\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\text{eff}} + c_0\hat{\mathbf{k}}) \otimes \chi \frac{d\Omega(\mathbf{k})}{|\bar{\mathbf{u}} + c_0\hat{\mathbf{k}}|}$$

with $\chi$ being the vector-valued solution of the equation

$$\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\text{eff}} + c_0\hat{\mathbf{k}} = \mathcal{Q}\chi. \tag{58}$$

By (58) the diffusion matrix $\mathbf{D}$ may be expressed as

$$\mathbf{D}(h) = -\frac{1}{\Omega_h} \int_{\mathcal{S}_h^+} \mathcal{Q}\chi \otimes \chi \frac{d\Omega(\mathbf{k})}{|\bar{\mathbf{u}} + c_0\hat{\mathbf{k}}|}.$$

Thus the matrix $\mathbf{D}$ is non-negative since the operator $\mathcal{Q}$ is non-positive definite ([8]).

### 4.4 Diffusion approximation with flow-straining

In this section we consider the long time limit of the perturbation of Howe's equation (51): the case of small Mach number, $Ma = \epsilon \ll 1$. We write the mean flow as $\bar{\mathbf{u}} = \varepsilon c_0 \hat{\mathbf{u}}$ where $\hat{\mathbf{u}} = \bar{\mathbf{u}}/|\bar{\mathbf{u}}|$. The appropriate equation is eq. (43) with (48)-(49).

As shown in Appendix C, the long-time, amplification/attenuation effect due to a small mean flow is a second order effect. To capture the second order effect we consider the diffusive scaling

$$t \to \frac{t}{\epsilon^2}, \quad \mathbf{x} \to \frac{\mathbf{x}}{\epsilon}, \quad \mathbf{k} \to \mathbf{k} \tag{59}$$

for (43) and derive a phase-space diffusion equation for the leading order term in the expansion $\langle W \rangle = \bar{W}_0 + \varepsilon \bar{W}_1 + \cdots$:

$$\frac{\partial \bar{W}_0}{\partial t} + \mathbf{B}(\omega) \cdot \nabla_{\mathbf{x}} \bar{W}_0 + A(\omega)\bar{W}_0 = \nabla_{\mathbf{x}} \cdot \mathbf{D} \nabla_{\mathbf{x}} \bar{W}_0. \tag{60}$$

with $\omega = c_0|\mathbf{k}|$. The diffusion matrix $\mathbf{D} = [D_{ij}]$ is given by

$$D_{ij}(\omega) = -\frac{1}{4\pi} \int d\Omega(\hat{\mathbf{k}}) \chi_i \mathcal{Q}_0 \chi_j \tag{61}$$

and is positive definite. Here $\mathcal{Q}_0$ is defined by (52). The drift $\mathbf{B}$ is given by

$$\begin{aligned} \mathbf{B} &= -\frac{|\mathbf{k}|^3}{4\pi} \int d\Omega(\hat{\mathbf{k}})(\hat{\mathbf{u}} \cdot \hat{\mathbf{k}} - \hat{\mathbf{u}} \cdot \hat{\mathbf{p}})\hat{\mathbf{p}} \cdot \nabla_{\mathbf{p}} [\sigma(\mathbf{k}, \mathbf{p})(\chi(\mathbf{p}) - \chi(\mathbf{k}))] & (62) \\ &\quad -\frac{1}{3} c_0 \hat{\mathbf{u}} + \frac{1}{2\pi} \int d\Omega(\hat{\mathbf{k}}) \hat{\mathbf{k}} \phi(\mathbf{k}) & (63) \end{aligned}$$

and the attenuation/amplification coefficient is

$$\begin{aligned} A &= -\frac{|\mathbf{k}|^3}{4\pi} \int d\Omega(\hat{\mathbf{k}})(\hat{\mathbf{u}} \cdot \hat{\mathbf{k}} - \hat{\mathbf{u}} \cdot \hat{\mathbf{p}})\hat{\mathbf{p}} \cdot \nabla_{\mathbf{p}} [\sigma(\mathbf{k}, \mathbf{p})(\phi(\mathbf{p}) - \phi(\mathbf{k}))] \\ &\quad + \frac{1}{4\pi} \int d\Omega(\hat{\mathbf{k}}) \phi(\mathbf{k}) \mathcal{Q}_0 \phi(\mathbf{k}) + \frac{1}{4\pi} \int d\Omega(\hat{\mathbf{k}})(5\hat{\mathbf{u}} \cdot \hat{\mathbf{k}} A_1(\mathbf{k}) - A_2(\mathbf{k})). \quad (64) \end{aligned}$$

Here $\sigma$ is given by (48) and $\phi, \chi = [\chi_j]$ are solutions of the following equations, respectively,

$$\begin{aligned} -A_1(\mathbf{k}) &= \mathcal{Q}_0 \phi(\mathbf{k}) & (65) \\ c_0 \hat{k}_j &= \mathcal{Q}_0 \chi_j(\mathbf{k}) & (66) \end{aligned}$$

with

$$A_1(\mathbf{k}) = \frac{|\mathbf{k}|^{d-1}}{c_0(2\pi)^{d-1}} \int d\Omega(\hat{\mathbf{p}})(\hat{\mathbf{R}}_{\mathbf{v}}(|\mathbf{k}|(\hat{\mathbf{p}}-\hat{\mathbf{k}}))\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})((\hat{\mathbf{p}}-\hat{\mathbf{k}})\cdot\hat{\mathbf{u}}),$$

which is the leading order term in the small Mach number expansion of $A(\mathbf{k})$ (45), and

$$A_2(\mathbf{k}) = \frac{|\mathbf{k}|^{d-1}}{2c_0} \int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})((\hat{\mathbf{k}}-\hat{\mathbf{p}})\cdot\hat{\mathbf{u}})\left[\hat{\mathbf{p}}\cdot\hat{\mathbf{u}}(3\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}-1)+\hat{\mathbf{k}}\cdot\hat{\mathbf{u}}(1+\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})\right]$$
$$-\frac{|\mathbf{k}|^{d-2}}{c_0}\int_{|\mathbf{p}|=|\mathbf{k}|}\frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})((\hat{\mathbf{p}}-\hat{\mathbf{k}})\cdot\hat{\mathbf{u}})^2\hat{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\left[|\mathbf{p}|^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})\right].$$

Eq. (60) describes the transport mechanisms as projected on the physical space. This reduction is possible because the decoupling of different acoustic frequencies due to a vanishing Mach number. Eq. (60) is simpler to solve than (43) because the dimension is halved once the two auxillary equations (65) and (66) are solved in the phase space. Finally we remark that although the derivation of the diffusion equation is lengthy it can be done rigorously because we start with the transport equation rather than the linearized Euler equation.

## 5 Conclusion

We study the sound wave propagation in a random flow whose mean flow is large compared with its fluctuation in the infinite three-dimensional space. In such case, the temporal variation of turbulent fluctuation is neglegible. We consider the intermediate regime, where the range of acoustic wave numbers overlaps with the range of turbulent wave numbers, between the short wave regime of geometric acoustics or paraxial approximation and the long wave regime of homogenization.

We have used the multi-scale expansions systematically to derive the radiative transport equations that describe the evolution of the correlation of acoustic field and the turbulent scattering and mode-coupling of sound waves. The radiative transport equations describe naturally the mechanisms of turbulent scattering and mode-coupling. We have shown that, because of the flow-straining term, the turbulent scattering becomes nonconservative in general and, depending on the propagation direction, a sound wave can gain as well lose energy though mode coupling (i.e. the flow can radiate sound as a result of flow-straining).

