Self-averaging of Wigner transforms in random media

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Abstract

We establish the self-averaging properties of the Wigner transform of a mixture of states in the regime when the correlation length of the random medium is much longer than the wave length but much shorter than the propagation distance. The main ingredients in the proof are the error estimates for the semiclassical approximation of the Wigner transform by the solution of the Liouville equations, and the limit theorem for two-particle motion along the characteristics of the Liouville equations. The results are applied to a mathematical model of the time-reversal experiments for the acoustic waves, and self-averaging properties of the re-transmitted wave are proved.

1 Introduction

1.1 The Wigner transform of mixtures of states

The Wigner transform is a useful tool in the analysis of the semi-classical limits of non-dissipative evolution equations as well as in the high frequency wave propagation [14, 17, 20]. It is defined as follows: given a family of functions $f_\varepsilon(t,\mathbf{x})$ uniformly bounded in $L^\infty([0,T];L^2(\mathbb{R}^d))$ its Wigner transform is

$$\tilde{W}_\varepsilon(t,\mathbf{x},\mathbf{k}) = \int e^{i\mathbf{k}\cdot\mathbf{y}} f_\varepsilon(t,\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2}) f_\varepsilon^*(t,\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2}) \frac{d\mathbf{y}}{(2\pi)^d}. \tag{1.1}$$

The family $\tilde{W}_\varepsilon$ is uniformly bounded in the space of Schwartz distributions $\mathcal{S}'(\mathbb{R}^d\times\mathbb{R}^d)$, and all its limit points are non-negative measures of bounded total mass [14, 17]. It is customary to interpret a limit Wigner measure W as the energy density in the phase space, since the limit points of $n_\varepsilon = |f_\varepsilon|^2$ are of the form $n(t,\mathbf{x}) = \int W(t,\mathbf{x},\mathbf{k})d\mathbf{k}$ provided that the family f_ε is ε-oscillatory and compact at infinity [14]. However, while neither n_ε nor its limit $n(t,\mathbf{x})$ satisfy a closed equation, both $\tilde{W}_\varepsilon$ and W usually obey an evolution equation when the family $f_\varepsilon(t,\mathbf{x})$ arises from a time-dependent PDE. This makes the Wigner transform a useful tool in the study of semiclassical and high frequency limits, especially in random media [1, 2, 10, 18, 20]. However, a priori bounds on the Wigner transform $\tilde{W}_\varepsilon$ other than those mentioned above are usually difficult to obtain. It has been observed in [17] that the Wigner transform of a mixture of states

$$W_\varepsilon(\mathbf{x},\mathbf{k}) = \int e^{i\mathbf{k}\cdot\mathbf{y}} f_\varepsilon(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2};\zeta) f_\varepsilon^*(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2};\zeta) \frac{d\mathbf{y}d\mu(\zeta)}{(2\pi)^d},$$

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enjoys better regularity properties. The family f_ε above depends on an additional "state" parameter $\zeta \in S$, where S is a state space equipped with a non-negative bounded measure $d\mu(\zeta)$. Typically this corresponds to introducing random initial data for f_ε at $t=0$ and estimating the expectation of $\tilde{W}_\varepsilon$ with respect to this randomness. This improved regularity has been used, for instance, in [18, 22] in the analysis of the average of the Wigner transform of mixtures of states in random media and in [17] in order to obtain an asymptotic expansion for the Wigner transform of a mixture of states.

The purpose of this paper is to analyze the self-averaging properties of moments of the mixed Wigner transform of the form $\int W_\varepsilon(t,\mathbf{x},\mathbf{k})S(\mathbf{k})d\mathbf{k}$, where $S(\mathbf{k})$ is a test function, and the family $f_\varepsilon(t,\mathbf{x};\zeta)$ satisfies the acoustic equations. This problem arises naturally in the mathematical study of the experiments in time-reversal of acoustic waves that we will describe in detail below. However, apart from the time-reversal application, the statistical stability of such moments provides an important key to understanding the physical applicability of the limit equations for the Wigner transform in random media in the situations when results for each realization are more relevant than the statistically averaged quantities.

We start with the wave equation in dimension $d \geq 3$

$$\frac{1}{c^2(\mathbf{x})}\frac{\partial^2\phi}{\partial t^2} - \Delta\phi = 0 \tag{1.2}$$

and assume that the wave speed has the form $c(\mathbf{x}) = c_0 + \sqrt{\delta}c_1(\mathbf{x})$. Here $c_0 > 0$ is the constant sound speed of the uniform background medium, while the small parameter $\delta \ll 1$ measures the strength of the mean zero random perturbation c_1. Rescaling the spatial and temporal variables $\mathbf{x} = \mathbf{x}'/\delta$ and $t = t'/\delta$ we obtain (after dropping the primes) equation (1.2) with rapidly fluctuating wave speed

$$c_\delta(\mathbf{x}) = c_0 + \sqrt{\delta}c_1\left(\frac{\mathbf{x}}{\delta}\right). \tag{1.3}$$

It is convenient to re-write (1.2) as the system of acoustic equations for the "pressure" $p = \frac{1}{c}\phi_t$ and "acoustic velocity" $\mathbf{u} = -\nabla\phi$:

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(c_\delta(\mathbf{x})p\right) = 0 \\ &\frac{\partial p}{\partial t} + c_\delta(\mathbf{x})\nabla\cdot\mathbf{u} = 0. \end{aligned} \tag{1.4}$$

The energy density for (1.4) is $E(t,\mathbf{x}) = |\mathbf{u}|^2 + p^2$: $\int E(t,\mathbf{x})d\mathbf{x} = \text{const}$ is independent of time. We will denote for brevity $\mathbf{v} = (\mathbf{u}, p) \in \mathbb{C}^{d+1}$ and write (1.4) in the more general form of a first order linear symmetric hyperbolic system

$$\frac{\partial \mathbf{v}_\varepsilon^\delta}{\partial t} + A_\delta(\mathbf{x})D^j\frac{\partial}{\partial x^j}\left(A_\delta(\mathbf{x})\mathbf{v}_\varepsilon^\delta(\mathbf{x})\right) = 0. \tag{1.5}$$

In the present case, the symmetric matrices A_δ and D^j are defined by

$$A_\delta(\mathbf{x}) = \mathrm{diag}(1,1,1,c_\delta(\mathbf{x})), \qquad \text{and} \qquad D^j = \mathbf{e}_j\otimes\mathbf{e}_{d+1} + \mathbf{e}_{d+1}\otimes\mathbf{e}_j,\ j=1,\ldots,d. \tag{1.6}$$

Notice that the matrices D^j are independent of $\mathbf{x}$. Here $\mathbf{e}_m \in \mathbb{R}^{d+1}$ is the standard orthonormal basis: $(\mathbf{e}_m)_k = \delta_{mk}$. The dispersion matrix for (1.5) is

$$P_0^\delta(\mathbf{x},\mathbf{k}) = iA_\delta(\mathbf{x})k_jD^jA_\delta(\mathbf{x}) = ic_\delta(\mathbf{x})k_jD^j = ic_\delta(\mathbf{x})\left[\tilde{\mathbf{k}}\otimes\mathbf{e}_{d+1} + \mathbf{e}_{d+1}\otimes\tilde{\mathbf{k}}\right],\quad \tilde{\mathbf{k}} = \sum_{j=1}^d k_j\mathbf{e}_j. \tag{1.7}$$

The self-adjoint matrix $(-iP_0^\delta)$ has an eigenvalue $\lambda_0 = 0$ of multiplicity $d-1$, and two simple eigenvalues $\lambda_\pm^\delta(\mathbf{x},\mathbf{k}) = \pm c_\delta(\mathbf{x})|\mathbf{k}|$. The corresponding eigenvectors are

$$\mathbf{b}_m^0 = \left(\mathbf{k}_m^\perp, 0\right), \ m = 1,\dots,d-1; \ \ \mathbf{b}_\pm = \frac{1}{\sqrt{2}}\left(\frac{\tilde{\mathbf{k}}}{|\mathbf{k}|} \pm \mathbf{e}_{d+1}\right), \tag{1.8}$$

where $\mathbf{k}_m^\perp \in \mathbb{R}^d$ is the orthonormal basis of vectors orthogonal to $\mathbf{k}$.

We assume that the initial data $\mathbf{v}_0(\mathbf{x};\zeta) = \mathbf{v}_\varepsilon^\delta(0,\mathbf{x};\zeta) = (-\varepsilon\nabla\phi_0^\varepsilon, 1/c_\delta\dot{\phi}_0^\varepsilon)$ for (1.5) is an ε-oscillatory and compact at infinity family of functions uniformly bounded in $L^2(\mathbb{R}^d)$ [14] for each "realization" ζ of the initial data. The scale ε of oscillations is much smaller than the correlation length δ of the medium: $\varepsilon \ll \delta \ll 1$. The $(d+1)\times(d+1)$ Wigner matrix of a mixture of solutions of (1.5) is defined by

$$W_\varepsilon^\delta(t,\mathbf{x},\mathbf{k}) = \int\limits_{\mathbb{R}^d\times S} e^{i\mathbf{k}\cdot\mathbf{y}}\mathbf{v}_\varepsilon^\delta(t,\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2};\zeta)\mathbf{v}_\varepsilon^{\delta*}(t,\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2};\zeta)\frac{d\mathbf{y}d\mu(\zeta)}{(2\pi)^d}.$$

The non-negative measure $d\mu$ has bounded total mass: $\int_S d\mu(\zeta) < \infty$. It is well-known [14, 17] that for each fixed $\delta > 0$ (and even without introduction of a mixture of states) one may pass to the limit $\varepsilon \to 0$ and show that W_ε^δ converges weakly in $\mathcal{S}'(\mathbb{R}^d\times\mathbb{R}^d)$ to

$$\bar{W}^\delta(t,\mathbf{x},\mathbf{k}) = u_+^\delta(t,\mathbf{x},\mathbf{k})\mathbf{b}_+(\mathbf{k})\otimes\mathbf{b}_+(\mathbf{k}) + u_-^\delta(t,\mathbf{x},\mathbf{k})\mathbf{b}_-(\mathbf{k})\otimes\mathbf{b}_-(\mathbf{k}).$$

The scalar amplitudes $u_\pm^\delta$ satisfy the Liouville equations:

$$\frac{\partial u_\pm^\delta}{\partial t} + \nabla_{\mathbf{k}}\lambda_\pm^\delta\cdot\nabla_{\mathbf{x}}u_\pm^\delta - \nabla_{\mathbf{x}}\lambda_\pm^\delta\cdot\nabla_{\mathbf{k}}u_\pm^\delta = 0. \tag{1.9}$$

Furthermore, one may formally pass to the limit $\delta \to 0$ in (1.9) and show that (see [3]) $\mathbb{E}\{u_\pm^\delta\}$ converge to the solution of

$$\frac{\partial \bar{u}_\pm}{\partial t} \pm c_0\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}\bar{u}_\pm = \frac{\partial}{\partial k_m}\left[|\mathbf{k}|^2 D_{mn}(\hat{\mathbf{k}})\frac{\partial\bar{u}_\pm}{\partial k_n}\right]. \tag{1.10}$$

Here $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$, and the diffusion matrix D is given by

$$D_{mn} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\partial^2 R(c_0 s\hat{\mathbf{k}})}{\partial x_n\partial x_m}ds, \tag{1.11}$$

where $R(\mathbf{x})$ is the correlation function of c_1: $\mathbb{E}\{c_1(\mathbf{y})c_1(\mathbf{x}+\mathbf{y})\} = R(\mathbf{x})$.

The purpose of this paper is to make the passage to the limit $\varepsilon,\delta \to 0$ rigorous for a mixture of states (and eliminate the consecutive limits $\varepsilon \to 0$ then $\delta \to 0$) and establish the self-averaging properties of moments of the form $s_\varepsilon^\delta(t,\mathbf{x}) = \int W_\varepsilon^\delta(t,\mathbf{x},\mathbf{k})S(\mathbf{k})d\mathbf{k}$, where $S \in L^2(\mathbb{R}^d)$ is a given test function.

The assumption that $\varepsilon \ll \delta$ is formalized as follows. We let $\mathcal{K}_\mu = \left\{(\varepsilon,\delta): \ \delta \geq |\ln\varepsilon|^{-2/3+\mu}\right\}$, with $0 < \mu < 2/3$ and assume that $(\varepsilon,\delta) \in \mathcal{K}_\mu$ for some $\mu \in (0,2/3)$. From now on, μ is a given fixed number in $(0,2/3)$.

1.2 The random medium

We make the following assumptions on the random field $c_1(\mathbf{x})$. Let $(\Omega, \mathcal{C}, \mathbb{P})$ be a certain probability space, and let $\mathbb{E}$ denote the expectation with respect to $\mathbb{P}$ and $\|\cdot\|_p$ denote the respective L^p norm for any $p \in [1, +\infty]$. We suppose further that $c_1 : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is a measurable, strictly stationary, mean-zero random field, that is pathwise C^4-smooth and satisfies

$$D_i := \operatorname*{ess-sup}_{\omega\in\Omega} |\nabla^i_{\mathbf{x}} c_1(\mathbf{x};\omega)| < +\infty, \quad i = 0, 1, \ldots, 4. \tag{1.12}$$

We assume in addition that c_1 is exponentially ϕ-mixing. More precisely, for any $R > 0$ we let $\mathcal{C}^i_R := \sigma\{c_1(\mathbf{x}) : |\mathbf{x}| \le R\}$ and $\mathcal{C}^e_R := \sigma\{c_1(\mathbf{x}) : |\mathbf{x}| \ge R\}$. We also define

$$\phi(\rho) := \sup[\,|\mathbb{P}(B) - \mathbb{P}(B|A)| : R > 0,\ A \in \mathcal{C}^i_R,\ B \in \mathcal{C}^e_{R+\rho}\,],$$

for all $\rho > 0$. We suppose that there exists a constant $C_1 > 0$ such that

$$\phi(\rho) \le 2\mathrm{e}^{-C_1\rho}, \quad \forall\, \rho > 0. \tag{1.13}$$

We let also

$$R(\mathbf{y}) = \mathbb{E}[c_1(\mathbf{y})c_1(\mathbf{0})], \quad \mathbf{y} \in \mathbb{R}^d$$

be the covariance function of the field $c_1(\cdot)$ and note that (1.13) implies that there exists a constant $C_2 > 0$ such that

$$|\nabla^m_{\mathbf{y}} R(\mathbf{y})| \le C_2 \mathrm{e}^{-|\mathbf{y}|/C_2}, \quad \forall\, \mathbf{y} \in \mathbb{R}^d,\ m = 0, \cdots, 4. \tag{1.14}$$

Finally we assume that $R \in C^\infty(\mathbb{R}^d)$, this condition will be used only to establish the hypoellipticity of (1.10). Notice that sufficiently regular random fields with finite correlation length satisfy the hypotheses of this section. The exponential ϕ-mixing assumption was used in [15] to analyze the solutions of Liouville equations with random coefficients. Their techniques lay at the core of our proof of the mixing properties presented in our main result, Theorem 1.1, below.

1.3 The main result

We assume that the initial Wigner transform $W^\delta_\varepsilon(0, \mathbf{x}, \mathbf{k})$ is uniformly bounded in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ and

$$W^\delta_\varepsilon(0, \mathbf{x}, \mathbf{k}) \to W_0(\mathbf{x}, \mathbf{k}) \text{ strongly in } L^2(\mathbb{R}^d \times \mathbb{R}^d) \text{ as } \mathcal{K}_\mu \ni (\varepsilon, \delta) \to 0. \tag{1.15}$$

We also assume that the limit $W_0 \in C_c(\mathbb{R}^d \times \mathbb{R}^d)$ with a support that satisfies

$$\operatorname{supp} W_0(\mathbf{x}, \mathbf{k}) \subseteq X = \left\{(\mathbf{x}, \mathbf{k}) : |\mathbf{x}| \le C, \quad C^{-1} \le |\mathbf{k}| \le C\right\}. \tag{1.16}$$

Note that (1.15) may not hold for a pure state since $\|\tilde{W}_\varepsilon\|_2 = (2\pi\varepsilon)^{-d/2}\|f_\varepsilon\|^2_2$ [17]. We will later present examples where it does hold for a mixture of states. Furthermore, we assume that W_0 has the form

$$W_0(\mathbf{x}, \mathbf{k}) = u^0_+ \mathbf{b}_+ \otimes \mathbf{b}_+ + u^0_- \mathbf{b}_- \otimes \mathbf{b}_- \tag{1.17}$$

and let

$$\bar{W}(t, \mathbf{x}, \mathbf{k}) = \bar{u}_+(t, \mathbf{x}, \mathbf{k})\mathbf{b}_+(\mathbf{k}) \otimes \mathbf{b}_+(\mathbf{k}) + \bar{u}_-(t, \mathbf{x}, \mathbf{k})\mathbf{b}_-(\mathbf{k}) \otimes \mathbf{b}_-(\mathbf{k}). \tag{1.18}$$

The functions $\bar{u}_\pm$ satisfy the Fokker-Planck equation (1.10) with initial data $u^0_\pm$ as in (1.17). The main result of this paper is the following theorem.

Theorem 1.1 *Let us assume that the random field $c_1(\mathbf{x})$ satisfies the assumptions given in Section 1.2 and that the initial data $W_\varepsilon^\delta(0,\mathbf{x},\mathbf{k})$ satisfies (1.15) and (1.16). Let $S(\mathbf{k}) \in L^2(\mathbb{R}^d)$ be a test function, and define the moments*

$$s_\varepsilon^\delta(t,\mathbf{x}) = \int W_\varepsilon^\delta(t,\mathbf{x},\mathbf{k})S(\mathbf{k})d\mathbf{k} \qquad \text{and} \qquad \bar{s}(t,\mathbf{x}) = \int \bar{W}(t,\mathbf{x},\mathbf{k})S(\mathbf{k})d\mathbf{k},$$

where $\bar{W}$ is given by (1.18). Then for each $t > 0$ we have

$$\mathbb{E}\left\{\int |s_\varepsilon^\delta(t,\mathbf{x}) - \bar{s}(t,\mathbf{x})|^2 d\mathbf{x}\right\} \to 0 \tag{1.19}$$

as $\mathcal{K}_\mu \ni (\varepsilon,\delta) \to 0$.

Theorem 1.1 means that the moments s_ε^δ converges to a deterministic limit. The main application of Theorem 1.1 we have in mind is the mathematical modeling of refocusing in the time-reversal experiments we present in Section 2.

Our results may be generalized in a fairly straightforward manner to other wave equations that may be written in the form (1.5), which include acoustic equations with variable density and compressibility, electromagnetic and elastic equations [20].

The papers is organized as follows. The mathematical framework of the time-reversal experiment as well as the main result concerning the self-averaging properties of the time reversed signal, Theorem 2.1, are presented in Section 2. Section 3 contains the derivation of the Liouville equations in the L^2-framework. Some straightforward but tedious calculations from this Section are presented in Appendices A, B and C. The limit theorem for the two-point motion along the characteristics of the Liouville equations, Theorem 4.4, is presented in Section 4. Theorem 1.1 follows from this result. The proof of Theorem 4.4 is contained in Section 5.

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2 Refocusing in the time-reversal experiments

2.1 Mathematical formulation of the time-reversal experiments

Refocusing of time-reversed acoustic waves is a remarkable mathematical property of wave propagation in complex media that has been discovered and intensively studied by experimentalists in the last decade (see [12, 16] and also [8] for further references to the physical literature). A typical experiment may be described schematically as follows. A point source emits a localized signal. The signal is recorded in time by an array of receivers. It is then reemitted into the medium *reversed in time* so that the part of the signal recorded last is reemitted first and vice versa. There are two striking experimental observations. First the repropagated signal tightly refocuses at the location of the original source when the medium is sufficiently heterogeneous even with a recording array of small size. This is to be compared with the extremely poor refocusing that would occur if the heterogeneous medium were replaced by a homogeneous medium. Second, the repropagated signal is self-averaging. This means that the refocused signal is essentially independent of the realization of a random medium with given statistics, assuming that we model the heterogeneous medium as a random medium.

The first mathematical study of a time-reversal experiment has been performed in [11] in the framework of one-dimensional layered random media. The one-dimensional case has been further studied in [21], and a three-dimensional layered medium was considered in [13]. The time-reversal experiments in an ergodic domain have been analyzed mathematically in [6]. The basic ideas that explain the role of randomness in the refocusing beyond the one-dimensional case were first outlined in [8] in the parabolic approximation of the wave equation, that was further analyzed in [2, 19]. Time-reversal in the general framework of multidimensional wave equations in random media has been studied formally in [3, 4]. One of the purposes of this paper is to present the rigorous proof of some of the results announced in [3].

The re-transmission scheme introduced in [3, 4] is as follows. Consider the system of acoustic equations (1.4) (or, equivalently, (1.5) for the pressure p and the acoustic velocity $\mathbf{u}(t,\mathbf{x})$). The initial data for (1.5) is assumed to be localized in space:

$$\mathbf{v}_\varepsilon(0,\mathbf{x}) = \mathbf{S}_0\left(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon}\right) = \left(-\nabla\phi_0\left(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon}\right), \frac{1}{c_\delta}\dot{\phi}_0\left(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon}\right)\right). \tag{2.1}$$

Here $\mathbf{x}_0 \in \mathbb{R}^d$ is the location of the original source, and $\mathbf{S}_0 \in \mathcal{S}(\mathbb{R}^d)$ is the source shape function. The small parameter $\varepsilon \ll 1$ measures the spatial localization of the source. The signal $\mathbf{v}_\varepsilon^\delta(t,T)$ is recorded at some time $t = T$, processed at the recording array and re-emitted into the medium. The new signal $\tilde{\mathbf{v}}_\varepsilon^\delta$ is the solution of (1.5) on the time interval $T \le t \le 2T$ with the Cauchy data

$$\tilde{\mathbf{v}}_\varepsilon^\delta(T,\mathbf{x}) = \Gamma[f_\varepsilon \star [\chi\mathbf{v}_\varepsilon^\delta(T)](\mathbf{x})\chi(\mathbf{x}). \tag{2.2}$$

The initial data (2.2) reflects the process of recording of the signal at the array and its smoothing by the recording process. The kernel $f_\varepsilon(\mathbf{x}) = \varepsilon^{-d}f(\mathbf{x}/\varepsilon)$ represents the smoothing. The array function $\chi(\mathbf{x})$ is either the characteristic function of the set of the receivers, or some non-uniform function supported on this set. We will assume for simplicity that $f(|\mathbf{y}|)$ is radially symmetric, and, moreover,

$$\chi \in C_c(\mathbb{R}^d), \quad \hat{f} \in C_c(\mathbb{R}^d), \quad \mathrm{supp}\hat{f}(\mathbf{k}) \subseteq \left\{0 < C^{-1} \le |\mathbf{k}| \le C < \infty\right\} \tag{2.3}$$

where

$$\hat{f}(\mathbf{k}) = \int e^{-i\mathbf{k}\cdot\mathbf{y}}f(\mathbf{y})d\mathbf{y}$$

is the Fourier transform of f. The matrix Γ corresponds to the linear transformation of the signal. The pure time-reversal corresponds to keeping pressure unchanged but reversing the acoustic velocity so that $\Gamma = \Gamma_0 := \mathrm{diag}(-1,-1,-1,1)$. However, this is only one possible transformation, and while we restrict Γ to the above choice our results may be extended to more general matrices Γ, or even allow Γ be a pseudo-differential operator of the form $\Gamma(\mathbf{x},\varepsilon D)$.

The re-propagated field near the source at time $t = 2T$ is defined as a function of the local coordinate $\boldsymbol{\xi}$ and of the source location $\mathbf{x}_0$:

$$\mathbf{v}_\varepsilon^{\delta,B}(\boldsymbol{\xi};\mathbf{x}_0) = \tilde{\mathbf{v}}_\varepsilon^\delta(2T,\mathbf{x}_0+\varepsilon\boldsymbol{\xi}).$$

2.2 The re-propagated signal and the Wigner transform

Let us assume that the random field c_1 satisfies the assumptions of Theorem 1.1 outlined in Section 1.2. Then Theorem 1.1 implies the following result.

Theorem 2.1 *Under the assumptions made above the re-propagated field* $\mathbf{v}_\varepsilon^{\delta,B}(\boldsymbol{\xi},\mathbf{x}_0)$ *converges as* $\mathcal{K}_\mu \ni (\varepsilon,\delta)\to 0$

$$\mathbf{v}^B(\boldsymbol{\xi},\mathbf{x}_0)=\int e^{i\mathbf{k}\cdot\boldsymbol{\xi}}[u_+(T,\mathbf{x}_0,\mathbf{k})\langle\hat{S}_0(\mathbf{k}),\mathbf{b}_-(\mathbf{k})\rangle\mathbf{b}_+(\mathbf{k})+u_-(T,\mathbf{x}_0,\mathbf{k})\langle\hat{S}_0(\mathbf{k}),\mathbf{b}_+(\mathbf{k})\rangle\mathbf{b}_-(\mathbf{k})]\frac{d\mathbf{k}}{(2\pi)^d}$$

in the sense that

$$\sup_{\boldsymbol{\xi}\in\mathbb{R}^d}\mathbb{E}\left\{\int|\mathbf{v}_\varepsilon^{\delta,B}(\boldsymbol{\xi},\mathbf{x}_0)-\mathbf{v}^B(\boldsymbol{\xi},\mathbf{x}_0)|^2d\mathbf{x}_0\right\}\to 0. \tag{2.4}$$

The functions $u_\pm(t,\mathbf{x},\mathbf{k})$ *are the solutions of the Fokker-Planck equation (1.10) with initial data* $a_\pm(0,\mathbf{x},\mathbf{k})=|\chi(\mathbf{x})|^2\hat{f}(\mathbf{k})$.

The proof of Theorem 2.1 is based on Theorem 1.1 and a representation of the re-propagated signal in terms of the Wigner transform of a mixture of solutions of the acoustic wave equations. The latter arises as follows. Let $\mathbf{Q}_\varepsilon^\delta(t,\mathbf{x};\mathbf{q})$ be the matrix-valued solution of (1.5) with initial data

$$\mathbf{Q}_\varepsilon^\delta(0,\mathbf{x};\mathbf{q})=\chi(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}/\varepsilon}I, \tag{2.5}$$

where I is the $(d+1)\times(d+1)$ identity matrix, $\chi(\mathbf{x})$ is the array function, and $\mathbf{q}\in\mathbb{R}^d$ is a fixed vector. It plays the role of the "state" of the initial data. Physically $\mathbf{Q}_\varepsilon^\delta$ describes evolution of a wave that is emitted by the recorders-transducers with a wave vector $\mathbf{q}$. The Wigner transform of the family $\mathbf{Q}_\varepsilon^\delta(t,\mathbf{x};\mathbf{q})$ is

$$\tilde{W}_\varepsilon^\delta(t,\mathbf{x},\mathbf{k};\mathbf{q})=\int e^{i\mathbf{k}\cdot\mathbf{y}}\mathbf{Q}_\varepsilon^\delta(t,\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2};\mathbf{q})\mathbf{Q}_\varepsilon^{\delta*}(t,\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2};\mathbf{q})\frac{d\mathbf{y}}{(2\pi)^d}. \tag{2.6}$$

The corresponding "mixed" Wigner transform is

$$W_\varepsilon^\delta(t,\mathbf{x},\mathbf{k})=\int\tilde{W}_\varepsilon^\delta(t,\mathbf{x},\mathbf{k};\mathbf{q})\hat{f}(\mathbf{q})d\mathbf{q}. \tag{2.7}$$

Then the re-propagated signal is described as follows in terms of W_ε^δ.

