STABILITY OF THE P TO S ENERGY RATIO IN THE DIFFUSIVE REGIME

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Abstract. We use the transport equations for elastic waves derived in Ryzhik et.al. (1995) to obtain results about mode conversion that are of interest in seismology. These transport equations take into account the S wave polarization and the P to S mode conversion. We also discuss the regime of deep coda waves where the diffusion approximation is valid. We show that the ratio of the P and S wave energy density equilibrates in a unique way that is independent of the details of the scattering. We note the relevance of this to the P/Lg energy stabilization observed in Hansen et.al. (1990).

1. Introduction

The propagation of elastic wave energy in the presence of inhomogeneities can be described by radiative transport equations. This description provides a good approximation when (i) typical wavelengths are short compared to the propagation distance, (ii) correlation lengths are comparable to wavelengths so that the inhomogeneities have appreciable effect and (iii) the fluctuations are weak. The relevant transport equations are derived in Ryzhik et.al. (1995) starting from the elastic wave equations in an unbounded medium. They are a coupled system for the angularly resolved P wave energy density $a^P(t, \mathbf{x}, \mathbf{k})$ and the S wave coherence matrix $W^S(t, \mathbf{x}, \mathbf{k})$. Here $\mathbf{k}$ is the wave vector, t is time and $\mathbf{x}$ is position. They account for the S wave polarization and the P to S and S to P wave energy conversion.

The transport equations explain the dominance of S waves in the deep coda, observed in seismograms and discussed in Aki (1992). The reason for this phenomenon is that in the diffusive regime, over distances long compared to the transport mean free path and over time long compared to the transport mean free time, the P to S energy conversion by the random inhomogeneities equilibrates in a universal way, **independent of the details of the scattering**. There is an equipartition of energy (Ryzhik et.al. (1995)) that leads to the relation

$$\mathcal{E}_P(t, \mathbf{x}) = \frac{v_S^3}{2v_P^3}\mathcal{E}_S(t, \mathbf{x}). \tag{1.1}$$

Here $\mathcal{E}_P$ and $\mathcal{E}_S$ are the P and S spatial energy densities, and v_P and v_S are the P and S wave speeds, respectively. For typical values of the P and S speeds this relation becomes $\mathcal{E}_S \sim 10\mathcal{E}_P$, which is in general agreement with observations.

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This equipartition law is derived when P to S mode conversion is generated by volume scattering. It was observed in Hansen et.al. (1990) that the P/Lg ratio stabilizes for elastic waves in the crustal region. This stabilization may be similar to that described in (1.1) and could follow from a radiative transport theory which takes into account the free surface and the crustal waveguide structure.

2. Radiative Transport Equations

The theory of radiative transport was originally developed to describe how light energy propagates through a turbulent atmosphere. It is based upon a linear transport equation for the angularly resolved energy density and was first derived phenomenologically at the beginning of this century (Chandrasekhar, 1960; van der Hulst, 1980). We show in Ryzhik et.al. (1995) how this theory can be derived from the governing equations for light and for other waves of any type, in a randomly inhomogeneous medium. Our results take into account nonuniformity of the background medium on the scale large compared to the wavelength, scattering by random inhomogeneities on a scale of the wavelength, the effect of polarization, the coupling of different types of waves, etc. The main new application is to elastic waves, in which shear waves exhibit polarization effects while the compressional waves do not, and the two types of waves are coupled. We also analyze solutions of the transport equations at long times and long distances and show that they have diffusive behavior.

Transport equations arise because a wave with wave vector $\mathbf{k}'$ at a point $\mathbf{x}$ in a randomly inhomogeneous medium may be scattered into any direction $\hat{\mathbf{k}}$ with wave vector $\mathbf{k}$. Therefore one must consider the angularly resolved, wave vector dependent, scalar energy density $a(t, \mathbf{x}, \mathbf{k})$ defined for all $\mathbf{k}$ at each point $\mathbf{x}$ and time t. Energy conservation is expressed by the transport equation

$$\begin{aligned}\frac{\partial a(t,\mathbf{x},\mathbf{k})}{\partial t} &+ \nabla_{\mathbf{k}}\omega(\mathbf{x},\mathbf{k})\cdot\nabla_{\mathbf{x}}a(t,\mathbf{x},\mathbf{k}) - \nabla_{\mathbf{x}}\omega(\mathbf{x},\mathbf{k})\cdot\nabla_{\mathbf{k}}a(t,\mathbf{x},\mathbf{k}) \\ &= \int_{\mathbb{R}^3}\sigma(\mathbf{x},\mathbf{k},\mathbf{k}')a(t,\mathbf{x},\mathbf{k}')d\mathbf{k}' - \Sigma(\mathbf{x},\mathbf{k})a(t,\mathbf{x},\mathbf{k}). \qquad (2.1)\end{aligned}$$

Here $\omega(\mathbf{x}, \mathbf{k})$ is the frequency at $\mathbf{x}$ of the wave with wave vector $\mathbf{k}$, $\sigma(\mathbf{x}, \mathbf{k}, \mathbf{k}')$ is the differential scattering cross-section, the rate at which energy with wave vector $\mathbf{k}'$ is converted to wave energy with wave vector $\mathbf{k}$ at position $\mathbf{x}$, and

$$\Sigma(\mathbf{x},\mathbf{k}) = \int \sigma(\mathbf{x},\mathbf{k}',\mathbf{k})d\mathbf{k}' \qquad (2.2)$$

is the total scattering cross-section. Both σ and Σ are nonnegative and σ is usually symmetric in $\mathbf{k}$ and $\mathbf{k}'$. For an acoustic wave the differential scattering cross-section is given by