Further, we have calculated the attenuation/amplification coefficients due to mode coupling and the scattering with flow-straining, respectively. These coefficients depict interesting dependence on the propagating direction and the wave length of sound wave. The emphasis is on the effects of the mean flow and the turbulent flow-straining. The anisotropy of the coefficients is due to the presence of mean flow.

Finally, we have obtained the diffusion equations which describe the transport process in the physical space, and, thus, further reduced the dimension of the flow-acoustic equations.

# Appendices

## A Derivation of the radiative transport equations

We derive the radiative transport equations (28) in this appendix. The order $O(\varepsilon^{-1})$ terms in (15) imply that the leading term in the asymptotic expansion (20) has the form (26). The order $O(\varepsilon^{-1/2})$ terms imply that

$$\bar{\mathbf{u}} \cdot \nabla_{\mathbf{z}} \mathbf{W}_1 + \boldsymbol{i}\mathbf{C}^{-1} k_j \mathbf{D}^j \mathbf{W}_1 - \boldsymbol{i}\mathbf{W}_1 k_j \mathbf{D}^j \mathbf{C}^{-1} + \frac{1}{2}\mathbf{C}^{-1}\mathbf{D}^j \frac{\partial \mathbf{W}_1}{\partial z_j} + \frac{1}{2}\frac{\partial \mathbf{W}_1}{\partial z_j}\mathbf{D}^j \mathbf{C}^{-1} \tag{67}$$
$$= \boldsymbol{i}\int \frac{d\mathbf{p}}{(2\pi)^d} e^{\boldsymbol{i}\mathbf{p}\cdot\mathbf{z}} (\mathbf{k}\cdot\hat{\mathbf{v}}(\mathbf{p})) \left[\mathbf{W}_0(\mathbf{k}+\frac{\mathbf{p}}{2}) - \mathbf{W}_0(\mathbf{k}-\frac{\mathbf{p}}{2})\right]$$
$$-\int \frac{d\mathbf{p}}{(2\pi)^d} e^{\boldsymbol{i}\mathbf{p}\cdot\mathbf{z}} \left[\hat{\mathbf{G}}(\mathbf{p})\mathbf{W}_0(\mathbf{k}-\frac{\mathbf{p}}{2}) + \mathbf{W}_0(\mathbf{k}+\frac{\mathbf{p}}{2})\hat{\mathbf{G}}^*(\mathbf{p})\right].$$

and in the order $\varepsilon^0$ we get

$$\frac{\partial \mathbf{W}_0}{\partial t} + \bar{\mathbf{u}}\cdot\nabla_{\mathbf{x}}\mathbf{W}_0 + \bar{\mathbf{u}}\cdot\nabla_{\mathbf{z}}\mathbf{W}_2 + \boldsymbol{i}\mathbf{C}^{-1}k_j\mathbf{D}^j\mathbf{W}_2 - \boldsymbol{i}\mathbf{W}_2 k_j\mathbf{D}^j\mathbf{C}^{-1} + \frac{1}{2}\mathbf{C}^{-1}\mathbf{D}^j\frac{\partial \mathbf{W}_0}{\partial x_j}$$
$$+\frac{1}{2}\frac{\partial \mathbf{W}_0}{\partial x_j}\mathbf{D}^j\mathbf{C}^{-1} + \frac{1}{2}\mathbf{C}^{-1}\mathbf{D}^j\frac{\partial \mathbf{W}_2}{\partial z_j} + \frac{1}{2}\frac{\partial \mathbf{W}_2}{\partial z_j}\mathbf{D}^j\mathbf{C}^{-1}$$
$$= \boldsymbol{i}\int \frac{d\mathbf{p}}{(2\pi)^d} e^{\boldsymbol{i}\mathbf{p}\cdot\mathbf{z}} (\mathbf{k}\cdot\hat{\mathbf{v}}(\mathbf{p})) \left[\mathbf{W}_1(\mathbf{k}+\frac{\mathbf{p}}{2}) - \mathbf{W}_1(\mathbf{k}-\frac{\mathbf{p}}{2})\right]$$
$$-\frac{1}{2}\int \frac{d\mathbf{p}}{(2\pi)^d} e^{\boldsymbol{i}\mathbf{p}\cdot\mathbf{z}} \hat{v}_j(\mathbf{p}) \left[\frac{\partial \mathbf{W}_1(\mathbf{k}-\frac{\mathbf{p}}{2})}{\partial z_j} + \frac{\partial \mathbf{W}_1(\mathbf{k}+\frac{\mathbf{p}}{2})}{\partial z_j}\right] \tag{68}$$
$$-\int \frac{d\mathbf{p}}{(2\pi)^d} e^{\boldsymbol{i}\mathbf{p}\cdot\mathbf{z}} \left[\hat{\mathbf{G}}(\mathbf{p})\mathbf{W}_1(\mathbf{k}-\frac{\mathbf{p}}{2}) + \mathbf{W}_1(\mathbf{k}+\frac{\mathbf{p}}{2})\hat{\mathbf{G}}^*(\mathbf{p})\right]. \tag{69}$$

The term $\mathbf{W}_1$ can be solved for in terms of $\mathbf{W}_0$ from (67). Inserting the resulting expression into (69) and averaging we can get a closed equation for $\langle \mathbf{W}_0 \rangle$. This is done as follows. We solve (67) for $\mathbf{W}_1(t,\mathbf{x},\mathbf{z},\mathbf{k})$ by taking the Fourier transform of this equation in $\mathbf{z}$ and representing

$$\hat{\mathbf{W}}_1(t,\mathbf{x},\mathbf{p},\mathbf{k}) = \sum_{\alpha,\beta,n,m} F^{\alpha\beta}_{nm}(t,\mathbf{x},\mathbf{p},\mathbf{k})\mathbf{b}^{\alpha,n}(\mathbf{k}+\frac{\mathbf{p}}{2}) \otimes \mathbf{b}^{\beta,m}(\mathbf{k}-\frac{\mathbf{p}}{2}),$$

where $\mathbf{b}^{\alpha,n}(\mathbf{k})$ are defined by (22). We insert (70) and (26), multiply the resulting equation by the matrix $\mathbf{C}\mathbf{b}^{\beta,m}(\mathbf{k}-\frac{\mathbf{p}}{2}) \otimes (\mathbf{C}\mathbf{b}^{\alpha,n}(\mathbf{k}+\frac{\mathbf{p}}{2}))$ and take the trace. Now we get

$$F^{\alpha\beta}_{nm}(t,\mathbf{x},\mathbf{p},\mathbf{k}) \tag{70}$$
$$= \frac{W^{\alpha}_{nj}(\mathbf{k}+\frac{\mathbf{p}}{2})\left[(\mathbf{k}\cdot\hat{\mathbf{v}}(\mathbf{p}))T^{\beta\alpha}_{mj}(\mathbf{k}-\frac{\mathbf{p}}{2},\mathbf{k}+\frac{\mathbf{p}}{2}) + \boldsymbol{i}(\hat{\mathbf{G}}^*(\mathbf{p})\mathbf{C}\mathbf{b}^{\beta,m}(\mathbf{k}-\frac{\mathbf{p}}{2})\cdot\mathbf{b}^{\alpha,j}(\mathbf{k}+\frac{\mathbf{p}}{2}))\right]}{\omega_\alpha(\mathbf{k}+\frac{\mathbf{p}}{2}) - \omega_\beta(\mathbf{k}-\frac{\mathbf{p}}{2}) + \bar{\mathbf{u}}\cdot\mathbf{p} - \boldsymbol{i}\theta}$$
$$-\frac{W^{\beta}_{jm}(\mathbf{k}-\frac{\mathbf{p}}{2})\left[(\mathbf{k}\cdot\hat{\mathbf{v}}(\mathbf{p}))T^{\beta\alpha}_{jn}(\mathbf{k}-\frac{\mathbf{p}}{2},\mathbf{k}+\frac{\mathbf{p}}{2}) - \boldsymbol{i}(\mathbf{C}\hat{\mathbf{G}}(\mathbf{p})\mathbf{b}^{\beta,j}(\mathbf{k}-\frac{\mathbf{p}}{2})\cdot\mathbf{b}^{\alpha,n}(\mathbf{k}+\frac{\mathbf{p}}{2}))\right]}{\omega_\alpha(\mathbf{k}+\frac{\mathbf{p}}{2}) - \omega_\beta(\mathbf{k}-\frac{\mathbf{p}}{2}) + \bar{\mathbf{u}}\cdot\mathbf{p} - \boldsymbol{i}\theta}$$