Lemma 2.2 *The re-propagated signal may be expressed as*

$$\mathbf{v}_\varepsilon^{\delta,B}(\boldsymbol{\xi},\mathbf{x}_0)=\int e^{i\mathbf{k}\cdot(\boldsymbol{\xi}-\mathbf{y})}W_\varepsilon^\delta(T,\mathbf{x}_0+\frac{\varepsilon(\boldsymbol{\xi}+\mathbf{y})}{2},\mathbf{k})\Gamma_0\mathbf{S}_0(\mathbf{y})\frac{d\mathbf{k}d\mathbf{y}}{(2\pi)^d}. \tag{2.8}$$

Proof. Let $G(t,\mathbf{x};\mathbf{y})$ be the Green's matrix of (1.5), that is, solution of (1.5) with the initial data $G(0,\mathbf{x};\mathbf{y})=I\delta(\mathbf{x}-\mathbf{y})$. Then the signal arriving to the recorders-transducers array is

$$\mathbf{v}_\varepsilon^\delta(T,\mathbf{x})=\int G(T,\mathbf{x};\mathbf{y})\mathbf{v}_\varepsilon^\delta(0,\mathbf{y})d\mathbf{y}=\int G(T,\mathbf{x};\mathbf{y})\mathbf{S}_0\left(\frac{\mathbf{y}-\mathbf{x}_0}{\varepsilon}\right)d\mathbf{y}$$

and the re-emitted signal is

$$\tilde{\mathbf{v}}_\varepsilon^\delta(T,\mathbf{z})=\int\Gamma_0 f_\varepsilon(\mathbf{z}-\mathbf{z}')\chi(\mathbf{z})\chi(\mathbf{z}')\mathbf{v}_\varepsilon^\delta(T,\mathbf{z}')d\mathbf{z}'.$$

Therefore we obtain

$$\begin{aligned}\mathbf{v}_\varepsilon^{\delta,B}(\boldsymbol{\xi},\mathbf{x}_0)&=\int G(T,\mathbf{x}_0+\varepsilon\boldsymbol{\xi};\mathbf{z})\tilde{\mathbf{v}}_\varepsilon(T,\mathbf{z})d\mathbf{z} &(2.9)\\ &=\int G(T,\mathbf{x}_0+\varepsilon\boldsymbol{\xi};\mathbf{z})\Gamma_0 f\left(\frac{\mathbf{z}-\mathbf{z}'}{\varepsilon}\right)\chi(\mathbf{z})\chi(\mathbf{z}')G(T,\mathbf{z}';\mathbf{y})\mathbf{S}_0\left(\frac{\mathbf{y}-\mathbf{x}_0}{\varepsilon}\right)\frac{d\mathbf{z}d\mathbf{z}'d\mathbf{y}}{\varepsilon^d}.\end{aligned}$$

However, we also have

$$\Gamma_0 G(t,\mathbf{x};\mathbf{y})\Gamma_0 = G^*(t,\mathbf{y};\mathbf{x}). \tag{2.10}$$

This is seen as follows: a solution of (1.5) satisfies

$$\mathbf{v}(t,\mathbf{x}) = \int G(t-s,\mathbf{x};\mathbf{y})\mathbf{v}(s,\mathbf{y})d\mathbf{y}$$

for all $0 \le s \le t$. Differentiating the above equation with respect to s, using (1.5) for $\mathbf{v}(s,\mathbf{y})$ and integrating by parts we obtain

$$0 = \int \left(-\frac{\partial G(t-s,\mathbf{x};\mathbf{y})}{\partial t} + \frac{\partial}{\partial y_j}\left(G(t-s,\mathbf{x};\mathbf{y})A_\delta(\mathbf{y})\right) D^j A_\delta(\mathbf{y}) \right) \mathbf{v}(s,\mathbf{y}).$$

Passing to the limit $s \to 0$ and using the fact that the initial data $\mathbf{v}_0(\mathbf{y})$ is arbitrary we obtain

$$\frac{\partial G(t,\mathbf{y};\mathbf{x})}{\partial t} - \frac{\partial}{\partial x_j}\left(G(t,\mathbf{y};\mathbf{x})A_\delta(\mathbf{x})\right) D^j A_\delta(\mathbf{x}) = 0. \tag{2.11}$$

Furthermore, the matrix $G^*(t,\mathbf{x};\mathbf{y})$ satisfies

$$\frac{\partial G^*(t,\mathbf{x};\mathbf{y})}{\partial t} + \frac{\partial}{\partial x_j}\left(G^*(t,\mathbf{x};\mathbf{y})A_\delta(\mathbf{x})\right) D^j A_\delta(\mathbf{x}) = 0.$$

Multiplying (2.11) by Γ_0 on the left and on the right, and using the commutation relations

$$\Gamma_0 A_\delta = A_\delta \Gamma_0, \quad \Gamma_0 D^j = -D^j \Gamma_0. \tag{2.12}$$

we deduce (2.10). Then, since $\Gamma_0^2 = I$, (2.9) may be re-written as

$$\begin{aligned}\mathbf{v}_\varepsilon^{\delta,B}(\boldsymbol{\xi},\mathbf{x}_0) &= \int G(T,\mathbf{x}_0+\varepsilon\boldsymbol{\xi};\mathbf{z})\chi(\mathbf{z})e^{i\mathbf{q}\cdot\mathbf{z}/\varepsilon}\chi(\mathbf{z}')e^{-i\mathbf{q}\cdot\mathbf{z}'/\varepsilon}G^*(T,\mathbf{x}_0+\varepsilon\mathbf{y};\mathbf{z}')\Gamma_0\hat{f}(\mathbf{q})\mathbf{S}_0(\mathbf{y})\frac{d\mathbf{z}d\mathbf{z}'d\mathbf{y}d\mathbf{q}}{(2\pi)^d} \\ &= \int \mathbf{Q}_\varepsilon^\delta(T,\mathbf{x}_0+\varepsilon\boldsymbol{\xi};\mathbf{q})\mathbf{Q}_\varepsilon^{\delta*}(T,\mathbf{x}_0+\varepsilon\mathbf{y};\mathbf{q})\hat{f}(\mathbf{q})\Gamma_0\mathbf{S}_0(\mathbf{y})\frac{d\mathbf{y}d\mathbf{q}}{(2\pi)^d}.\end{aligned} \tag{2.13}$$

and (2.8) follows. □

The following lemma allows us to drop the term of order ε in the argument of W_ε in (2.8).

Lemma 2.3 *Let us define* $\tilde{\mathbf{v}}_\varepsilon^{\delta,B}(\boldsymbol{\xi};\mathbf{x}_0) = \int e^{i\mathbf{k}\cdot(\boldsymbol{\xi}-\mathbf{y})}W_\varepsilon^\delta(T,\mathbf{x}_0,\mathbf{k})\Gamma_0\mathbf{S}_0(\mathbf{y})\frac{d\mathbf{k}d\mathbf{y}}{(2\pi)^d}$. *There exists a deterministic function* $C(\varepsilon,\delta)$ *so that*

$$\sup_{\boldsymbol{\xi}} \|\mathbf{v}_\varepsilon^{\delta,B} - \tilde{\mathbf{v}}_\varepsilon^{\delta,B}\|_{L^2_{\mathbf{x}_0}} \le C(\varepsilon,\delta) \tag{2.14}$$

and $C(\varepsilon,\delta) \to 0$ *as* $\mathcal{K}_\mu \ni (\varepsilon,\delta) \to 0$.

The proof of Lemma 2.3 is presented in Appendix A. Note that Theorem 1.1 may be applied directly to the moment

$$\tilde{\mathbf{v}}_\varepsilon^\delta(\boldsymbol{\xi};\mathbf{x}_0) = \int e^{i\mathbf{k}\cdot\boldsymbol{\xi}}W_\varepsilon^\delta(T,\mathbf{x}_0,\mathbf{k})\Gamma_0\hat{\mathbf{S}}_0(\mathbf{k})\frac{d\mathbf{k}}{(2\pi)^d}$$

and the conclusion of Theorem 2.1 follows.

3 The high frequency analysis

In this section we study the deterministic high-frequency behavior of the Wigner transform and estimate the error between the Wigner transform and the solution of the Liouville equations. It is well known [14, 17, 20] that in the high frequency regime the weak limit of the Wigner transform as $\varepsilon \to 0$ and $\delta > 0$ is fixed, is described by the classical Liouville equations in the phase space. Here, we do not pass to the limit $\varepsilon \to 0$ at δ fixed but rather control the error introduced by the semi-classical approximation. As explained in the introduction, this is possible because we are dealing with the Wigner transform of a mixture of states that may have strong limits [17] rather than the Wigner transform of pure states, which converges only weakly.

3.1 Convergence on the initial data

We first show that the assumptions on the convergence of the initial data in Theorem 1.1 are purely academic, and in particular are satisfied in the time-reversal application. We note that the L^2-norm of a pure Wigner transform $\tilde{W}_\varepsilon(t, \mathbf{x}, \mathbf{k}; \mathbf{q})$ of a single wave function, such as (2.6), blows up as $\varepsilon \to 0$ in $L^2(\mathbb{R}^d)$, because

$$\begin{aligned} \|\tilde{W}_\varepsilon(t;\zeta)\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} &:= \int \mathrm{Tr}[\tilde{W}_\varepsilon(t,\mathbf{x},\mathbf{k};\zeta)\tilde{W}^*_\varepsilon(t,\mathbf{x},\mathbf{k};\zeta)]d\mathbf{x}d\mathbf{k} = (2\pi\varepsilon)^{-d/2}\|Q^\varepsilon(t,\mathbf{q})\|^2_{L^2(\mathbb{R}^d)} \\ &= (2\pi\varepsilon)^{-d/2}\|\chi\|_{L^2(\mathbb{R}^d)}. \end{aligned} \quad (3.1)$$

Therefore (1.15) may not hold for a pure state. Two examples when assumption (1.15) holds are given by the following lemma, which may be verified by a straightforward calculation. The first one arises when the initial data is random, and the second comes from the time-reversal application.

Lemma 3.1 *Assumption (1.15) is satisfied in the following two cases:*
(1) Statistical averaging: the initial data is $\mathbf{v}_0^\varepsilon(\mathbf{x};\zeta) = \psi(\mathbf{x})V(\mathbf{x}/\varepsilon;\zeta)$, *where* $V(\mathbf{y};\zeta)$ *is a mean zero, scalar spatially homogeneous random process with a rapidly decaying two-point correlation function* $R(\mathbf{z})$: $E\{V(\mathbf{y})V(\mathbf{y}+\mathbf{z})\} = \int V(\mathbf{y};\zeta)V(\mathbf{y}+\mathbf{z};\zeta)d\mu(\zeta) = R(\mathbf{z}) \in L^2(\mathbb{R}^d)$, *and* $\psi(\mathbf{x}) \in C_c(\mathbb{R}^d)$. *The limit Wigner distribution is given by* $W_0(\mathbf{x},\mathbf{k}) = |\psi(\mathbf{x})|^2\tilde{R}(\mathbf{k})$, *where* $\tilde{R}(\mathbf{k})$ *is the inverse Fourier transform of* $R(\mathbf{y})$.
(2) Smoothing of oscillations: the initial data is $\mathbf{v}_0^\varepsilon(\mathbf{x};\zeta) = \psi(\mathbf{x})e^{i\zeta\cdot\mathbf{x}/\varepsilon}$, *where* $\psi(\mathbf{x}) \in C_c(\mathbb{R}^d)$. *The measure* μ *is* $d\mu(\zeta) = g(\zeta)d\zeta$, $\zeta \in \mathbb{R}^d$, *and* $g \in L^2(\mathbb{R}^d)$. *The limit Wigner distribution is* $W_0(\mathbf{x},\mathbf{k}) = |\psi(\mathbf{x})|^2 g(\mathbf{k})$.

Proof. We only verify case (2), the other case being similar:

$$W_\varepsilon^0(\mathbf{x},\mathbf{k}) = \int e^{i\mathbf{k}\cdot\mathbf{y}}\psi(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\psi^*(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\hat{g}(\mathbf{y})\frac{d\mathbf{y}}{(2\pi)^d}$$

so that

$$\|W_\varepsilon^0 - W_0\|_2^2 = \int \left(\psi(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\psi^*(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2}) - |\psi(\mathbf{x})|^2\right)^2 |\hat{g}(\mathbf{y})|^2\frac{d\mathbf{x}d\mathbf{y}}{(2\pi)^d} = \int I_\varepsilon(\mathbf{y})|\hat{g}(\mathbf{y})|^2\frac{d\mathbf{y}}{(2\pi)^d}. \quad (3.2)$$

However, we have $\int |I_\varepsilon(\mathbf{y})| \le 4\|\psi\|^4_{L^4}$ and

$$I_\varepsilon(\mathbf{y}) = \int \left(\psi(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\psi^*(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2}) - |\psi(\mathbf{x})|^2\right)^2 d\mathbf{x} \to 0$$

as $\varepsilon \to 0$ since $\psi \in C_c(\mathbb{R}^d)$, pointwise in $\mathbf{y}$. Therefore $\|W_\varepsilon^0 - W_0\|_2 \to 0$ by the Lebesgue dominated convergence theorem. □

Note that if $g(\zeta)$ and ψ in part (2) of Lemma 3.1 are sufficiently regular, then $\|W_0^\varepsilon - W_0\|_2 = O(\varepsilon)$ so that one may get the order of convergence in (1.15).

3.2 Approximation by the Liouville equations

We now estimate directly the error between the mixed Wigner transform and its semi-classical approximation. The dispersion matrix $P_0^\delta(\mathbf{x},\mathbf{k}) = ic_\delta(\mathbf{x})k_j D^j$ may be diagonalized as

$$-iP_0^\delta(\mathbf{x},\mathbf{k}) = \sum_{q=0}^{2} \lambda_q^\delta(\mathbf{x},\mathbf{k})\Pi_q(\mathbf{k}), \qquad \sum_{q=0}^{2} \Pi_q(\mathbf{k}) = I. \tag{3.3}$$

Here Π_q is the projection matrix onto the eigenspace corresponding to the eigenvalue λ_q^δ. Notice that the eigenspaces are independent of the spatial position $\mathbf{x}$, hence of the parameter δ; see (1.7)-(1.8).

As we have mentioned before, for a fixed $\delta > 0$ the Wigner transform $W_\varepsilon^\delta(t,\mathbf{x},\mathbf{k})$ converges weakly as $\varepsilon \to 0$ to its semi-classical limit $U^\delta(t,\mathbf{x},\mathbf{k})$ given by

$$U^\delta(t,\mathbf{x},\mathbf{k}) = \sum_q u_q^\delta(t,\mathbf{x},\mathbf{k})\Pi_q(\mathbf{k}). \tag{3.4}$$

The functions u_q^δ satisfy the Liouville equations

$$\frac{\partial u_q^\delta}{\partial t} + \nabla_{\mathbf{k}}\lambda_q^\delta \cdot \nabla_{\mathbf{x}} u_q^\delta - \nabla_{\mathbf{x}}\lambda_q^\delta \cdot \nabla_{\mathbf{k}} u_q^\delta = 0 \tag{3.5}$$

with initial data $u_q^\delta(0,\mathbf{x},\mathbf{k}) = \mathrm{Tr}\Pi_q W_0(\mathbf{x},\mathbf{k})\Pi_q$. Our goal is to estimate the difference between W_ε^δ and U^δ in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Let us denote by $\gamma_q^\delta(\mathbf{x},\mathbf{k})$ the largest eigenvalue of the matrix $(F_q^\delta F_q^{\delta *})^{1/2}$, where

$$F_q^\delta = \begin{pmatrix} -\dfrac{\partial^2\lambda_q^\delta}{\partial k_i \partial x_j} & -\dfrac{\partial^2\lambda_q^\delta}{\partial k_i \partial k_j} \\ \dfrac{\partial^2\lambda_q^\delta}{\partial x_i \partial x_j} & \dfrac{\partial^2\lambda_q^\delta}{\partial x_i \partial k_j} \end{pmatrix}.$$

Note that (1.12) implies that $\gamma_1^\delta(\mathbf{x},\mathbf{k}) = \gamma_2^\delta(\mathbf{x},\mathbf{k}) = \gamma^\delta(\mathbf{x},\mathbf{k})$, while $\gamma_0 = 0$. The initial data u_q^0 is supported on a compact set S because W_0 is (see (1.16)). Then the set

$$\mathcal{S} = \bigcup_{t\geq 0,\ \delta\in(0,1]} \mathrm{supp}\ u_q^\delta(t,\mathbf{x},\mathbf{k})$$

is bounded because the speed $c_\delta(\mathbf{x})$ is uniformly bounded from above and below for δ sufficiently small (1.12). Therefore we have $\gamma^\delta(\mathbf{x},\mathbf{k}) \leq C/\delta^{3/2}$ with a deterministic constant $C > 0$. We denote $\bar\gamma_\delta = \sup_{\mathcal{S}} \gamma^\delta(\mathbf{x},\mathbf{k})$. We have the following approximation theorem.

Theorem 3.2 *Let the acoustic speed $c_\delta(\mathbf{x})$ be of the form (1.3) and satisfy assumptions (1.12). We assume that the Wigner transform W_ε^δ satisfies (1.15) and that (1.16) holds. Moreover, we assume that the initial limit Wigner transform W_0 is of the form*

$$W_0(\mathbf{x},\mathbf{k}) = \sum_q u_q^0(\mathbf{x},\mathbf{k})\Pi_q(\mathbf{k}). \tag{3.6}$$

Let $U^\delta(t,\mathbf{x},\mathbf{k}) = \sum_p u_p^\delta(t,\mathbf{x},\mathbf{k})\Pi_p(\mathbf{k})$, where the functions u_p^δ satisfy the Liouville equations (3.5) with initial data $u_q^0(\mathbf{x},\mathbf{k})$. Then we have

$$\|W_\varepsilon^\delta(t,\mathbf{x},\mathbf{k}) - U^\delta(t,\mathbf{x},\mathbf{k})\|_2 \leq C(\delta)[\varepsilon\|W_0\|_{H^2}e^{2\bar\gamma_\delta t} + \varepsilon^2\|W_0\|_{H^3}e^{3\bar\gamma_\delta t}] + \|W_\varepsilon^\delta(0) - W_0\|_2, \tag{3.7}$$

where $C(\delta)$ is a rational function of δ with deterministic coefficients that may depend on the constant $C > 0$ in the bound (1.16) on the support of W_0.

Theorem 3.2 shows that the semi-classical approximation is valid for times $T \ll |\ln \varepsilon|/\bar{\gamma}_\delta$. This is reminiscent of the Ehrenfest time of validity of the semi-classical approximation in quantum mechanics, see [5, 9] for recent mathematical results in this direction for the Schrödinger operators. The pre-factor constants on the right side of (3.7) are not optimal but sufficient for the purposes of our analysis.

The assumption that initially W_0 has no terms of the form $\Pi_p \Pi_q$ with $p \neq q$ is necessary in general for the Liouville equation to provide an approximation to W_ε^δ in the strong sense. This may be seen on the simple example of the solution

$$u_\varepsilon(t, x) = ae^{i(\mathbf{q}\cdot\mathbf{x}-ct)/\varepsilon} + be^{i(\mathbf{q}\cdot\mathbf{x}+ct)/\varepsilon}$$

of the wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

with a constant speed c. The cross-terms in the Wigner distribution $W_\varepsilon = [|a|^2 + |b|^2 + ab^* e^{-2ict/\varepsilon} + a^* b e^{2ict/\varepsilon}]\delta(\mathbf{k} - \mathbf{q})$ vanish only in the weak sense as a function of t but not strongly.

The Wigner distribution that arises in the time-reversal application has an initial data that is described by part (2) of Lemma 3.1:

$$W_0(\mathbf{x}, \mathbf{k}) = |\chi(\mathbf{x})|^2 \hat{f}(\mathbf{k}) I$$

and satisfies the assumption (3.6) with $u_q^0(\mathbf{x}, \mathbf{k}) = |\chi(\mathbf{x})|^2 \hat{f}(\mathbf{k})$ for all eigenspaces because of the second equation in (3.3). The error introduced by the replacement of the initial data in (3.7) in that case is given by (3.2) and is $O(\varepsilon)$ provided that χ and f are sufficiently regular.

The proof of Theorem 3.2 is quite straightforward though tedious. We first obtain the evolution equation for W_ε^δ in Section 3.3, and show that it preserves the L^2-norm. This allows us to replace the initial data in the equation for W_ε^δ by W_0 at the expense of the last term in (3.7). We obtain the Liouville equations (3.5) in Section 3.4 and estimate the right side of (3.7) in terms of the H^3-norm of its solution. Finally, in Appendix C we obtain the necessary estimates for the solution of the Liouville equation.

3.3 The evolution equation for the Wigner transform

The L^2-norm of the Wigner transform $\tilde{W}(t, \mathbf{x}, \mathbf{k}; \zeta)$ of a pure state, or a fixed ζ, is preserved in time as follows from the preservation of the L^2-norm of solutions of (1.5). We obtain now an evolution equation for the Wigner transform W_ε of mixed states and show that its L^2-norm is also preserved. It is convenient to define the skew-symmetric matrix symbol

$$P_\varepsilon^\delta(\mathbf{x}, \mathbf{k}) = P_0^\delta(\mathbf{x}, \mathbf{k}) + \varepsilon P_1^\delta(\mathbf{x}), \tag{3.8}$$

where P_0^δ is defined by (1.7) and the symbol P_1^δ depends only on $\mathbf{x}$:

$$P_1^\delta(\mathbf{x}) = A_\delta(\mathbf{x}) D^j \frac{1}{2}\frac{\partial A_\delta}{\partial x_j}(\mathbf{x}) - \frac{1}{2}\frac{\partial A_\delta}{\partial x_j}(\mathbf{x}) D^j A_\delta(\mathbf{x}) = \frac{1}{2}\frac{\partial c_\delta}{\partial x_j}(\mathbf{x}) \left[\mathbf{e}_j \otimes \mathbf{e}_{d+1} - \mathbf{e}_{d+1} \otimes \mathbf{e}_j\right]. \tag{3.9}$$

The latter equality follows from (1.6) and calculations of the form

$$D^j \frac{\partial A_\delta}{\partial x_j}(\mathbf{x}) = \frac{\partial c_\delta}{\partial x_j}(\mathbf{x}) \mathbf{e}_j \otimes \mathbf{e}_{d+1}. \tag{3.10}$$

The following lemma describes the evolution of the Wigner transform W_ε^δ.

Lemma 3.3 *The Wigner transform $W_\varepsilon^\delta(t,\mathbf{x},\mathbf{k})$ satisfies the evolution equation*

$$\varepsilon\frac{\partial W_\varepsilon^\delta}{\partial t}+\mathcal{L}_\varepsilon^\delta W_\varepsilon^\delta=0 \tag{3.11}$$

with initial data $W_\varepsilon^\delta(0,\mathbf{x},\mathbf{k})$. The operator $\mathcal{L}_\varepsilon^\delta$ is given by

$$\mathcal{L}_\varepsilon^\delta f(\mathbf{x},\mathbf{k})=\int\left(P_\varepsilon^\delta(\mathbf{y},\mathbf{q})e^{i\phi}f(\mathbf{z},\mathbf{p})-f(\mathbf{z},\mathbf{p})e^{-i\phi}P_\varepsilon^\delta(\mathbf{y},\mathbf{q})\right)\frac{d\mathbf{z}d\mathbf{p}d\mathbf{y}d\mathbf{q}}{(\pi\varepsilon)^{2d}}, \tag{3.12}$$

where $\phi(\mathbf{x},\mathbf{z},\mathbf{k},\mathbf{p},\mathbf{y},\mathbf{q})=\frac{2}{\varepsilon}((\mathbf{p}-\mathbf{k})\cdot\boldsymbol{y}+(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}+(\mathbf{k}-\mathbf{q})\cdot\mathbf{z})$. The integral of the trace and the L^2-norm of the Wigner transform W_ε are preserved:

$$\int \mathrm{Tr}\mathrm{W}_\varepsilon^\delta(\mathrm{t},\mathbf{x},\mathbf{k})d\mathbf{x}d\mathbf{k}=\int \mathrm{Tr}\mathrm{W}_\varepsilon^\delta(0,\mathbf{x},\mathbf{k})d\mathbf{x}d\mathbf{k} \tag{3.13}$$

and

$$\int \mathrm{Tr}[\mathrm{W}_\varepsilon^\delta(\mathrm{t},\mathbf{x},\mathbf{k})\mathrm{W}_\varepsilon^{\delta*}(\mathrm{t},\mathbf{x},\mathbf{k})]d\mathbf{x}d\mathbf{k}=\int \mathrm{Tr}[\mathrm{W}_\varepsilon^\delta(0,\mathbf{x},\mathbf{k})\mathrm{W}_\varepsilon^{\delta*}(0,\mathbf{x},\mathbf{k})]d\mathbf{x}d\mathbf{k}. \tag{3.14}$$

This lemma is verified by a direct calculation that we present for the convenience of the reader in Appendix B.

Note that the solution of (3.11) with self-adjoint initial data remains self-adjoint and the L^2-norm is preserved. Therefore, we have the following corollary.

Corollary 3.4 *Let $W_0(\mathbf{x},\mathbf{k})$ be a strong limit of $W_\varepsilon(0)$ in L^2, which exists by assumption (1.15). Then the solutions $W_\varepsilon^\delta(t,\mathbf{x},\mathbf{k})$ and $\bar{W}_\varepsilon^\delta(t,\mathbf{x},\mathbf{k})$ of (3.11) with initial conditions $W_\varepsilon^0(\mathbf{x},\mathbf{k})$ and $W_0(\mathbf{x},\mathbf{k})$, respectively, satisfy*

$$\|\bar{W}_\varepsilon^\delta(t)-W_\varepsilon^\delta(t)\|_2=\|W_\varepsilon^0-W_0\|_2\to 0 \ \textit{as}\ \varepsilon\to 0.$$

This shows that in the analysis of (3.11), we can replace strongly converging initial conditions by their limit, and consider then the limit of $\bar{W}_\varepsilon(t,\mathbf{x},\mathbf{k})$ as $\varepsilon\to 0$ with fixed initial conditions. This is done in the following section.

3.4 Derivation of the Liouville equations

We consider in this section the solution $\bar{W}_\varepsilon^\delta(t,\mathbf{x},\mathbf{k})$ of the evolution equation (3.11) with fixed initial data $W_0(\mathbf{x},\mathbf{k})$ and show that it may be approximated by the solution of the Liouville equation. We split the operator $\mathcal{L}_\varepsilon^\delta=\mathcal{L}_\varepsilon^{\delta,0}+\varepsilon\mathcal{L}_\varepsilon^{\delta,1}$, where

$$\mathcal{L}_\varepsilon^{\delta,j} f(\mathbf{x},\mathbf{k})=\int\left(P_j^\delta(\mathbf{y},\mathbf{q})e^{i\phi}f(\mathbf{z},\mathbf{p})-f(\mathbf{z},\mathbf{p})e^{-i\phi}P_j^\delta(\mathbf{y},\mathbf{q})\right)\frac{d\mathbf{z}d\mathbf{p}d\mathbf{y}d\mathbf{q}}{(\pi\varepsilon)^{2d}},\quad j=0,1,$$

and the symbols P_j^δ are given by (1.7) and (3.9). The operator $\mathcal{L}_\varepsilon^{\delta,0}$ is given explicitly by

$$\begin{aligned}\mathcal{L}_\varepsilon^{\delta,0} f(\mathbf{x},\mathbf{k})&=\int e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{y}}\left[c_\delta(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})ip_jD^jf(\mathbf{x},\mathbf{p})-c_\delta(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})f(\mathbf{x},\mathbf{p})ip_jD^j\right]\frac{d\mathbf{p}d\mathbf{y}}{(2\pi)^d}\\ +\frac{\varepsilon}{2}\int e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{y}}&\left(D^j\frac{\partial}{\partial x_j}\left[c_\delta(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})f(\mathbf{x},\mathbf{p})\right]+\frac{\partial}{\partial x_j}\left[c_\delta(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})f(\mathbf{x},\mathbf{p})\right]D^j\right)\frac{d\mathbf{p}d\mathbf{y}}{(2\pi)^d}=\mathcal{L}_{\varepsilon,\delta}^{01}+\mathcal{L}_{\varepsilon,\delta}^{02}.\end{aligned}$$

We recast the operator $\mathcal{L}_{\varepsilon,\delta}^{01}$ as

$$\begin{aligned}\mathcal{L}_{\varepsilon,\delta}^{01} f(\mathbf{x},\mathbf{k})=&c_\delta(\mathbf{x})\left[ik_jD^jf(\mathbf{x},\mathbf{k})-f(\mathbf{x},\mathbf{k})ik_jD^j\right]-\frac{\varepsilon}{2}\frac{\partial c_\delta(\mathbf{x})}{\partial x_m}\frac{\partial}{\partial k_m}\left(k_jD^jf(\mathbf{x},\mathbf{k})+f(\mathbf{x},\mathbf{k})k_jD^j\right)\\ &+\varepsilon\mathcal{R}_{\varepsilon,\delta}^{01}f\end{aligned}$$

with the correction term

$$\mathcal{R}^{01}_{\varepsilon,\delta}f(\mathbf{x},\mathbf{k}) = \frac{1}{\varepsilon}\int e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{y}}\Big[\Big(c_\delta(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2}) - c_\delta(\mathbf{x}) + \frac{\varepsilon}{2}\mathbf{y}\cdot\nabla c_\delta(\mathbf{x})\Big)\,ip_jD^jf(\mathbf{x},\mathbf{p})$$
$$-\Big(c_\delta(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2}) - c_\delta(\mathbf{x}) - \frac{\varepsilon}{2}\mathbf{y}\cdot\nabla c_\delta(\mathbf{x})\Big)\,f(\mathbf{x},\mathbf{p})ip_jD^j\Big]\frac{d\mathbf{p}d\mathbf{y}}{(2\pi)^d}.$$

Similarly, we have

$$\mathcal{L}^{02}_{\varepsilon,\delta}f(\mathbf{x},\mathbf{k}) = \frac{\varepsilon}{2}D^j\frac{\partial}{\partial x_j}\left(c_\delta(\mathbf{x})f(\mathbf{x},\mathbf{k})\right) + \frac{\varepsilon}{2}\frac{\partial}{\partial x_j}\left(c_\delta(\mathbf{x})f(\mathbf{x},\mathbf{k})\right)D^j + \varepsilon\mathcal{R}^{02}_{\varepsilon,\delta}f$$

with

$$\mathcal{R}^{02}_{\varepsilon,\delta}f(\mathbf{x},\mathbf{k}) = \frac{1}{2}\int e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{y}}\Big(D^j\frac{\partial}{\partial x_j}\Big[\Big\{c_\delta(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2}) - c_\delta(\mathbf{x})\Big\}f(\mathbf{x},\mathbf{p})\Big]$$
$$+\frac{\partial}{\partial x_j}\Big[\Big\{c_\delta(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2}) - c_\delta(\mathbf{x})\Big\}f(\mathbf{x},\mathbf{p})\Big]D^j\Big)\frac{d\mathbf{p}d\mathbf{y}}{(2\pi)^d}.$$

The operator $\mathcal{L}^{\delta,1}_\varepsilon$ is given explicitly by

$$\begin{aligned}\mathcal{L}^{\delta,1}_\varepsilon f(\mathbf{x},\mathbf{k}) &= \int e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{y}}\Big[P^\delta_1(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})f(\mathbf{x},\mathbf{p}) - f(\mathbf{x},\mathbf{p})P^\delta_1(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\Big]\frac{d\mathbf{p}d\mathbf{y}}{(2\pi)^d}\\ &= P^\delta_1(\mathbf{x})f(\mathbf{x},\mathbf{k}) - f(\mathbf{x},\mathbf{k})P^\delta_1(\mathbf{x}) + \mathcal{R}^1_{\varepsilon,\delta}f(\mathbf{x},\mathbf{k})\end{aligned}\tag{3.15}$$

with the correction $\mathcal{R}^1_{\varepsilon,\delta}$ defined by

$$\mathcal{R}^1_{\varepsilon,\delta}f(\mathbf{x},\mathbf{k}) = \int e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{y}}\Big[\Big(P^\delta_1(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2}) - P^\delta_1(\mathbf{x})\Big)f(\mathbf{x},\mathbf{p}) - f(\mathbf{x},\mathbf{p})\Big(P^\delta_1(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2}) - P^\delta_1(\mathbf{x})\Big)\Big]\frac{d\mathbf{p}d\mathbf{y}}{(2\pi)^d}.\tag{3.16}$$

Putting together the above expressions, we obtain the following equation for $\bar{W}^\delta_\varepsilon$

$$\frac{\partial\bar{W}^\delta_\varepsilon}{\partial t} = \frac{1}{\varepsilon}\mathcal{L}^\delta_\varepsilon\bar{W}^\delta_\varepsilon = \frac{[\bar{W}^\delta_\varepsilon, P^\delta_0]}{\varepsilon} + [\bar{W}^\delta_\varepsilon, P^\delta_1] + \frac{1}{2i}(\{\bar{W}^\delta_\varepsilon, P^\delta_0\} - \{P^\delta_0, \bar{W}^\delta_\varepsilon\}) - \mathcal{R}^\delta_\varepsilon\bar{W}^\delta_\varepsilon\tag{3.17}$$

with $\mathcal{R}^\delta_\varepsilon = \mathcal{R}^{01}_{\varepsilon,\delta} + \mathcal{R}^{02}_{\varepsilon,\delta} + \mathcal{R}^1_{\varepsilon,\delta}$. Here $\{f,g\}$ is the standard Poisson bracket

$$\{f,g\} = \nabla_{\mathbf{k}}f\cdot\nabla_{\mathbf{x}}g - \nabla_{\mathbf{x}}f\cdot\nabla_{\mathbf{k}}g$$

and $[A,B] = AB - BA$ is the commutator. We now introduce the expansion

$$\bar{W}^\delta_\varepsilon = U^\delta + \varepsilon U^\delta_1 + U^\delta_{2,\varepsilon}.\tag{3.18}$$

We insert this ansatz into (3.17) and equating like powers of ε obtain at the order ε^{-1}

$$[P^\delta_0, U^\delta] = 0,\tag{3.19}$$

which is equivalent to

$$U^\delta = \sum_{q=0}^{2}\Pi_qU^\delta\Pi_q = \sum_{q=0}^{2}U^\delta_q,\tag{3.20}$$

where $U_q^\delta = \Pi_q U^\delta \Pi_q$, and for $q = 1, 2$ one has $U_q^\delta = u_q^\delta \Pi_q$ with $u_q = \mathrm{Tr} U_q^\delta$. The matrices Π_q are projections on the eigenspaces of P_0^δ, as in (3.3). This means that the matrix U^δ does not have off-diagonal contributions in the eigenbasis of P_0^δ. The equation of order $O(\varepsilon^0)$ is given by

$$\frac{\partial U^\delta}{\partial t} = [U_1^\delta, P_0^\delta] + [U^\delta, P_1^\delta] + \frac{1}{2i}(\{U^\delta, P_0^\delta\} - \{P_0^\delta, U^\delta\}). \tag{3.21}$$

Multiplying the above equation on both sides by Π_q yields

$$\frac{\partial \Pi_q U^\delta \Pi_q}{\partial t} = \Pi_q [U^\delta, P_1^\delta] \Pi_q + \frac{1}{2i} \Pi_q \big(\{U^\varepsilon, P_0^\delta\} - \{P_0^\delta, U^\delta\}\big) \Pi_q. \tag{3.22}$$

This is nothing but equation (6.16) of reference [14] for the Wigner matrix without consideration of mixtures of states. The only difference is that the leading order term P_0 depends on the parameter δ. This, of course, does not change the algebra, and following [14] one obtains a system of decoupled Liouville equations for $u_q^\delta = \mathrm{Tr} U_q^\delta$, $q = 1, 2$,

$$\frac{\partial u_q^\delta}{\partial t} + \left\{\lambda_q^\delta, u_q^\delta\right\} = 0 \tag{3.23}$$

with initial data $u_q^\delta(0, \mathbf{x}, \mathbf{k}) = \mathrm{Tr}[\Pi_q(\mathbf{k}) W_0(\mathbf{x}, \mathbf{k}) \Pi_q(\mathbf{k})]$. The zero eigenvalue component of the matrix U^δ, that is, $U_0(t, \mathbf{x}, \mathbf{k}) = \Pi_0(\mathbf{k}) W_0(\mathbf{x}, \mathbf{k}) \Pi_0(\mathbf{k})$, does not change in time.