$$\begin{aligned}\sigma(\mathbf{x},\mathbf{k},\mathbf{k}') &= \Big((\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')^2\hat{R}_{\rho\rho}(\mathbf{k}-\mathbf{k}') + 2(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')\hat{R}_{\rho\kappa}(\mathbf{k}-\mathbf{k}') + \hat{R}_{\kappa\kappa}(\mathbf{k}-\mathbf{k}')\Big) \\ &\cdot\frac{\pi v^2(\mathbf{x})|\mathbf{k}|^2}{2}\delta(v(\mathbf{x})|\mathbf{k}| - v(\mathbf{x})|\mathbf{k}'|), \qquad (2.3)\end{aligned}$$

where $\hat{R}_{\rho\rho}$, $\hat{R}_{\rho\kappa}$ and $\hat{R}_{\kappa\kappa}$ are the power spectra of the fluctuations of the density ρ and compressibility κ defined in Ryzhik et.al. (1995). The left side of (2.1) is the total time derivative of $a(t, \mathbf{x}, \mathbf{k})$ at a point moving along a ray in phase space $(\mathbf{x}, \mathbf{k})$, which means that the frequency of the ray is adjusting to the appropriate local value. The right side of (2.1) represents the effects of scattering.

The transport equation (2.1) is conservative because

$$\iint a(t, \mathbf{x}, \mathbf{k}) d\mathbf{x} d\mathbf{k} = \text{const}$$

when the total scattering cross-section is given by (2.2). For simplicity we assumed no intrinsic attenuation in Ryzhik et.al. (1995). However, attenuation is easily accounted for by letting the total scattering cross-section be the sum of two terms

$$\Sigma(\mathbf{x}, \mathbf{k}) = \Sigma_{sc}(\mathbf{x}, \mathbf{k}) + \Sigma_{ab}(\mathbf{x}, \mathbf{k}) \tag{2.4}$$

where $\Sigma_{sc}(\mathbf{x}, \mathbf{k})$ is the total cross-section due to scattering and is given by (2.2) and $\Sigma_{ab}(\mathbf{x}, \mathbf{k})$ is the attenuation rate.

The reason that power spectral densities of the inhomogeneities determine the scattering cross-section (2.3) is seen most easily from a Born expansion of the wave equations when the inhomogeneities are weak. This is because the single scattering approximation of (2.1) and the second moments of the single scattering approximation for the underlying wave equations (the Born expansion) must be the same. The latter are determined by the power spectra of the inhomogeneities. In the same manner we can explain the appearance of the delta function in the cross-section (2.3) when the random inhomogeneities do not depend on time and therefore the frequencies are unchanged by the scattering. The transport equation (2.1) arises also when the waves are scattered by discrete scatterers that are randomly distributed in the medium. In this case the scattering cross-section (2.3) is the same as the cross-section of a single scatterer times the density of scatterers. We deal only with continuous random media in Ryzhik et.al. (1995).

Equation (2.1) has been derived from equations governing particular wave motions by various authors, such as Stott (1968), Watson and Peacher (1970), Watson (1969), Watson (1970), Law and Watson (1970), Barabanenkov et.al. (1972), Besieris and Tappert (1973), Howe (1973), Ishimaru (1978) and Besieris et.al. (1982) with a recent survey presented in Barabanenkov et.al. (1991). These derivations also determine the functions $\omega(\mathbf{x}, \mathbf{k})$ and $\sigma(\mathbf{x}, \mathbf{k}, \mathbf{k})$ and show how a is related to the wave field. In Ryzhik et.al. (1995) we derive (2.1) and these functions as a special case of a more general theory.

We expect that radiative transport equations will provide a good description of wave energy transport when, as mentioned in the Introduction, (i) typical wavelengths are short compared to macroscopic features of the medium (high frequency approximation), (ii) correlation lengths of the inhomogeneities are comparable to wavelengths and (iii) the fluctuations of the inhomogeneities are weak. In general, we do not know the correlation lengths of the inhomogeneities. Condition (ii) is, therefore, important because it allows strong interaction between the waves and the inhomogeneities, which is the most interesting and difficult case to analyze. In addition to these three conditions, the inhomogeneities must not be too anisotropic because it is well known that in layered random media, for example, we have wave localization even with weak fluctuations, which is quite different from wave transport phenomena (Asch et.al., 1991). When the fluctuations are strong we can have wave localization even when the inhomogeneities are isotropic (Froelich and Spencer, 1983; Sheng, 1995).

In a homogeneous background medium where $\omega = v|\mathbf{k}|$ and v is the uniform speed of the wave and absorption is weak, the solutions of (2.1) exhibit diffusive behavior at times and distances that are long compared to a typical transport mean free time $1/\Sigma$ and a typical transport mean free path $|\nabla_{\mathbf{k}}\omega|/\Sigma$, respectively. The differential scattering cross-section is always rotationally invariant so that

$$\sigma(\mathbf{k}, \mathbf{k}') = \tilde{\sigma}(|\mathbf{k}|, \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}')\delta(|\mathbf{k}| - |\mathbf{k}'|) \tag{2.5}$$

which means that scattering from a wave with wave vector $\mathbf{k}'$ into a wave with wave vector $\mathbf{k}$ depends only on the angle between the two wave vectors. In the diffusion regime the phase space energy density $a(t, \mathbf{x}, \mathbf{k})$ is approximately independent of the direction of the wave vector $\mathbf{k}$, $a(t, \mathbf{x}, \mathbf{k}) \sim \bar{a}(t, \mathbf{x}, |\mathbf{k}|)$. In the simplest case of a uniform background medium $\bar{a}$ satisfies the diffusion equation

$$\frac{\partial \bar{a}}{\partial t} = \nabla_{\mathbf{x}} \cdot (D\nabla_{\mathbf{x}}\bar{a}) - \nu\bar{a}. \tag{2.6}$$