$T^{\alpha\beta}_{mj}(\mathbf{k},\mathbf{p}) = \mathbf{C}\mathbf{b}^{\beta,m}(\mathbf{k})\cdot\mathbf{b}^{\alpha,j}(\mathbf{p})$ as in (35). To avoid vanishing denominator, a regularization parameter $\theta$ is introduced. The limit $\theta \to 0$ will be taken at the end of the derivation.

To get an effective equation we average equations (69), multiply throughout the averaged equation by the matrix $\mathbf{C}^{\tau,mn}(\mathbf{k}) = \mathbf{Cb}^{\tau,m}(\mathbf{k}) \otimes (\mathbf{Cb}^{\tau,n}(\mathbf{k}))$, and take the trace. We assume that $\mathbf{W}_2(t,\mathbf{x},\mathbf{z},\mathbf{k})$ is sublinear in the fast spatial variable $\mathbf{z}$ and thus $\langle \nabla_{\mathbf{z}} \mathbf{W}_2 \rangle = 0$. We note that

$$\mathrm{Tr}\mathbf{C}^{\tau,mn}(\mathbf{k})(\mathbf{L}(\mathbf{k})\mathbf{W}_2 - \mathbf{W}_2\mathbf{L}^*(\mathbf{k})) = 0$$

because of (22).

We get the left side of the equation:

$$LHS = \frac{\partial \langle W^\tau_{nm} \rangle}{\partial t} + (\bar{\mathbf{u}} + \nabla_{\mathbf{k}}\omega_\tau) \cdot \nabla_{\mathbf{x}} \langle W^\tau_{nm} \rangle.$$

The right side is a sum of several terms:

$$RHS = I^\tau_{1,nm} + I^\tau_{2,nm} + I^\tau_{3,nm}, \tag{71}$$

where

$$I^\tau_{1,nm} = -\frac{1}{2} \langle \mathrm{Tr} \int \frac{d\mathbf{p}}{(2\pi)^d} e^{\boldsymbol{i}\mathbf{p}\cdot\mathbf{z}} \hat{v}_j(\mathbf{p}) \left( \frac{\partial \mathbf{W}_1(\mathbf{k} - \frac{\mathbf{p}}{2})}{\partial z_j} + \frac{\partial \mathbf{W}_1(\mathbf{k} + \frac{\mathbf{p}}{2})}{\partial z_j} \right) \mathbf{C}^{\tau,mn}(\mathbf{k}) \rangle,$$

$$I^\tau_{2,nm} = \langle \mathrm{Tr} \int \frac{d\mathbf{p}}{(2\pi)^d} e^{\boldsymbol{i}\mathbf{p}\cdot\mathbf{z}} \left( \boldsymbol{i}(\mathbf{k} \cdot \hat{\mathbf{v}}(\mathbf{p}))\mathbf{W}_1(\mathbf{k} + \frac{\mathbf{p}}{2}) - \mathbf{W}_1(\mathbf{k} + \frac{\mathbf{p}}{2})\hat{\mathbf{G}}^*(\mathbf{p}) \right) \mathbf{C}^{\tau,mn}(\mathbf{k}) \rangle,$$

and

$$I^\tau_{3,nm} = -\langle \mathrm{Tr} \int \frac{d\mathbf{p}}{(2\pi)^d} e^{\boldsymbol{i}\mathbf{p}\cdot\mathbf{z}} \left( \boldsymbol{i}(\mathbf{k} \cdot \hat{\mathbf{v}}(\mathbf{p}))\mathbf{W}_1(\mathbf{k} - \frac{\mathbf{p}}{2}) + \hat{\mathbf{G}}(\mathbf{p})\mathbf{W}_1(\mathbf{k} - \frac{\mathbf{p}}{2}) \right) \mathbf{C}^{\tau,mn}(\mathbf{k}) \rangle.$$

First we observe that $I^\tau_{1,nm}$ contains (after averaging) terms of the form $p_j \hat{R}_{jl}(\mathbf{p})$ which vanish because $\mathbf{v}(\mathbf{x})$ is incompressible. Thus we have

$$I^\tau_{1,nm} = 0.$$

We also have the symmetry

$$I^\tau_{2,nm} = I^{\tau *}_{3,mn}. \tag{72}$$

So we have only $I^\tau_{2,nm}$ to compute. We have

$$\begin{aligned} I^\tau_{2,nm} \;=\; & \sum_{\alpha,\beta} \langle \iint \frac{d\mathbf{p}d\mathbf{q}e^{\boldsymbol{i}(\mathbf{p}+\mathbf{q})\cdot\mathbf{z}}}{(2\pi)^d} F^{\alpha\beta}_{n_1,m_1}(\mathbf{q}, \mathbf{k} + \frac{\mathbf{p}}{2}) \\ & \times \Big[ \boldsymbol{i}(\mathbf{k} \cdot \hat{\mathbf{v}}(\mathbf{p}))(\mathbf{Cb}^{\tau,m}(\mathbf{k}) \cdot \mathbf{b}^{\beta,m_1}(\mathbf{k} + \frac{\mathbf{p}}{2} - \frac{\mathbf{q}}{2}))(\mathbf{Cb}^{\tau,n}(\mathbf{k}) \cdot \mathbf{b}^{\alpha,n_1}(\mathbf{k} + \frac{\mathbf{p}}{2} + \frac{\mathbf{q}}{2})) \\ & - (\hat{\mathbf{G}}^*(\mathbf{p})\mathbf{Cb}^{\tau,m}(\mathbf{k}) \cdot \mathbf{b}^{\beta,m_1}(\mathbf{k} + \frac{\mathbf{p}}{2} - \frac{\mathbf{q}}{2}))(\mathbf{Cb}^{\tau,n}(\mathbf{k}) \cdot \mathbf{b}^{\alpha,n_1}(\mathbf{k} + \frac{\mathbf{p}}{2} + \frac{\mathbf{q}}{2})) \Big] \rangle. \end{aligned} \tag{73}$$