We have to show that the terms U_1^δ and $U_{2,\varepsilon}^\delta$ in (3.18) are small. In order to uniquely characterize U_1^δ, we assume that it is orthogonal to the terms of the form (3.20), that is,

$$U_1^\delta = \sum_{p \neq q} \Pi_p U_1^\delta \Pi_q. \tag{3.24}$$

Then, (3.20) and (3.21) imply that

$$\Pi_m U_1^\delta \Pi_p = \frac{1}{i(\lambda_m^\delta - \lambda_p^\delta)} \Pi_m \mathcal{B}(U^\delta) \Pi_p, \tag{3.25}$$

where

$$\mathcal{B}(U^\delta) = [U^\delta, P_1^\delta] + \frac{1}{2i}(\{U^\delta, P_0^\delta\} - \{P_0^\delta, U^\delta\}).$$

We now analyze the term $U_{2,\varepsilon}^\delta$ in (3.18) and show that it vanishes in the limit $\varepsilon \to 0$. The equation for the $U_{2,\varepsilon}^\delta$ is

$$\frac{\partial U_{2,\varepsilon}^\delta}{\partial t} = \frac{1}{\varepsilon} \mathcal{L}_\varepsilon^\delta U_{2,\varepsilon}^\delta + S_\varepsilon, \tag{3.26}$$

where

$$S_\varepsilon = \varepsilon\Big([U_1^\delta, P_1^\delta] + \frac{1}{2i}(\{U_1^\delta, P_0^\delta\} - \{P_0^\delta, U_1^\delta\})\Big) - \varepsilon \frac{\partial U_1^\delta}{\partial t} - \mathcal{R}_\varepsilon^\delta(U^\delta + \varepsilon U_1^\delta). \tag{3.27}$$

The initial condition for (3.26) is $U_{2,\varepsilon}^\delta(0, \mathbf{x}, \mathbf{k}) = -\varepsilon U_1^\delta(0, \mathbf{x}, \mathbf{k})$ because of (3.6), which implies that $W_0(0, \mathbf{x}, \mathbf{k}) = U^\delta(0, \mathbf{x}, \mathbf{k})$. We now use the fact that $\mathcal{L}_\varepsilon^\delta$ is skew-symmetric to obtain the bound

$$\|U_{2,\varepsilon}^\delta(t)\|_2 \leq \varepsilon \|U_1^\delta(0)\|_2 + \int_0^t \|S_\varepsilon(s)\|_2 ds. \tag{3.28}$$

The analysis of the convergence of the difference of $\bar{W}_\varepsilon^\delta$ and U^δ to zero thus relies on estimating the error term S_ε. The relevant bounds are provided by the following two lemmas. Here we denote $\|f\|_{\dot{H}^s} = \|D^s f\|_{L^2}$.

Lemma 3.5 *There exists a constant $C > 0$ that depends on the constant in the bound (1.16) on the support of W_0, and on the constants D_i in (1.12) so that*

$$\|S_\varepsilon\|_2 \le C\left[\frac{\varepsilon}{\delta^3}\|U^\delta\|_{H^2} + \frac{\varepsilon^2}{\delta^{11/2}}\|U^\delta\|_{H^3}\right]. \tag{3.29}$$

Lemma 3.6 *The $\dot H^s(\mathbb{R}^d \times \mathbb{R}^d)$-norm, $s = 1, 2, 3$, of $U^\delta(t)$ is bounded by*

$$\|u_q^\delta(t)\|_{\dot H^s} \le C_s\|u_q^0\|_{\dot H^s}\exp(s\bar\gamma_\delta t). \tag{3.30}$$

Here $u_q^0 = \mathrm{Tr}[\Pi_q W_0 \Pi_q]$, the initial data for the Liouville equation (3.23), the constant C_s is a deterministic rational function of δ.

Note that the prefactors of the type δ^{-m} in Lemma 3.5 are not as important as the terms $\|U^\delta\|_{H^s}$ since the latter grow exponentially in $\bar\gamma_\delta \sim \delta^{-3/2}$ according to Lemma 3.6.

Proof of Lemma 3.5. Observe that thanks to (3.27), we have

$$\|S_\varepsilon\|_2 \le C\left[\frac{\varepsilon}{\sqrt{\delta}}\|U_1^\delta\|_{H^1} + \varepsilon\left\|\frac{\partial U_1^\delta}{\partial t}\right\|_2 + \|\mathcal{R}_\varepsilon^\delta(U^\delta + \varepsilon U_1^\delta)\|\right]. \tag{3.31}$$

We have the following bound for U_1^δ:

$$\|U_1^\delta\|_{H^s} \le \frac{C}{\delta^{2s}}\|U^\delta\|_{H^{s+1}} \tag{3.32}$$

with a constant $C > 0$ that depends only on the constant in the bound (1.16) on the support of W_0 and on the constants D_i in (1.12). Indeed, expression (3.24) implies that $\|U_1^\delta\|_{H^s} \le C\delta^{\frac{1}{2}-s}\|\mathcal{B}(U^\delta)\|_{H^s}$, while we have $\|\mathcal{B}(U^\delta)\|_{H^s} \le C\delta^{-s-\frac{1}{2}}\|U^\delta\|_{H^{s+1}}$ so that (3.32) follows. This bound is by no means optimal but will be sufficient for our purposes. Furthermore, we have

$$\left\|\frac{\partial U_1^\delta}{\partial t}\right\|_2 \le C\left\|\mathcal{B}\left(\frac{\partial U^\delta}{\partial t}\right)\right\|_2 \le \frac{C}{\delta^3}\|U^\delta\|_{H^2}. \tag{3.33}$$

In order to complete the bound (3.29) for S_ε we show that

$$\|\mathcal{R}_\varepsilon^\delta f\|_2 \le \frac{C\varepsilon}{\delta^{3/2}}\sum_j[\|k_j f\|_{H^2} + \|f\|_{H^2}]. \tag{3.34}$$

We only consider $\mathcal{R}_{\varepsilon,\delta}^{01}$ as the corresponding bounds for the operators $\mathcal{R}_{\varepsilon,\delta}^{02}$ and $\mathcal{R}_{\varepsilon,\delta}^{1}$ are obtained similarly. We split $\mathcal{R}_{\varepsilon,\delta}^{01}$ as $\mathcal{R}_{\varepsilon,\delta}^{01} = I_{01} - II_{01}$. We have

$$\begin{aligned} I_{01}f &= \frac{1}{\varepsilon}\int e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{y}}\left(c_\delta(\mathbf{x} - \frac{\varepsilon\mathbf{y}}{2}) - c_\delta(\mathbf{x}) + \frac{\varepsilon}{2}\mathbf{y}\cdot\nabla c_\delta(\mathbf{x})\right) ip_j D^j f(\mathbf{x},\mathbf{p})\frac{d\mathbf{p}d\mathbf{y}}{(2\pi)^d} \\ &= \frac{1}{4\varepsilon}\int_0^\varepsilon(\varepsilon - s)\int e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{y}}y_l y_m\frac{\partial^2 c_\delta(\mathbf{x} - \frac{s\mathbf{y}}{2})}{\partial x_l\partial x_m}ip_j D^j f(\mathbf{x},\mathbf{p})\frac{d\mathbf{p}d\mathbf{y}}{(2\pi)^d}ds = \frac{1}{4\varepsilon}\int_0^\varepsilon(\varepsilon - s)\tilde I_{01}(s)f ds. \end{aligned}$$

Moreover, we obtain that

$$\int |\tilde{I}_{01}(s)f(\mathbf{x},\mathbf{k})|^2 d\mathbf{x}d\mathbf{k} = \mathrm{Tr}\int e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{y}-i(\mathbf{k}-\mathbf{q})\cdot\mathbf{z}} y_l y_m z_{l'} z_{m'} \frac{\partial^2 c_\delta(\mathbf{x}-\frac{s\mathbf{y}}{2})}{\partial x_l \partial x_m}\frac{\partial^2 c_\delta(\mathbf{x}-\frac{s\mathbf{z}}{2})}{\partial x_{l'}\partial x_{m'}}$$
$$\times p_j q_r D^j f(\mathbf{x},\mathbf{p}) f^*(\mathbf{x},\mathbf{q}) D^r \frac{d\mathbf{p}d\mathbf{y}d\mathbf{q}d\mathbf{z}d\mathbf{x}d\mathbf{k}}{(2\pi)^{2d}}$$
$$= \mathrm{Tr}\int e^{i(\mathbf{q}-\mathbf{p})\cdot\mathbf{y}} y_l y_m y_{l'} y_{m'} \frac{\partial^2 c_\delta(\mathbf{x}-\frac{s\mathbf{y}}{2})}{\partial x_l \partial x_m}\frac{\partial^2 c_\delta(\mathbf{x}-\frac{s\mathbf{y}}{2})}{\partial x_{l'}\partial x_{m'}} p_j q_r D^j f(\mathbf{x},\mathbf{p}) f^*(\mathbf{x},\mathbf{q}) D^r \frac{d\mathbf{p}d\mathbf{y}d\mathbf{q}d\mathbf{x}}{(2\pi)^d}$$
$$\le \frac{C}{\delta^3}\sum_j \|k_j f\|^2_{H^2}.$$

Therefore the Minkowski inequality implies that $\|I_{01}f\|_2 \le C\varepsilon\delta^{-3/2}\sum_j \|k_j f\|_{H^2}$, and the same bound holds for II_{01}. The operators $\mathcal{R}^{02}_\varepsilon$ and $\mathcal{R}^1_\varepsilon$ may be bounded in a similar way as $\|\mathcal{R}^{02}_\varepsilon f\|_{L^2} + \|\mathcal{R}^1_\varepsilon f\|_{L^2} \le C\varepsilon\delta^{-3/2}\|f\|_{H^2}$. Therefore we have the bound (3.34) and then (3.29) follows from (3.31)-(3.34). □

Theorem 3.2 now follows from the bound (3.32) for U^δ_1, the bound (3.28) for $U^\delta_{2,\varepsilon}$, and Lemmas 3.6 and 3.5. It only remains to prove Lemma 3.6, which is done in Appendix C.

4 The Liouville equations in a random medium

We formulate in this section the main result concerning the convergence of the expectation of the solution of the Liouville equation (1.9) to the solution of the phase space diffusion equation (1.10) in the limit $\delta \to 0$. We also show that the values of the solution of the Liouville equation at different points in the phase space become independent in this limit. This allows us to establish the self-averaging property in Theorem 1.1.

4.1 Preliminaries

We let $\mathcal{C}_m := C([0,+\infty);(\mathbb{R}^d)^m)$, and for any $R_1,\cdots,R_m > 0$ we denote by $\mathcal{C}_m(R_1,\cdots,R_m) := C\left([0,+\infty);S^{d-1}_{R_1}\times\cdots\times S^{d-1}_{R_m}\right)$, where S^{d-1}_R is the sphere in $\mathbb{R}^d$ of radius $R>0$ centered at $\mathbf{0}$. We also let $\pi_t : \mathcal{C}_m \to (\mathbb{R}^d)^m$, $t>0$, be the canonical mapping $\pi_t(K) = (K_1(t),\cdots,K_m(t))$, $K = (K_1,\cdots,K_m)\in\mathcal{C}_m$. For any $u \le v$ we denote by $\mathcal{M}^{u,v}_m$ be the σ-algebra of subsets of $\mathcal{C}_m$ generated by π_t, $t\in[u,v]$, and let $\mathcal{M}_m := \mathcal{M}^{0,+\infty}_m$ and $\mathcal{T}_m$ be the filtered measurable space $\left(\mathcal{C}_m,\mathcal{M}_m,\left(\mathcal{M}^{0,t}_m\right)_{t\ge0}\right)$. For any set $A\in\mathcal{B}(\mathbb{R}^d)$ we denote $\mathcal{C}(A) := \sigma\{c_1(\mathbf{x}) : \mathbf{x}\in A\}$.

We suppose further that $c_1 : \mathbb{R}^d\times\Omega\to\mathbb{R}$ is a scalar , measurable, strictly stationary, zero mean random field that satisfies assumptions presented in Section 1.2, that is, it satisfies the almost sure bounds (1.12), is exponentially ϕ-mixing (1.13), and has a C^∞-correlation function $R(\mathbf{x})$.

We define the differential operator

$$\mathcal{L}F(\mathbf{k}) = \sum_{p,q=1}^d |\mathbf{k}|^2 D_{p,q}(\hat{\mathbf{k}})\partial^2_{k_p,k_q}F(\mathbf{k}) + \sum_{p=1}^d |\mathbf{k}|E_p(\hat{\mathbf{k}})\partial_{k_p}F(\mathbf{k}), \quad F\in C^\infty_0(\mathbb{R}^d\setminus\{\mathbf{0}\}) \tag{4.1}$$

with the diffusion matrix D given by (1.11) and the drift $\mathbf{E}$ defined by

$$E_p(\hat{\mathbf{k}}) = -c_0\sum_{q=1}^d \int_0^{+\infty} s\,\partial^3_{x_p,x_q,x_q} R(c_0 s\hat{\mathbf{k}})\,ds, \quad \forall p = 1,\cdots,d.$$

A simple calculation shows that $\mathcal{L}$ is a generator of a diffusion on $S_{k_0}^{d-1}$ given by Itô S.D.E.

$$\begin{cases} d\boldsymbol{k}(t) = |\boldsymbol{k}(t)| \left(\mathbf{E}(\hat{\boldsymbol{k}}(t))\, dt + \sqrt{2}\, \mathbf{D}^{1/2}(\hat{\boldsymbol{k}}(t))\, d\mathbf{B}(t) \right) \\ \boldsymbol{k}(0) = \mathbf{k}_0 \neq \mathbf{0}. \end{cases} \tag{4.2}$$

Here $\mathbf{E} = (E_1, \cdots, E_d)$ and $\mathbf{B}(\cdot)$ is a d-dimensional standard Brownian motion.

Remark 4.1 A simple calculation shows that the diffusion $\boldsymbol{k}(\cdot)$ given by (4.2) is symmetric. Indeed the generator can be written in the form

$$\mathcal{L}F(\mathbf{k}) = \sum_{p,q=1}^{d} \partial_{k_p} \left(|\mathbf{k}|^2 D_{p,q}(\hat{\mathbf{k}}) \partial_{k_q} F(\mathbf{k}) \right), \quad F \in C_0^\infty(\mathbb{R}^d \setminus \{\mathbf{0}\}).$$

For any $\mathbf{k} \neq \mathbf{0}$ we denote by $\mathfrak{Q}_{\mathbf{k}}$ the law of such a diffusion starting at $\mathbf{k}$, which is supported in $\mathcal{C}_1(k)$, $k = |\mathbf{k}|$.

Remark 4.2 The matrix $\mathbf{D} := [D_{p,q}]$ is degenerate since $\mathbf{D}(\hat{\mathbf{k}})\mathbf{k} = \mathbf{0}$ for all $\mathbf{k} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$. It can be shown however that under fairly general assumptions its rank equals $d-1$.

Proposition 4.3 *Suppose that $\hat{R}(\mathbf{0}) > 0$. Then, the rank of $\mathbf{D}$ equals $d-1$.*

Proof. Suppose that $c_0 = 1$ and let $H_{\mathbf{k}} := [\mathbf{p} \in \mathbb{R}^d : \mathbf{p} \cdot \hat{\mathbf{k}} = 0]$ be the hyperplane orthogonal to $\mathbf{k}$. Then,

$$D_{ml}(\hat{\mathbf{k}}) = -\frac{1}{2} \int_{-\infty}^{\infty} \partial^2_{x_m, x_l} R(s\hat{\mathbf{k}})\, ds = \frac{1}{2} \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}^d} e^{is\hat{\mathbf{k}} \cdot \mathbf{p}} p_m p_l \hat{R}(\mathbf{p}) d\mathbf{p} \right) ds$$
$$= \frac{1}{2^d \pi^{d-1}} \int_{H_{\mathbf{k}}} p_m p_l \hat{R}(\mathbf{p}) d\mathbf{p}$$

and hence for any $\boldsymbol{\xi} \in \mathbb{R}^d$ we have

$$(\mathbf{D}(\hat{\mathbf{k}})\boldsymbol{\xi}, \boldsymbol{\xi}) = D_{ml}(\hat{\mathbf{k}}) \xi_m \xi_l = \frac{1}{2^d \pi^{d-1}} \int_{H_{\mathbf{k}}} (\mathbf{p} \cdot \boldsymbol{\xi})^2 \hat{R}(\mathbf{p}) d\mathbf{p}. \tag{4.3}$$

Suppose that $\boldsymbol{\xi} \in H_{\mathbf{k}}$. Then, since $\hat{R}(\mathbf{p}) \geq 0$ the left hand side of (4.3) is nonnegative. We claim that in fact $(\mathbf{D}(\hat{\mathbf{k}})\boldsymbol{\xi}, \boldsymbol{\xi}) > 0$. Indeed, if otherwise then, since $\hat{R}$ is continuous, we would have $\hat{R}(\mathbf{p})(\mathbf{p} \cdot \boldsymbol{\xi})^2 = 0$ for all $\mathbf{p} \in H_{\mathbf{k}}$, which is impossible due to the fact that $\hat{R}(\mathbf{0}) > 0$ and the set $H_{\boldsymbol{\xi}} \cap H_{\mathbf{k}}$ has the linear dimension $d-2$. □

The above argument shows that $\mathbf{D}(\hat{\mathbf{k}})$ is of rank $d-1$ if there exists $\mathbf{p}_0 \in H_{\mathbf{k}}$ such that $\hat{R}(\mathbf{p}_0) > 0$. On the other hand, if $\hat{R}(\mathbf{p}) = 0$ for all $\mathbf{p}$ in the plane $H_{\mathbf{k}}$ then $\mathbf{D}(\hat{\mathbf{k}})\boldsymbol{\xi} = 0$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$. Therefore the matrix $\mathbf{D}(\hat{\mathbf{k}})$ either has rank $d-1$, or vanishes identically. Another condition ensuring the latter does not happen is the radial symmetry of $R(\cdot)$.

4.2 Two particle model

We would like to show that solution $u^\delta(t, \mathbf{x}, \mathbf{k})$ of (1.9) decorrelates in the limit $\delta \to 0$ at two different points, that is, that

$$\mathbb{E}\left\{ u^\delta(t, \mathbf{x}_1, \mathbf{k}_1) u^\delta(t, \mathbf{x}_2, \mathbf{k}_2) \right\} - \mathbb{E}\left\{ u^\delta(t, \mathbf{x}_1, \mathbf{k}_1) \right\} \mathbb{E}\left\{ u^\delta(t, \mathbf{x}_2, \mathbf{k}_2) \right\} \to 0 \text{ as } \delta \to 0 \tag{4.4}$$

provided that $\mathbf{k}_1 \neq \mathbf{k}_2$. Recall that $u^\delta(t,\mathbf{x},\mathbf{k})$ may be represented as

$$u_q^\delta(T,\mathbf{x},\mathbf{k}) = u_q^0(\mathbf{X}^\delta(T,\mathbf{x},\mathbf{k}), -\mathbf{K}^\delta(T,\mathbf{x},\mathbf{k})),$$

where u_q^0 is the initial data for (1.9), and

$$\begin{aligned} \frac{d\mathbf{X}^\delta(t)}{dt} &= \frac{\partial\lambda_q^\delta}{\partial\mathbf{k}}(\mathbf{X}^\delta(t),\mathbf{K}^\delta(t)), \qquad \mathbf{X}^\delta(0) = \mathbf{x} \\ \frac{d\mathbf{K}^\delta(t)}{dt} &= -\frac{\partial\lambda_q^\delta}{\partial\mathbf{x}}(\mathbf{X}^\delta(t),\mathbf{K}^\delta(t)), \qquad \mathbf{K}^\delta(0) = -\mathbf{k}. \end{aligned} \tag{4.5}$$

In order to establish (4.4) we have to consider motion of two particles that may start at the same physical point but are moving in different directions. The equations of motion for two particles are governed by the Hamiltonian system

$$\left\{ \begin{array}{l} \frac{d\boldsymbol{x}_m^{(\delta)}(t;\mathbf{x}_m,\mathbf{k}_m)}{dt} = \nabla_{\mathbf{k}}\lambda_q^\delta\left(\boldsymbol{x}_m^{(\delta)}(t;\mathbf{x}_m,\mathbf{k}_m), \boldsymbol{k}_m^{(\delta)}(t;\mathbf{x}_m,\mathbf{k}_m)\right) \\ \frac{d\boldsymbol{k}_m^{(\delta)}(t;\mathbf{x}_m,\mathbf{k}_m)}{dt} = -\nabla_{\mathbf{x}}\lambda_q^\delta\left(\boldsymbol{x}_m^{(\delta)}(t;\mathbf{x}_m,\mathbf{k}_m), \boldsymbol{k}_m^{(\delta)}(t;\mathbf{x}_m,\mathbf{k}_m)\right) \\ \boldsymbol{x}_m^{(\delta)}(0;\mathbf{x}_m,\mathbf{k}_m) = \mathbf{x}_m, \quad \boldsymbol{k}_m^{(\delta)}(0;\mathbf{x}_m,\mathbf{k}_m) = \mathbf{k}_m, \quad m = 1,2. \end{array} \right. \tag{4.6}$$

We will assume that $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{0}$, and

$$\mathbf{k}_1 \neq \mathbf{0}, \quad \mathbf{k}_2 \neq \mathbf{0} \text{ and } \hat{\mathbf{k}}_1 \neq \hat{\mathbf{k}}_2. \tag{4.7}$$

The above system can be rewritten in the form

$$\left\{ \begin{array}{l} \frac{d\boldsymbol{x}_m^{(\delta)}(t;\mathbf{x}_m,\mathbf{k}_m)}{dt} = \left[c_0 + \sqrt{\delta}c_1\left(\frac{\boldsymbol{x}_m^{(\delta)}(t;\mathbf{x}_m,\mathbf{k}_m)}{\delta}\right)\right]\hat{\boldsymbol{k}}_m^{(\delta)}(t;\mathbf{x}_m,\mathbf{k}_m) \\ \frac{d\boldsymbol{k}_m^{(\delta)}(t;\mathbf{x}_m,\mathbf{k}_m)}{dt} = -\frac{1}{\sqrt{\delta}}\nabla_{\mathbf{x}}c_1\left(\frac{\boldsymbol{x}_m^{(\delta)}(t;\mathbf{x}_m,\mathbf{k}_m)}{\delta}\right)|\boldsymbol{k}_m^{(\delta)}(t;\mathbf{x}_m,\mathbf{k}_m)| \\ \boldsymbol{x}_m^{(\delta)}(0;\mathbf{x}_m,\mathbf{k}_m) = \mathbf{0}, \quad \boldsymbol{k}_m^{(\delta)}(0;\mathbf{x}_m,\mathbf{k}_m) = \mathbf{k}_m, \quad m = 1,2. \end{array} \right. \tag{4.8}$$

The main result of this section is the following.

Theorem 4.4 *Suppose that the random field $c_1(\cdot)$ satisfies the assumptions in Section 1.2 and that $d \geq 3$. Then, the laws of processes $(\boldsymbol{k}_1^{(\delta)}(\cdot), \boldsymbol{x}_1^{(\delta)}(\cdot), \boldsymbol{k}_2^{(\delta)}(\cdot), \boldsymbol{x}_2^{(\delta)}(\cdot))$ determined by (4.6), converge weakly in $\mathcal{C}_4$, as $\delta \to 0$, to the law of $(\boldsymbol{k}_1(\cdot), \boldsymbol{x}_1(\cdot), \boldsymbol{k}_2(\cdot), \boldsymbol{x}_2(\cdot))$, where $\boldsymbol{k}_j(\cdot)$, $j = 1,2$ are independent symmetric diffusions given by (4.2) starting at $\mathbf{k}_j$, $j = 1,2$ respectively and*

$$\boldsymbol{x}_j(t) = -c_0\int_0^t \hat{\boldsymbol{k}}_j(s)ds, \quad j = 1,2.$$

Theorem 1.1 is a simple corollary of Theorems 3.2 and 4.4.
Proof of Theorem 1.1. First we observe that

$$\int\left|\int(W_\varepsilon^\delta(t,\mathbf{x},\mathbf{k}) - U^\delta(t,\mathbf{x},\mathbf{k})S(\mathbf{k})d\mathbf{k}\right|^2 d\mathbf{x} \leq \|S\|_{L^2}^2\int|W_\varepsilon^\delta(t,\mathbf{x},\mathbf{k}) - U^\delta(t,\mathbf{x},\mathbf{k})|^2 d\mathbf{k}d\mathbf{x} \to 0$$

as $(\varepsilon,\delta) \to 0$ in $\mathcal{K}_\mu$ and this convergence is uniform in realizations of the random medium provided that the bounds (1.12) are satisfied. Therefore it suffices to study $\tilde{s}^\delta(\mathbf{x}) = \int U^\delta(t,\mathbf{x},\mathbf{k})S(\mathbf{k})d\mathbf{k}$. We observe that

$$\mathbb{E}\left\{\int \|\tilde{s}^\delta(\mathbf{x})-\bar{s}(\mathbf{x})\|^2 d\mathbf{x}\right\} = \mathbb{E}\left\{\int \left\| \int (U^\delta(t,\mathbf{x},\mathbf{k})-\bar{W}(t,\mathbf{x},\mathbf{k}))S(\mathbf{k})d\mathbf{k}\right\|^2 d\mathbf{x}\right\}$$

$$= \mathbb{E}\left\{S^*(\mathbf{k})_1 \int (U^{\delta *}(t,\mathbf{x},\mathbf{k}_1)-\bar{W}^*(t,\mathbf{x},\mathbf{k}_1))(U^\delta(t,\mathbf{x},\mathbf{k}_2)-\bar{W}(t,\mathbf{x},\mathbf{k}_2))S(\mathbf{k}_2)d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{x}\right\}$$

with $\bar{s}(\mathbf{x})$ and $\bar{W}(t,\mathbf{x},\mathbf{k})$ as in the formulation of Theorem 1.1. Theorem 4.4 implies that

$$\mathbb{E}\left\{U^\delta(t,\mathbf{x},\mathbf{k})\right\} \to \bar{W}(t,\mathbf{x},\mathbf{k}), \quad \mathbb{E}\left\{U^\delta(t,\mathbf{x},\mathbf{k}_1)U^\delta(t,\mathbf{x},\mathbf{k}_2)\right\} \to \bar{W}(t,\mathbf{x},\mathbf{k}_1)\bar{W}(t,\mathbf{x},\mathbf{k}_2)$$

pointwise in $\mathbf{x}$ and $\mathbf{k}$. Recall that the functions $U^\delta(t,\mathbf{x},\mathbf{k})$ and $\bar{W}(t,\mathbf{x},\mathbf{k})$ are uniformly compactly supported and bounded in L^∞. Therefore the Lebesgue dominated convergence implies that

$$\mathbb{E}\left\{\int \|\tilde{s}^\delta(\mathbf{x})-\bar{s}(\mathbf{x})\|^2 d\mathbf{x}\right\} \to 0$$

and the proof of Theorem 1.1 is complete. □

5 Proof of Theorem 4.4

Before we present the proof of this result we wish to spend a few words to lay out its main ideas. They are based in large part on the ideas of [15] where the phase space diffusion equation for the limit of the expectation of the solution of the Liouville equation with the Hamiltonian $H^\delta(\mathbf{x},\mathbf{k}) = k^2/2+\sqrt{\delta}V(\mathbf{x}/\delta)$ has been obtained. The two-particle case introduces some additional difficulties into the problem. Our first step in the proof, in Section 5.1 below, is to replace the processes $(\boldsymbol{k}_1^\delta(\cdot),\boldsymbol{k}_2^\delta(\cdot))$ by $(\boldsymbol{l}_1^\delta(\cdot),\boldsymbol{l}_2^\delta(\cdot))$ that agree with $(\boldsymbol{k}_1^\delta(\cdot),\boldsymbol{k}_2^\delta(\cdot))$ up to certain stopping times. These times are determined by the stopping rules, introduced by multiplying the Hamiltonian $\lambda^\delta(\mathbf{x},\mathbf{k})$ by several cut-off functions. Their role is to prevent the trajectory of each particle to self-intersect and also not to allow the particles to get too close to each other. We shall prove tightness of such modified processes by showing that for any bounded, positive and continuous function F one can find a constant $C>0$ such that $F(\boldsymbol{l}_1^\delta(t),\boldsymbol{l}_2^\delta(t))+Ct$, $t\ge 0$ are sub-martingales (see e.g. [23] Theorem 1.4.6), cf (5.29). This fact will be established thanks to the decorrelation properties of the random field $\nabla_{\mathbf{x}} c_1(\cdot)$. More precisely, the latter imply mixing lemmas contained in Section 5.2. The second ingredient of the proof is a perturbative argument that allows us to replace the trajectory $\boldsymbol{x}_i^{(\delta)}(\cdot)$ (in fact its modification $\boldsymbol{y}_i^{(\delta)}(\cdot)$ that arises from the replacement of $\boldsymbol{k}^\delta$ by $\boldsymbol{l}^\delta$) by a linear approximation over the time interval that is much longer than the correlation time (that we recall is of order $O(\delta)$) yet is sufficiently short so we can control the accuracy of the approximation, cf. Lemma 5.4. In order to ensure that the approximate motion (under linear approximation) is not transverse to the direction of the field at a given time, which could prevent us from using the decorrelation properties of the field, but is rather propelled forward, we have to introduce another stopping time rule, cf. the condition on the scalar product of wave number directions contained in (5.5).