The constant diffusion coefficient is

$$D(|\mathbf{k}|) = \frac{vl^*}{3}, \tag{2.7}$$

where the diffusion mean free path $l^*(|\mathbf{k}|)$ is

$$l^* = v\left(2\pi \int_{-1}^{1} \tilde{\sigma}(|\mathbf{k}|, \mu)(1 - \mu)d\mu\right)^{-1}. \tag{2.8}$$

with the variable of integration μ equal to the cosine of the angle between incident and scattered directions. Note that the diffusion mean free path l^* is in general longer than the transport mean free path l, defined by

$$l = \frac{v}{\Sigma} = v\left(2\pi \int \tilde{\sigma}(|\mathbf{k}|, \mu)d\mu\right)^{-1}.$$

The two are equal in the case of isotropic scattering. In (2.6) the absorption coefficient

$$\nu(|\mathbf{k}|) = \frac{1}{4\pi} \int \Sigma_{ab}(\mathbf{k})d\Omega(\hat{\mathbf{k}}) \tag{2.9}$$

is the attenuation rate $\Sigma_{ab}(\mathbf{k})$ averaged over the angular directions $\hat{\mathbf{k}}$.

The diffusion approximation comes about as follows. The phase space energy density becomes approximately independent of the direction $\hat{\mathbf{k}}$ in the far field regime, or, equivalently, for deep or late coda waves. Mathematically this is because the state independent of $\hat{\mathbf{k}}$ is the only equilibrium state for the scattering operator on the right side of equation (2.1). We integrate equation (2.1) with respect to the directions $\hat{\mathbf{k}}$ and use (2.2) to obtain

$$\frac{\partial \bar{a}}{\partial t} + \nabla_{\mathbf{x}} \cdot \boldsymbol{\psi} = -\nu(|\mathbf{k})\bar{a}, \tag{2.10}$$

where ν is given by (2.9),

$$\bar{a}(t, \mathbf{x}, |\mathbf{k}|) = \frac{1}{4\pi} \int a(t, \mathbf{x}, \mathbf{k}) d\Omega(\hat{\mathbf{k}}) \tag{2.11}$$

is the average energy density and

$$\boldsymbol{\psi}(t, \mathbf{x}, |\mathbf{k}|) = \frac{v}{4\pi} \int \hat{\mathbf{k}} a(t, \mathbf{x}, \mathbf{k}) d\Omega(\hat{\mathbf{k}}) \tag{2.12}$$

is the flux. Next we multiply (2.1) by $\hat{\mathbf{k}}$ and integrate with respect to $\hat{\mathbf{k}}$

$$\frac{1}{v}\frac{\partial \psi_i}{\partial t} + \frac{v}{4\pi} \int \hat{\mathbf{k}}_i \hat{\mathbf{k}}_j \frac{\partial a}{\partial x^j} d\Omega(\hat{\mathbf{k}}) = -\frac{1}{l^*}\psi_i + \frac{1}{4\pi} \int \hat{\mathbf{k}}_i \Sigma_{ab}(\mathbf{k}) a(\mathbf{k}) d\Omega(\hat{\mathbf{k}}). \tag{2.13}$$

Close to equilibrium a varies slowly in time so we can neglect the term $\partial \psi_i / \partial t$ in (2.13), which is comparable to $\bar{a}_{tt}$. Absorption is weak and so we can also drop the absorption term in (2.13). Note finally that when a is approximately independent of direction $\hat{\mathbf{k}}$, $a \sim \bar{a}$ and the integral on the left of (2.13) is just $(v/3)\nabla_{\mathbf{x}} a$ so (2.13) becomes

$$\boldsymbol{\psi} = -\frac{v l^*}{3} \nabla_{\mathbf{x}} a, \tag{2.14}$$

where l^* is given by (2.8). We insert (2.14) into (2.10), replace $\bar{a}$ by a and obtain the diffusion equation (2.6) with the diffusion coefficient (2.7). A more systematic derivation of the diffusion approximation for transport equations is given in Ryzhik et.al. (1995).

Diffusion approximations for scalar transport equations are very well known (Case and Zweifel, 1967), including their behavior near boundaries (Larsen and Keller, 1974; Bensoussan et.al., 1979). We show that diffusion approximations are also valid for the more general transport equations that arise for elastic waves.

Before going to elastic waves we note that there is another way to obtain the diffusive behavior. We introduce

$$\mathcal{J}(t, \mathbf{x}, |\mathbf{k}|) = \int a(t, \mathbf{x}, \mathbf{k}) d\Omega(\hat{\mathbf{k}}), \tag{2.15}$$

the energy density integrated over the angles. When the scattering is isotropic and independent of $\mathbf{x}$ the differential scattering cross-section is given by

$$\sigma(\mathbf{k}, \mathbf{k}') = \tilde{\sigma}(|\mathbf{k}|)\delta(|\mathbf{k}| - |\mathbf{k}'|). \tag{2.16}$$

Consider, for example, the time-independent version of the transport equation (2.1) with a source. Then $\mathcal{J}$ satisfies the time-independent integral equation

$$\mathcal{J}(\mathbf{x}, |\mathbf{k}|) = \int_{\mathbb{R}^3} [\eta_{sc}\mathcal{J}(\mathbf{x}', |\mathbf{k}|) + \frac{1}{v} f(\mathbf{x}')] \frac{e^{-\eta_{tot}|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|^2} d\mathbf{x}', \tag{2.17}$$

where $f(\mathbf{x})$ is the source energy density function, $\eta_{sc} = \tilde{\sigma}/v$ is the inverse of the scattering mean free path, and $\eta_{tot} = \Sigma/v$, where Σ is the total scattering cross-section (2.4). This equation can be solved explicitly and in the case of a δ-function source the solution in the absence of absorption behaves like $1/r$, which is diffusive behavior. The absorption term Σ_{ab} modifies this solution, making it decay exponentially with a decay rate which vanishes when $\Sigma_{ab} = 0$. This is analogous to the effect of the νa term in the diffusion equation (2.6).