We insert expression (70) for $F^{\alpha\beta}_{n_1,m_1}$ and (16) for $\hat{\mathbf{G}}(\mathbf{p})$ into (73). We evaluate this average by assuming the hypothesis of scales separation: $W^\alpha_{ij}$ varies on a scale much slower than the velocity fluctuation so that, up to the leading order, $\langle W^\alpha_{ij} \rangle$ factors out. We get

$$\begin{aligned} I^\tau_{2,nm} \;=\; & \sum_{\alpha,\beta} \int \frac{d\mathbf{p}}{(2\pi)^d} \delta_{\tau\alpha}\delta_{nn_1} \hat{R}_{sl}(\mathbf{p}) \left[ \boldsymbol{i}\tilde{k}_s T^{\tau\beta}_{mm_1}(\mathbf{k}, \mathbf{k}+\mathbf{p}) - \boldsymbol{i}(\tilde{\mathbf{p}} \cdot \mathbf{b}^{\beta m_1}(\mathbf{k}+\mathbf{p}))(\mathbf{Cb}^{\tau m}(\mathbf{k}))_s \right] \\ & \times \left\{ \frac{\langle W^\alpha_{n_1 j} \rangle(\mathbf{k})[\tilde{k}_l T^{\beta\alpha}_{m_1 j}(\mathbf{k}+\mathbf{p}, \mathbf{k}) + (\tilde{\mathbf{p}} \cdot \mathbf{b}^{\alpha j}(\mathbf{k}))(\mathbf{Cb}^{\beta m_1})_l(\mathbf{k}+\mathbf{p})]}{\omega_\alpha(\mathbf{k}) - \omega_\beta(\mathbf{k}+\mathbf{p}) - \bar{\mathbf{u}} \cdot \mathbf{p} - \boldsymbol{i}\theta} \right. \\ & \left. - \frac{\langle W^\beta_{j m_1} \rangle(\mathbf{k}+\mathbf{p})[\tilde{k}_l T^{\beta\alpha}_{j n_1}(\mathbf{k}+\mathbf{p}, \mathbf{k}) - (\tilde{\mathbf{p}} \cdot \mathbf{b}^{\beta j}(\mathbf{k}+\mathbf{p}))(\mathbf{Cb}^{\alpha n_1})_l(\mathbf{k})]}{\omega_\alpha(\mathbf{k}) - \omega_\beta(\mathbf{k}+\mathbf{p}) - \bar{\mathbf{u}} \cdot \mathbf{p} - \boldsymbol{i}\theta} \right\}. \end{aligned} \tag{74}$$

We define the tensor

$$S^{\tau\beta}_{jml}(\mathbf{k},\mathbf{p}) = \tilde{k}_j T^{\tau\beta}_{ml}(\mathbf{k},\mathbf{p}) - ((\tilde{\mathbf{p}}-\tilde{\mathbf{k}})\cdot\mathbf{b}^{\beta,l}(\mathbf{p}))(\mathbf{C}\mathbf{b}^{\tau,m}(\mathbf{k}))_j,$$

which, like $T^{\tau\beta}_{ml}(\mathbf{k},\mathbf{p})$ in (34), is completely determined by the dispersion matrix $\mathbf{L}(\mathbf{k})$ but independent of the flows. For the system (1)-(2) we have the explicit expression for $S^{\tau\beta}_{jml}$ as in Section 3.2. Then we may rewrite (74), using also incompressibility of the fluctuations $\mathbf{v}(\mathbf{y})$, as

$$\begin{aligned} I^{\tau}_{2,nm} &= \sum_\beta \boldsymbol{i}\int \frac{d\mathbf{p}}{(2\pi)^d}\frac{\langle W^\tau_{ns}\rangle(\mathbf{k})\hat{R}_{jl}(\mathbf{p}-\mathbf{k})S^{\tau\beta}_{jmr}(\mathbf{k},\mathbf{p})S^{\beta\tau}_{lrs}(\mathbf{p},\mathbf{k})}{\omega_\tau(\mathbf{k})-\omega_\beta(\mathbf{p})+\bar{\mathbf{u}}\cdot(\mathbf{k}-\mathbf{p})-\boldsymbol{i}\theta} \\ &\quad -\boldsymbol{i}\int \frac{d\mathbf{p}}{(2\pi)^d}\frac{\langle W^\beta_{rs}\rangle(\mathbf{p})\hat{R}_{jl}(\mathbf{p}-\mathbf{k})S^{\tau\beta}_{jms}(\mathbf{k},\mathbf{p})S^{\tau\beta}_{lnr}(\mathbf{k},\mathbf{p})}{\omega_\tau(\mathbf{k})-\omega_\beta(\mathbf{p})+\bar{\mathbf{u}}\cdot(\mathbf{k}-\mathbf{p})-\boldsymbol{i}\theta}. \end{aligned} \tag{75}$$

Note that, because $\mathbf{v}$ is incompressible, the tensor $\hat{\mathbf{R}}$

$$\hat{\mathbf{R}} = \begin{pmatrix} \hat{\mathbf{R}}_{\mathbf{v}} & \hat{\mathbf{R}}_{\mathbf{v},p} \\ \hat{\mathbf{R}}^*_{\mathbf{v},p} & \hat{R}_p \end{pmatrix}$$

satisfies

$$\hat{\mathbf{R}}(\mathbf{k})\tilde{\mathbf{k}} = \begin{pmatrix} \mathbf{0} \\ \hat{\mathbf{R}}_{\mathbf{v},p}(\mathbf{k})\cdot\mathbf{k} \end{pmatrix},$$

and

$$(\hat{\mathbf{R}}_{\mathbf{v},p}(\mathbf{k})\cdot\mathbf{k})\delta(\mathbf{k}+\mathbf{q}) = \langle\hat{P}(\mathbf{k})\hat{\mathbf{v}}(\mathbf{q})\rangle\cdot\mathbf{k} = -\langle\hat{P}(\mathbf{k})\hat{\mathbf{v}}(\mathbf{q})\cdot\mathbf{q}\rangle = 0.$$

We combine (75) and the symmetry (72) to evaluate the right side of (71) in the limit $\theta\to 0$ and we obtain

$$\begin{aligned} RHS &= \sum_\beta\int\frac{d\mathbf{p}}{(2\pi)^{d-1}}\hat{R}_{jl}(\mathbf{p}-\mathbf{k})S^{\tau\beta}_{jms}(\mathbf{k},\mathbf{p})S^{\tau\beta}_{lnr}(\mathbf{k},\mathbf{p})\langle W^\beta_{rs}\rangle(\mathbf{p})\delta(\omega_\tau(\mathbf{k})-\omega_\beta(\mathbf{p})+\bar{\mathbf{u}}\cdot(\mathbf{k}-\mathbf{p})) \\ &\quad -(\langle\mathbf{W}^\tau\rangle(\mathbf{k})\Sigma^\tau(\mathbf{k}))_{nm} - (\Sigma^{\tau*}(\mathbf{k})\langle\mathbf{W}^\tau\rangle(\mathbf{k}))_{nm} \end{aligned}$$

with $\Sigma^\tau$ given by (29).

## B Derivation of the diffusion equation without flow-straining

Here we derive the diffusion equation (57). Equation (53), after the change of coordinates (55), becomes

$$\begin{aligned} &\epsilon^2\frac{\partial\langle W\rangle}{\partial s} + \epsilon(\bar{\mathbf{u}}-\bar{\mathbf{u}}_{\text{eff}}+c_0\hat{\mathbf{k}})\cdot\nabla_{\mathbf{y}}\langle W\rangle \\ &= \frac{1}{(2\pi)^{d-1}}\int_{\omega_a^+(\mathbf{k})=\omega_a^+(\mathbf{p})}\frac{d\Omega(\mathbf{p})}{|\bar{\mathbf{u}}+c_0\hat{\mathbf{p}}|}(\hat{\mathbf{R}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k})\frac{1}{4}(1+\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2[\langle W\rangle(\mathbf{p})-\langle W\rangle(\mathbf{k})]. \end{aligned} \tag{76}$$

Inserting the expansion $\langle W\rangle = \bar{W}_0+\epsilon\bar{W}_1+\epsilon^2\bar{W}_2+...$ into (76) we obtain, in the order $O(1)$,

$$\frac{1}{(2\pi)^{d-1}}\int_{\omega_a^+(\mathbf{k})=\omega_a^+(\mathbf{p})}\frac{d\Omega(\mathbf{p})}{|\bar{\mathbf{u}}+c_0\hat{\mathbf{p}}|}(\hat{\mathbf{R}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k})\frac{1}{4}(1+\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2[\bar{W}_0(\mathbf{p})-\bar{W}_0(\mathbf{k})] = 0.$$