Conducting the proof of tightness we also identify a certain martingale property of any limiting law of $(\boldsymbol{l}_1^\delta(\cdot),\boldsymbol{l}_2^\delta(\cdot))$, as $\delta\to 0$ that holds up to the aforementioned stopping time. By proving that this time goes to infinity with the removal of the cut-offs we are able to prove both the weak convergence of the laws of $(\boldsymbol{k}_1^\delta(\cdot),\boldsymbol{k}_2^\delta(\cdot))$ and identify a well-posed martingale problem associated with the limiting measure. This step is done in Section 5.4.

With no loss of generality we shall assume throughout this section that $c_0=1$.

5.1 The cut-off functions

Let $p, q > 0$ and $k \geq 0$ be integers. Let M be chosen in such a way that

$$M \geq |\mathbf{k}_1| \vee |\mathbf{k}_2| \quad \text{and} \quad |\mathbf{k}_1| \wedge |\mathbf{k}_2| \geq M^{-1}. \tag{5.1}$$

Let $\hat{\mathbf{k}}_1 \neq \hat{\mathbf{k}}_2$ be such as in the statement of Theorem 4.4. Denote

$$\mathbf{K}_N := \left\{ (\hat{\mathbf{k}}, \hat{\mathbf{k}}') : \ (\hat{\mathbf{k}}, \hat{\mathbf{k}}_1)_{\mathbb{R}^d} \geq 1 - \frac{1}{N+1}, \ (\hat{\mathbf{k}}', \hat{\mathbf{k}}_2)_{\mathbb{R}^d} \geq 1 - \frac{1}{N+1} \right\} \tag{5.2}$$

and choose N a positive integer such that

$$\gamma_N := \inf \left\{ |\hat{\mathbf{k}} - \hat{\mathbf{k}}'| : (\hat{\mathbf{k}}, \hat{\mathbf{k}}') \in \mathbf{K}_N \right\} > 0, \tag{5.3}$$

that is, the cones of aperture $1/(N+1)$ centered at $\hat{\mathbf{k}}_1$ and $\mathbf{k}_2$ are separated. As a consequence of (5.3) we may choose a positive integer q so that

$$\lambda_N(p) := \inf \left\{ \left| \frac{1}{p}\hat{\mathbf{k}} - \rho \hat{\mathbf{k}}' \right| \wedge \left| \frac{1}{p}\hat{\mathbf{k}}' - \rho \hat{\mathbf{k}} \right| : \rho \in \left[0, \frac{1}{p}\right], \quad (\hat{\mathbf{k}}, \hat{\mathbf{k}}') \in \mathbf{K}_N \right\} \geq \frac{4}{q}. \tag{5.4}$$

We define now several auxiliary functions that will be used to introduce the cut-offs in the dynamics. The function $\psi : \mathbb{R}^d \times (S_1^{d-1})^2 \to [0,1]$ is C^∞ and has the property that

$$\psi(\mathbf{k}, \mathbf{l}_1, \mathbf{l}_2) = \begin{cases} 1, & \text{if } \ \hat{\mathbf{k}} \cdot \mathbf{l}_1 \geq 1 - \frac{1}{N+1} \ \text{ and } \ \hat{\mathbf{k}} \cdot \mathbf{l}_2 \geq 1 - \frac{1}{N+1} \\ & \quad \text{and} \quad M^{-1} \leq |\mathbf{k}| \leq M \\ 0, & \text{if } \ \hat{\mathbf{k}} \cdot \mathbf{l}_1 \leq 1 - \frac{2}{N+1} \ \text{ or } \ \hat{\mathbf{k}} \cdot \mathbf{l}_2 \leq 1 - \frac{2}{N+1} \\ & \quad \text{or} \quad |\mathbf{k}| \leq (2M)^{-1} \quad \text{or} \quad |\mathbf{k}| \geq 2M. \end{cases} \tag{5.5}$$

The function $\phi_k : \mathbb{R}^d \times \mathcal{C}_1 \to [0,1]$ is C^∞ for a fixed path $K(t)$ and satisfies

$$\phi_k(\mathbf{y}; K) = \begin{cases} 1, & \text{if} \quad \inf\limits_{0 \leq t \leq t_{k-1}^{(p)}} \left| \mathbf{y} - \int_0^t K(s) ds \right| \geq \frac{2}{q} \\ 0, & \text{if} \quad \inf\limits_{0 \leq t \leq t_{k-1}^{(p)}} \left| \mathbf{y} - \int_0^t K(s) ds \right| \leq \frac{1}{q}. \end{cases} \tag{5.6}$$

Here $t_k^{(p)} := kp^{-1}$ and, by convention, $K(s) := K(0)$, $s \leq 0$. The function $\xi_k : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{C}_2 \to [0,1]$ is smooth when the paths $K_1(\cdot), K_2(\cdot) \in \mathcal{C}_1$ are fixed. We let

$$p_1 := 2^q [8(1 + D_0)] p \tag{5.7}$$

and $s_k^{(p_1)} := kp_1^{-1}$ be a sub-partition of t_k, and define

$$\xi_k(\mathbf{y}_1, \mathbf{y}_2; K_1(\cdot), K_2(\cdot)) = \begin{cases} 1, & \text{if} \quad \inf\limits_{0 \leq t \leq s_k^{(p_1)}} \left| \mathbf{y}_1 - \int_0^t K_2(s) ds \right| \geq \frac{2}{q} \\ & \text{and} \quad \inf\limits_{0 \leq t \leq s_k^{(p_1)}} \left| \mathbf{y}_2 - \int_0^t K_1(s) ds \right| \geq \frac{2}{q} \\ 0, & \text{if} \quad \inf\limits_{0 \leq t \leq s_k^{(p_1)}} \left| \mathbf{y}_1 - \int_0^t K_2(s) ds \right| \leq \frac{1}{q} \\ & \text{or} \quad \inf\limits_{0 \leq t \leq s_k^{(p_1)}} \left| \mathbf{y}_2 - \int_0^t K_1(s) ds \right| \leq \frac{1}{q}. \end{cases} \tag{5.8}$$

For $j = 1,2$ we set

$$\Phi_j(t, \mathbf{y}; K(\cdot)) := \begin{cases} 1, & \text{if } 0 \le t < t_1^{(p)} \\ \phi_k(\mathbf{y}; K(\cdot)), & \text{if } t_k^{(p)} \le t < t_{k+1}^{(p)}. \end{cases} \tag{5.9}$$

Each $\Phi_j(\cdot)$ shall be used to modify the dynamics of the corresponding particle in order to avoid a possibility of self-intersections of its trajectory. The cut-off function

$$\Psi(t, \mathbf{k}; K(\cdot)) := \begin{cases} \psi\left(\mathbf{k}, \hat{K}\left(t_{k-1}^{(p)}\right), \hat{K}\left(t_k^{(p)}\right)\right) & \text{for } t \in [t_k^{(p)}, t_{k+1}^{(p)}) \text{ and } k \ge 1 \\ \psi(\mathbf{k}, \hat{K}(0), \hat{K}(0)) & \text{for } t \in [0, t_1^{(p)}) \end{cases} \tag{5.10}$$

will allow us to control the direction of the particle motion over each interval of the partition as well as not to allow the trajectory to escape to the regions where the change of velocity can be uncontrollable. The cut-off

$$\Xi(t, \mathbf{y}_1, \mathbf{y}_2; K_1(\cdot), K_2(\cdot)) = \begin{cases} 1, & \text{if } 0 \le t < t_1^{(p)} \\ \xi_k(\mathbf{y}_1, \mathbf{y}_2; K_1(\cdot), K_2(\cdot)), & \text{if } s_k^{(p_1)} \le t < s_{k+1}^{(p_1)} \text{ and } t_1^{(p)} \le s_k^{(p_1)} \end{cases} \tag{5.11}$$

is introduced in order not to allow the two trajectories to come too close to each other. Note that this cut-off is "switched on" only after time $t = t_1^{(p)}$ so as to allow the two particles to separate initially. After this time it is updated every $1/p_1$ time step, that is, more frequently that the cut-offs that control the self-intersections of each trajectory that are updated only at each $1/p$ time step.

The following lemma can be checked by a direct calculation. Both here and in what follows we denote by $D_{\bullet,\beta}$ the partial with respect to the β component of the given vector variable.

Lemma 5.1 *Let $\mathbf{m} = (m_1, \cdots, m_d)$ be a multi-index with nonnegative integer valued components, $m = \sum_{p=1}^{d} m_p$. There exist constants C_3, $C_4 > 0$ depending only on M, N, p, q, m such that*

$$|D_{\mathbf{y}}^{\mathbf{m}} \Phi_j(t, \mathbf{y})| \le C_3, \quad |D_{\mathbf{y}_j}^{\mathbf{m}} \Xi(t, \mathbf{y}_1, \mathbf{y}_2)| \le C_4, \quad j = 1,2.$$

Let $K = (K_1, K_2) \in \mathcal{C}_2$ and denote

$$\Theta_j(s, \boldsymbol{y}_1, \boldsymbol{y}_2, \mathbf{l}; K) := \Psi(s, \mathbf{l}; K_j) \Phi_j\left(s, \boldsymbol{y}_j; K_j\right) \Xi\left(s, \boldsymbol{y}_1, \boldsymbol{y}_2; K\right), \tag{5.12}$$

$$\Lambda_j(s, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_1', \mathbf{y}_2', \mathbf{l}; K) := \Theta_j(s, \mathbf{y}_1, \mathbf{y}_2, \mathbf{l}; K) \Theta_j(s, \mathbf{y}_1', \mathbf{y}_2', \mathbf{l}; K). \tag{5.13}$$

We also introduce a random transformation of paths $\tilde{K}(\cdot) = (\tilde{K}_1(\cdot), \tilde{K}_2(\cdot))$ for any $K \in \mathcal{C}_2$ given by

$$\tilde{K}_j(t) = \left[1 + \sqrt{\delta} c_1\left(\frac{K_j(t)}{\delta}\right)\right] \hat{K}_j(t), \quad t \ge 0. \tag{5.14}$$

Finally, let us set

$$F_j(t, \mathbf{y}_1, \mathbf{y}_2, \mathbf{l}; K) = \Theta_j(t, \delta\mathbf{y}_1, \delta\mathbf{y}_2, \mathbf{l}; K) \nabla_{\mathbf{y}_j} c_1(\mathbf{y}_j) |\mathbf{l}|, \quad j = 1,2. \tag{5.15}$$

The modified two particle system with the cut-offs that we will consider is given by

$$\begin{cases} \frac{d\boldsymbol{y}_j^{(\delta)}(t)}{dt} = \left[1 + \sqrt{\delta} c_1\left(\frac{\boldsymbol{y}_j^{(\delta)}(t; \mathbf{x}_j, \mathbf{k}_j)}{\delta}\right)\right] \hat{\boldsymbol{l}}_j^{(\delta)}(t; \mathbf{x}_j, \mathbf{k}_j) \\ \frac{d\boldsymbol{l}_j^{(\delta)}(t)}{dt} = -\frac{1}{\sqrt{\delta}} F_j\left(t, \frac{\boldsymbol{y}_1^{(\delta)}(t)}{\delta}, \frac{\boldsymbol{y}_2^{(\delta)}(t)}{\delta}, \boldsymbol{l}_j^{(\delta)}(t); \tilde{\boldsymbol{l}}^{(\delta)}(\cdot)\right) \\ \boldsymbol{y}_j^{(\delta)}(0) = \mathbf{0}, \quad \boldsymbol{l}_j^{(\delta)}(0) = \mathbf{k}_j, \quad j = 1,2, \end{cases} \tag{5.16}$$

where the path $\tilde{\boldsymbol{l}}^{(\delta)}(\cdot) = (\tilde{\boldsymbol{l}}_1^{(\delta)}(\cdot), \tilde{\boldsymbol{l}}_2^{(\delta)}(\cdot))$ is obtained from $\boldsymbol{l}(\cdot)$ by the transformation (5.14). We will denote by $Q^\delta(\cdot; M, N, p, q)$ the law of $(\boldsymbol{l}_1^{(\delta)}(\cdot), \boldsymbol{y}_1^{(\delta)}(\cdot), \boldsymbol{l}_2^{(\delta)}(\cdot), \boldsymbol{y}_2^{(\delta)}(\cdot))$ for a given $\delta > 0$ over $\mathcal{C}_4$.

5.2 The Mixing Lemmas

For any $t \geq 0$ we denote by $\mathcal{F}_t$ the σ-algebra generated by $(\boldsymbol{l}_1^{(\delta)}(s), \boldsymbol{y}_1^{(\delta)}(s), \boldsymbol{l}_2^{(\delta)}(s), \boldsymbol{y}_2^{(\delta)}(s))$, $s \leq t$. Throughout this section we assume that $X_1, X_2 : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d^2} \to \mathbb{R}$ are certain continuous functions, Z is a random variable and g_1, g_2 are $\mathbb{R}^d$-valued random vectors. We suppose further that Z, g_1, g_2, are $\mathcal{F}_t$-measurable, while X_1, X_2 are random fields of the form $X_i(\mathbf{x}) = X_i\left(c_1(\mathbf{x}), \nabla_{\mathbf{x}} c_1(\mathbf{x}), \nabla^2_{\mathbf{x}} c_1(\mathbf{x})\right)$, satisfy $\lim_{|\mathbf{x}|\to 0} \|X_i(\mathbf{x}) - X_i(\mathbf{0})\|_\infty = 0, \quad i = 1, 2$. We also let

$$U(\theta_1, \theta_2) := \mathbb{E}[X_1(\theta_1) X_2(\theta_2)], \quad (\theta_1, \theta_2) \in (\mathbb{R}^d)^2. \tag{5.17}$$

The following mixing lemmas will be of crucial importance for us in the sequel.

Lemma 5.2 *Assume that $r, t \geq 0$ and*

$$\inf_{u \leq t} \left| g_i - \frac{\boldsymbol{y}_j^{(\delta)}(u)}{\delta} \right| \geq \frac{r}{\delta} \tag{5.18}$$

$\mathbb{P}$–a.s. on the set $Z \neq 0$ for $i, j = 1, 2$. Then, we have

$$|\mathbb{E}[X_1(g_1) X_2(g_2) Z] - \mathbb{E}[U(g_1, g_2) Z]| \leq 2\phi\left(\frac{r}{2\delta}\right) \|X_1\|_\infty \|X_2\|_\infty \|Z\|_1. \tag{5.19}$$

Proof. The proof is a modification of the proof of Lemma 2 of [15] so we only highlight its main points. Choose an arbitrary $\eta > 0$. By a suitable modification of g_1, g_2 on the event $Z = 0$, so that the modified r.v. remain $\mathcal{F}_t$–measurable, we can guarantee that (5.18) holds $\mathbb{P}$–a.s. Let

$$\mathbf{i} = (i_1, \cdots, i_d) \in \mathbb{Z}^d \text{ and } C_{\mathbf{i}} := [i_1/2^{M_1}, (i_1+1)/2^{M_1}) \times \cdots \times [i_d/2^{M_1}, (i_d+1)/2^{M_1})$$

and

$$c_{\mathbf{i}} := ((2i_1+1)/2^{M_1+1}, \cdots, (2i_d+1)/2^{M_1+1}).$$

Here $M_1 > 0$ is a sufficiently large integer so that

$$\|X_i(\mathbf{x}) - X_i(c_{\mathbf{i}})\|_\infty \leq \eta, \quad \forall\, \mathbf{i} \in \mathbb{Z}^d,\ \mathbf{x} \in C_{\mathbf{i}},\ i = 1, 2 \tag{5.20}$$

and $2^{-M_1} < r/(20\delta)$. We let

$$D^0_{\mathbf{i},\mathbf{j}} := [\mathbf{z} :\ dist(\mathbf{z}, C_{\mathbf{i}} \cup C_{\mathbf{j}}) > 3r(4\delta)^{-1}]$$

$$D_{\mathbf{i},\mathbf{j}} := [\mathbf{z} :\ dist(\mathbf{z}, C_{\mathbf{i}} \cup C_{\mathbf{j}}) > r(2\delta)^{-1}]$$

and

$$\mathfrak{Y}_t^{(\delta)} := \left[\frac{1}{\delta}(\boldsymbol{y}_1^{(\delta)}(s), \boldsymbol{y}_2^{(\delta)}(s)) : s \leq t\right].$$

Let us denote by $I_{\mathbf{i},\mathbf{j}}$ the indicator of the event $[(g_1, g_2) \in C_{\mathbf{i}} \times C_{\mathbf{j}}]$ and the event $A_{\mathbf{i},\mathbf{j}} = [\omega : \mathfrak{Y}_t^{(\delta)}(\omega) \subseteq D^0_{\mathbf{i},\mathbf{j}}]$. Note that

$$\mathbb{E}[X_1(g_1) X_2(g_2) Z] = \sum_{\mathbf{i},\mathbf{j}} \mathbb{E}\left[X_1(g_1) X_2(g_2) Z I_{\mathbf{i},\mathbf{j}} \chi_{A_{\mathbf{i},\mathbf{j}}}\right]. \tag{5.21}$$

Using precisely the same argument as the one contained in p. 31 of [15] we prove that $Z I_{\mathbf{i},\mathbf{j}} \chi_{A_{\mathbf{i},\mathbf{j}}}$ is $\mathcal{C}(D_{\mathbf{i},\mathbf{j}})$–measurable for each $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$. Note however that the right hand side of (5.21) is equal, up to a term of order $O(\eta)$, to

$$\sum_{\mathbf{i},\mathbf{j}} \mathbb{E}\left[X_1(c_{\mathbf{i}}) X_2(c_{\mathbf{j}}) Z I_{\mathbf{i},\mathbf{j}} \chi_{A_{\mathbf{i},\mathbf{j}}}\right]. \tag{5.22}$$

The random variable $X_1(c_{\mathbf{i}})X_2(c_{\mathbf{j}})$ is however $\mathcal{C}(C_{\mathbf{i}}\cup C_{\mathbf{j}})$–measurable. Therefore we can write, see e.g. [7] p.171, that

$$\sum_{\mathbf{i},\mathbf{j}}\left|\mathbb{E}\left[X_1(c_{\mathbf{i}})X_2(c_{\mathbf{j}})ZI_{\mathbf{i},\mathbf{j}}\chi_{A_{\mathbf{i},\mathbf{j}}}\right]-U(c_{\mathbf{i}},c_{\mathbf{j}})\mathbb{E}\left[ZI_{\mathbf{i},\mathbf{j}}\chi_{A_{\mathbf{i},\mathbf{j}}}\right]\right| \tag{5.23}$$

$$\le\sum_{\mathbf{i},\mathbf{j}}\phi\left(\frac{r}{2\delta}\right)\left|\mathbb{E}\left[ZI_{\mathbf{i},\mathbf{j}}\chi_{A_{\mathbf{i},\mathbf{j}}}\right]\right|\|X_1\|_\infty\|X_2\|_\infty.$$

However, $U(c_{\mathbf{i}},c_{\mathbf{j}})$ equals, up to a term of order $O(\eta)$, to $U(g_1,g_2)$ on the event corresponding to $I_{\mathbf{i},\mathbf{j}}$. The conclusion of Lemma 5.2 follows upon the passage to the limit $M_1\to+\infty$ and $\eta\downarrow 0$. □

Lemma 5.3 *Assume that r,t are as in the previous lemma. Let $\mathbb{E}X_1=0$. Furthermore, we assume that g_2 satisfies* (5.18),

$$\inf_{u\le t}\left|g_1-\frac{\boldsymbol{y}_j^{(\delta)}(u)}{\delta}\right|\ge\frac{r+r_1}{\delta},\quad j=1,2 \tag{5.24}$$

and

$$|g_1-g_2|\ge\frac{r_1}{\delta}, \tag{5.25}$$

for some $r_1\ge 0$, $\mathbb{P}$-a.s. on the event $Z\neq 0$. Then we have

$$|\mathbb{E}[X_1(g_1)X_2(g_2)\,Z]-\mathbb{E}[U(g_1,g_2)Z]|\le C_5\phi^{1/2}\left(\frac{r}{2\delta}\right)\phi^{1/2}\left(\frac{r_1}{2\delta}\right)\|X_1\|_\infty\|X_2\|_\infty\|Z\|_1 \tag{5.26}$$

for some absolute constant $C_5>0$ Here the function U is given by (5.17).

Proof. We prove that the left hand side of (5.26) is bounded by

$$C_6\phi\left(\frac{r_1}{2\delta}\right)\|X_1\|_\infty\|X_2\|_\infty\|Z\|_1. \tag{5.27}$$

This together with the result of the previous lemma imply (5.26).

Let $\eta>0$ and M_1 be as in the proof of Lemma 5.2, and in addition $2^{-M_1}<r_1/(20\delta)$. Note that $X_2(c_{\mathbf{j}})ZI_{\mathbf{i},\mathbf{j}}\chi_{A_{\mathbf{i},\mathbf{j}}}$ (in the notation of the proof of Lemma 5.2) is $\mathcal{C}(D_{\mathbf{i},\mathbf{j}}\cup C_{\mathbf{j}})$–measurable. In addition, we have $dist(C_{\mathbf{i}},D_{\mathbf{i},\mathbf{j}}\cup C_{\mathbf{j}})>r_1(2\delta)^{-1}$ thus, using the mixing coefficient as in e.g. [7] p.171 we can estimate

$$\sum_{\mathbf{i},\mathbf{j}}\left|\mathbb{E}\left[X_1(c_{\mathbf{i}})X_2(c_{\mathbf{j}})ZI_{\mathbf{i},\mathbf{j}}\chi_{A_{\mathbf{i},\mathbf{j}}}\right]\right|\le 2\phi\left(\frac{r_1}{2\delta}\right)\|X_1\|_\infty\|X_2\|_\infty\|Z\|_1.$$

On the other hand, we have $I_{\mathbf{i},\mathbf{j}}\neq 0$ only if $|c_{\mathbf{i}}-c_{\mathbf{j}}|\ge r_1(2\delta)^{-1}$, which in turn implies that

$$|U(c_{\mathbf{i}},c_{\mathbf{j}})|\le C_7\phi\left(\frac{r_1}{2\delta}\right)\|X_1\|_\infty\|X_2\|_\infty,$$

with the constant C_7 independent of $\eta>0$.Summarizing, we have shown that

$$\sum_{\mathbf{i},\mathbf{j}}\left|U(c_{\mathbf{i}},c_{\mathbf{j}})\mathbb{E}\left[ZI_{\mathbf{i},\mathbf{j}}\chi_{A_{\mathbf{i},\mathbf{j}}}\right]\right|\le C_8\phi\left(\frac{r_1}{2\delta}\right)\|X_1\|_\infty\|X_2\|_\infty\|Z\|_1,$$

with the constant C_8 independent of $\eta>0$. Letting $\eta\to 0$ and using (5.20) we conclude (5.26). □

5.3 Tightness and the martingale property of limiting measures

In this section we prove tightness of the family $Q^\delta(\cdot;M,N,p,q)$, $\delta\in(0,1]$ and show that any weak limit point $Q(\cdot;M,N,p,q)$ of this family as $\delta\to 0$, has a certain martingale property.

Let $\mathcal{L}^{M,N,p,q}$ be a random partial differential operator defined on $C_0^\infty((\mathbb{R}^d)^2)$ as follows. For any $K=(K_1,K_2)\in\mathcal{C}_2$ and $G\in C_0^\infty((\mathbb{R}^d)^2)$ we set $Y=(Y_1,Y_2)\in\mathcal{C}_2$,

$$Y_i(t)=\int\limits_0^t K_i(s)ds,\quad i=1,2, \tag{5.28}$$

$$\Theta_i(t;K):=\Phi_{i,*}(t;K_i)\Psi_*(t;K_i)\Xi_*(t;K),$$

where

$$\Phi_{i,*}(t;K_i):=\Phi_i\left(t,Y_i(t);K_1\right),\quad \Psi_*(t;K_i):=\Psi(t,K_i(t);K_i),\quad \Xi_*(t;K):=\Xi\left(t,Y_1(t),Y_2(t);K\right).$$

We let

$$(\mathcal{L}^{M,N,p,q}G)(\mathbf{k}_1,\mathbf{k}_2;K):=\Theta_1^2(t;K)\mathcal{L}_{\mathbf{k}_1}G(\mathbf{k}_1,\mathbf{k}_2)+\Theta_2^2(t;K)\mathcal{L}_{\mathbf{k}_2}G(\mathbf{k}_1,\mathbf{k}_2),$$

with $\mathcal{L}_{\mathbf{k}_i}$, $i=1,2$ given by (4.1).

Let $\zeta\in C_b((\mathbb{R}^d)^{2n})$ be an arbitrary nonnegative function, let $0\le t_1<\cdots<t_n\le t<u$ and define $\zeta(K):=\zeta(K(t_1),\cdots,K(t_n))$. We will show that for any function $G\in C_0^\infty((\mathbb{R}^d)^2)$ there exists a deterministic constant $C_9>0$ such that

$$\begin{aligned}&\left|\mathbb{E}\left\{\left[G(\boldsymbol{l}_1^{(\delta)}(u),\boldsymbol{l}_2^{(\delta)}(u))-G(\boldsymbol{l}_1^{(\delta)}(t),\boldsymbol{l}_2^{(\delta)}(t))\right]\zeta(\boldsymbol{l}_1^{(\delta)}(\cdot),\boldsymbol{l}_2^{(\delta)}(\cdot))\right\}\right|\\ &\le C_9(u-t)\mathbb{E}[\zeta(\boldsymbol{l}_1^{(\delta)}(\cdot),\boldsymbol{l}_2^{(\delta)}(\cdot))],\quad \forall\zeta(\cdot),\delta\in(0,1].\end{aligned} \tag{5.29}$$

The choice of the constant C_9 may depend on a particular function G but should be the same for all the spatial translates of G, and may not depend on the test function ζ. This, according to Theorem 1.4.6 of [23], implies tightness of the laws of $(\boldsymbol{l}_1^{(\delta)}(\cdot),\boldsymbol{l}_2^{(\delta)}(\cdot))$, $\delta\in(0,1]$ over $\mathcal{C}_2$.

Additionally, we prove that if $Q(\cdot;M,N,p,q)$ is any limiting law of $Q^{\delta_n}(\cdot;M,N,p,q)$, as $\delta_n\to 0$ then

$$\begin{aligned}&\lim_{n\to+\infty}\mathbb{E}\left\{\left[G(\boldsymbol{l}_1^{(\delta_n)}(u),\boldsymbol{l}_2^{(\delta_n)}(u))-G(\boldsymbol{l}_1^{(\delta_n)}(t),\boldsymbol{l}_2^{(\delta_n)}(t))\right]\zeta(\boldsymbol{l}_1^{(\delta_n)}(\cdot),\boldsymbol{l}_2^{(\delta_n)}(\cdot))\right\}\\ &=\int\left\{\left[\int\limits_t^u(\mathcal{L}^{M,N,p,q}G)(K(s);K)ds\right]\zeta(K)\right\}Q(dK;M,N,p,q)\end{aligned} \tag{5.30}$$

for any $u>t$. This property will be used in the next section to identify the limiting law of $(\boldsymbol{k}_1^{(\delta)}(\cdot),\boldsymbol{k}_2^{(\delta)}(\cdot))$, as $\delta\to 0$.