When the differential scattering cross-section is not of the form (2.16) and scattering is not isotropic then instead of a single integral equation (2.17) we get a system of equations for higher angular moments of the intensity $a(\mathbf{x}, \mathbf{k})$ (Chandrasekhar, 1960). The resulting system is harder to analyze, its explicit solution is complicated and the diffusion approximation is not so transparent. This intermediate step of obtaining equations for integrated quantities is fortunately unnecessary since the solutions of (2.1) converge rapidly to the diffusion approximation for any rotationally invariant differential scattering cross-section of the form (2.5). Therefore in the deep coda regime we need only consider solutions of the diffusion equation.

3. Transport Theory for Elastic Waves

Diffusion theory was used in seismology by Wesley (1965), Nakamura (1977) and others. Dainty and Toksöz (1977) derived the diffusion equation for acoustics and suggested the form of the diffusion coefficient for elastic waves. R.S. Wu (1985) used extensively the stationary version of the transport equation (2.1), with a source and with isotropic differential scattering cross-section to model propagation of S waves with multiple scattering. Neither the polarization of S waves nor the P to S wave conversion was taken into account. Transport theory was used to effectively separate scattering from absorption attenuation. The diffusive regime for elastic waves was discussed in Wu and Aki (1988) where the seismic data for Hindu-Kush region was analyzed and it was found that absorption dominates over scattering in this region. Multiple scattering in the earth's crust was considered in Toksöz et.al. (1988) and scalar transport theory was used to separate scattering from absorption effects for Rg waves and other waves. Mayeda et.al. (1991) compared data from southern California to the predictions of radiative transport theory for the dependence of the total energy on distance and found very good agreement between the two. McSweeney et.al. (1991) found the same results for southcentral Alaska. Fehler et.al. (1992) applied multiple lapse-time window analysis to the Kanto-Tokai region and compared the data with predictions from transport theory. Multiple scattering in the time domain was considered by Hoshiba (1991) using a Monte Carlo method to model seismic wave propagation in a random medium. The time dependent, scalar transport equation (2.17) was considered by Zeng et.al. (1991). Zeng (1991) constructed approximate, hybrid single-scattering–diffusive solutions to the time-dependent transport equation and showed that it was a good approximation to the full numerical solution.

In all these papers the vector nature of the underlying elastic wave motion was not taken into consideration. Mode conversion for surface waves was considered in a phenomenological way by Chen and Aki (1993) and general mode conversion between longitudinal compressional or P waves and transverse shear or S waves was considered by Sato (1994) and by Zeng (1993). However, the transport equations proposed phenomenologically in these papers do not account for polarization of the shear waves. Sato (1994) and Zeng (1993) incorporated the P to S conversion in the integral equation (2.17) and not in its angularly resolved form (2.1). Starting from the elastic wave equations in a random medium we derive in Ryzhik et.al. (1995) a system of transport equations that accounts correctly for P to S mode conversion and for polarization effects.

Longitudinal P waves propagate with local speed $v_P(\mathbf{x}) = \sqrt{(2\mu(\mathbf{x}) + \lambda(\mathbf{x}))/\rho(\mathbf{x})}$ and transverse shear or S waves that can be polarized propagate with local speed $v_S(\mathbf{x}) = \sqrt{\mu(\mathbf{x})/\rho(\mathbf{x})}$. The corresponding dispersion relations are $\omega_P = v_P|\mathbf{k}|$ and $\omega_S = v_S|\mathbf{k}|$, respectively. The P and S wave modes interact in an inhomogeneous medium because a P wave with a wavenumber $|\mathbf{k}|$, when scattered can generate an S wave with wavenumber $|\mathbf{p}|$ with the same frequency; that is, $v_P(\mathbf{x})|\mathbf{k}| = v_S(\mathbf{x})|\mathbf{p}|$, and vice versa. These scattering processes conserve energy. Therefore the transport equations for P and S waves must be coupled. The transport equation for the P wave energy density should be a scalar equation similar to (2.1) with an additional term that accounts for S to P energy conversion. The S waves can be polarized and are similar in this respect to electromagnetic waves. The phenomenological radiative transport theory for the latter was developed by Chandrasekhar (1960). An important element in that theory is that the transport of energy of polarized waves is described not by a scalar density but by a 2×2 coherence matrix. The coherence matrix has information not only about the total energy of the wave but also about its distribution between the two polarizations and about the cross polarization. This allows for the consideration of polarization effects in the framework of transport theory. In the case of elastic waves, the transport equation for the S wave coherence matrix should be like Chandrasekhar's equation (1960, equation (212)) with an additional term that accounts for P to S energy conversion. We show in Ryzhik et.al. (1995) that this is in fact the case and we determine explicitly the form of the scattering cross-sections in terms of the power spectral densities of the material inhomogeneities. We show also how the quantities satisfying the transport equations are related to the underlying field properties.