Then it follows from the Krein-Rutman theorem [8] that $\bar{W}_0$ is constant on the ellipsoids $\mathcal{S}^{\pm}_h, \forall h\in R$:

$$\bar{W}_0 = \bar{W}_0(s,\mathbf{y},c_0|\mathbf{k}|+\bar{\mathbf{u}}\cdot\mathbf{k}). \tag{77}$$

The order $O(\epsilon)$ term in (53) gives

$$(\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\mathrm{eff}} + c_0\hat{\mathbf{k}}) \cdot \nabla_{\mathbf{y}} \bar{W}_0 = \mathcal{Q}\bar{W}_1(\mathbf{k}) \tag{78}$$

with the operator $\mathcal{Q}$ as defined by (54). The solvability condition for $\bar{W}_1$ from (78) is

$$\int_{\omega_a^+(\mathbf{k})=h} (\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\mathrm{eff}}(h) + c_0\hat{\mathbf{k}}) \cdot \nabla_{\mathbf{y}} \bar{W}_0(\mathbf{k}) \frac{d\Omega(\mathbf{k})}{|\bar{\mathbf{u}} + c_0\hat{\mathbf{k}}|} = 0, \quad \forall h \in R.$$

Using (77) we obtain the effective drift $\bar{\mathbf{u}}_{\mathrm{eff}}$ as given in (56).

To solve (78) we first solve for the corrector $\chi(\mathbf{k})$

$$\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\mathrm{eff}} + c_0\hat{\mathbf{k}} = \mathcal{Q}\chi \tag{79}$$

Then the first order term $\bar{W}_1$ is given by

$$\bar{W}_1(t, \mathbf{x}, \mathbf{k}) = \chi(\mathbf{k}) \cdot \nabla_{\mathbf{y}} \bar{W}_0 \tag{80}$$

The order $O(\varepsilon^2)$ terms in (53) give

$$\frac{\partial \bar{W}_0}{\partial s} + (\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\mathrm{eff}} + c_0\hat{\mathbf{k}}) \cdot \nabla_{\mathbf{y}} \bar{W}_1 = \mathcal{Q}\bar{W}_2$$

which is solvable provided that

$$\int_{\mathcal{S}_h^+} \left( \frac{\partial \bar{W}_0}{\partial s} + (\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\mathrm{eff}} + c_0\hat{\mathbf{k}}) \cdot \nabla_{\mathbf{y}} \bar{W}_1 \right) \frac{d\Omega(\mathbf{k})}{|\bar{\mathbf{u}} + c_0\hat{\mathbf{k}}|} = 0, \quad \forall h \in R.$$

Using (80) we have

$$\int_{\mathcal{S}_h^+} \left( \frac{\partial \bar{W}_0}{\partial s} + (\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\mathrm{eff}} + c_0\hat{\mathbf{k}}) \cdot \nabla_{\mathbf{y}} [\chi \cdot \nabla_{\mathbf{y}} \bar{W}_0] \right) \frac{d\Omega(\mathbf{k})}{|\bar{\mathbf{u}} + c_0\hat{\mathbf{k}}|} = 0, \quad \forall h \in R.$$

Thus the limiting diffusion equation for $\bar{W}_0$ is (57).

# C Derivation of the diffusion approximation with flow-straining

We derive in the section the diffusion approximation with straining (60). In the long run the deviation from the equilibrium state with equi-distribution of energy on frequency surfaces, is small. It is more convenient to use a slightly different form the transport equation (43), which, after rescaling, becomes

$$\begin{aligned} & \epsilon^2 \frac{\partial \langle W \rangle}{\partial t} + \epsilon c_0(\hat{\mathbf{k}} + \epsilon \hat{\mathbf{u}}) \cdot \nabla_{\mathbf{x}} \langle W \rangle - L'_\epsilon(\mathbf{k}) \langle W \rangle \\ = & \int \frac{d\mathbf{p}}{(2\pi)^{d-1}} \sigma(\mathbf{k}, \mathbf{p})(\langle W \rangle(\mathbf{p}) - \langle W \rangle(\mathbf{k})) \delta(c_0|\mathbf{k}| - c_0|\mathbf{p}| + \epsilon c_0 \hat{\mathbf{u}} \cdot (\mathbf{k} - \mathbf{p})), \end{aligned} \tag{81}$$

where

$$L'_\epsilon(\mathbf{k}) = \int \frac{d\mathbf{p}}{(2\pi)^{d-1}} \sigma'(\mathbf{k}, \mathbf{p}) \delta(c_0|\mathbf{k}| - c_0|\mathbf{p}| + \epsilon c_0 \hat{\mathbf{u}} \cdot (\mathbf{k} - \mathbf{p})) \tag{82}$$

which is different from (50) since $\sigma(\mathbf{k}, \mathbf{p}) \neq \sigma(\mathbf{p}, \mathbf{k})$ in general. The cross-sections $\sigma(\mathbf{k}, \mathbf{p})$ and $\sigma'(\mathbf{k}, \mathbf{p})$ are given by (48) and (49). Let us first consider the coefficient $L'_\epsilon(\mathbf{k})$ defined above.

We use the delta function expansion

$$\int d\mathbf{p} f(\mathbf{p})\delta(c_0|\mathbf{k}| - c_0|\mathbf{p}| + \epsilon c_0 \hat{\mathbf{u}}\cdot(\mathbf{k}-\mathbf{p})) \tag{83}$$
$$= \int d\mathbf{p} f(\mathbf{p})\delta(c_0|\mathbf{k}| - c_0|\mathbf{p}|) - \epsilon \int d\mathbf{p} f(\mathbf{p})(\hat{\mathbf{p}}\cdot\hat{\mathbf{u}})\delta(c_0|\mathbf{k}| - c_0|\mathbf{p}|)$$
$$+\epsilon\frac{|\mathbf{k}|}{c_0}\int_{|\mathbf{p}|=|\mathbf{k}|} d\Omega(\hat{\mathbf{p}})(\hat{\mathbf{u}}\cdot(\hat{\mathbf{k}}-\hat{\mathbf{p}}))\hat{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\left(|\mathbf{p}|^2 f(\mathbf{p})\right) + O(\epsilon^2),$$

for any smooth, fast decaying test functions $f(\mathbf{p})$. Using (83) we write

$$L'_\epsilon(\mathbf{k}) = \epsilon A_1 + \epsilon^2 A_2 + \dots. \tag{84}$$

The scalars $A_{1,2}$ are given by

$$A_1(\mathbf{k}) = \frac{|\mathbf{k}|^{d-1}}{c_0}\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})((\hat{\mathbf{p}}-\hat{\mathbf{k}})\cdot\hat{\mathbf{u}})$$

and

$$A_2(\mathbf{k}) = \frac{|\mathbf{k}|^{d-1}}{2c_0}\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})((\hat{\mathbf{k}}-\hat{\mathbf{p}})\cdot\hat{\mathbf{u}})\left[\hat{\mathbf{p}}\cdot\hat{\mathbf{u}}(3\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}-1) + \hat{\mathbf{k}}\cdot\hat{\mathbf{u}}(1+\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})\right]$$
$$-\frac{|\mathbf{k}|^{d-2}}{c_0}\int_{|\mathbf{p}|=|\mathbf{k}|}\frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})((\hat{\mathbf{p}}-\hat{\mathbf{k}})\cdot\hat{\mathbf{u}})^2\hat{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\left[|\mathbf{p}|^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})\right].$$

Note that we have

$$\int d\Omega(\hat{\mathbf{k}})A_1(\mathbf{k}) = 0 \tag{85}$$

and hence the amplification/attenuation effect is only second order in the Mach number $\epsilon$ and hence diffusive scaling leads to a non-trivial limit. Similarly, we expand the right hand side of (81).