Throughout the remainder of this section we suppress writing both the superscript δ and the cut-off parameters M,N,p,q of the respective measures. With no loss of generality we assume that there exists k_1 such that $s_{k_1}^{(p_1)}\le t<u\le s_{k_1+1}^{(p_1)}$, cf. (5.7). Given $s\ge\sigma>0$, we define the linear approximation

$$\boldsymbol{L}_j(\sigma,s):=\boldsymbol{y}_j(\sigma)+(s-\sigma)\hat{\boldsymbol{l}}_j(\sigma),$$

and

$$\boldsymbol{R}_j(v,\sigma,s):=(1-v)\boldsymbol{L}_j(\sigma,s)+v\boldsymbol{y}_j(s),\quad j=1,2.$$

The following simple lemma can be verified by a direct calculation.

Lemma 5.4 *Suppose that $s \geq \sigma$. Then,*

$$|\boldsymbol{y}_j(s) - L_j(\sigma, s)| \leq \frac{D_1(s-\sigma)^2}{2\sqrt{\delta}} + D_0\sqrt{\delta}(s-\sigma), \quad \forall\, \delta > 0,\ j = 1, 2.$$

Remark 5.5 Throughout this argument we use

$$\sigma(s) := \max[t, s - \delta^{1-\gamma_1}] \text{ for some } \gamma_1 \in (0, 1/8). \tag{5.31}$$

The above lemma proves that for this choice of σ the linear approximation $L_j(\sigma, s)$ of the particle position given by $\boldsymbol{y}_j(s)$ is exact, up to a term of order

$$O(\delta^{3/2-2\gamma_1}). \tag{5.32}$$

We begin now the proof of (5.29). Our strategy is based on the perturbation method: the trajectory is approximated by the iterated linear approximation sufficiently many times so that the error becomes deterministically small. The terms that involve the linear approximation are potentially large but are handled with the help of the mixing lemmas. Note that

$$\begin{aligned} &G(\boldsymbol{l}_1^{(\delta)}(u), \boldsymbol{l}_2^{(\delta)}(u)) - G(\boldsymbol{l}_1^{(\delta)}(t), \boldsymbol{l}_2^{(\delta)}(t)) \\ &= -\frac{1}{\sqrt{\delta}} \sum_{j,\alpha} \int_t^u D_{\mathbf{l}_j,\alpha} G(\boldsymbol{l}_1(s), \boldsymbol{l}_2(s)) F_{j,\alpha}\left(s, \frac{\boldsymbol{y}_1(s)}{\delta}, \frac{\boldsymbol{y}_2(s)}{\delta}, \boldsymbol{l}_j(s)\right) ds. \end{aligned} \tag{5.33}$$

We can rewrite (5.33) in the form

$$I^{(1)} + I^{(2)} + I^{(3)}, \tag{5.34}$$

where

$$\begin{aligned} I^{(1)} &:= -\frac{1}{\sqrt{\delta}} \sum_{j,\alpha} \int_t^u D_{\mathbf{l}_j,\alpha} G(\boldsymbol{l}_1(\sigma), \boldsymbol{l}_2(\sigma)) F_{j,\alpha}\left(s, \frac{\boldsymbol{y}_1(s)}{\delta}, \frac{\boldsymbol{y}_2(s)}{\delta}, \boldsymbol{l}_j(\sigma)\right) ds, \\ I^{(2)} &:= \frac{1}{\delta} \sum_{j,\alpha} \sum_{i,\beta} \int_t^u \int_\sigma^s D_{\mathbf{l}_j,\alpha} G(\boldsymbol{l}_1(\rho), \boldsymbol{l}_2(\rho)) D_{\mathbf{l}_j,\beta} F_{j,\alpha}\left(s, \frac{\boldsymbol{y}_1(s)}{\delta}, \frac{\boldsymbol{y}_2(s)}{\delta}, \boldsymbol{l}_j(\rho)\right) \\ &\times F_{j,\beta}\left(\rho, \frac{\boldsymbol{y}_1(\rho)}{\delta}, \frac{\boldsymbol{y}_2(\rho)}{\delta}, \boldsymbol{l}_j(\rho)\right) ds\, d\rho, \\ I^{(3)} &:= \frac{1}{\delta} \sum_{j,\alpha} \sum_{i,\beta} \int_t^u \int_\sigma^s D_{\mathbf{l}_i,\beta} D_{\mathbf{l}_j,\alpha} G(\boldsymbol{l}_1(\rho), \boldsymbol{l}_2(\rho)) \\ &\times F_{j,\alpha}\left(s, \frac{\boldsymbol{y}_1(s)}{\delta}, \frac{\boldsymbol{y}_2(s)}{\delta}, \boldsymbol{l}_j(\rho)\right) F_{i,\beta}\left(\rho, \frac{\boldsymbol{y}_1(\rho)}{\delta}, \frac{\boldsymbol{y}_2(\rho)}{\delta}, \boldsymbol{l}_i(\rho)\right) ds\, d\rho. \end{aligned}$$

5.3.1 Term $\mathbb{E}[I^{(1)}\zeta]$.

The term $I^{(1)}$ can be rewritten in the form

$$J^{(1)} + J^{(2)},$$

where

$$J^{(1)} := -\frac{1}{\sqrt{\delta}} \sum_{j,\alpha} \int_t^u D_{\mathbf{l}_j,\alpha} G(\boldsymbol{l}_1(\sigma), \boldsymbol{l}_2(\sigma)) F_{j,\alpha}\left(s, \frac{\boldsymbol{L}_1(\sigma, s)}{\delta}, \frac{\boldsymbol{L}_2(\sigma, s)}{\delta}, \boldsymbol{l}_j(\sigma)\right) ds,$$

and

$$J^{(2)} := -\frac{1}{\delta^{3/2}}\sum_{j,\alpha}\sum_{i,\beta}\int_t^u\int_0^1 D_{\mathbf{l}_j,\alpha}G(\boldsymbol{l}_1(\sigma),\boldsymbol{l}_2(\sigma))D_{\mathbf{y}_i,\beta}F_{j,\alpha}\left(s,\frac{\boldsymbol{R_1}(\mathbf{v},\sigma,\mathbf{s})}{\delta},\frac{\boldsymbol{R}_2(v,\sigma,s)}{\delta},\boldsymbol{l}_j(\sigma)\right)$$
$$\times\,(y_{i,\beta}(s)-L_{i,\beta}(\sigma,s))\,ds\,dv. \tag{5.35}$$

Note that we have replaced $\boldsymbol{y}_j$ by its linearization $\boldsymbol{L}_j$ in the term $J^{(1)}$. The linear approximation is always "propelled forward", which allows us to use Lemma 5.2 to handle the term $\mathbb{E}[J^{(1)}\zeta]$. Suppose that k is such that $s,t\in[t_k^{(p)},t_{k+1}^{(p)})$, recall also that $s,t,u\in[s_{k_1}^{(p_1)},s_{k_1+1}^{(p_1)})$, and let us fix one trajectory by setting, for instance, $j=1$. We will use Lemma 5.2 with $X_1(\mathbf{x})=-\nabla_{\mathbf{x}}c_1(\mathbf{x})$, $X_2(\mathbf{x})\equiv\mathbf{1}$,

$$Z=\Theta_1\left(s_{k_1}^{(p_1)},\frac{\boldsymbol{L}_1(\sigma,s)}{\delta},\frac{\boldsymbol{L}_2(\sigma,s)}{\delta},\boldsymbol{l}_1(\sigma)\right)|\boldsymbol{l}_1(\sigma)|D_{\mathbf{l}_1}G(\boldsymbol{l}_1(\sigma),\boldsymbol{l}_2(\sigma))\zeta$$

and $g_1=\boldsymbol{L}_1(\sigma,s)\delta^{-1}$, cf. (5.12). We need to verify (5.18). Suppose therefore that $Z\neq0$. For $\rho\in[0,t_{k-1}^{(p)}]$ we have $|\boldsymbol{L}_1(\sigma,s)-\boldsymbol{y}_1(\rho)|\geq(2q)^{-1}$, provided that $0<\delta<(2q)^{-1/(1-\gamma_1)}$. For $\rho\in[t_{k-1}^{(p)},\sigma]$ we have

$$\begin{aligned}&(\boldsymbol{L}_1(\sigma,s)-\boldsymbol{y}_1(\rho))\cdot\hat{\boldsymbol{l}}_1\left(t_{k-1}^{(p)}\right) &(5.36)\\ &\geq(s-\sigma)\hat{\boldsymbol{l}}_1(\sigma)\cdot\hat{\boldsymbol{l}}_1\left(t_{k-1}^{(p)}\right)+\int_\rho^\sigma\left[1+\sqrt{\delta}c_1\left(\frac{\boldsymbol{y}_1(\rho_1)}{\delta}\right)\right]\hat{\boldsymbol{l}}_1(\rho_1)\cdot\hat{\boldsymbol{l}}_1\left(t_{k-1}^{(p)}\right)d\rho_1\\ &\geq(s-\sigma)\left(1-\frac{2}{N+1}\right)+(1-\sqrt{\delta}D_0)(s-\rho)\left(1-\frac{2}{N+1}\right)\geq(s-\sigma)\left(1-\frac{2}{N+1}\right),\end{aligned}$$

provided that $\delta<1/D_0^2$. We see from (5.36) that (5.18) is satisfied with $r=\left(1-\frac{2}{N+1}\right)(s-\sigma)$ and $j=1$.

We verify next that g_1 is also separated from $\boldsymbol{y}_2(\rho)\delta^{-1}$, $\rho\in[0,\sigma]$. Consider two cases. First, when $s,t\in[0,t_1^{(p)})$, using condition (5.3) we obtain then that there exists $\gamma_N'>0$ depending only on N such that

$$\left|g_1-\frac{\boldsymbol{y}_2(\rho)}{\delta}\right|\geq\frac{\gamma_N'(s-\sigma)}{\delta}.$$

Suppose then that $s,t\geq1/p$ and $s,t\in[s_{k_1}^{(p_1)},s_{k_1+1}^{(p_1)})$. Then we have for $\rho\in[0,s_{k_1}^{(p_1)}]$, with p_1 given by (5.7), $|\boldsymbol{L}_1(\sigma,s)-\boldsymbol{y}_2(\rho)|\geq(2q)^{-1}$, provided that δ is as above. For $\rho\in[s_{k_1}^{(p_1)},\sigma]$ we get, thanks to (5.7),

$$\begin{aligned}|\boldsymbol{L}_1(\sigma,s)-\boldsymbol{y}_2(\rho)|&\geq\left|\boldsymbol{L}_1(\sigma,s)-\boldsymbol{y}_2\left(s_{k_1}^{(p_1)}\right)\right|-\left|\boldsymbol{y}_2\left(s_{k_1}^{(p_1)}\right)-\boldsymbol{y}_2(\rho)\right|\\&\geq\frac{1}{2q}-\frac{1+D_0}{p_1}\geq\frac{1}{4q}\geq(s-\sigma)\left(1-\frac{2}{N+1}\right),\end{aligned}$$

provided that $\delta<(4q)^{-(1-\gamma_1)}$.

Using Lemma 5.2 we estimate

$$\begin{aligned}\left|\mathbb{E}[J^{(1)}\zeta]\right|&\leq\frac{MD_0}{\sqrt{\delta}}\|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)}\mathbb{E}[\zeta]\int_t^u\phi\left(C_{10}\frac{s-\sigma}{\delta}\right)ds &(5.37)\\&\leq C_{11}(\delta)(u-t)\|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)}\mathbb{E}[\zeta],\end{aligned}$$

where $C_{10} := \min[\gamma'_N, 1/2(1-2/(N+1))]$, and $C_{11}(\delta)$ depends only on δ and vanishes as $\delta \to 0$.

On the other hand, the term $J^{(2)}$ defined by (5.35) may be written as

$$J^{(2)} = J_1^{(2)} + J_2^{(2)},$$

where

$$\begin{aligned} J_1^{(2)} := & -\frac{1}{\delta^{3/2}} \sum_{j,\alpha} \sum_{i,\beta} \int_t^u D_{\mathbf{l}_j,\alpha} G(\boldsymbol{l}_1(\sigma), \boldsymbol{l}_2(\sigma)) \\ & \times D_{\mathbf{y}_i,\beta} F_{j,\alpha}\left(s, \frac{\boldsymbol{L}_1(\sigma,s)}{\delta}, \frac{\boldsymbol{L}_2(\sigma,s)}{\delta}, \boldsymbol{l}_j(\sigma)\right)(y_{i,\beta}(s) - L_{i,\beta}(\sigma,s))\, ds \end{aligned}$$

and

$$\begin{aligned} J_2^{(2)} := & -\frac{1}{\delta^{5/2}} \sum_{j,\alpha} \sum_{\substack{i,\beta \\ k,\gamma}} \int_t^u \int_0^1 \int_0^1 D_{\mathbf{y}_k,\gamma} D_{\mathbf{y}_i,\beta} F_{j,\alpha}\left(s, \frac{\boldsymbol{R}_1(\theta v,\sigma,s)}{\delta}, \frac{\boldsymbol{R}_2(\theta v,\sigma,s)}{\delta}, \boldsymbol{l}_j(\sigma)\right) \\ & \times D_{\mathbf{l}_j,\alpha} G(\boldsymbol{l}_1(\sigma), \boldsymbol{l}_2(\sigma)) v\, (y_{i,\beta}(s) - L_{i,\beta}(\sigma,s))(y_{k,\gamma}(s) - L_{k,\gamma}(\sigma,s))\, ds\, dv\, d\theta. \end{aligned} \tag{5.38}$$

The second term may be handled easily with the help of Lemma 5.4 and (5.32). We have

$$|\mathbb{E}[J_2^{(2)}\zeta]| \le C_{12} D_2 \mathbb{E}[\zeta] \|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)} (u-t) \delta^{-5/2} \delta^{3-4\gamma_1} \le C_{13} \delta^{1/2-4\gamma_1} (u-t) \mathbb{E}[\zeta] \|\nabla G\|_{L^\infty(\mathbb{R}^{2d})}. \tag{5.39}$$

In order to estimate $J_1^{(2)}$ we split it as

$$J_1^{(2)} = J_{1,1}^{(2)} + J_{1,2}^{(2)} \tag{5.40}$$

where

$$\begin{aligned} J_{1,1}^{(2)} := & -\frac{1}{\delta^{3/2}} \sum_{j,\alpha} \sum_{i,\beta} \int_t^u \int_\sigma^s D_{\mathbf{l}_j,\alpha} G(\boldsymbol{l}_1(\sigma), \boldsymbol{l}_2(\sigma)) D_{\mathbf{y}_i,\beta} F_{j,\alpha}\left(s, \frac{\boldsymbol{L}_1(\sigma,s)}{\delta}, \frac{\boldsymbol{L}_2(\sigma,s)}{\delta}, \boldsymbol{l}_j(\sigma)\right) \\ & \times (s-\rho_1) \frac{d}{d\rho_1} \hat{l}_{i,\beta}(\rho_1)\, ds\, d\rho_1, \end{aligned} \tag{5.41}$$

and

$$\begin{aligned} J_{1,2}^{(2)} := & -\frac{1}{\delta} \sum_{j,\alpha} \sum_{i,\beta} \int_t^u \int_\sigma^s D_{\mathbf{l}_j,\alpha} G(\boldsymbol{l}_1(\sigma), \boldsymbol{l}_2(\sigma)) D_{\mathbf{y}_i,\beta} F_{j,\alpha}\left(s, \frac{\boldsymbol{L}_1(\sigma,s)}{\delta}, \frac{\boldsymbol{L}_2(\sigma,s)}{\delta}, \boldsymbol{l}_j(\sigma)\right) \\ & \times c_1\left(\frac{\boldsymbol{y}_i(\rho)}{\delta}\right) \hat{l}_{i,\beta}(\rho)\, ds\, d\rho, \end{aligned} \tag{5.42}$$

with

$$\frac{d}{d\rho_1} \hat{l}_{i,\beta}(\rho_1) = |\boldsymbol{l}(\rho_1)|^{-1} \left[\frac{d}{d\rho_1} l_{i,\beta}(\rho_1) - (\hat{\boldsymbol{l}}_i(\rho_1), \frac{d}{d\rho_1} \boldsymbol{l}_i(\rho_1))_{\mathbb{R}^d} \hat{l}_{i,\beta}(\rho_1)\right]. \tag{5.43}$$

We deal with $J_{1,2}^{(2)}$ first. It may be split as $J_{1,2}^{(2)} = J_{1,2,1}^{(2)} + J_{1,2,2}^{(2)} + J_{1,2,3}^{(2)}$, where

$$\begin{aligned} J_{1,2,1}^{(2)} := & -\frac{1}{\delta} \sum_{j,\alpha} \sum_{i,\beta} \int_t^u \int_\sigma^s D_{\mathbf{l}_j,\alpha} G(\boldsymbol{l}_1(\sigma), \boldsymbol{l}_2(\sigma)) D_{\mathbf{y}_i,\beta} F_{j,\alpha}\left(s, \frac{\boldsymbol{L}_1(\sigma,s)}{\delta}, \frac{\boldsymbol{L}_2(\sigma,s)}{\delta}, \boldsymbol{l}_j(\sigma)\right) \\ & \times c_1\left(\frac{L_i(\sigma,\rho)}{\delta}\right) \hat{l}_{i,\beta}(\sigma)\, ds\, d\rho, \end{aligned} \tag{5.44}$$

$$J^{(2)}_{1,2,2} := -\frac{1}{\delta^2}\sum_{j,\alpha}\sum_{i,\beta,\gamma}\int_t^u\int_\sigma^s\int_0^1 D_{\mathbf{l}_j,\alpha}G(\boldsymbol{l}_1(\sigma),\boldsymbol{l}_2(\sigma))D_{\mathbf{y}_i,\beta}F_{j,\alpha}\left(s,\frac{\boldsymbol{L}_1(\sigma,s)}{\delta},\frac{\boldsymbol{L}_2(\sigma,s)}{\delta},\boldsymbol{l}_j(\sigma)\right)$$
$$\times\,(D_{\mathbf{y}_i,\gamma}c_1)\left(\frac{R_i(v,\sigma,\rho)}{\delta}\right)(y_{i,\gamma}(\rho)-L_{i,\gamma}(\sigma,\rho))\hat{l}_{i,\beta}(\rho)\,ds\,d\rho\,dv$$

and

$$J^{(2)}_{1,2,3} := -\frac{1}{\delta}\sum_{j,\alpha}\sum_{i,\beta}\int_t^u\int_\sigma^s\int_\sigma^\rho D_{\mathbf{l}_j,\alpha}G(\boldsymbol{l}_1(\sigma),\boldsymbol{l}_2(\sigma))D_{\mathbf{y}_i,\beta}F_{j,\alpha}\left(s,\frac{\boldsymbol{L}_1(\sigma,s)}{\delta},\frac{\boldsymbol{L}_2(\sigma,s)}{\delta},\boldsymbol{l}_j(\sigma)\right)$$
$$\times\,c_1\left(\frac{\boldsymbol{L}_i(\sigma,s)}{\delta}\right)\frac{d}{d\rho_1}\hat{l}_{i,\beta}(\rho_1)\,ds\,d\rho\,d\rho_1.$$

By virtue of Lemma 5.4, (5.32) and the definition (5.15), we obtain easily

$$|\mathbb{E}[J^{(2)}_{1,2,2}\zeta]| = O(\delta^{1/2-2\gamma_1})\|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)}(u-t)\mathbb{E}\zeta,\qquad\text{as }\delta\to 0. \tag{5.45}$$

The same argument also shows that $|\mathbb{E}[J^{(2)}_{1,2,3}\zeta]|$ is of the order of magnitude of the right hand side of (5.45).

Using Lemma 5.1 and the definition (5.15) we conclude that

$$D_{\mathbf{y}_i}\Theta_i\left(s,\frac{\boldsymbol{L}_1(\sigma,s)}{\delta},\frac{\boldsymbol{L}_2(\sigma,s)}{\delta},\boldsymbol{l}_i(\sigma)\right) = O(\delta).$$

Therefore, $|\mathbb{E}[J^{(2)}_{1,2,1}\zeta]|$ is equal, up to a term of order $O(\delta^{1-\gamma_1})(u-t)\|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)}\mathbb{E}\zeta$, to

$$-\frac{1}{\delta}\sum_{i,\alpha,\beta}\int_t^u\int_\sigma^s\mathbb{E}\Big[D_{\mathbf{l}_i,\alpha}G(\boldsymbol{l}_1(\sigma),\boldsymbol{l}_2(\sigma))\Theta_i\left(s^{(p_1)}_{k_1},\frac{\boldsymbol{L}_1(\sigma,s)}{\delta},\frac{\boldsymbol{L}_2(\sigma,s)}{\delta},\boldsymbol{l}_i(\sigma)\right) \tag{5.46}$$
$$\times\,D_{\mathbf{y}_i,\beta}D_{\mathbf{y}_i,\alpha}c_1\left(\frac{\boldsymbol{L}_i(\sigma,s)}{\delta}\right)c_1\left(\frac{\boldsymbol{L}_i(\sigma,\rho)}{\delta}\right)|\boldsymbol{l}_i(\sigma)|\hat{l}_{i,\beta}(\sigma)\,\zeta\Big]\,ds\,d\rho.$$

Let $\delta<(2p_1)^{1/(1-\gamma_1)}$ and fix i. We may apply Lemma 5.3, with

$$Z = D_{\mathbf{l}_i,\alpha}G(\boldsymbol{l}_1(\sigma),\boldsymbol{l}_2(\sigma))\Theta_i\left(s^{(p_1)}_{k_1},\frac{\boldsymbol{L}_1(\sigma,s)}{\delta},\frac{\boldsymbol{L}_2(\sigma,s)}{\delta},\boldsymbol{l}_i(\sigma)\right)|\boldsymbol{l}_i(\sigma)|\hat{l}_{i,\beta}(\sigma)\zeta,$$
$$X_1 := D_{\mathbf{y}_i,\beta}D_{\mathbf{y}_i,\alpha}c_1(\mathbf{x}),\quad X_2 := c_1(\mathbf{x}),$$
$$g_1 := \frac{\boldsymbol{L}_i(\sigma,s)}{\delta},\quad g_2 := \frac{\boldsymbol{L}_i(\sigma,\rho)}{\delta},\quad r = C_{13}(\rho-\sigma),\ r_1 = C_{13}(s-\rho),$$

where $C_{13}>0$ depends only on N. We conclude that

$$\Bigg|\mathbb{E}\left[J^{(2)}_{1,2,1}\zeta\right]+\frac{1}{\delta}\sum_{i,\alpha,\beta}\int_t^u\int_\sigma^s\mathbb{E}\Big[\Theta_i\left(\sigma,\frac{\boldsymbol{L}_1(\sigma,s)}{\delta},\frac{\boldsymbol{L}_2(\sigma,s)}{\delta},\boldsymbol{l}_i(\sigma)\right)D_{\mathbf{l}_i,\alpha}G(\boldsymbol{l}_1(\sigma),\boldsymbol{l}_2(\sigma)) \tag{5.47}$$
$$\times|\boldsymbol{l}_i(\sigma)|\hat{l}_{i,\beta}(\sigma)\partial^2_{\alpha,\beta}R\left(\frac{\boldsymbol{L}_i(\sigma,s)-\boldsymbol{L}_i(\sigma,\rho)}{\delta}\right)\zeta\Big]\,ds\,d\rho\Bigg|$$
$$\le C_{14}\delta^{-1}\|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)}\mathbb{E}[\zeta]\int_t^u\int_\sigma^s\phi^{1/2}\left(\frac{C_{13}(\rho-\sigma)}{2\delta}\right)\phi^{1/2}\left(\frac{C_{13}(s-\rho)}{2\delta}\right)ds\,d\rho.$$

Here we used the fact that

$$\Theta_i\left(\sigma, \frac{\boldsymbol{L}_1(\sigma,s)}{\delta}, \frac{\boldsymbol{L}_2(\sigma,s)}{\delta}, \boldsymbol{l}_i(\sigma)\right) = \Theta_i\left(s_{k_1}^{(p_1)}, \frac{\boldsymbol{L}_1(\sigma,s)}{\delta}, \frac{\boldsymbol{L}_2(\sigma,s)}{\delta}, \boldsymbol{l}_i(\sigma)\right).$$

The right hand side of (5.47) is of the form $C_{15}(\delta)(u-t)\|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)}\mathbb{E}\zeta$, where $C_{15}(\delta)$ vanishes, as $\delta \to 0$. The second term appearing on the left hand side equals to

$$\begin{aligned} &-\sum_{i,\alpha,\beta}\int_t^u \mathbb{E}\left\{\Theta_i\left(\sigma, \frac{\boldsymbol{L}_1(\sigma,s)}{\delta}, \frac{\boldsymbol{L}_2(\sigma,s)}{\delta}, \boldsymbol{l}_i(\sigma)\right) D_{\mathbf{l}_i,\alpha}G(\boldsymbol{l}_1(\sigma),\boldsymbol{l}_2(\sigma))\right. \\ &\times\left.\left[\int_\sigma^s \frac{d}{d\rho}\partial_\alpha R\left(\frac{s-\rho}{\delta}\hat{\mathbf{l}}_i(\sigma)\right)d\rho\right]|\boldsymbol{l}_i(\sigma)|\zeta\right\}ds \\ &= \sum_{i,\alpha,\beta}\int_t^u \mathbb{E}\left[\Theta_i\left(\sigma, \frac{\boldsymbol{L}_1(\sigma,s)}{\delta}, \frac{\boldsymbol{L}_2(\sigma,s)}{\delta}, \boldsymbol{l}_i(\sigma)\right) D_{\mathbf{l}_i,\alpha}G(\boldsymbol{l}_1(\sigma),\boldsymbol{l}_2(\sigma))\partial_\alpha R\left(\frac{s-\sigma}{\delta}\hat{\mathbf{l}}_i(\sigma)\right)|\boldsymbol{l}_i(\sigma)|\zeta\right]ds \end{aligned} \tag{5.48}$$

thanks to the fact that $\nabla_{\boldsymbol{y}}R(\mathbf{0}) = \mathbf{0}$. The term appearing on the right hand side of (5.48) vanishes as $\delta \to 0$ and, in consequence we have shown that $|\mathbb{E}[J_{1,2,1}^{(2)}\zeta]| = C_{16}(\delta)(u-t)\|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)}\mathbb{E}\zeta$, where $C_{16}(\delta)$ vanishes, as $\delta \to 0$.