The coupled transport equations for the P wave energy density $a^P(t, \mathbf{x}, \mathbf{k})$ and the 2×2 coherence matrix $W^S(t, \mathbf{x}, \mathbf{k})$ for the S waves have the forms

$$
\begin{aligned}
&\frac{\partial a^P}{\partial t} + \nabla_{\mathbf{k}}\omega^P \cdot \nabla_{\mathbf{x}} a^P - \nabla_{\mathbf{x}}\omega^P \cdot \nabla_{\mathbf{k}} a^P \\
&\quad = \int \sigma^{PP}(\mathbf{k}, \mathbf{k}') a^P(\mathbf{k}') d\mathbf{k}' - \Sigma^{PP}(\mathbf{k}) a^P(\mathbf{k}) \\
&\quad + \int \sigma^{PS}(\mathbf{k}, \mathbf{k}') [W^S(\mathbf{k}')] d\mathbf{k}' - \Sigma^{PS}(\mathbf{k}) a^P(\mathbf{k})
\end{aligned}
\tag{3.1a}
$$

and

$$\begin{aligned}\frac{\partial W^S}{\partial t} &+ \nabla_{\mathbf{k}}\omega^S \cdot \nabla_{\mathbf{x}} W^S - \nabla_{\mathbf{x}}\omega^S \cdot \nabla_{\mathbf{k}} W^S + WN - NW \\ &= \int \sigma^{SS}(\mathbf{k},\mathbf{k}')[W^S(\mathbf{k}')]d\mathbf{k}' - \Sigma^{SS}(\mathbf{k})W^S(\mathbf{k}) \\ &+ \int \sigma^{SP}(\mathbf{k},\mathbf{k}')[a^P(\mathbf{k}')]d\mathbf{k}' - \Sigma^{SP}(\mathbf{k})W^S(\mathbf{k}).\end{aligned} \tag{3.1b}$$

The differential scattering cross-section $\sigma^{PP}(\mathbf{k},\mathbf{k}')$ for P to P scattering is similar to (2.3) for scattering of scalar waves and the differential scattering tensor $\sigma^{SS}(\mathbf{k},\mathbf{k}')$ is similar to Chandrasekhar's tensor (Chandrasekhar, 1960). They have the forms

$$\sigma^{PP}(\mathbf{k},\mathbf{k}') = \sigma_{pp}(\mathbf{k},\mathbf{k}')\delta(v_P|\mathbf{k}| - v_P|\mathbf{k}'|) \tag{3.2}$$

and

$$\begin{aligned}\sigma^{SS}(\mathbf{k},\mathbf{k}')[W(\mathbf{k}')] = \{&\sigma_{ss}^{TT}T(\mathbf{k},\mathbf{k}')W(\mathbf{k}')T(\mathbf{k}',\mathbf{k}) + \sigma_{ss}^{\Gamma\Gamma}\Gamma(\mathbf{k},\mathbf{k}')W(\mathbf{k}')\Gamma(\mathbf{k}',\mathbf{k}) \\ &+\sigma_{ss}^{\Gamma T}[T(\mathbf{k},\mathbf{k}')W(\mathbf{k}')\Gamma(\mathbf{k}',\mathbf{k}) + \Gamma(\mathbf{k},\mathbf{k}')W(\mathbf{k}')T(\mathbf{k}',\mathbf{k})]\} \\ &\cdot \delta(v_S|\mathbf{k}| - v_S|\mathbf{k}'|).\end{aligned} \tag{3.3}$$

The 2×2 matrices $T(\mathbf{k},\mathbf{k}')$ and $\Gamma(\mathbf{k},\mathbf{k}')$ are defined by

$$T_{ij}(\mathbf{k},\mathbf{k}') = \mathbf{z}^{(i)}(\mathbf{k}) \cdot \mathbf{z}^{(j)}(\mathbf{k}') \tag{3.4}$$

and

$$\Gamma_{ij}(\mathbf{k},\mathbf{k}') = (\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}')(\mathbf{z}^{(i)}(\mathbf{k}) \cdot \mathbf{z}^{(j)}(\mathbf{k}')) + (\hat{\mathbf{k}} \cdot \mathbf{z}^{(j)}(\mathbf{k}'))(\hat{\mathbf{k}}' \cdot \mathbf{z}^{(i)}(\mathbf{k})). \tag{3.5}$$

Here $(\hat{\mathbf{k}}, \mathbf{z}^{(1)}(\mathbf{k}), \mathbf{z}^{(2)}(\mathbf{k}))$ is the orthonormal propagation triple consisting of the direction of propagation $\hat{\mathbf{k}}$ and two transverse unit vectors $\mathbf{z}^{(1)}(\mathbf{k}), \mathbf{z}^{(2)}(\mathbf{k})$, which in polar coordinates are

$$\hat{\mathbf{k}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \ \mathbf{z}^{(1)}(\mathbf{k}) = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \ \mathbf{z}^{(2)}(\mathbf{k}) = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}. \tag{3.6}$$

The scalar functions σ_{pp} and σ_{ss} are given in terms of power spectral densities of the inhomogeneities in Ryzhik et.al. (1995). The total scattering cross-sections Σ^{PP} and Σ^{SS} are the integrals of the corresponding differential scattering cross-sections, as in (2.2), since we assume that there is no intrinsic dissipation.

Note that the geometric structure of the matrices T and Γ is built into the transport equations and is independent of the random inhomogeneities. This means that the phenomenological models of isotropic S to S scattering suggested in Zeng (1993) and Sato (1994) are not valid and polarization of S waves is important.