We assume the expansion $\langle W\rangle = \bar{W}_0^\epsilon + \epsilon\bar{W}_1 + \epsilon^2\bar{W}_2 + \dots$ with the ansatz for the leading term

$$\bar{W}_0^\epsilon = \bar{W}_0(t,\mathbf{x},c_0|\mathbf{k}| + \epsilon c_0\hat{\mathbf{u}}\cdot\mathbf{k}) = \bar{W}_0(t,\mathbf{x},\omega) + \epsilon c_0\hat{\mathbf{u}}\cdot\mathbf{k}\frac{\partial\bar{W}_0(c_0|\mathbf{k}|)}{\partial\omega} + \dots,$$

Here $\omega = c_0|\mathbf{k}|$ denotes the frequency variable. Then the contribution of the leading term $\bar{W}_0^\epsilon$ to the right side of (81) vanishes identically

$$\int \frac{d\mathbf{p}}{(2\pi)^{d-1}}\sigma(\mathbf{k},\mathbf{p})\left[\bar{W}_0^\epsilon(\mathbf{p}) - \bar{W}_0^\epsilon(\mathbf{k})\right]\delta(c_0|\mathbf{k}| - c_0|\mathbf{p}| + \epsilon c_0\hat{\mathbf{u}}\cdot(\mathbf{k}-\mathbf{p})) = 0$$

since $\bar{W}_0^\epsilon$ is constant on the frequency ellipsoids. The leading order term in (81) is $O(\epsilon)$:

$$c_0\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}\bar{W}_0(\omega) - A_1(\mathbf{k})\bar{W}_0(c_0|\mathbf{k}|) = \mathcal{Q}_0\bar{W}_1 \tag{86}$$

with $\mathcal{Q}_0$ as defined by (52). The solvability condition for (86) is that the integral of the left side over the frequency sphere $\mathcal{S}_h^+ = \{\mathbf{k} : c_0|\mathbf{k}| = h\}$, $\forall h > 0$ vanishes. We note that

$$\int c_0\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}\bar{W}_0(c_0|\mathbf{k}|)\delta(c_0|\mathbf{k}| - \omega)dk = 0, \quad \forall\omega\geq 0$$

and also (85) holds. Thus, (86) is solvable by the Fredholm alternative.

Let $\phi(\mathbf{k})$ and $\chi_j(\mathbf{k})$ be solutions of (65) and (66), respectively. Then $\bar{W}_1$ is given by

$$\bar{W}_1 = \sum_{j=1}^{d} \chi_j(\mathbf{k}) \frac{\partial \bar{W}_0}{\partial x_j} + \phi(\mathbf{k}) \bar{W}_0. \qquad (87)$$

Substituting (84), (C) in (81), we find the order $O(\epsilon^2)$ term:

$$\frac{\partial \bar{W}_0}{\partial t} + c_0 \hat{\mathbf{u}} \cdot \nabla_{\mathbf{x}} \bar{W}_0 + c_0 \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} \bar{W}_1 + c_0^2 \hat{\mathbf{u}} \cdot \mathbf{k} \hat{k}_j \frac{\partial^2 \bar{W}_0}{\partial x_j \partial \omega} - c_0 \hat{\mathbf{u}} \cdot \mathbf{k} A_1(\mathbf{k}) \frac{\partial \bar{W}_0}{\partial \omega} - A_1 \bar{W}_1 - A_2 \bar{W}_0$$
$$-\mathcal{Q}_1 \bar{W}_1 = \mathcal{Q}_0 \bar{W}_2 \qquad (88)$$

since the terms on the right that involve $\bar{W}_0$ and its derivatives vanish identically by contruction. The operator $\mathcal{Q}_1$ on the left side of (88) is defined by

$$\begin{aligned} & \mathcal{Q}_1 f \\ = & -\frac{|\mathbf{k}|^{d-1}}{c_0} \int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}} (\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2 (f(\mathbf{p}) - f(\mathbf{k})) \\ & + \frac{|\mathbf{k}|^{d-2}}{c_0} \int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}} (\hat{\mathbf{u}}\cdot\hat{\mathbf{k}} - \hat{\mathbf{u}}\cdot\hat{\mathbf{p}}) \hat{\mathbf{p}} \cdot \nabla_{\mathbf{p}} \left[ |\mathbf{p}|^2 (\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(f(\mathbf{p}) - f(\mathbf{k})) \left( \hat{\mathbf{k}}\cdot\hat{\mathbf{p}} + \frac{1}{2} - \frac{|\mathbf{p}|}{2|\mathbf{k}|} \right)^2 \right]. \end{aligned}$$

As before $\bar{W}_2$ is solvable from (88) if the left side of (88) has the zero mean on every frequency surface $c_0|\mathbf{k}| = \omega$. Let us now compute all the terms in (88) after averaging over this sphere of wave vectors:

$$\int d\mathbf{k} \frac{\partial \bar{W}_0}{\partial t} \delta(c_0|\mathbf{k}| - \omega) = \frac{4\pi |\mathbf{k}|^2}{c_0} \frac{\partial \bar{W}_0(\omega)}{\partial t}$$

and

$$\int d\mathbf{k} c_0 \hat{\mathbf{u}} \cdot \nabla_{\mathbf{x}} \bar{W}_0 \delta(c_0|\mathbf{k}| - \omega) = 4\pi |\mathbf{k}|^2 \hat{\mathbf{u}} \cdot \nabla_{\mathbf{x}} \bar{W}_0 = \frac{4\pi |\mathbf{k}|^2}{c_0} \mathbf{B}_0 \cdot \nabla_{\mathbf{x}} \bar{W}_0.$$

By (87),

$$\begin{aligned} & \int d\mathbf{k} c_0 \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} \bar{W}_1(\mathbf{k}) \delta(c_0|\mathbf{k}| - \omega) = |\mathbf{k}|^2 \int d\Omega(\hat{\mathbf{k}}) \hat{k}_n \chi_j(\mathbf{k}) \frac{\partial^2 \bar{W}_0}{\partial x_j \partial x_n} + |\mathbf{k}|^2 \int d\Omega(\hat{\mathbf{k}}) \hat{k}_n \phi(\mathbf{k}) \frac{\partial \bar{W}_0}{\partial x_n} \\ & =: -\frac{4\pi |\mathbf{k}|^2}{c_0} D_{nj} \frac{\partial^2 \bar{W}_0}{\partial x_n \partial x_j} + \frac{4\pi |\mathbf{k}|^2}{c_0} \mathbf{B}_1 \cdot \nabla_{\mathbf{x}} \bar{W}_0, \end{aligned}$$

and

$$\int d\mathbf{k} c_0^2 \hat{\mathbf{u}} \cdot \mathbf{k} \hat{k}_j \frac{\partial^2 \bar{W}_0}{\partial x_j \partial \omega} \delta(c_0|\mathbf{k}| - \omega) = \frac{4\pi c_0 |\mathbf{k}|^3}{3} \hat{u}_j \frac{\partial^2 \bar{W}_0}{\partial x_j \partial \omega} =: \frac{4\pi |\mathbf{k}|^2}{c_0} F_j^0 \frac{\partial^2 \bar{W}_0}{\partial x_j \partial \omega}. \qquad (89)$$