We now estimate $J_{1,1}^{(2)}$ given by (5.41). Note that according to (5.43) and (5.16) we have

$$J_{1,1}^{(2)} = J_{1,1,1}^{(2)} + J_{1,1,2}^{(2)},$$

where

$$\begin{aligned} J_{1,1,1}^{(2)} &:= \frac{1}{\delta^2}\sum_{j,\alpha}\sum_{i,\beta}\int_t^u\int_\sigma^s D_{\mathbf{l}_j,\alpha}G(\boldsymbol{l}_1(\sigma),\boldsymbol{l}_2(\sigma))D_{\mathbf{y}_i,\beta}F_{j,\alpha}\left(s, \frac{\boldsymbol{L}_1(\sigma,s)}{\delta}, \frac{\boldsymbol{L}_2(\sigma,s)}{\delta}, \boldsymbol{l}_j(\sigma)\right) \\ &\times(s-\rho_1)\Gamma_{i,\beta}\left(\rho_1, \frac{\boldsymbol{y}_1(\rho_1)}{\delta}, \frac{\boldsymbol{y}_2(\rho_1)}{\delta}, \boldsymbol{l}_i(\sigma)\right)ds\,d\rho_1, \end{aligned}$$

with

$$\Gamma_i(\rho,\boldsymbol{y}_1,\boldsymbol{y}_2,\mathbf{l}) := |\mathbf{l}|^{-1}\left[F_i(\rho,\boldsymbol{y}_1,\boldsymbol{y}_2,\mathbf{l}) - \left(\hat{\mathbf{l}}, F_i(\rho,\boldsymbol{y}_1,\boldsymbol{y}_2,\mathbf{l})\right)_{\mathbb{R}^d}\hat{\mathbf{l}}\right],$$

while

$$\begin{aligned} J_{1,1,2}^{(2)} &:= \frac{1}{\delta^2}\sum_{j,\alpha}\sum_{i,\beta}\int_t^u\int_\sigma^s\int_\sigma^{\rho_1} D_{\mathbf{l}_j,\alpha}G(\boldsymbol{l}_1(\sigma),\boldsymbol{l}_2(\sigma))D_{\mathbf{y}_i,\beta}F_{j,\alpha}\left(s, \frac{\boldsymbol{L}_1(\sigma,s)}{\delta}, \frac{\boldsymbol{L}_2(\sigma,s)}{\delta}, \boldsymbol{l}_j(\sigma)\right) \\ &\times(s-\rho_1)\frac{d}{d\rho_2}\Gamma_{i,\beta}\left(\rho_1, \frac{\boldsymbol{y}_1(\rho_1)}{\delta}, \frac{\boldsymbol{y}_2(\rho_1)}{\delta}, \boldsymbol{l}_i(\rho_2)\right)ds\,d\rho_1\,d\rho_2. \end{aligned} \tag{5.49}$$

A straightforward computation, using Lemma 5.4 (note that $\frac{d}{d\rho_2}\Gamma_{i,\beta} \sim \delta^{-1/2}$ in (5.49)), shows that

$$|\mathbb{E}[J_{1,1,2}^{(2)}\zeta]| \le O(\delta^{1/2-3\gamma_1})(u-t)\|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)}\mathbb{E}[\zeta].$$

An application of Lemma 5.4, in the same fashion as it was done in the calculations concerning the terms $\mathbb{E}[J_{1,2,2}^{(2)}\zeta]$ and $\mathbb{E}[J_{1,2,3}^{(2)}\zeta]$, yields that $\mathbb{E}[J_{1,1,1}^{(2)}\zeta]$ is equal, up to a term of the order $C_{17}(\delta)(u-$

$t)\|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)}\mathbb{E}[\zeta]$, where $\lim\limits_{\delta\to 0} C_{17}(\delta)=0$, to

$$\frac{1}{\delta^2}\sum_{i,\alpha,\beta}\int_t^u\int_\sigma^s (s-\rho_1)\mathbb{E}\Big[D_{\mathbf{l}_i,\alpha}G(\boldsymbol{l}_1(\sigma),\boldsymbol{l}_2(\sigma))D_{\mathbf{y}_i,\beta}F_{i,\alpha}\left(s,\frac{\boldsymbol{L}_1(\sigma,s)}{\delta},\frac{\boldsymbol{L}_2(\sigma,s)}{\delta},\boldsymbol{l}_i(\sigma)\right) \tag{5.50}$$
$$\times\,\Gamma_{i,\beta}\left(\rho_1,\frac{\boldsymbol{L}_1(\sigma,\rho_1)}{\delta},\frac{\boldsymbol{L}_2(\sigma,\rho_1)}{\delta},\boldsymbol{l}_i(\sigma)\right)\zeta\Big]\,ds\,d\rho_1.$$

We denote

$$V_{i,\beta}(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_1',\boldsymbol{y}_2',\mathbf{l}):=\left(\sum_{\gamma,q}\partial^3_{\beta,\gamma,q}R(\boldsymbol{y}_i-\boldsymbol{y}_i')\hat{l}_q\hat{l}_\gamma-\sum_\gamma\partial^3_{\beta,\gamma,\gamma}R(\boldsymbol{y}_i-\boldsymbol{y}_i')\right)|\mathbf{l}|.$$

Applying Lemmas 5.1 and 5.3, as in (5.46) (5.47), we conclude that (5.50) is equal, up to a term of order $C_{18}(\delta)(u-t)\|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)}\mathbb{E}[\zeta]$, where $\lim\limits_{\delta\to 0}C_{18}(\delta)=0$, to

$$\frac{1}{\delta^2}\sum_{i,\alpha}\int_t^u\int_\sigma^s(s-\rho_1)\mathbb{E}\Big\{D_{\mathbf{l}_i,\alpha}G(\boldsymbol{l}_1(\sigma),\boldsymbol{l}_2(\sigma))\Lambda_i(\sigma,P_i;\tilde{\boldsymbol{l}}(\cdot))V_{i,\alpha}(P_i)\,\zeta\Big\}\,ds\,d\rho_1, \tag{5.51}$$

with Λ_i defined by (5.13), and $P_i=(\boldsymbol{L}_1(\sigma,s),\boldsymbol{L}_2(\sigma,s),\boldsymbol{L}_1(\sigma,\rho_1),\boldsymbol{L}_2(\sigma,\rho_1),\boldsymbol{l}_i(\sigma))$. Note, however, that for $s\in[s^{(p_1)}_{k_1},s^{(p_1)}_{k_1+1}]$

$$\Xi\left(s,\boldsymbol{L}_1(\sigma,s),\boldsymbol{L}_2(\sigma,s)\right)=\Xi\left(s^{(p_1)}_{k_1},\boldsymbol{L}_1(\sigma,s),\boldsymbol{L}_2(\sigma,s)\right)$$

and

$$\left|\Xi\left(\sigma,\boldsymbol{L}_1(\sigma,s),\boldsymbol{L}_2(\sigma,s)\right)-\Xi\left(\sigma,\boldsymbol{y}_1(\sigma),\boldsymbol{y}_2(\sigma)\right)\right|\le C\sum_{p=1}^2\left|L_p(\sigma,s)-\boldsymbol{y}_p(\sigma)\right|\le C(s-\sigma)\le C\delta^{1-\gamma_1}.$$

A similar estimate holds also for the terms containing $\boldsymbol{L}_i(\sigma,\rho_1)$ and we conclude that the expression in (5.51) is equal, up to a term of order $C_{19}(\delta)(u-t)\|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)}\mathbb{E}[\zeta]$, where $\lim\limits_{\delta\to0}C_{19}(\delta)=0$, to

$$\frac{1}{\delta^2}\sum_{i,\alpha}\int_{t+\delta^{1-\gamma_1}}^u\mathbb{E}\left\{D_{\mathbf{l}_j,\alpha}G(\boldsymbol{l}_1(\sigma),\boldsymbol{l}_2(\sigma))\overline{\Theta}_i^2(\sigma)\left[\int_\sigma^s(s-\rho_1)V_{i,\alpha}(P_i)\,d\rho_1\right]\zeta\right\}ds, \tag{5.52}$$

with

$$\overline{\Theta}_i(\sigma):=\Theta_i(\sigma,\boldsymbol{y}_1(\sigma),\boldsymbol{y}_2(\sigma),\boldsymbol{l}(\sigma);\tilde{\boldsymbol{l}}(\cdot)).$$

Note that, for $s>t+\delta^{1-\gamma_1}$ we have

$$\frac{1}{\delta^2}\sum_{i,\alpha}\int_{s-\delta^{1-\gamma_1}}^s(s-\rho_1)V_{i,\alpha}(P_i)\,d\rho_1=\frac{1}{\delta^2}\sum_{i,\alpha}|\boldsymbol{l}_i(\sigma)|\int_{s-\delta^{1-\gamma_1}}^s(s-\rho_1) \tag{5.53}$$
$$\times\left[\sum_{\gamma,q}\partial^3_{\alpha,\gamma,q}R\left(\frac{s-\rho_1}{\delta}\hat{\boldsymbol{l}}_i(\sigma)\right)\hat{l}_{i,q}(\sigma)\hat{l}_{i,\gamma}(\sigma)-\sum_\gamma\partial^3_{\alpha,\gamma,\gamma}R\left(\frac{s-\rho_1}{\delta}\hat{\boldsymbol{l}}_i(\sigma)\right)\right]d\rho_1$$
$$=\sum_{i,\alpha}|\boldsymbol{l}_i(\sigma)|\int_0^{\delta^{-\gamma_1}}\rho_1\left[\sum_{\gamma,q}\partial^3_{\alpha,\gamma,q}R\left(\rho_1\hat{\boldsymbol{l}}_i(\sigma)\right)\hat{l}_{i,q}(\sigma)\hat{l}_{i,\gamma}(\sigma)-\sum_\gamma\partial^3_{\alpha,\gamma,\gamma}R\left(\rho_1\hat{\boldsymbol{l}}_i(\sigma)\right)\right]d\rho_1.$$

Using the fact that

$$\sum_q \partial^3_{\alpha,\gamma,q} R\left(\rho_1\,\hat{\boldsymbol{l}}_i(\sigma)\right)\hat{l}_{i,q}(\sigma) = \frac{d}{d\rho_1}\partial^2_{\alpha,\gamma} R\left(\rho_1\,\hat{\boldsymbol{l}}_i(\sigma)\right)$$

we obtain, upon the integration by parts performed in the first term on the utmost right hand side of (5.53), that this term equals to

$$\sum_{i,\alpha,\gamma} |\boldsymbol{l}_i(\sigma)| \left[\delta^{-\gamma_1}\partial^2_{\alpha,\gamma} R\left(\delta^{-\gamma_1}\,\hat{\boldsymbol{l}}_i(\sigma)\right)\hat{l}_{i,\gamma}(\sigma) - \int\limits_0^{\delta^{-\gamma_1}} \partial^2_{\alpha,\gamma} R\left(\rho_1\,\hat{\boldsymbol{l}}_i(\sigma)\right)\hat{l}_{i,\gamma}(\sigma)\, d\rho_1 \right. \tag{5.54}$$

$$\left. - \int\limits_0^{\delta^{-\gamma_1}} \rho_1 \partial^3_{\alpha,\gamma,\gamma} R\left(\rho_1\,\hat{\boldsymbol{l}}_i(\sigma)\right)\, d\rho_1 \right]$$

$$= \sum_{i,\alpha,\gamma} |\boldsymbol{l}_i(\sigma)| \left[\delta^{-\gamma_1}\partial^2_{\alpha,\gamma} R\left(\delta^{-\gamma_1}\,\hat{\boldsymbol{l}}_i(\sigma)\right)\hat{l}_{i,\gamma}(\sigma) - \partial_\alpha R\left(\delta^{-\gamma_1}\,\hat{\boldsymbol{l}}_i(\sigma)\right) - \int\limits_0^{\delta^{-\gamma_1}} \rho_1 \partial^3_{\alpha,\gamma,\gamma} R\left(\rho_1\,\hat{\boldsymbol{l}}_i(\sigma)\right)\, d\rho_1 \right].$$

We have used here the fact that $\nabla R(\mathbf{0}) = \mathbf{0}$ and

$$\sum_\gamma \partial_{\alpha,\gamma} R\left(\rho_1\,\hat{\boldsymbol{l}}_i(\sigma)\right)\hat{l}_{i,\gamma}(\sigma) = \frac{d}{d\rho_1}\partial_\alpha R\left(\rho_1\,\hat{\boldsymbol{l}}_i(\sigma)\right).$$

Summarizing the work done in this section, we have shown that

$$|\mathbb{E}[I^{(1)}\zeta]| \le C_{20}(u-t)\|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)}\mathbb{E}\zeta, \tag{5.55}$$

where the constant C_{20} does not depend on δ and G.

5.3.2 The terms $\mathbb{E}[I^{(2)}\zeta]$ and $\mathbb{E}[I^{(3)}\zeta]$

The calculations concerning these terms essentially follow the respective steps performed in the previous section so we only highlight their main points. First, we note that because $\boldsymbol{l}_i(\rho) - \boldsymbol{l}_i(\sigma) \sim \delta^{1/2-\gamma_1}$ we have that $\mathbb{E}[I^{(2)}\zeta]$ is, up to a term $C_{21}(\delta)(u-t)\|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)}\mathbb{E}[\zeta]$, where $\lim_{\delta\to 0} C_{21}(\delta) = 0$, equal to

$$\frac{1}{\delta}\sum_{j,\alpha,\beta}\int\limits_t^u\int\limits_\sigma^s \mathbb{E}\left[D_{\mathbf{l}_j,\alpha}G(\boldsymbol{l}_1(\sigma),\boldsymbol{l}_2(\sigma))D_{\mathbf{l}_j,\beta}F_{j,\alpha}\left(s,\frac{\boldsymbol{y}_1(s)}{\delta},\frac{\boldsymbol{y}_2(s)}{\delta},\boldsymbol{l}_j(\sigma)\right)\right. \tag{5.56}$$

$$\left.\times F_{j,\beta}\left(\rho,\frac{\boldsymbol{y}_1(\rho)}{\delta},\frac{\boldsymbol{y}_2(\rho)}{\delta},\boldsymbol{l}_j(\sigma)\right)\zeta\right] ds\, d\rho.$$

Replacing ρ by σ as the argument of $\boldsymbol{l}_1(\cdot)$, $\boldsymbol{l}_2(\cdot)$ in (5.56) needs a correction that is of order of magnitude $O(\delta^{1/2-2\gamma_1})(u-t)\|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)}\mathbb{E}[\zeta]$, since $\gamma_1 \in (0,1/8]$. Next we note that (5.56) equals

$$\begin{aligned}
&\frac{1}{\delta}\sum_{j,\alpha,\beta}\int_t^u\int_\sigma^s \mathbb{E}\Big[D_{\mathbf{l}_j,\alpha}G(\boldsymbol{l}_1(\sigma),\boldsymbol{l}_2(\sigma))D_{\mathbf{l}_j,\beta}F_{j,\alpha}\left(s,\frac{\boldsymbol{L}_1(\sigma,s)}{\delta},\frac{\boldsymbol{L}_2(\sigma,s)}{\delta},\boldsymbol{l}_j(\rho)\right) \\
&\times F_{j,\beta}\left(\rho,\frac{\boldsymbol{L}_1(\sigma,\rho)}{\delta},\frac{\boldsymbol{L}_2(\sigma,\rho)}{\delta},\boldsymbol{l}_j(\sigma)\right)\zeta\Big]\,ds\,d\rho \\
&+\frac{1}{\delta^2}\sum_{i,\gamma}\sum_{j,\alpha,\beta}\int_t^u\int_\sigma^s\int_0^1 \mathbb{E}\Big[D_{\mathbf{l}_j,\alpha}G(\boldsymbol{l}_1(\rho),\boldsymbol{l}_2(\rho))D_{\mathbf{y}_i,\gamma}D_{\mathbf{l}_j,\beta}F_{j,\alpha}\left(s,\frac{\boldsymbol{R}_1(v,\sigma,s)}{\delta},\frac{\boldsymbol{R}_2(v,\sigma,s)}{\delta},\boldsymbol{l}_j(\sigma)\right) \\
&\times F_{j,\beta}\left(\rho,\frac{\boldsymbol{L}_1(\sigma,\rho)}{\delta},\frac{\boldsymbol{L}_2(\sigma,\rho)}{\delta},\boldsymbol{l}_j(\sigma)\right)(y_{i,\gamma}(s)-L_{i,\gamma}(\sigma,s))\zeta\Big]\,ds\,d\rho\,dv \\
&+\frac{1}{\delta^2}\sum_{i,\gamma}\sum_{j,\alpha,\beta}\int_t^u\int_\sigma^s\int_0^1 \mathbb{E}\Big[D_{\mathbf{l}_j,\alpha}G(\boldsymbol{l}_1(\rho),\boldsymbol{l}_2(\rho))D_{\mathbf{y}_i,\gamma}D_{\mathbf{l}_j,\beta}F_{j,\alpha}\left(s,\frac{\boldsymbol{y}_1(s)}{\delta},\frac{\boldsymbol{y}_2(s)}{\delta},\boldsymbol{l}_j(\sigma)\right) \\
&\times F_{j,\beta}\left(\rho,\frac{\boldsymbol{R}_1(v,\sigma,\rho)}{\delta},\frac{\boldsymbol{R}_2(v,\sigma,\rho)}{\delta},\boldsymbol{l}_j(\sigma)\right)(y_{i,\gamma}(s)-L_{i,\gamma}(\sigma,\rho))\zeta\Big]\,ds\,d\rho\,dv.
\end{aligned} \tag{5.57}$$

A simple argument using Lemma 5.4, (5.31) and (5.32) shows that the second and third terms of (5.57) are both of order of magnitude $O(\delta^{1/2-3\gamma_1})(u-t)\|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)}\mathbb{E}[\zeta]$.

The first term, on the other hand, can be handled with the help of Lemma 5.3 in the same fashion as we have dealt with the term $J^{(2)}_{1,2,1}$, given by (5.44) of Section 5.3.1, and we obtain that

$$|\mathbb{E}[I^{(2)}\zeta]| \le C_{22}(\delta)(u-t)\|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)}\mathbb{E}[\zeta], \tag{5.58}$$

where $\lim_{\delta\to 0} C_{22}(\delta) = 0$.

Finally, concerning the limit of $\mathbb{E}[I^{(3)}\zeta]$ we note that by Lemma 5.4 we have

$$\mathbb{E}[I^{(3)}\zeta] \le C_{23}(\delta)(u-t)\|\nabla G\|_{L^\infty((\mathbb{R}^d)^2)}\mathbb{E}[\zeta] + \sum_{i,j}\mathcal{I}_{i,j} \tag{5.59}$$

where $\lim_{\delta\to 0} C_{23}(\delta) = 0$ and

$$\begin{aligned}
\mathcal{I}_{i,j} := \frac{1}{\delta}\sum_{\alpha,\beta}\int_t^u\int_\sigma^s \mathbb{E}\Big\{&D_{\mathbf{l}_i,\beta}D_{\mathbf{l}_j,\alpha}G(\boldsymbol{l}_1(\sigma),\boldsymbol{l}_2(\sigma)) \\
&\times F_{j,\alpha}\left(s,\frac{\boldsymbol{L}_1(\sigma,s)}{\delta},\frac{\boldsymbol{L}_2(\sigma,s)}{\delta},\boldsymbol{l}_j(\sigma)\right)F_{i,\beta}\left(\rho,\frac{\boldsymbol{L}_1(\sigma,\rho)}{\delta},\frac{\boldsymbol{L}_2(\sigma,\rho)}{\delta},\boldsymbol{l}_i(\sigma)\right)\zeta\Big\}\,ds\,d\rho,
\end{aligned}$$

First, let $i \ne j$ and $2\delta^{1-\gamma_1}M \le (2q)^{-1}$. Suppose also that $s \ge t_1^{(p)}$. We have then

$$|\boldsymbol{L}_i(\sigma,s) - \boldsymbol{L}_j(\sigma,\rho)| \ge \frac{1}{q} - 2M(s-\sigma) \ge \frac{1}{2q}$$

on the event (with fixed α,β)

$$\Theta_j\left(s,\frac{\boldsymbol{L}_1(\sigma,s)}{\delta},\frac{\boldsymbol{L}_2(\sigma,s)}{\delta},\boldsymbol{l}_j(\sigma)\right)\Theta_i\left(s,\frac{\boldsymbol{L}_1(\sigma,\rho)}{\delta},\frac{\boldsymbol{L}_2(\sigma,\rho)}{\delta},\boldsymbol{l}_i(\sigma)\right)$$

$$\times D_{\mathbf{l}_i,\beta} D_{\mathbf{l}_j,\alpha} G(\boldsymbol{l}_1(\sigma), \boldsymbol{l}_2(\sigma)) |\boldsymbol{l}_j(\sigma)||\boldsymbol{l}_i(\sigma)| \neq 0.$$

When, on the other hand, $s, \rho \in [0, t_1^{(p)}]$, then we conclude from (5.3) that

$$|\boldsymbol{L}_i(\sigma, s) - \boldsymbol{L}_j(\sigma, \rho)| \geq \gamma_N' s \geq \gamma_N'(s - \sigma).$$

Therefore $|\mathcal{I}_{i,j}|$ can be then estimated via Lemma 5.3 and Lemma 5.1 by

$$C_{23} D_1^2 M^2 \|D_{\mathbf{l}_1} D_{\mathbf{l}_2} G\|_{L^\infty} \left[\delta^{-\gamma_1} \phi^{1/2}\left(\frac{\gamma_N'}{\delta^{\gamma_1}}\right) + \delta^{1-2\gamma_1}\right] \mathbb{E}[\zeta]. \tag{5.60}$$

It obviously vanishes, as $\delta \to 0$. The second term in (5.60) arises from the contribution of $s < t + \delta^{1-\gamma_1}$.

When $i = j$ we can use Lemma 5.3 in order to obtain

$$|\mathcal{I}_{i,i}| \leq C_{24}(u - t) \|\nabla^2 G\|_{L^\infty((\mathbb{R}^d)^2)} \mathbb{E}[\zeta].$$

Summarizing, we conclude that

$$|\mathbb{E}[I^{(3)} \zeta]| \leq C_{25}(u - t) \|\nabla^2 G\|_{L^\infty((\mathbb{R}^d)^2)} \mathbb{E}[\zeta],$$

where C_{25} can be chosen independently of δ and G. Hence we conclude (5.29) and tightness follows.

Suppose now that Q is any limiting measure of $Q^{(\delta_n)}$ for a certain sequence $\delta_n \to 0$, as $n \to +\infty$. Coming back to (5.52) we conclude, using calculation (5.53)–(5.54), that the limit, as $\delta \to 0$, of the expression on the left hand side of (5.52) equals to

$$\sum_{i,\alpha} \int \left[\int_t^u a_\alpha^{(i)}(s) D_{\mathbf{l}_i,\alpha} G(K_1(s), K_2(s)) |K_i(s)| \overline{\Lambda}_i(s)\, \zeta(K)\, ds\right] Q(dK), \tag{5.61}$$

where

$$\overline{\Lambda}_i(s) := \Lambda_i\left(s, Y_1(s), Y_2(s), K_1(s), K_2(s); K(\cdot), \hat{K}(\cdot)\right),$$

$$Y_i(s) := \mathbf{x}_i + \int_0^s \hat{K}_i(\rho)\, d\rho, \quad i = 1, 2,$$

$$a_\alpha^{(i)}(s) := -\sum_\gamma \int_0^{+\infty} \rho_1 \partial^3_{\alpha,\gamma,\gamma} R\left(\rho_1 \hat{K}_i(s)\right) d\rho_1.$$

Similarly, we calculate the limit, as $\delta \to 0$, of $\mathbb{E}[I^{(3)} \zeta]$. We know that only the limits of the terms $\mathcal{I}_{i,i}$ contribute. A straightforward computation shows that

$$\lim_{\delta \to 0} \sum_i \mathcal{I}_{i,i} = \sum_{i,\alpha,\beta} \int \left[\int_t^u c_{\alpha,\beta}^{(i)}(s) D_{\mathbf{l}_i,\alpha} D_{\mathbf{l}_i,\beta} G(K_1(s), K_2(s)) \overline{H}^{(i)}(s)\, \zeta(K)\, ds\right] Q(dK),$$

where

$$c_{\alpha,\beta}^{(i)}(s) := -|K_i(s)|^2 \int_0^{+\infty} \partial^2_{\alpha,\beta} R(\rho \hat{K}_i(s))\, d\rho,$$

$$\overline{H}^{(i)}(s) := \Theta_i^2\left(s, Y_1(s), Y_2(s), K_i(s)\right).$$

Summarizing, we have shown that any limiting measure Q satisfies (5.30).

5.4 The removal of cut-offs and the proof of weak convergence of $(\boldsymbol{k}_1^{(\delta)}(\cdot), \boldsymbol{k}_2^{(\delta)}(\cdot))$

Let $\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2} := \mathfrak{Q}_{\mathbf{k}_1} \otimes \mathfrak{Q}_{\mathbf{k}_2}$ be the law of two independent copies of the diffusion given by (4.1) over $\mathcal{C}_2(k_1,k_2)$ starting respectively at $\mathbf{k}_1$ and $\mathbf{k}_2$. For a fixed M let $\mathfrak{Q}_{\mathbf{k}_1}^{(M)}$ be the law over $\mathcal{C}_1$ of any diffusion starting at a given $\mathbf{k}_1 \in \mathbb{R}^d$ with the generator $\mathcal{L}^{(M)}$ given by

$$\mathcal{L}^{(M)}F(\mathbf{k}) = \sum_{p,q} a_{p,q}^{(M)}(\mathbf{k})\partial^2_{k_p,k_q}F(\mathbf{k}) + \sum_p b_p^{(M)}(\mathbf{k})\partial_{k_p}F(\mathbf{k}), \quad F \in C_0^\infty(\mathbb{R}^d).$$

Here $a_{p,q}^{(M)}(\cdot)$, $b_p^{(M)}(\cdot)$ are bounded and twice continuously differentiable, $a_{p,q}^{(M)}(\mathbf{k}) = |\mathbf{k}|^2 D_{p,q}(\hat{\mathbf{k}})$, $b_p^{(M)}(\mathbf{k}) = |\mathbf{k}|E_p(\hat{\mathbf{k}})$ for $M^{-1} \le |\mathbf{k}| \le M$. By virtue of Theorems 5.2.3 and 5.3.2 of [23] we conclude that $\mathfrak{Q}_{\mathbf{k}_1}^{(M)}$ is the unique probability measure such that

$$F(K(t)) - F(\mathbf{k}_1) - \int_0^t \mathcal{L}^{(M)}F(K(s))ds, \quad t \ge 0$$

is an $\left(\mathcal{M}_1^{0,t}\right)_{t\ge 0}$-martingale for any $F \in C_b^2(\mathbb{R}^d)$. We define $\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}^{(M)} := \mathfrak{Q}_{\mathbf{k}_1}^{(M)} \otimes \mathfrak{Q}_{\mathbf{k}_2}^{(M)}$.

Let us briefly describe the strategy of the proof of weak convergence of $(\boldsymbol{k}_1^{(\delta)}(\cdot), \boldsymbol{k}_2^{(\delta)}(\cdot))$. First, for any $K \in \mathcal{C}_2$ we define a certain stopping time $W(K; M, N, p, q)$, see (5.63). The crucial property of that time is that the dynamics given by (4.8) agrees with the dynamics of the truncated system (5.16) up to $W(\cdot; M, N, p, q)$. We also show that any limiting measure $Q(\cdot; M, N, p, q)$ satisfies, up to the stopping time, the martingale problem associated with the diffusion given by $\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}$. This property allows to identify $Q(\cdot; M, N, p, q)$ with $\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}$ on the σ–algebra $\mathcal{M}_2^{0,W}$ corresponding to the stopping time. The final step is to show that for sufficiently large N, so that (5.3) is satisfied, and sufficiently large M, as in (5.1), the stopping time $W(\cdot; M, N, p, q)$ converges to infinity in $\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}$ as $q \to +\infty$ and $p \to +\infty$ (in that order), see (5.64). The weak convergence statement is a consequence of this property of the stopping time and it is shown in the calculation following (5.80).

We introduce the following $(\mathcal{M}_2^{0,t})_{t\ge 0}$–stopping times. As before, for any $K = (K_1, K_2)$ such that $K(t) \ne \mathbf{0}$ for all $t \ge 0$ we define

$$Y_j(t) := \int_0^t \hat{K}_j(s)ds. \tag{5.62}$$

For such a K we let $S(N,p) := \lim_{n\uparrow+\infty} S_n(N,p)$, where

$$\begin{aligned} S_n(N,p) := \inf\Big\{ & t \ge 0: \text{ for some } k \ge 0 \text{ we have } t \in \left[t_k^{(p)}, t_{k+1}^{(p)}\right) \\ & \text{and } \hat{K}_i(t_j^{(p)}) \cdot \hat{K}_i(t) < 1 - \frac{2}{N+1} + \frac{1}{n}, \text{ for some } i \in \{1,2\} \text{ or } j \in \{k-1,k\}\Big\}. \end{aligned}$$

If K is such that it becomes $\mathbf{0}$ for some t we adopt of the convention that $S(N,p) = +\infty$. We let further $T(M) := \lim_{n\uparrow+\infty} T_n(M)$, where

$$\begin{aligned} T_n(M) := \inf\Big\{ & t \ge 0: |K_i(t)| < \frac{1}{M} + \frac{1}{n}, \text{ for some } i \in \{1,2\}, \\ & \text{or } |K_i(t)| > M - \frac{1}{n}, \text{ for some } i \in \{1,2\}\Big\}. \end{aligned}$$

Finally, for any $R_1, R_2 > 0$ and $K = (K_1, K_2) \in \mathcal{C}_2(R_1, R_2)$ we let $U(p,q;K) := \lim_{n\uparrow+\infty} U_n(p,q;K)$, $V(p,q;K) := \lim_{n\uparrow+\infty} V_n(p,q;K)$, where

$$U_n(p,q;K) := \inf\Big\{t \ge 0 : \text{ for some } k \ge 1,\ i \in \{1,2\} \text{ we have } u \in [0, t_{k-1}^{(p)}],\ t \in \Big[t_k^{(p)}, t_{k+1}^{(p)}\Big) \\ \text{such that } |Y_i(t) - Y_i(u)| < \frac{1}{q} + \frac{1}{n}\Big\},$$

$$V_n(p,q;K) := \inf\left\{t \ge \frac{1}{p} : \inf_{0\le u\le t} |Y_1(t) - Y_2(u)| < \frac{1}{q} + \frac{1}{n}, \text{ or } \inf_{0\le u\le t} |Y_2(t) - Y_1(u)| < \frac{1}{q} + \frac{1}{n}\right\}.$$

We adopt the convention that any of the above defined stopping times is infinite if the respective set of times over which it is determined is empty.

Suppose that $T_0 > 0$ is an arbitrary deterministic time. Let

$$W(M,N,p,q) := S(N,p) \wedge T(M) \wedge U(p,q) \wedge V(p,q) \text{ and} \tag{5.63}$$
$$B(M,N,p,q) := \{S(N,p) \wedge U(p,q) \wedge V(p,q) \le T(M) \wedge T_0\}.$$

We have $B \in \mathcal{M}_2^{0,W}$. According to Theorem 6.1.2 of [23] the measures $\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}$, $\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}^{(M)}$, $Q(\cdot;M,N,p,q)$ agree, when restricted to $\mathcal{M}_2^{0,W}$. In what follows we show that

$$\lim_{p\to+\infty}\lim_{q\to+\infty} \mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}[W(M,N,p,q) < +\infty] = 0. \tag{5.64}$$

The condition

$$T_0 < W(\boldsymbol{k}_1^{(\delta)}(\cdot), \boldsymbol{k}_2^{(\delta)}(\cdot); M,N,p,q) = W(\boldsymbol{l}_1^{(\delta)}(\cdot), \boldsymbol{l}_2^{(\delta)}(\cdot); M,N,p,q) \tag{5.65}$$

implies $(\boldsymbol{k}_1^{(\delta)}(s), \boldsymbol{k}_2^{(\delta)}(s)) = (\boldsymbol{l}_1^{(\delta)}(s), \boldsymbol{l}_2^{(\delta)}(s))$ for $s \in [0, T_0]$. We will use both (5.64) and (5.65) to establish weak convergence of the laws of $(\boldsymbol{k}_1^{(\delta)}(\cdot), \boldsymbol{k}_2^{(\delta)}(\cdot))$ over $C([0,T_0];(\mathbb{R}^d)^2)$.

We start with the following simple observation.