The coupling matrix N, which arises due to variations of the background on a large scale and vanishes in the case of a uniform medium, is given by

$$N(\mathbf{x},\mathbf{k}) = \sum_{i=1}^{3} \frac{\partial v(\mathbf{x})}{\partial x^i}|\mathbf{k}|\mathbf{z}^{(1)}(\mathbf{k}) \cdot \frac{\partial \mathbf{z}^{(2)}(\mathbf{k})}{\partial k_i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.7}$$

The scattering cross-sections for the S to P and P to S coupling terms, σ^{PS} and σ^{SP}, respectively, have the forms

$$\sigma^{PS}(\mathbf{k},\mathbf{k}')[W^S(\mathbf{k}')] = \mathrm{Tr}[\sigma_{ps}(\mathbf{k},\mathbf{k}')G(\mathbf{k},\mathbf{k}')W^S(\mathbf{k}')]\delta(v_P|\mathbf{k}| - v_S|\mathbf{k}'|) \tag{3.8}$$

$$\sigma^{SP}(\mathbf{k},\mathbf{k}')[a^P(\mathbf{k}')] = \sigma_{ps}(\mathbf{k}',\mathbf{k})G(\mathbf{k}',\mathbf{k})a^P(\mathbf{k}')\delta(v_S|\mathbf{k}| - v_P|\mathbf{k}'|) \tag{3.9}$$

with the 2×2 matrix G given by

$$G_{ij}(\mathbf{k},\mathbf{k}') = (\hat{\mathbf{k}}\cdot\mathbf{z}^{(i)}(\mathbf{k}'))(\hat{\mathbf{k}}\cdot\mathbf{z}^{(j)}(\mathbf{k}')) \tag{3.10}$$

and Tr denoting the trace of a matrix. The geometric nature of G shows that S to P scattering is not isotropic, in general, independently of the nature of the inhomogeneities. The scalar function σ_{ps} is given explicitly in terms of power spectral densities of the inhomogeneities in Ryzhik et.al. (1995) and in the Appendix. The delta function in (3.8) and (3.9) appears because P waves with wave number $|\mathbf{k}|$ when scattered generate S waves with wave number $v_P|\mathbf{k}|/v_S$ and vice versa, since the frequency of the waves is not changed by scattering.

The geometrical meaning of the 2×2 matrices T, Γ and G that appear in the differential scattering cross-sections (3.3) and (3.8) and (3.9) is similar to that of T which appears in Chandrasekhar's equations (Chandrasekhar, 1960). They arise from a single scattering event of P and S waves with wave vector $\mathbf{k}'$ that scatter to P and S waves with wave vector $\mathbf{k}$ and from the fact that the transport equations deal with quadratic field quantities.

Equations (3.1 a,b) have an important reciprocity property between P to S and S to P scattering. It is expressed by

$$\sigma^{PS}(\mathbf{k},\mathbf{k}') = \mathrm{Tr}\sigma^{SP}(\mathbf{k}',\mathbf{k}). \tag{3.11}$$

This property is important because it is responsible for the form of the equilibrium in scattering in the far field regime. That is, it makes the P and S wave energies equilibrate in the ratio (1.1) as we explain below in detail. Aki (1992) used reciprocity of single scattering, and the asymptotic forms $1/v_P^2 r$ and $1/v_S^2 r$ of point source P and S wave solutions of the elastic equations in a uniform medium, to show that

$$\frac{g_{PS}}{g_{SP}} = \frac{2v_P^4}{v_S^4}. \tag{3.12}$$

Here g_{PS} and g_{SP} are the P to S and S to P energy conversion coefficients *for a single scattering*, respectively (the factor of 2 omitted in Aki (1992) was corrected in Korneev and Johnson (1993)). The main implication of (3.12) is that S waves will dominate after many scatterings because typically $g_{PS}/g_{SP} \sim 18$. The dominance of S waves is also observed in the seismological data (Aki, 1992). Zeng (1993) phenomenologically incorporated (3.12) (without the factor 2) in the radiative transport theory and considered a system of coupled equations (2.17) with scattering coefficients chosen according to (3.12). He observed the dominance of S wave energy in the numerical solutions but, of course, this is expected since it was put into the equations by adopting (3.12).

The passage from single scattering to the multiple scattering phenomena is not so direct. For example, the far field energy density for single scattering behaves like r^{-2}, but with multiple scattering it behaves like r^{-1}. The analog of (3.12) with multiple scattering is the ratio of the transport mean free paths

$$l_{PS} = \frac{v_P}{\Sigma^{PS}},\ l_{SP} = \frac{v_S}{\Sigma^{SP}} \tag{3.13}$$

where the total scattering cross-section Σ^{PS} and Σ^{SP} are given by

$$\Sigma^{PS}(\mathbf{k}) = \int \mathrm{Tr}\sigma^{SP}(\mathbf{k}', \mathbf{k})[I]d\mathbf{k}'$$
$$\Sigma^{SP}(\mathbf{k}) = \int \sigma^{PS}(\mathbf{k}', \mathbf{k})[I]d\mathbf{k}'.$$

We show in the Appendix that

$$\frac{l_{SP}(v_P\mathbf{k}/v_S)}{l_{PS}(\mathbf{k})} = \frac{2v_P^2}{v_S^2}, \tag{3.14}$$

where the wave vectors of the P and S waves are chosen so that they could be scattered into each other. This relation holds in general and is a consequence of the reciprocity relation (3.11), as is (3.12). However, neither (3.13) nor (3.14) implies the equipartition law (1.1). This law holds in the diffusive regime and it is discussed in detail in Ryzhik et.al. (1995) and below, (3.17). Note that the diffusive approximation is established there without the intermediate step of deriving a system of integral equations analogous to (2.17). We are unable to derive a closed system of equations like those suggested in Zeng (1993) and Sato (1994) for the energy densities integrated over angles. The complications that arise are the same as in the case of the scalar transport equation with non-isotropic scattering. The main difference in the case of the elastic transport equations is that because of the polarization of the S waves the S to S and P to S scattering cross-sections are intrinsically non-isotropic. Any phenomenological transport theory similar to the one suggested in Sato (1994) and Zeng (1993) must therefore take into account this anisotropy. This means that the resulting system of integral equations for the angular moments will have many more unknowns than the energies. However, this system of equations is not important for analysis of the deep coda behavior since in this regime the solution of the radiative transport equations (3.1a) and (3.1b) converges rapidly to the solution of the diffusion equation and so the latter is more relevant than the elastic analog of (2.17).