So we have $\mathbf{B}_1 = [B_{1,j}]$, $A_0$ with

$$\begin{aligned} B_{1,j} &= \frac{1}{4\pi} \int d\Omega(\hat{\mathbf{p}}) \phi \mathcal{Q}_0 \chi_j \\ A_0 &= \frac{1}{4\pi} \int d\Omega(\hat{\mathbf{p}}) \phi \mathcal{Q}_0 \phi \end{aligned}$$

The next term on the left side is

$$-c_0 \int d\mathbf{k} \hat{\mathbf{u}} \cdot \mathbf{k} A_1(\mathbf{k}) \delta(c_0|\mathbf{k}| - \omega) \frac{\partial \bar{W}_0}{\partial \omega} = -|\mathbf{k}|^3 \int d\Omega(\hat{\mathbf{k}}) \hat{\mathbf{u}} \cdot \hat{\mathbf{k}} A_1(\mathbf{k}) \frac{\partial \bar{W}_0}{\partial \omega} =: \frac{4\pi |\mathbf{k}|^2}{c_0} C_1 \frac{\partial \bar{W}_0}{\partial \omega}. \qquad (90)$$

The next term on the left in (88) is

$$
\begin{aligned}
& - \int d\mathbf{k} A_1(\mathbf{k}) \bar{W}_1(\mathbf{k}) \delta(c_0|\mathbf{k}| - \omega) \\
= \quad & -\frac{|\mathbf{k}|^2}{c_0} \int d\Omega(\hat{\mathbf{k}}) A_1(\mathbf{k}) \chi_j(\mathbf{k}) \frac{\partial \bar{W}_0}{\partial x_j} - \frac{|\mathbf{k}|^2}{c_0} \int d\Omega(\hat{\mathbf{k}}) A_1(\mathbf{k}) \phi(\mathbf{k}) \bar{W}_0 \\
=: \quad & \frac{4\pi|\mathbf{k}|^2}{c_0} \mathbf{B}_2 \cdot \nabla_{\mathbf{x}} \bar{W}_0 + \frac{4\pi|\mathbf{k}|^2}{c_0} A_0 \bar{W}_0.
\end{aligned}
$$

Thus, $\mathbf{B}_2 = [B_{2,j}]$ with

$$
B_{2,j} = \frac{1}{4\pi} \int d\Omega(\hat{\mathbf{p}}) \mathcal{Q}_0 \chi_j \phi.
$$

The second to the last term on the left of (88) is

$$
-\frac{|\mathbf{k}|^2}{c_0} \int d\Omega(\hat{\mathbf{k}}) A_2(\mathbf{k}) \bar{W}_0(\omega) =: \frac{4\pi|\mathbf{k}|^2}{c_0} A_1 \bar{W}_0
$$

with $A_1$ becomes part of the attenuation/amplification coefficient. The last term on the left side of (88) we have to compute is

$$
- \int d\mathbf{k} \mathcal{Q}_1[\bar{W}_1] \delta(c_0|\mathbf{k}| - \omega) = - \int d\mathbf{k} \mathcal{Q}_1 \left[ \chi_j \frac{\partial \bar{W}_0}{\partial x_j} \right] \delta(c_0|\mathbf{k}| - \omega) - \int d\mathbf{k} \mathcal{Q}_1[\phi \bar{W}_0] \delta(c_0|\mathbf{k}| - \omega) =: I + II.
$$

The first term may be written as

$$
\begin{aligned}
I \quad = \quad & \frac{|\mathbf{k}|^4}{c_0^2} \int d\Omega(\hat{\mathbf{k}}) \int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}} (\hat{R}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2 (\chi_j(\mathbf{p}) - \chi_j(\mathbf{k})) \frac{\partial \bar{W}_0}{\partial x_j}(\omega) \\
& - \frac{|\mathbf{k}|^3}{c_0^2} \int d\Omega(\hat{\mathbf{k}}) \int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}} (\hat{\mathbf{u}}\cdot\hat{\mathbf{k}} - \hat{\mathbf{u}}\cdot\hat{\mathbf{p}}) \hat{\mathbf{p}} \cdot \nabla_{\mathbf{p}} \Bigg[ |\mathbf{p}|^2 (\hat{R}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k}) \chi_j(\mathbf{p}) \frac{\partial \bar{W}_0}{\partial x_j}(c_0|\mathbf{p}|) \\
& \qquad - \chi_j(\mathbf{k}) \frac{\partial \bar{W}_0}{\partial x_j}(c_0|\mathbf{k}|) \left( \hat{\mathbf{k}}\cdot\hat{\mathbf{p}} + \frac{1}{2} - \frac{|\mathbf{p}|}{2|\mathbf{k}|} \right)^2 \Bigg] \\
=: \quad & I_1 + I_2.
\end{aligned}
$$

Using (66) we have

$$
\begin{aligned}
I_1 \quad & = \quad -\frac{|\mathbf{k}|^2}{c_0} \int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}} \mathcal{Q}_0 \chi_j(\mathbf{p}) (\hat{\mathbf{p}}\cdot\hat{\mathbf{u}}) \frac{\partial \bar{W}_0}{\partial x_j} \\
& = \quad -|\mathbf{k}|^2 \int d\Omega(\hat{\mathbf{p}}) (\hat{\mathbf{u}}\cdot\hat{\mathbf{p}}) \hat{\mathbf{p}} \cdot \nabla_{\mathbf{x}} \bar{W}_0 = -\frac{4\pi|\mathbf{k}|^2}{3} \hat{\mathbf{u}} \cdot \nabla_{\mathbf{x}} \bar{W}_0
\end{aligned}
$$

and

$$
\begin{aligned}
& I_2 = -\frac{|\mathbf{k}|^3}{c_0^2} \int d\Omega(\hat{\mathbf{k}}) \int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}} (\hat{\mathbf{u}}\cdot\hat{\mathbf{k}} - \hat{\mathbf{u}}\cdot\hat{\mathbf{p}}) \hat{\mathbf{p}} \cdot \nabla_{\mathbf{p}} \Big[ |\mathbf{p}|^2 (\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k}) \\
& \times \left( \chi_j(\mathbf{p}) \frac{\partial \bar{W}_0(c_0|\mathbf{p}|)}{\partial x_j} - \chi_j(\mathbf{k}) \frac{\partial \bar{W}_0(c_0|\mathbf{k}|)}{\partial x_j} \right) \left( \hat{\mathbf{k}}\cdot\hat{\mathbf{p}} + \frac{1}{2} - \frac{|\mathbf{p}|}{2|\mathbf{k}|} \right)^2 \Big] \\
& = -\frac{|\mathbf{k}|^3}{c_0^2} \int d\Omega(\hat{\mathbf{k}}) \int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}} (\hat{\mathbf{u}}\cdot\hat{\mathbf{k}} - \hat{\mathbf{u}}\cdot\hat{\mathbf{p}}) \Bigg\{ |\mathbf{k}|^2 (\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k}) \chi_j(\mathbf{p}) (\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2 c_0 \frac{\partial^2 \bar{W}_0}{\partial x_j \partial \omega} \\
& + \hat{\mathbf{p}} \cdot \nabla_{\mathbf{p}} \left[ |\mathbf{p}|^2 (\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k}) (\chi_j(\mathbf{p}) - \chi_j(\mathbf{k})) \left( \hat{\mathbf{k}}\cdot\hat{\mathbf{p}} + \frac{1}{2} - \frac{|\mathbf{p}|}{2|\mathbf{k}|} \right)^2 \right] \frac{\partial \bar{W}_0(c_0|\mathbf{k}|)}{\partial x_j} \Bigg\} \\
& =: \frac{4\pi|\mathbf{k}|^2}{c_0} \left[ F_j^1 \frac{\partial^2 \bar{W}_0}{\partial x_j \partial \omega} + \mathbf{B}_3 \cdot \nabla_{\mathbf{x}} \bar{W}_0 \right].
\end{aligned}
\tag{91}
$$