Lemma 5.6 *With the choice of M as in* (5.1) *we have*

$$\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}[T(M) = +\infty] = 1.$$

Proof. A simple calculation using Itô formula and Remark 4.1 shows that $d|\mathbf{k}_j(t)|^2 = 0$, $j = 1,2$ which proves the lemma. □

Lemma 5.7 *Under the assumptions of Theorem* 4.4 *we have*

$$\lim_{q\to+\infty} U(p,q) = +\infty, \forall\, p, \quad \mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2} - a.s. \tag{5.66}$$

Proof. The proof is essentially the repetition of the argument from [15] pp. 60-61 so we only highlight its main points. It suffices only to show that

$$\lim_{q\to+\infty} U^{(i)}(p,q) = +\infty, \forall\, p, \quad \mathfrak{Q}_{\mathbf{k}_i} - a.s., \tag{5.67}$$

where $U^{(i)}(p,q) := \lim_{n\uparrow+\infty} U_n^{(i)}(p,q)$,

$$U_n^{(i)}(p,q) := \inf\Big\{t \ge 0 : \text{ for some } k \ge 1, \text{ we have } u \in [0, t_{k-1}^{(p)}],\ t \in \Big[t_k^{(p)}, t_{k+1}^{(p)}\Big)$$

$$\text{such that } \left| \int_u^t \hat{K}_i(s)ds \right| < \frac{1}{q} + \frac{1}{n} \Big\}.$$

However, (5.67) can be proved with the help of the argument contained in pp. 60-61 [15] so we omit the details here. We obtain from (5.67)

$$\lim_{q \to +\infty} U(p,q) = +\infty, \forall p, \quad \mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2} - a.s.$$

However $U(p,q) = U^{(1)}(p,q) \wedge U^{(2)}(p,q)$ and (5.66) follows. □

Let us denote by

$$\mathfrak{Y}_t^{(j)} := \bigcup_{0 \le s \le t} Y_j(s) \tag{5.68}$$

and by $B_r(\mathfrak{Y}_t^{(j)}) := [\mathbf{x} : \ \operatorname{dist}(\mathbf{x}, \mathfrak{Y}_t^{(j)}) \le r]$ the sausage, up to time t, of diameter $r > 0$ around trajectory $Y_j(\cdot)$.

The next lemma shows that $S(N,p)$ becomes infinite as $p \to \infty$ for each N.

Lemma 5.8 *We have*

$$\lim_{p \to +\infty} S(N,p) = +\infty, \forall \ N, \quad \mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2} - a.s. \tag{5.69}$$

Proof. The conclusion of the lemma is a consequence of the uniform continuity of paths of the diffusion on any finite time interval $[0,T]$, which implies that

$$\lim_{p \to +\infty} \min_{t_k^{(p)} \in [0,T]} \min_{t \in [t_k^{(p)}, t_{k+1}^{(p)}]} \hat{K}_j(t) \cdot \hat{K}_j(t_k^{(p)}) = 1, \quad j = 1, 2.$$

□

Our next lemma shows that $V(p,q)$ becomes infinite with $p, q \to +\infty$.

Lemma 5.9 *Suppose that N is as in (5.3) and $T_1, \eta > 0$ are arbitrary. Then, one can find p_0, q_0 such that*

$$\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}[S(N,p) \wedge V(p,q) \le T_1] < \eta, \quad \forall\, p \ge p_0, \ q \ge q_0. \tag{5.70}$$

In order to prove this lemma we will need an auxiliary property of $(K_1(\cdot), Y_1(\cdot))$. Let $k_1 = |\mathbf{k}_1|$. Note that the process $(K_1(\cdot), Y_1(\cdot))$ is a diffusion on $\mathbb{R}^d \times \mathbb{R}^d$, actually supported on $S_{k_1}^{d-1}(\mathbf{0}) \times \mathbb{R}^d$, over $(\mathcal{T}_1, \mathfrak{Q}_{\mathbf{k}_1})$. Its generator is given by

$$\mathcal{N}(\mathbf{k}, \mathbf{x}) := \mathcal{L}F(\mathbf{k}, \mathbf{x}) + \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} F(\mathbf{k}, \mathbf{x}).$$

We denote by $P(t, \mathbf{k}, \mathbf{x}; \cdot)$ its transition probability. It satisfies the Fokker-Planck equation

$$\int_0^{+\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\partial_t - \mathcal{N})\varphi(t, \mathbf{k}', \mathbf{y}) P(t, \mathbf{k}, \mathbf{x}; dt, d\mathbf{k}', d\mathbf{y}) = 0, \quad \forall \varphi \in C_0^\infty((0,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d). \tag{5.71}$$

Lemma 5.10 *Let $t > 0$, $(\mathbf{k}, \mathbf{x}) \in S_k^{d-1} \times \mathbb{R}^d$ $(k = |\mathbf{k}|)$. Then, $P(t, \mathbf{k}, \mathbf{x}; \cdot)$ is absolutely continuous with respect to the Lebesgue measure on $S_k^{d-1} \times \mathbb{R}^d$, with the transition probability density $p(t, \mathbf{k}, \mathbf{x}, \cdot, \cdot)$ that is a C^∞-function. In particular, for any $T, K, \eta > 0$ there exists a constant $C > 0$ such that*

$$\max_{t \in [\eta, T]} \max_{(\mathbf{k}, \mathbf{x}) \in S_k^{d-1} \times \overline{B}_K(\mathbf{0})} P(t, \mathbf{k}, \mathbf{x}; S^{d-1} \times A) \le C|A| \tag{5.72}$$

for any $A \subset \overline{B}_K(\mathbf{0})$ and $A \in \mathcal{B}(\mathbb{R}^d)$.

Proof. Let $k := |\mathbf{k}|$ and $S_i^{(\pm)} : B_k^{d-1}(\mathbf{0}) \to S^{d-1}$ be given by

$$S_i^{(\pm)}(\mathbf{l}) := \underbrace{(l_1, \cdots, \pm\sqrt{k^2 - l^2}, \cdots, l_{d-1})}_{i\text{-th component}}, \quad \mathbf{l} = (l_1, \cdots, l_{d-1}) \in B_k^{d-1}(\mathbf{0}),\ l := |\mathbf{l}|.$$

Define the measure $P_i^{(\pm)}(t, B \times A) := P(t, \mathbf{k}, \mathbf{x}; S_i^{(\pm)}(B) \times A)$, $A \in \mathcal{B}(\mathbb{R}^d)$, $B \in \mathcal{B}(B_k^{d-1}(\mathbf{0}))$. The conclusion of the lemma holds if we can show that each measure P_i possesses a C^∞ smooth density. In what follows we consider only the case $i = d$ and denote $S := S_d^{(+)}$, $P_S := P_d^{(+)}$. Note that the respective measure satisfies the equation

$$(\partial_t - \tilde{\mathcal{N}}^*) P_S(t, \cdot) = 0$$

in the distribution sense, with

$$\tilde{\mathcal{N}} F(\mathbf{l}, \mathbf{x}) := k^2 \sum_{p,q=1}^{d-1} \tilde{D}_{p,q}(\mathbf{l}) \partial^2_{l_p, l_q} F(\mathbf{l}, \mathbf{x}) + k \sum_{p=1}^{d-1} \tilde{E}_p(\mathbf{l}) \partial_{l_p} F(\mathbf{l}, \mathbf{x})$$

$$+ \frac{1}{k} \sum_{p=1}^{d-1} l_p \partial_{x_p} F(\mathbf{k}, \mathbf{x}) + \sqrt{1 - \left(\frac{l}{k}\right)^2}\, \partial_{x_d} F(\mathbf{k}, \mathbf{x}),$$

where $\tilde{D}_{p,q}(\mathbf{l}) = D_{p,q}(k^{-1}\mathbf{l}, k^{-1}\sqrt{k^2 - l^2})$, $\tilde{E}_p(\mathbf{l}) = E_p(k^{-1}\mathbf{l}, k^{-1}\sqrt{k^2 - l^2})$. It suffices therefore to prove that $\partial_t - \tilde{\mathcal{N}}^*$ is hypoelliptic in order to prove the Lemma. Note that

$$(\partial_t - \tilde{\mathcal{N}}^*) F = \sum_{p=1}^{d} X_p^2 F + X_0 F + a(\mathbf{l}) F, \quad \forall\, F \in C_0^\infty(B_k^{d-1})$$

where

$$X_p(\mathbf{l}) := k \sum_{q=1}^{d-1} \hat{D}^{1/2}_{p,q}(\mathbf{l}) \partial_{l_q}, \quad p = 1, \cdots, d-1,$$

$$X_0(\mathbf{l}) := \partial_t - \frac{1}{k} \sum_{q=1}^{d-1} l_q \partial_{x_q} - \sqrt{1 - \left(\frac{l}{k}\right)^2}\, \partial_{x_d} + \sum_{q=1}^{d-1} a_q(\mathbf{l}) \partial_{l_q}$$

and $a(\cdot)$, $a_p(\cdot)$, $p = 1, \cdots, d-1$ are C^∞-smooth functions. It suffices therefore to prove that for any $(t, \mathbf{l}, \mathbf{x}) \in \mathbb{R} \times S^{d-1} \times \mathbb{R}^d$ the linear subspace $\mathfrak{L}_{t,\mathbf{l},\mathbf{x}}$ of the tangent space to $\mathbb{R} \times S^{d-1} \times \mathbb{R}^d$, spanned by the vector fields belonging to the Lie algebra $\mathfrak{L}$ generated by $X_0, X_1, \cdots, X_d$, is of dimension $2d$. The $(d-1) \times (d-1)$ matrix $\tilde{\mathbf{D}}(\cdot) := [\tilde{D}_{p,q}(\mathbf{l})]$, as well as $\tilde{\mathbf{D}}^{1/2}(\cdot)$, is non-degenerate in $B_k^{d-1}(\mathbf{0})$ due to Proposition 4.3 (actually it degenerates in the limit when $\boldsymbol{l}$ approaches $\partial B_k^{d-1}(\mathbf{0})$). Hence the vectors $\partial_{l_p} \in \mathfrak{L}_{t,\mathbf{l},\mathbf{x}}$, $p = 1, \cdots, d-1$.

Note also that

$$[X_0, X_p] = \sum_{q=1}^{d-1} \hat{D}^{1/2}_{p,q}(\mathbf{l}) \left(\partial_{x_q} + \frac{l_q}{\sqrt{k^2 - l^2}} \partial_{x_d} \right) + \sum_{q=1}^{d-1} b_q(\mathbf{l}) \partial_{l_q},$$

where $b_p(\cdot),\ p=1,\cdots,d-1$ are C^∞-smooth functions. Hence, $\partial_{x_q}+l_q(k^2-l^2)^{-1/2}\partial_{x_d}\in\mathfrak{L}_{t,\mathbf{l},\mathbf{x}}$, $q=1,\cdots,d-1$. Furthermore,

$$\sum_{p=1}^{d-1}[[X_0,X_p],X_p]=-k\left[\mathrm{tr}\tilde{\mathbf{D}}(\mathbf{l})+(\tilde{\mathbf{D}}(\mathbf{l})\mathbf{l},\mathbf{l})_{\mathbb{R}^d}(k^2-l^2)^{-1}\right](k^2-l^2)^{-1/2}\partial_{x_d}$$
$$+\sum_{q=1}^{d-1}d_q(\mathbf{l})\left(\partial_{x_q}+\frac{l_q}{\sqrt{k^2-l^2}}\partial_{x_d}\right)+\sum_{q=1}^{d-1}c_q(\mathbf{l})\partial_{l_q},$$

where $c_p(\cdot),d_p(\cdot),\ p=1,\cdots,d-1$ are C^∞-smooth functions. We can conclude therefore that $\partial_{x_d}\in\mathfrak{L}_{t,\mathbf{l},\mathbf{x}}$, hence also $\partial_{x_p}\in\mathfrak{L}_{t,\mathbf{l},\mathbf{x}}$, $p=1,\cdots,d-1$ and finally we also get $\partial_t\in\mathfrak{L}_{t,\mathbf{l},\mathbf{x}}$, so that the proof of Lemma 5.10 is complete. □

Proof of Lemma 5.9. Let

$$A(N,p):=[S(N,p)\ge T_1+1].$$

Choose p sufficiently large so that

$$\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}[A(N,p)]\ge 1-\eta/2.$$

This can be done thanks to the continuity property of diffusion paths. For any $(K_1(\cdot),K_2(\cdot))\in A(N,p)$ we have

$$\left|Y_1\left(\frac{1}{p}\right)-Y_2(\rho)\right|\ge\lambda_N(p),\ \text{and}\ \left|Y_2\left(\frac{1}{p}\right)-Y_1(\rho)\right|\ge\lambda_N(p)\tag{5.73}$$

for all $\rho\in[0,1/p]$, according to (5.4) (see (5.62) for the definition of $Y_i(\cdot)$, $i=1,2$). Recall also that $\mathfrak{Y}_t^{(i)}(K_i)$, $i=1,2$ are defined by (5.68).

Let $V^{(1)}(p,q;K):=\lim_{n\to+\infty}V_n^{(1)}(p,q;K)$, where

$$V_n^{(1)}(p,q;K):=\inf\left\{t\ge\frac{1}{p}:\ \mathrm{dist}(Y_1(t),\mathfrak{Y}_t^{(2)})<\frac{1}{q}+\frac{1}{n}\right\}$$

and likewise we introduce $V^{(2)}(p,q;K):=\lim_{n\to+\infty}V_n^{(2)}(q;K)$, with

$$V_n^{(2)}(p,q;K):=\inf\left\{t\ge\frac{1}{p}:\ \mathrm{dist}(Y_2(t),\mathfrak{Y}_t^{(1)})<\frac{1}{q}+\frac{1}{n}\right\}$$

Note that $V(p,q;K):=V^{(1)}(p,q;K)\wedge V^{(2)}(p,q;K)$.

The conclusion of Lemma 5.9 is then a consequence of the following.

Lemma 5.11 *For any N sufficiently large so that* (5.3) *holds and $p\ge1$ we have*

$$\lim_{q\to+\infty}V^{(i)}(p,q;K)=+\infty\quad\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}-a.s.\ on\ A(N,p),\quad\forall\mathbf{k}_1,\mathbf{k}_2\ne\mathbf{0},\ i=1,2.\tag{5.74}$$

Proof. With no loss of generality we assume that $i=1$. Note that obviously

$$V^{(1)}(p,q';K)\ge V^{(1)}(p,q;K)\text{ for }q'\ge q\text{ and all }K\in\mathcal{C}_2(k_1,k_2).$$

For any K_2 we denote by

$$A(N,p;K_2):=[K_1:(K_1,K_2)\in A(N,p)].$$

It suffices to show that for $\mathfrak{Q}_{\mathbf{k}_2}$–a.s. K_2 we have

$$\lim_{q\to+\infty} V^{(1)}(p,q;K_1,K_2)=+\infty, \quad \mathfrak{Q}_{\mathbf{k}_1}-a.s. \text{ on } A(N,p;K_2).$$

Let us denote

$$B(t,\mathbf{x};K_2):=[K_1:|Y_1(t;\mathbf{x})-Y_2(\rho)|\geq\lambda_N(p),\,\rho\in[0,1/p]].$$

Note that $A(N,p;K_2)\subseteq B\left(\frac{1}{p},\mathbf{0};K_2\right)$, according to (5.73).

Let $T_1>0$ be arbitrary. We show that

$$\lim_{q\to+\infty}\mathfrak{Q}_{\mathbf{k}_1}\left[V^{(1)}(p,q;\cdot,K_2)\leq T_1,\,B\left(\frac{1}{p},\mathbf{0};K_2\right)\right]=0, \quad \mathfrak{Q}_{\mathbf{k}_2}-\text{a.s. in } K_2. \tag{5.75}$$

The expression under the limit in (5.75) can be estimated by

$$\mathfrak{Q}_{\mathbf{k}_1}\left[\inf_{u\in[0,T_1]}\text{dist }(Y_1(u),\mathfrak{Y}^{(2)}_{T_1})\leq\frac{1}{q},\,B\left(\frac{1}{p},\mathbf{0};K_2\right)\right]$$

$$=\iint\limits_{S^{d-1}_{k_1}\times[1/p\geq|\mathbf{x}|\geq\lambda_n(p)]} P\left(\frac{1}{p},\mathbf{k}_1,\mathbf{0},d\mathbf{k},d\mathbf{x}\right)\mathfrak{Q}_{\mathbf{k}}\left[\inf_{u\in[0,T_1-1/p]}\text{dist }(Y_1(u;\mathbf{x}),\mathfrak{Y}^{(2)}_{T_1})\leq\frac{1}{q},\,B(0,\mathbf{x};K_2)\right].$$

Here we used the Markov property of the process (K_1,Y_1). (5.75) follows if we can show that

$$\lim_{q\to+\infty}\mathfrak{Q}_{\mathbf{k}}\left[\inf_{u\in[0,T_1-1/p]}\text{dist }(Y_1(u;\mathbf{x}),\mathfrak{Y}^{(2)}_{T_1})\leq\frac{1}{q}\right]=0 \tag{5.76}$$

for every $\mathbf{k}\in S^{d-1}_{k_1}$ and $\mathbf{x}$ satisfying $1/p\geq|\mathbf{x}|\geq\lambda_N(p)$, $\mathfrak{Q}_{\mathbf{k}_2}-$ a.s. in K_2.

Suppose first that $\eta_1:=\frac{1}{2}\text{dist}(\mathbf{x},\mathfrak{Y}^{(2)}_{T_1})>0$. Then,

$$\mathfrak{Q}_{\mathbf{k}}\left[\inf_{0\leq u\leq\eta_1}\text{dist}\left(Y_1(u;\mathbf{x}),\mathfrak{Y}^{(2)}_{T_1}\right)\geq\frac{1}{q}\right]=1, \quad \forall\, q\geq 4\eta^{-1}. \tag{5.77}$$

Note that the expression under the limit on the left hand side of (5.76) can be estimated by

$$\mathfrak{Q}_{\mathbf{k}}\left[\inf_{\eta_1\leq j/q\leq T_1}\text{dist}\left(Y_1(j/q;\mathbf{x}),\mathfrak{Y}^{(2)}_{T_1}\right)\leq\frac{2}{q}\right] \tag{5.78}$$

$$\leq(T_1+1)q\max_{\eta_1\leq j/q\leq T_1}\mathfrak{Q}_{\mathbf{k}}\left[Y_1(j/q;\mathbf{x})\in B_{2/q}(\mathfrak{Y}^{(2)}_{T_1})\right].$$

The right hand side of (5.78) can be estimated, with the help of Lemma 5.10, by

$$C(\eta_1)(T_1+1)q^{2-d}, \quad \forall\, q\geq 4\eta_1^{-1}$$

(recall that $Y_2(\cdot)$ is of C^1-class, with $|Y_2'(\cdot)|\leq 1$) and (5.75) follows, provided we can prove that

$$\mathfrak{Q}_{\mathbf{k}_2}\left[\text{dist}\left(\mathbf{x},\mathfrak{Y}^{(2)}_{T_1}\right)=0\right]=0 \tag{5.79}$$

for $1/p\geq|\mathbf{x}|\geq\lambda_N(p)$ Recall that $|\mathbf{x}-Y_2(\rho)|\geq\lambda_N(p)$, $\rho\in[0,1/p]$. For any $\rho>0$ we can estimate therefore the left hand side of (5.79) by

$$\frac{T_1+1}{\rho}\max_{1/p\leq j\rho\leq T_1}\mathfrak{Q}_{\mathbf{k}_2}\left[|Y_2(j\rho)-\mathbf{x}|\leq 2\rho\right]\leq C(p)(T_1+1)\rho^{d-1}$$

for some constant $C(p)>0$ depending only on p. Since the last inequality holds for all $\rho>0$ we conclude (5.79). □

An immediate consequence of Lemmas 5.6, 5.7 and 5.9 is the following.

Corollary 5.12 *For any $M, \varepsilon > 0$ there exist find sufficiently large p, q and N so that*

$$\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}[B(M,N,p,q)] < \varepsilon.$$

Choose any $T_0 > 0$ and F a bounded and continuous functional over $\mathcal{C}_2$ that is $\mathcal{M}_2^{0,T_0}$-measurable. We show that

$$\limsup_{\delta\to 0} \mathbb{E}\left[F(\boldsymbol{k}_1^{(\delta)}(\cdot), \boldsymbol{k}_2^{(\delta)}(\cdot))\right] \le \int F(K(\cdot))\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}(dK). \tag{5.80}$$

This, in fact, implies weak convergence of the laws of $(\boldsymbol{k}_1^{(\delta)}(\cdot), \boldsymbol{k}_2^{(\delta)}(\cdot))$ over $\mathcal{C}_2$, as $\delta \to 0$.

Fix $\eta > 0$ and choose $M > 0$ such that $M - 1$ satisfies (5.1). Then, by virtue of Lemma 5.6

$$\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}[T(M-1) \le T_0] = 0. \tag{5.81}$$

Let p, q be such that

$$\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}[B(M,N,p,q)] \le \eta. \tag{5.82}$$

Note that $\overline{B}(M-1, N-1, p, q-1) \subseteq B(M,N,p,q)$.

Let $\delta_n \to 0$, then we can choose a subsequence, that we still denote as (δ_n), such that the laws of $(\boldsymbol{l}_1^{(\delta_n)}(\cdot), \boldsymbol{l}_2^{(\delta_n)}(\cdot))$ converge over $\mathcal{C}_2$, as $n \to +\infty$, to a certain $Q(\cdot; M, N, p, q)$. We have

$$\begin{aligned}
&\limsup_{n\to+\infty} \mathbb{E}\left[F(\boldsymbol{k}_1^{(\delta_n)}(\cdot), \boldsymbol{k}_2^{(\delta_n)}(\cdot))\right] \qquad (5.83)\\
&\le \limsup_{n\to+\infty} \mathbb{E}\left[F(\boldsymbol{l}_1^{(\delta_n)}(\cdot), \boldsymbol{l}_2^{(\delta_n)}(\cdot)); W(\boldsymbol{l}_1^{(\delta_n)}(\cdot), \boldsymbol{l}_2^{(\delta_n)}(\cdot); M-1, N-1, p, q-1) > T_0\right]\\
&+ \limsup_{n\to+\infty} \left|\mathbb{E}\left[F(\boldsymbol{k}_1^{(\delta_n)}(\cdot), \boldsymbol{k}_2^{(\delta_n)}(\cdot)); W(\boldsymbol{l}_1^{(\delta_n)}(\cdot), \boldsymbol{l}_2^{(\delta_n)}(\cdot); M-1, N-1, p, q-1) \le T_0\right]\right|.
\end{aligned}$$

The second term on the right hand side of (5.83) can be estimated by

$$\|F\|_{L^\infty}\left(\limsup_{n\to+\infty} \mathbb{P}\left[T(\boldsymbol{l}_1^{(\delta_n)}(\cdot), \boldsymbol{l}_2^{(\delta_n)}(\cdot); M-1) \le T_0\right] + \mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}\left[\overline{B}(M-1, N-1, p, q-1)\right]\right) \tag{5.84}$$

Note also that

$$\begin{aligned}
&\limsup_{n\to+\infty} \left|\mathbb{P}\left[T(\boldsymbol{l}_1^{(\delta_n)}(\cdot), \boldsymbol{l}_2^{(\delta_n)}(\cdot); M-1) > T_0\right] - \mathbb{P}\left[W(\boldsymbol{l}_1^{(\delta_n)}(\cdot), \boldsymbol{l}_2^{(\delta_n)}(\cdot); M-1, N-1, p, q-1) > T_0\right]\right|\\
&\le \limsup_{n\to+\infty} \mathbb{P}\left[(\boldsymbol{l}_1^{(\delta_n)}(\cdot), \boldsymbol{l}_2^{(\delta_n)}(\cdot)) \in \overline{B}(M-1, N-1, p, q-1)\right] \qquad (5.85)\\
&\le \mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}\left[\overline{B}(M-1, N-1, p, q-1)\right] \le \eta
\end{aligned}$$

and hence

$$\begin{aligned}
&\limsup_{n\to+\infty} \mathbb{P}\left[T(\boldsymbol{l}_1^{(\delta_n)}(\cdot), \boldsymbol{l}_2^{(\delta_n)}(\cdot); M-1) \le T_0\right] \qquad (5.86)\\
&= 1 - \liminf_{n\to+\infty} \mathbb{P}\left[T(\boldsymbol{l}_1^{(\delta_n)}(\cdot), \boldsymbol{l}_2^{(\delta_n)}(\cdot); M-1) > T_0\right] \le \mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}\left[W(K; M-1, N-1, p, q-1) \le T_0\right] + \eta.
\end{aligned}$$

The first expression on the utmost right hand side of (5.86) is less than or equal to

$$\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}\left[T(K; M-1) \le T_0\right] + \mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}\left[\overline{B}(M-1, N-1, p, q-1)\right] \le \eta \tag{5.87}$$

according to (5.81) and (5.82). Summarizing, the expression in (5.84) can be estimated by $2\eta\|F\|_{L^\infty}$

The first term on the right hand side of (5.83) can be estimated by

$$\int F(K(\cdot))\mathbf{1}_{[W(K;M,N,p,q)>T_0]}\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}(dK) \le \int F(K(\cdot))\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}(dK)$$

$$+\|F\|_{L^\infty}\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}[W(K;M,N,p,q)\le T_0] \le \int F(K(\cdot))\mathfrak{Q}_{\mathbf{k}_1,\mathbf{k}_2}(dK) + 2\eta\|F\|_{L^\infty}.$$

The last estimate follows from an analogous estimate to (5.87). Summarizing, since $\eta > 0$ is arbitrary we conclude (5.80).

A Proof of Lemma 2.3

We define

$$d_\varepsilon^\delta(\boldsymbol{\xi},\mathbf{x}_0) = \mathbf{v}_\varepsilon^{\delta,B} - \tilde{\mathbf{v}}_\varepsilon^{\delta,B} = \int e^{i\mathbf{k}\cdot(\boldsymbol{\xi}-\mathbf{y})}\left[W_\varepsilon^\delta(\mathbf{x}_0+\frac{\varepsilon(\boldsymbol{\xi}+\mathbf{y})}{2},\mathbf{k}) - W_\varepsilon^\delta(\mathbf{x}_0,\mathbf{k})\right]\Gamma_0\mathbf{S}_0(\mathbf{y})\frac{d\mathbf{k}d\mathbf{y}}{(2\pi)^d} \tag{A.1}$$

and split $d_\varepsilon^\beta = \sum_{j=1}^3 d_\varepsilon^{\delta,j}$ according to the decomposition

$$W_\varepsilon^\delta(\mathbf{x}_0+\frac{\varepsilon(\boldsymbol{\xi}+\mathbf{y})}{2},\mathbf{k}) - W_\varepsilon^\delta(\mathbf{x}_0,\mathbf{k}) = \big(W_\varepsilon^\delta(\mathbf{x}_0+\frac{\varepsilon(\boldsymbol{\xi}+\mathbf{y})}{2},\mathbf{k}) - U^\delta(\mathbf{x}_0+\frac{\varepsilon(\boldsymbol{\xi}+\mathbf{y})}{2},\mathbf{k})\big)$$
$$+\big(U^\delta(\mathbf{x}_0+\frac{\varepsilon(\boldsymbol{\xi}+\mathbf{y})}{2},\mathbf{k}) - U^\delta(\mathbf{x}_0,\mathbf{k})\big) + \big(U^\delta(\mathbf{x}_0,\mathbf{k}) - W_\varepsilon^\delta(\mathbf{x}_0,\mathbf{k})\big).$$

Here $U^\delta = \sum u_q^\delta \Pi_q$ is the semi-classical approximation of W_ε^δ, with u_q^δ the solutions of the Liouville equations. The last term may be estimated as

$$\begin{aligned}\int \|d_\varepsilon^{\delta,3}(\boldsymbol{\xi},\mathbf{x}_0)\|^2 d\mathbf{x}_0 \quad &\le \quad \int \left\|\int e^{i\mathbf{k}\cdot\boldsymbol{\xi}}[U^\delta(\mathbf{x}_0,\mathbf{k}) - W_\varepsilon^\delta(\mathbf{x}_0,\mathbf{k})]\Gamma_0\hat{\mathbf{S}}_0(\mathbf{k})\frac{d\mathbf{k}}{(2\pi)^d}\right\|^2 d\mathbf{x}_0 \\ &\le \quad C\|\mathbf{S}_0\|_{L^2}^2\int \|U^\delta(\mathbf{x}_0,\mathbf{k}) - W_\varepsilon^\delta(\mathbf{x}_0,\mathbf{k})\|^2 d\mathbf{x}_0 d\mathbf{k} \to 0\end{aligned}$$

as $\mathcal{K}_\mu \ni (\varepsilon,\delta) \to 0$ with C independent of ξ. The Fourier transform of the first term in $\mathbf{x}_0$ is

$$\begin{aligned}\hat{d}_\varepsilon^{\delta,1}(\boldsymbol{\xi};\mathbf{p}) &= \int e^{-i\mathbf{p}\cdot\mathbf{x}_0+i\mathbf{k}\cdot(\boldsymbol{\xi}-\mathbf{y})} f_\varepsilon^\delta(\mathbf{x}_0+\frac{\varepsilon(\boldsymbol{\xi}+\mathbf{y})}{2},\mathbf{k})\Gamma_0\mathbf{S}_0(\mathbf{y})\frac{d\mathbf{y}d\mathbf{k}d\mathbf{x}_0}{(2\pi)^d} \\ &= \int e^{i\mathbf{k}\cdot(\boldsymbol{\xi}-\mathbf{y})+i\varepsilon\mathbf{p}\cdot(\boldsymbol{\xi}+\mathbf{y})/2}\hat{f}_\varepsilon^\delta(\mathbf{p},\mathbf{k})\Gamma_0\mathbf{S}_0(\mathbf{y})\frac{d\mathbf{y}d\mathbf{k}}{(2\pi)^d} = \int e^{i(\mathbf{k}+\varepsilon\mathbf{p}/2)\cdot\boldsymbol{\xi}}\hat{f}_\varepsilon^\delta(\mathbf{p},\mathbf{k})\Gamma_0\hat{\mathbf{S}}_0(\mathbf{k}-\frac{\varepsilon\mathbf{p}}{2})\frac{d\mathbf{k}}{(2\pi)^d},\end{aligned}$$

where $f_\varepsilon^\delta(\mathbf{x},\mathbf{k}) = W_\varepsilon^\delta(\mathbf{x},\mathbf{k}) - U^\delta(\mathbf{x},\mathbf{k})$. Therefore we have using the Cauchy-Schwartz inequality

$$\int \|\hat{d}_\varepsilon^{\delta,1}(\boldsymbol{\xi};\mathbf{p})\|^2 d\mathbf{p} \le C\|\mathbf{S}_0\|_{L^2}\|f_\varepsilon^\delta\|_{L^2} \to 0$$

as $\mathcal{K}_\mu \ni (\varepsilon,\delta) \to 0$ with C independent of ξ. Finally, the Fourier transform of $d_\varepsilon^{\delta,2}$ is

$$\begin{aligned}\hat{d}_\varepsilon^{\delta,2}(\boldsymbol{\xi};\mathbf{p}) \quad &= \quad \int e^{-i\mathbf{p}\cdot\mathbf{x}_0+i\mathbf{k}\cdot(\boldsymbol{\xi}-\mathbf{y})}[U^\delta(\mathbf{x}_0+\frac{\varepsilon(\boldsymbol{\xi}+\mathbf{y})}{2},\mathbf{k}) - U^\delta(\mathbf{x}_0,\mathbf{k})]\Gamma_0\mathbf{S}_0(\mathbf{y})\frac{d\mathbf{y}d\mathbf{k}d\mathbf{x}_0}{(2\pi)^d} \\ &= \quad \int e^{i\mathbf{k}\cdot(\boldsymbol{\xi}-\mathbf{y})}\left[e^{i\varepsilon\mathbf{p}\cdot(\boldsymbol{\xi}+\mathbf{y})/2} - 1\right]\hat{U}^\delta(\mathbf{p},\mathbf{k})\Gamma_0\mathbf{S}_0(\mathbf{y})\frac{d\mathbf{y}d\mathbf{k}}{(2\pi)^d} \\ &= \quad \int e^{i\mathbf{k}\cdot\boldsymbol{\xi}}\hat{U}^\delta(\mathbf{p},\mathbf{k})\Gamma_0\left[e^{i\varepsilon\mathbf{p}\cdot\boldsymbol{\xi}/2}\hat{\mathbf{S}}_0(\mathbf{k}-\frac{\varepsilon\mathbf{p}}{2}) - \hat{\mathbf{S}}_0(\mathbf{k})\right]\frac{d\mathbf{k}}{(2\pi)^d}.\end{aligned}$$