As in the case of the scalar transport equation (2.1) the elastic transport equations (3.1a) and (3.1b) simplify considerably in the regime where the diffusion approximation is valid; that is, when the transport mean free path is small compared to the propagation distance. In this regime the solution of the transport equations (3.1a) and (3.1b) is close to equilibrium. At equilibrium the P wave energy density $a^P(t, \mathbf{x}, \mathbf{k})$ is independent of the direction of the wave vector $\mathbf{k}$, so it depends only upon $|\mathbf{k}|$. Therefore when the field is near equilibrium we can write

$$a^P(t, \mathbf{x}, \mathbf{k}) \sim \phi(t, \mathbf{x}, |\mathbf{k}|). \tag{3.15a}$$

Similarly, at equilibrium the S wave coherence matrix $W^S(t,\mathbf{x},\mathbf{k})$ is independent of the direction of $\mathbf{k}$ and it is proportional to the identity matrix. Therefore near equilibrium

$$W^S(t,\mathbf{x},\mathbf{k}) \sim w(t,\mathbf{x},|\mathbf{k}|)I. \tag{3.15b}$$

Furthermore, at equilibrium the scalar function w in (3.15b) is related to the scalar function ϕ in (3.15a) by

$$w(t,\mathbf{x},|\mathbf{k}|) = \phi(t,\mathbf{x},|\mathbf{k}'|) \tag{3.16a}$$

with the wavenumbers $|\mathbf{k}|$ and $|\mathbf{k}'|$ corresponding to the same frequency

$$v_P|\mathbf{k}'| = v_S|\mathbf{k}|. \tag{3.16b}$$

This expresses the equidistribution of wave energy density between P and S waves with the same frequency.

Integrating (3.16a) over $\mathbf{k}$ yields

$$\mathcal{E}_P(t,\mathbf{x}) = \frac{v_S^3}{2v_P^3}\mathcal{E}_S(t,\mathbf{x}) \tag{3.17}$$

where $\mathcal{E}_P$ and $\mathcal{E}_S$ are the P and S wave spatial energy densities. They are related to a^P and W^S by

$$\mathcal{E}_P(t,\mathbf{x}) = \int a^P(t,\mathbf{x},\mathbf{k})d\mathbf{k}$$

and

$$\mathcal{E}_S(t,\mathbf{x}) = \int \mathrm{Tr}W^S(t,\mathbf{x},\mathbf{k})d\mathbf{k}.$$

The factor of 2 in (3.17) is due to the polarization of the S waves and shows that the vector nature of the S waves cannot be ignored. Relation (3.17) shows that the S waves dominate in the far field and $\mathcal{E}_S/\mathcal{E}_P \sim 10$. Accidentally this value is close to the value 9 given by the right side of (3.12) without the factor of 2, which is the form given by Aki (1992). However, the mechanism that produces this dominance is through the stabilization or equilibration of P to S mode conversion by multiple scattering. This stabilization, which is derived here from first principles, is reminiscent of the important empirical observation of Hansen et.al. (1990) regarding the stabilization of the Lg wave energy. The ratio of S to P energy depends only on the speed of the waves, while the stabilization of the Lg wave energy is presumably due to the fact that the Lg waves are slower than the P waves and, in addition, some analog of (1.1) holds.

To determine the scalar function $\phi(t,\mathbf{x},|\mathbf{k}|)$ in (3.15a) and (3.16a), we first integrate the transport equations (3.1a) and the trace of (3.1b) with respect to the direction $\hat{\mathbf{k}}$. Then we add the equations together. The right side of the resulting equation vanishes because energy is conserved by the totality of the scattering processes. Next we multiply the transport equations by $\hat{\mathbf{k}}$, integrate over angles and add them. As in the case of the scalar transport equation (equations (2.10) and (2.13)), after neglecting second order time derivatives we find that ϕ satisfies the diffusion equation (2.6). A systematic derivation of the diffusion approximation for elastic waves is given in Ryzhik et.al. (1995). The diffusion coefficient $D(|\mathbf{k}|)$ is given by

$$D(|\mathbf{k}|) = \frac{1}{\frac{2}{v_S^3} + \frac{1}{v_P^3}} \left(\frac{l_p^* v_P}{3 v_P^3} + \frac{2 l_s^* v_S}{3 v_S^3} \right). \tag{3.18}$$

It can be interpreted as the weighted mean of the "individual" diffusion coefficients of P and S waves. The weights are in the ratio $v_S^3/2v_P^3$ as in (1.1). When the fluctuations of the medium properties have flat spectral densities, the diffusion mean free paths are given by

$$l_p^* = \frac{l_p}{1 - < \cos\theta_p >}, \quad l_s^* = \frac{l_s}{1 - < \cos\theta_s >}.$$

Here $< \cos\theta_p >$ and $< \cos\theta_s >$ are the average cosines of the angles of the P to P and S to S scatterings, respectively, and l_p and l_s are the transport mean free paths

$$l_p = \frac{v_P}{\Sigma^{PP} + \Sigma^{PS}}$$

and

$$l_s = \frac{v_S}{\Sigma^{SS} + \Sigma^{SP}}.$$

In general the diffusion mean free paths are longer than the transport mean free paths.

The general form of the diffusion coefficient (3.18) for elastic waves was predicted by Dainty and Toksöz (1977) from physical considerations. Specifically, they correctly argued that D should be a weighted mean of the P and S wave diffusion coefficients. They suggested that the weights should be equal to the roughly one-to-ten observed far field P to S energy ratio as is the case from (3.18). This is based on the equipartition law (3.16a), which, along with the one-to-ten ratio of energies (3.17), does not depend on the details of the scattering. Dainty and Toksöz (1977) (pp.379-380) had expected that it would depend on the details. After our work was completed we became aware of Weaver's papers (Weaver (1982) and Weaver (1990)) where the equipartition law (1.1) and transport theory for elastic waves are derived by a different method.