Next we look at

$$II = -\int d\mathbf{k}\mathcal{Q}_1[\phi(\mathbf{k})\bar{W}_0(\mathbf{k})]\delta(c_0|\mathbf{k}| - \omega) = II_1 + II_2.$$

By (65) we have

$$\begin{aligned}
II_1 &= -\frac{|\mathbf{k}|^2}{c_0}\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}\mathcal{Q}_0\chi_j(\mathbf{p})(\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})\bar{W}_0 \\
&= \frac{|\mathbf{k}|^2}{c_0}\int d\Omega(\hat{\mathbf{k}})A_1(\mathbf{p})(\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})\bar{W}_0(\omega) = \frac{|\mathbf{k}|^2}{c_0}\int d\Omega(\hat{\mathbf{p}})(\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})A_1(\mathbf{p})\bar{W}_0 \\
&= -\frac{4\pi|\mathbf{k}|^2}{c_0}\frac{C_1}{\omega}\bar{W}_0 \\
&=: \frac{4\pi|\mathbf{k}|^2}{c_0}A_2\bar{W}_0
\end{aligned}$$

and

$$\begin{aligned}
II_2 &= -\frac{|\mathbf{k}|^3}{c_0^2}\int d\Omega(\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}} - \hat{\mathbf{u}}\cdot\hat{\mathbf{p}})\hat{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\Big[|\mathbf{p}|^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k}) \\
&\times(\phi(\mathbf{p})\bar{W}_0(c_0|\mathbf{p}|) - \phi(\mathbf{k})\bar{W}_0(c_0|\mathbf{k}|))\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} + \frac{1}{2} - \frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)^2\Big] \\
&= -\frac{|\mathbf{k}|^3}{c_0^2}\int d\Omega(\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}} - \hat{\mathbf{u}}\cdot\hat{\mathbf{p}})\Big\{|\mathbf{k}|^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})\phi(\mathbf{p})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2c_0\frac{\partial\bar{W}_0}{\partial\omega} \\
&+\hat{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\left[|\mathbf{p}|^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\phi(\mathbf{p}) - \phi(\mathbf{k}))\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} + \frac{1}{2} - \frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)^2\right]\bar{W}_0(c_0|\mathbf{k}|)\Big\} \\
&=: \frac{4\pi|\mathbf{k}|^2}{c_0}\left\{C_2\frac{\partial\bar{W}_0}{\partial\omega} + A_3\bar{W}_0\right\}. \qquad (92)
\end{aligned}$$

Now we start simplifying these expressions and putting them together. The terms involving the mixed second order derivative in $x_j$ and $\omega$ appear in (89) and (91). Their total contribution is zero:

$$\begin{aligned}
F_j &= F_j^0 + F_j^1 \\
&= \frac{c_0^2|\mathbf{k}|}{3}\hat{u}_j - \frac{|\mathbf{k}|^3}{4\pi}\int d\Omega(\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}} - \hat{\mathbf{u}}\cdot\hat{\mathbf{p}})(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2\chi_j(\mathbf{p}) \\
&= \frac{c_0^2|\mathbf{k}|}{3}\hat{u}_j - \frac{|\mathbf{k}|^3}{4\pi}\int d\Omega(\hat{\mathbf{k}})\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}\int \frac{d\Omega(\mathbf{p})}{(2\pi)^{d-1}}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2(\chi_j(\mathbf{p}) - \chi_j(\mathbf{k})) \\
&= \frac{c_0^2|\mathbf{k}|}{3}\hat{u}_j - \frac{c_0|\mathbf{k}|}{4\pi}\int d\Omega(\hat{\mathbf{k}})\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}\mathcal{Q}_0\chi_j(\mathbf{k}) \\
&= \frac{c_0^2|\mathbf{k}|}{3}\hat{u}_j - \frac{c_0^2|\mathbf{k}|}{4\pi}\int d\Omega(\hat{\mathbf{k}})\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}\hat{k}_j \\
&= \frac{c_0^2|\mathbf{k}|}{3}\hat{u}_j - \frac{c_0^2|\mathbf{k}|}{3}\hat{u}_j = 0, \qquad (93)
\end{aligned}$$

where we used (66). The drift in frequency $\partial\bar{W}_0/\partial\omega$ appears in (90) and (92). Its total contribution is also zero:

$$C_1 + C_2$$

$$
\begin{aligned}
&= C_1 - \frac{|\mathbf{k}|^3}{4\pi}\int d\Omega(\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}-\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2\phi(\mathbf{p}) \\
&= C_1 - \frac{|\mathbf{k}|^3}{4\pi}\int d\Omega(\hat{\mathbf{k}})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2(\phi(\mathbf{p})-\phi(\mathbf{k})) \\
&= C_1 + \frac{|\mathbf{k}|c_0}{4\pi}\int d\Omega(\hat{\mathbf{k}})A_1(\mathbf{k})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}) = 0 \qquad (94)
\end{aligned}
$$

where we used (65) and (90). Therefore different frequencies are decoupled.

Now we look at the drift in the spatial variable $\mathbf{x}$ and the amplification/attenuation coefficient. The following term in the expression of $\mathbf{B}_3$ can be simplified

$$
\begin{aligned}
& -\frac{2|\mathbf{k}|^2}{4\pi c_0}\int d\Omega(\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}-\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2(\chi_j(\mathbf{p})-\chi_j(\mathbf{k})) \\
=& -\frac{4|\mathbf{k}|^2}{4\pi c_0}\int d\Omega(\hat{\mathbf{k}})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\chi_j(\mathbf{p})-\chi_j(\mathbf{k})) \\
=& -\frac{1}{\pi}\int d\Omega(\hat{\mathbf{k}})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}})\mathcal{Q}_0\chi_j \\
=& -\frac{4}{3}c_0\hat{u}_j \qquad (95)
\end{aligned}
$$

as well as a similar term in the expression of $A_3$

$$
\begin{aligned}
& -\frac{2|\mathbf{k}|^2}{4\pi c_0}\int d\Omega(\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}-\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2(\phi(\mathbf{p})-\phi(\mathbf{k})) \\
=& -\frac{4|\mathbf{k}|^2}{4\pi c_0}\int d\Omega(\hat{\mathbf{k}})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\phi(\mathbf{p})-\phi(\mathbf{k})) \\
=& \frac{4}{4\pi}\int d\Omega(\hat{\mathbf{k}})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}})A_1(\mathbf{k}) = -4C_1/\omega. \qquad (96)
\end{aligned}
$$

Note also that $\mathbf{B}_2 = \mathbf{B}_1$. Then we collect the rest of the terms. In summary we get a diffusion equation for $\bar{W}_0$:

$$
\frac{\partial \bar{W}_0}{\partial t} + \mathbf{B}(\omega)\cdot\nabla_{\mathbf{x}}\bar{W}_0 + A(\omega)\bar{W}_0 = D_{ij}\frac{\partial^2 \bar{W}_0}{\partial x_i \partial x_j}
$$

with $\mathbf{B}, A, D_{ij}$ given by (62), (64), (61), respectively.

**Acknowledgment** We thank Prof. John Hunter for several stimulating discussions. We are grateful to Prof. J. Keller and the referees for helpful suggestions that led to improvement of the manuscript. The research is supported in part by National Science Foundation Grant DMS-9600119, DMS-9971322 and NSF MRSEC Program Grant DMR-9400379. Part of this research was done while L.R. was visiting MSRI.

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