4. Summary

The main results in Ryzhik et.al. (1995), presented here with seismological applications, are (i) the derivation from first principles of the correct radiative transport equations for elastic wave motion in unbounded media, (ii) the demonstration that polarization of shear waves is important and must be taken into consideration and (iii) the demonstration that in the diffusive regime there is a universal P to S wave energy stabilization. This energy equipartition phenomenon, although intuitively clear, was not known before and its precise form (1.1) is not easy to guess. It is perhaps the simplest instance of many different energy equipartition laws that are valid in other complex situations, such as the ones encountered in crustal wave propagation, that have not been discovered yet.

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5. Appendix

We shall now derive (3.14) from the definition of $\sigma_{ps}(\mathbf{k},\mathbf{k}')$ which is given by

$$\sigma_{ps}(\mathbf{k},\mathbf{k}') = \frac{\pi\mu}{2\rho}\{|\mathbf{k}'|^2\hat{R}_{\rho\rho}(|\mathbf{k}-\mathbf{k}'|) + 4|\mathbf{k}|^2(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')^2\hat{R}_{\mu\mu}(|\mathbf{k}-\mathbf{k}'|)$$
$$+4|\mathbf{k}||\mathbf{k}'|(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')\hat{R}_{\mu\rho}(|\mathbf{k}-\mathbf{k}'|)\}. \tag{5.1}$$

The total scattering cross-sections $\Sigma^{PS}(\mathbf{k})$ and $\Sigma^{SP}(\mathbf{k})$ are given by

$$\begin{aligned}\Sigma^{PS}(\mathbf{k}) &= \int \sigma_{ps}(\mathbf{k},\mathbf{k}')\mathrm{Tr}G(\hat{\mathbf{k}},\hat{\mathbf{k}}')\delta(v_S|\mathbf{k}'| - v_P|\mathbf{k}|)d\mathbf{k}' \\ &= \frac{\pi\mu v_P^4|\mathbf{k}|^4}{2\rho v_S^5}\int\left\{\hat{R}_{\rho\rho}(\zeta_1) + \frac{4v_S^2}{v_P^2}(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')^2\hat{R}_{\mu\mu}(\zeta_1) + 4\frac{v_S}{v_P}(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')\hat{R}_{\mu\rho}(\zeta_1)\right\} \\ &\cdot \mathrm{Tr}G(\hat{\mathbf{k}},\hat{\mathbf{k}}')d\Omega(\hat{\mathbf{k}}') = \frac{\pi\mu v_P^4|\mathbf{k}|^4}{2\rho v_S^5}I_1\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}\Sigma^{SP}(\mathbf{k}) &= \int \sigma_{ps}(\mathbf{k}',\mathbf{k})G(\mathbf{k}',\mathbf{k})\delta(v_S|\mathbf{k}| - v_P|\mathbf{k}'|)d\mathbf{k}' \\ &= \frac{\pi\mu v_S^2|\mathbf{k}|^4}{2\rho v_P^3}\int\left\{\hat{R}_{\rho\rho}(\zeta_2) + \frac{4v_S^2}{v_P^2}(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')^2\hat{R}_{\mu\mu}(\zeta_2) + \frac{4v_S}{v_P}(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')\hat{R}_{\mu\rho}(\zeta_2)\right\} \\ &\cdot G(\hat{\mathbf{k}}',\hat{\mathbf{k}})d\Omega(\hat{\mathbf{k}}') = \frac{\pi\mu v_S^2|\mathbf{k}|^4}{2\rho v_P^3}I_2,\end{aligned} \tag{5.3}$$

where $d\Omega(\hat{\mathbf{k}}')$ is the surface element on the unit sphere and

$$\zeta_1(\mathbf{k},\hat{\mathbf{k}}') = \sqrt{|\mathbf{k}|^2 + \frac{v_P^2}{v_S^2}|\mathbf{k}|^2 + 2\frac{v_P}{v_S}|\mathbf{k}|^2(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')}$$
$$\zeta_2(\mathbf{k},\hat{\mathbf{k}}') = \sqrt{|\mathbf{k}|^2 + \frac{v_P^2}{v_S^2}|\mathbf{k}|^2 + 2\frac{v_S}{v_P}|\mathbf{k}|^2(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')}. \tag{5.4}$$

Note that

$$\zeta_1(\mathbf{k},\hat{\mathbf{k}}') = \zeta_2(\frac{v_P}{v_S}\mathbf{k},\hat{\mathbf{k}}') \tag{5.5}$$

and that the integral I_2 in (5.4) is proportional to the identity matrix. This can be verified by direct computation for $\mathbf{k} = (0,0,1)$. Due to rotational invariance, this implies that for all $\mathbf{k}$,

$$I_2(\mathbf{k}) = I_2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

is proportional to the identity matrix. We also have

$$\mathrm{Tr}G(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = \mathrm{Tr}G(\hat{\mathbf{k}}', \hat{\mathbf{k}}) = 1 - (\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}')^2$$

and hence $I_1(\mathbf{k}) = 2I_2(v_P\mathbf{k}/v_S)$. Then

$$\frac{l_{PS}(\mathbf{k})}{l_{SP}(\frac{v_P\mathbf{k}}{v_S})} = \frac{v_P}{\Sigma^{PS}(\mathbf{k})} \frac{v_S}{\Sigma^{PS}(\frac{v_P\mathbf{k}}{v_S})} = \frac{v_S^2}{2v_P^2},$$

which is (3.14).

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