We write $e^{i\varepsilon\mathbf{p}\cdot\boldsymbol{\xi}/2} = (e^{i\varepsilon\mathbf{p}\cdot\boldsymbol{\xi}/2} - 1) + 1$ and decompose $\hat{d}_\varepsilon^{\delta,2}(\boldsymbol{\xi};\mathbf{p})$ as $I_1(\boldsymbol{\xi};\mathbf{p}) + I_2(\boldsymbol{\xi};\mathbf{p})$ accordingly. We have for the second term

$$\int \|I_2(\boldsymbol{\xi};\mathbf{p})\|^2 d\mathbf{p} \le C \int \Big(\int \|\hat{U}^\delta(\mathbf{p},\mathbf{k})\| \|\hat{\mathbf{S}}_0(\mathbf{k} - \frac{\varepsilon\mathbf{p}}{2}) - \hat{\mathbf{S}}_0(\mathbf{k})\| \frac{d\mathbf{k}}{(2\pi)^d} \Big)^2 d\mathbf{p}$$
$$\le C \int \Big(\int \|\hat{U}^\delta(\mathbf{p},\mathbf{k})\|^2 d\mathbf{k} \Big) \Big(\int \|\hat{\mathbf{S}}_0(\mathbf{l} - \frac{\varepsilon\mathbf{p}}{2}) - \hat{\mathbf{S}}_0(\mathbf{l})\|^2 d\mathbf{l} \Big) d\mathbf{p}.$$

Note that

$$\int \|\hat{\mathbf{S}}_0(\mathbf{l} - \frac{\varepsilon\mathbf{p}}{2}) - \hat{\mathbf{S}}_0(\mathbf{l})\|^2 d\mathbf{l} \le \varepsilon^2 \|\mathbf{p}\|^2 \int \Big(\int_0^1 \|\nabla \mathbf{S}_0(\mathbf{l} - \varepsilon\mathbf{p}s)\|^2 ds \Big) d\mathbf{l} \le \varepsilon^2 \|\mathbf{p}\|^2 \|\nabla \mathbf{S}_0\|_{L^2}^2$$

and hence

$$\int \|I_2(\boldsymbol{\xi};\mathbf{p})\|^2 d\mathbf{p} \le \varepsilon^2 \|\nabla \mathbf{S}_0\|_{L^2}^2 \|U^\delta\|_{H^1}^2 \to 0$$

as $\mathcal{K}_\mu \ni (\varepsilon,\delta) \to 0$ according to Lemma 3.6. It remains to bound the L^2 norm of $I_1(\mathbf{p};\boldsymbol{\xi})$. We derive two estimates according as ξ is small or large. The first estimate is

$$\begin{aligned} |I_1(\boldsymbol{\xi};\mathbf{p})| &\le C \int e^{i\mathbf{k}\cdot\boldsymbol{\xi}} |\hat{U}^\delta(\mathbf{p},\mathbf{k})| \varepsilon |\mathbf{p}\cdot\boldsymbol{\xi}| |\hat{\mathbf{S}}_0(\mathbf{k} - \frac{\varepsilon\mathbf{p}}{2})| \frac{d\mathbf{k}}{(2\pi)^d} \\ &\le C\varepsilon|\xi| \|\hat{U}^\delta(\mathbf{p},\mathbf{k})\|_{L^2_{\mathbf{k}}(\mathbf{p})|\mathbf{p}|} \|\mathbf{S}_0\|_2, \end{aligned}$$

so that

$$\int |I_1(\boldsymbol{\xi};\mathbf{p})|^2 d\mathbf{p} \le \varepsilon^2 |\boldsymbol{\xi}|^2 \|U^\delta(\mathbf{p},\mathbf{k})\|^2_{H^1(\mathbb{R}^d_{\mathbf{x}};L^2(\mathbb{R}^d_{\mathbf{k}}))} \|\mathbf{S}_0\|_2^2.$$

At the same time using integrations by parts we get

$$I_1(\boldsymbol{\xi};\mathbf{p}) = \int \frac{i}{|\boldsymbol{\xi}|} e^{i\mathbf{k}\cdot\boldsymbol{\xi}} \hat{\boldsymbol{\xi}} \cdot \nabla_{\mathbf{k}} \big(\hat{U}^\delta(\mathbf{p},\mathbf{k}) \Gamma_0 \hat{\mathbf{S}}_0(\mathbf{k} - \frac{\varepsilon\mathbf{p}}{2}) \big) (e^{i\varepsilon\mathbf{p}\cdot\boldsymbol{\xi}/2} - 1) \frac{d\mathbf{k}}{(2\pi)^d}.$$

This shows that

$$\int |I_1(\boldsymbol{\xi};\mathbf{p})|^2 d\mathbf{p} \le \frac{C}{|\boldsymbol{\xi}|^2} \|U\|^2_{L^2(\mathbb{R}^d_{\mathbf{x}};H^1(\mathbb{R}^d_{\mathbf{k}}))} \|(1+|\mathbf{x}|^2)^{1/2} \mathbf{S}_0\|_2^2.$$

With these estimates, we obtain that

$$\int |I_1(\boldsymbol{\xi};\mathbf{p})|^2 d\mathbf{p} \le C \min(h_\varepsilon^\delta |\boldsymbol{\xi}|^2, |\boldsymbol{\xi}|^{-2})$$

with $h_\varepsilon^\delta \to 0$ as $\mathcal{K}_\mu \ni (\varepsilon,\delta) \to 0$. This implies that $\int |I_1(\boldsymbol{\xi};\mathbf{p})|^2 d\mathbf{p} \to 0$, hence $\int |\hat{d}_\varepsilon^{\delta,2}(\boldsymbol{\xi};\mathbf{p})|^2 d\mathbf{p} \to 0$ as $\mathcal{K}_\mu \ni (\varepsilon,\delta) \to 0$ uniformly with respect to $\boldsymbol{\xi} \in \mathbb{R}^d$ and concludes the proof of Lemma 2.3.

B Proof of Lemma 3.3

We may recast (1.5) as

$$\frac{\partial \mathbf{v}_\varepsilon^\delta}{\partial t} + c_\delta(\mathbf{x}) D^j \frac{\partial \mathbf{v}}{\partial x^j} + \frac{\partial c_\delta}{\partial x^j} [\mathbf{e}_j \otimes \mathbf{e}_{d+1}] \mathbf{v}_\varepsilon^\delta = 0. \tag{B.1}$$

Thanks to calculations of the form (3.10), this is equivalent to the equation

$$\varepsilon \frac{\partial \mathbf{v}_\varepsilon^\delta}{\partial t} + P_\varepsilon^{\delta,W}(\mathbf{x}, \varepsilon D_{\mathbf{x}}) \mathbf{v}_\varepsilon^\delta = 0, \tag{B.2}$$

where the symbol P^δ_ε is given by (3.8). We recall that the pseudo-differential Weyl operator $P^W(\mathbf{x},\varepsilon D)$ associated to a symbol $P(\mathbf{x},\mathbf{k})$ is defined by Weyl's quantization rule

$$P^W(\mathbf{x},\varepsilon D_\mathbf{x})u = \int_{\mathbb{R}^{2d}} e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}} P(\frac{\mathbf{x}+\mathbf{y}}{2},\varepsilon\mathbf{k})u(\boldsymbol{y})\frac{d\mathbf{y}d\mathbf{k}}{(2\pi)^d}. \tag{B.3}$$

The fact that (B.2) is equivalent to (B.1) is verified by a straightforward calculation:

$$\begin{aligned}
&P_0^{\delta,W}(\mathbf{x},\varepsilon D)\mathbf{v}^\delta_\varepsilon(\mathbf{x}) = \int e^{i(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi}} P_0^\delta\left(\frac{\mathbf{x}+\mathbf{y}}{2},\varepsilon\boldsymbol{\xi}\right)\mathbf{v}^\delta_\varepsilon(\mathbf{y})\frac{d\boldsymbol{\xi}d\mathbf{y}}{(2\pi)^d}\\
&= i\varepsilon\int e^{i(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi}} c_\delta\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)D^j\xi_j\mathbf{v}^\delta_\varepsilon(\mathbf{y})\frac{d\boldsymbol{\xi}d\mathbf{y}}{(2\pi)^d} = \varepsilon\int c_\delta\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)D^j\mathbf{v}^\delta_\varepsilon(\mathbf{y})\left(-\frac{\partial}{\partial y_j}\delta(\mathbf{x}-\mathbf{y})\right)d\mathbf{y}\\
&= \varepsilon\frac{\partial}{\partial y_j}\left[c_\delta\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)D^j\mathbf{v}^\delta_\varepsilon(\mathbf{y})\right]\Big|_{\mathbf{y}=\mathbf{x}} = \varepsilon c_\delta(\mathbf{x})D^j\frac{\partial\mathbf{v}^\delta_\varepsilon(\mathbf{x})}{\partial x_j} + \frac{\varepsilon}{2}\frac{\partial c_\delta(\mathbf{x})}{\partial x_j}D^j\mathbf{v}^\delta_\varepsilon(\mathbf{x})\\
&= -\varepsilon\frac{\partial\mathbf{v}^\delta_\varepsilon}{\partial t} + \frac{\varepsilon}{2}\frac{\partial c_\delta(\mathbf{x})}{\partial x_j}[-\mathbf{e}_j\otimes\mathbf{e}_{d+1}+\mathbf{e}_{d+1}\otimes\mathbf{e}_j]\mathbf{v}^\delta_\varepsilon = -\varepsilon\frac{\partial\mathbf{v}^\delta_\varepsilon}{\partial t} - \varepsilon P_1^\delta(\mathbf{x})\mathbf{v}^\delta_\varepsilon(\mathbf{x})
\end{aligned}$$

and now (B.2) follows because $P_1^{\delta,W}(\mathbf{x})\mathbf{v}^\delta_\varepsilon(\mathbf{x}) = P_1^\delta(\mathbf{x})\mathbf{v}^\delta_\varepsilon(\mathbf{x})$ since $P_1^\delta(\mathbf{x})$ is independent of $\mathbf{k}$.

The associated Cauchy problem for the Wigner transform $\tilde{W}^\delta_\varepsilon$ with a fixed ζ is given by

$$\begin{aligned}
&\varepsilon\frac{\partial\tilde{W}^\delta_\varepsilon}{\partial t} + \tilde{W}[P^{\delta,W}_\varepsilon(\mathbf{x},\varepsilon D_\mathbf{x})\mathbf{v}^\delta_\varepsilon,\mathbf{v}^\delta_\varepsilon] + \tilde{W}[\mathbf{v}^\delta_\varepsilon,P^{\delta,W}_\varepsilon(\mathbf{x},\varepsilon D_\mathbf{x})\mathbf{v}^\delta_\varepsilon] = 0\\
&\tilde{W}^\delta_\varepsilon(0,\mathbf{x},\mathbf{k})) = \tilde{W}^\delta_\varepsilon(0,\mathbf{x},\mathbf{k};\zeta),
\end{aligned} \tag{B.4}$$

where the Wigner transform of two different fields is defined by

$$\tilde{W}[\phi_\varepsilon,\psi_\varepsilon](\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{y}}\phi_\varepsilon(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\psi^*_\varepsilon(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\frac{d\mathbf{y}}{(2\pi)^d}. \tag{B.5}$$

We deduce from the definitions of $\tilde{W}_\varepsilon$ and P^W_ε that

$$\begin{aligned}
&\tilde{W}[P^{\delta,W}_\varepsilon(\mathbf{x},\varepsilon D_\mathbf{x})\mathbf{v}^\delta_\varepsilon,\mathbf{v}^\delta_\varepsilon](\mathbf{x},\mathbf{k})\\
&= \int e^{i\mathbf{k}\cdot\mathbf{y}}(P^W_\varepsilon(\mathbf{x},\varepsilon D_\mathbf{x})\mathbf{v}^\delta_\varepsilon)(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\mathbf{v}^{\delta*}_\varepsilon(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\frac{d\mathbf{y}}{(2\pi)^d}\\
&= \int e^{i\mathbf{k}\cdot\mathbf{y}}e^{i(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2}-\mathbf{z})\cdot\mathbf{q}}P^\delta_\varepsilon(\frac{\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2}+\mathbf{z}}{2},\varepsilon\mathbf{q})\mathbf{v}^\delta_\varepsilon(\mathbf{z})\mathbf{v}^{\delta*}_\varepsilon(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\frac{d\mathbf{y}d\mathbf{z}d\mathbf{q}}{(2\pi)^{2d}}\\
&= \int e^{i\mathbf{k}\cdot\mathbf{y}}e^{i(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2}-\mathbf{z})\cdot\mathbf{q}}P^\delta_\varepsilon(\frac{\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2}+\mathbf{z}}{2},\varepsilon\mathbf{q})e^{-i\frac{\mathbf{p}}{\varepsilon}\cdot(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2}-\mathbf{z})}W^\delta_\varepsilon(\frac{\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2}+\mathbf{z}}{2},\mathbf{p})\frac{d\mathbf{p}d\mathbf{y}d\mathbf{z}d\mathbf{q}}{(2\pi)^{2d}}\\
&= \int e^{i\mathbf{y}\cdot(\mathbf{k}-\mathbf{p})}e^{i2(\mathbf{x}-\mathbf{z})\cdot(\mathbf{q}-\mathbf{p}/\varepsilon)}P^\delta_\varepsilon(\mathbf{z}-\frac{\varepsilon\mathbf{y}}{2},\varepsilon\mathbf{q})W^\delta_\varepsilon(\mathbf{z},\mathbf{p})\frac{d\mathbf{z}d\mathbf{p}d\mathbf{y}d\mathbf{q}}{(\pi\sqrt{2})^{2d}}\\
&= \int P^\delta_\varepsilon(\mathbf{y},\mathbf{q})W^\delta_\varepsilon(\mathbf{z},\mathbf{p})e^{\frac{2i}{\varepsilon}((\mathbf{p}-\mathbf{k})\cdot\mathbf{y}+(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}+(\mathbf{k}-\mathbf{q})\cdot\mathbf{z}}\frac{d\mathbf{z}d\mathbf{p}d\mathbf{y}d\mathbf{q}}{(\pi\varepsilon)^{2d}} = \int P^\delta_\varepsilon(\mathbf{y},\mathbf{q})W^\delta_\varepsilon(\mathbf{z},\mathbf{p})e^{i\phi}\frac{d\mathbf{z}d\mathbf{p}d\mathbf{y}d\mathbf{q}}{(\pi\varepsilon)^{2d}}.
\end{aligned} \tag{B.6}$$

Moreover, the matrix $\tilde{W}^\delta_\varepsilon$ is self-adjoint, while $W[f_\varepsilon,g_\varepsilon] = W^*_\varepsilon[g,f]$ for any pair of functions f and g, and the symbol P_ε is skew-symmetric. Thus (B.6) and (B.4) imply that the pure Wigner transform $\tilde{W}_\varepsilon$ satisfies (3.11), and hence so does W_ε. Moreover, the function ϕ satisfies an anti-symmetry relation

$$\phi(\mathbf{x},\mathbf{z},\mathbf{k},\mathbf{p};\mathbf{y},\mathbf{q}) = -\phi(\mathbf{z},\mathbf{x},\mathbf{p},\mathbf{k};\mathbf{y},\mathbf{q}).$$

Then, using the fact that W_ε is self-adjoint we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} \mathrm{Tr}((\mathcal{L}_\varepsilon^\delta W_\varepsilon^\delta) W_\varepsilon^{\delta *}) d\mathbf{x}d\mathbf{k} \\
&= \int_{\mathbb{R}^{6d}} \mathrm{Tr}(P_\varepsilon^\delta(\mathbf{y},\mathbf{q}) W_\varepsilon^\delta(\mathbf{z},\mathbf{p}) W_\varepsilon^\delta(\mathbf{x},\mathbf{k}) e^{i\phi} - W_\varepsilon(\mathbf{z},\mathbf{p}) P_\varepsilon^\delta(\mathbf{y},\mathbf{q}) W_\varepsilon^\delta(\mathbf{x},\mathbf{k}) e^{-i\phi}) \frac{d\mathbf{x}d\mathbf{k}d\mathbf{y}d\mathbf{z}d\mathbf{p}d\mathbf{q}}{(\pi\varepsilon)^{2d}} \\
&= \int_{\mathbb{R}^{6d}} \mathrm{Tr}(P_\varepsilon^\delta(\mathbf{y},\mathbf{q}) W_\varepsilon^\delta(\mathbf{z},\mathbf{p}) W_\varepsilon^\delta(\mathbf{x},\mathbf{k}) e^{i\phi} - P_\varepsilon^\delta(\mathbf{y},\mathbf{q}) W_\varepsilon^\delta(\mathbf{z},\mathbf{p}) W_\varepsilon^\delta(\mathbf{x},\mathbf{k}) e^{i\phi}) = 0,
\end{aligned}$$

where we interchanged $\mathbf{x} \leftrightarrow \mathbf{z}$ and $\mathbf{k} \leftrightarrow \mathbf{p}$ in the second term on the last line, and used the anti-symmetry of ϕ. This implies conservation of the L^2-norm (3.14). Note that (3.13) follows immediately from (3.11) and the proof of Lemma 3.3 is complete.

C Regularity of the Liouville equations

We prove Lemma 3.6 in this Appendix. We recall that the functions u_q^δ satisfy the evolution equations

$$\begin{aligned}
&\frac{\partial u_q^\delta}{\partial t} + \{\lambda_q^\delta, u_q^\delta\} = 0 \\
&u_q^0 = u_q(t=0) = \mathrm{Tr}[\Pi_q W_0 \Pi_q].
\end{aligned} \tag{C.1}$$

These equations can be solved by following the Hamiltonian flow generated by λ_q^δ. More precisely, let us define for T, $\mathbf{x}$, $\mathbf{k}$ given, the trajectories

$$\begin{aligned}
\frac{d\mathbf{X}(t)}{dt} &= -\frac{\partial \lambda_q^\delta}{\partial \mathbf{k}}(\mathbf{X}(t), \mathbf{K}(t)), \qquad \mathbf{X}(0) = \mathbf{x} \\
\frac{d\mathbf{K}(t)}{dt} &= \frac{\partial \lambda_q^\delta}{\partial \mathbf{x}}(\mathbf{X}(t), \mathbf{K}(t)), \qquad \mathbf{K}(0) = \mathbf{k}.
\end{aligned} \tag{C.2}$$

Then solution of (C.1) is given by

$$u_q^\delta(T,\mathbf{x},\mathbf{k}) = u_q^0(\mathbf{X}(T,\mathbf{x},\mathbf{k}), \mathbf{K}(T,\mathbf{x},\mathbf{k})). \tag{C.3}$$

The flow (4.5) preserves the Hamiltonian $\lambda_q^\delta(\mathbf{x},\mathbf{k})$ and the initial data u_q^0 is supported on a compact set S. Therefore the set

$$\mathcal{S} = \bigcup_{t\geq 0, \delta \in (0,1]} \mathrm{supp}\ u_q^\delta(t,\mathbf{x},\mathbf{k})$$

is compact because the speed $c_\delta(\mathbf{x})$ is uniformly bounded from above and below. Furthermore

$$\nabla u_q^\delta = D^{\delta *} \nabla u_q^0, \quad \|u_q(t)\|_{\dot{H}^1} \leq \|u_q^0\|_{\dot{H}^1} \|D^\delta(t)\|_\infty$$

where $D^\delta(t,\mathbf{x},\mathbf{k})$ is the Jacobian matrix, $D_j^{\delta,i} = \partial Z_i^\delta / \partial z_j$, with $\det D^\delta(t) \equiv 1$, $\mathbf{Z} = (\mathbf{X},\mathbf{K})$, and $\mathbf{z} = (\mathbf{x},\mathbf{k})$. To simplify notation, we do not write explicitly the dependence of D^δ and its derivatives with respect to the eigenvalue label q in the sequel. Here we define

$$\|D^\delta\|_\infty = \sup_{(\mathbf{x},\mathbf{k})\in\mathcal{S}} \left(\mathrm{Tr}[D^\delta(\mathbf{x},\mathbf{k}) D^{\delta *}(\mathbf{x},\mathbf{k})] \right)^{1/2}.$$

More generally, given a tensor $T_{j_1 j_2 \dots j_m}$ we denote

$$\|T\|_\infty = \sup_{(\mathbf{x},\mathbf{k})\in\mathcal{S}} \left(\sum_{j_1,\dots,j_m} |T_{j_1\dots j_m}|^2 \right)^{1/2}.$$

We will also use the matrix norm $|A|$ that is dual to the Euclidean norm on $\mathbb{R}^d$ and is equal to the square root of the largest eigenvalue of the matrix AA^*, and denote

$$|A|_\infty = \sup_{(\mathbf{x},\mathbf{k})\in\mathcal{S}} |A(\mathbf{x},\mathbf{k})|.$$

Furthermore, we have

$$\frac{\partial^2 u_q^\delta}{\partial z_j \partial z_p} = \frac{\partial^2 Z_m^\delta}{\partial z_j \partial z_p}\frac{\partial u_q^0}{\partial z_m} + \frac{\partial Z_m^\delta}{\partial z_j}\frac{\partial Z_r^\delta}{\partial z_p}\frac{\partial^2 u_q^0}{\partial z_m \partial z_r}$$

so that

$$\sum_{j,p}\left|\frac{\partial^2 u_q^\delta}{\partial z_j \partial z_p}\right|^2 \leq 2\frac{\partial^2 Z_m^\delta}{\partial z_j \partial z_p}\frac{\partial u_q^0}{\partial z_m}\frac{\partial^2 Z_l^\delta}{\partial z_j \partial z_p}\frac{\partial u_q^0}{\partial z_l} + 2\frac{\partial Z_m^\delta}{\partial z_j}\frac{\partial Z_r^\delta}{\partial z_p}\frac{\partial^2 u_q^0}{\partial z_m \partial z_r}\frac{\partial Z_l^\delta}{\partial z_j}\frac{\partial Z_s^\delta}{\partial z_p}\frac{\partial^2 u_q^0}{\partial z_l \partial z_s}$$

$$\leq 2\sum_{m,j,p}\left|\frac{\partial^2 Z_m^\delta}{\partial z_j \partial z_p}\right|^2 \|\nabla u_q^0\|^2 + 2\sum_{l,s}\left|\frac{\partial^2 u_q^0}{\partial z_l \partial z_s}\right|^2 \sum_{m,j}|D_{mj}^\delta|^2$$

and hence

$$\|u_q^\delta(t)\|_{\dot H^2} \leq 2\|u_q^0\|_{\dot H^1}\|D_2^\delta(t)\|_\infty + 2\|u_q^0\|_{\dot H^2}\|D^\delta(t)\|_\infty^2,$$

with $D_{2,jl}^{\delta,m} = \partial^2 Z_m^\delta/\partial z_j \partial z_l$. We observe that

$$\frac{\partial^3 u_q^\delta}{\partial z_j \partial z_p \partial z_s} = \frac{\partial^3 Z_m^\delta}{\partial z_j \partial z_p \partial z_s}\frac{\partial u_q^0}{\partial z_m} + \frac{\partial^2 Z_m^\delta}{\partial z_j \partial z_p}\frac{\partial Z_l^\delta}{\partial z_s}\frac{\partial^2 u_q^0}{\partial z_m \partial z_l} + \frac{\partial^2 Z_m^\delta}{\partial z_j \partial z_s}\frac{\partial Z_r^\delta}{\partial z_p}\frac{\partial^2 u_q^0}{\partial z_m \partial z_r}$$
$$+ \frac{\partial Z_m^\delta}{\partial z_j}\frac{\partial^2 Z_r^\delta}{\partial z_p \partial z_s}\frac{\partial^2 u_q^0}{\partial z_m \partial z_r} + \frac{\partial Z_m^\delta}{\partial z_j}\frac{\partial Z_r^\delta}{\partial z_p}\frac{\partial Z_l^\delta}{\partial z_s}\frac{\partial^3 u_q^0}{\partial z_m \partial z_r \partial z_l}.$$

Therefore we have

$$\sum_{j,p,s}\left|\frac{\partial^3 u_q^\delta}{\partial z_j \partial z_p \partial z_s}\right|^2 \leq 5\sum_{m,j,p,s}\left|\frac{\partial^3 Z_m}{\partial z_j \partial z_p \partial z_s}\right|^2 \|\nabla u_q^0\|^2 + 15\|D(t)\|_\infty^2\|D_2(t)\|_\infty^2 \sum_{m,n}\left|\frac{\partial^2 u_q^0}{\partial z_m \partial z_n}\right|^2$$
$$+ 5\|D(t)\|_\infty^6 \sum_{m,r,l}\left|\frac{\partial^3 u_q^0}{\partial z_m \partial z_r \partial z_l}\right|^2$$

so that

$$\|u_q^\delta(t)\|_{\dot H^3} \leq 3\left[\|u_q^0\|_{\dot H^1}\|D_3^\delta(t)\|_\infty + 3\|u_q^0\|_{\dot H^2}\|D_2^\delta(t)\|_{L^\infty}\|D^\delta(t)\|_\infty + \|u_q^0\|_{\dot H^3}\|D^\delta(t)\|_\infty^3\right]$$

with $D_{3,jlp}^{\delta,m} = \partial^3 Z_m^\delta/\partial z_j \partial z_l \partial z_p$.

It thus remains to estimate the matrices D^δ, D_2^δ, and D_3^δ. The matrix D^δ satisfies the differential equation

$$\frac{dD^\delta}{dt} = F^\delta D^\delta, \quad F^\delta = \begin{pmatrix} -\dfrac{\partial^2 \lambda_q^\delta}{\partial k_i \partial x_j} & -\dfrac{\partial^2 \lambda_q^\delta}{\partial k_i \partial k_j} \\ \dfrac{\partial^2 \lambda_q^\delta}{\partial x_i \partial x_j} & \dfrac{\partial^2 \lambda_q^\delta}{\partial x_i \partial k_j} \end{pmatrix}, \quad D^\delta(0) = I.$$

Therefore we have

$$
\begin{aligned}
&\frac{d}{dt}\mathrm{Tr}[D^\delta D^{\delta *}] = 2\mathrm{Tr}[F^\delta D^\delta D^{\delta *}] = 2\sum F^\delta_{il} D^{\delta,l}_m D^{\delta,m}_i \\
&\le 2\sum F^\delta_{il}\left(\sum_k |D^{\delta,l}_k|^2\right)^{1/2}\left(\sum_p |D^{\delta,p}_i|^2\right)^{1/2} \le 2|F|\mathrm{Tr}[D^\delta D^{\delta *}]
\end{aligned}
$$

so that $\|D^\delta(t)\|_\infty \le \exp\left(|F^\delta|_\infty t\right)$ and hence

$$
\|u^\delta_q(t)\|_{\dot H^1} \le \|u^0_q\|_{\dot H^1}\exp\left(|F^\delta|_\infty t\right).
$$

Differentiating (4.5) once again we obtain

$$
\frac{dD^{\delta,i}_{2,jk}}{dt} = \frac{\partial F^\delta_{il}}{\partial z_m} D^{\delta,m}_k D^{\delta,l}_j + F_{il} D^{\delta,l}_{2,jk},
$$

so that along each characteristic

$$
\frac{1}{2}\frac{d}{dt}\|D^\delta_2\|^2 = \frac{\partial F^\delta_{il}}{\partial z_m} D^{\delta,m}_k D^{\delta,l}_j D^\delta_{2,jk} + F^\delta_{il} D^{\delta,l}_{2,jk} D^{\delta,i}_{2,jk} \le \|F^\delta_2\|_\infty \|D^\delta\|^2 \|D^\delta_2\| + |F^\delta|_\infty \|D^\delta_2\|^2,
$$

where $F^\delta_{2,ijk} = \partial F^\delta_{ij}/\partial z_k$. Furthermore, initially at $t=0$ we have $D^\delta_2(0) = 0$. Therefore we obtain

$$
\|D^\delta_2(t)\|_{L^\infty} \le \frac{\|F^\delta_2\|_\infty}{|F^\delta|_\infty}\exp(2|F^\delta|_\infty t).
$$

and thus

$$
\|u^\delta_q(t)\|_{\dot H^2} \le 2\left(\frac{\|F^\delta_2\|_\infty}{|F^\delta|_\infty}\|u^0_q\|_{\dot H^1} + \|u^0_q\|_{\dot H^2}\right)\exp(2|F^\delta|_\infty t) \le 2\left(\frac{\|F^\delta_2\|_\infty}{|F^\delta|_\infty} + 1\right)\|u^0_q\|_{H^2}\exp(2|F^\delta|_\infty t).
$$

Similarly, the tensor D^δ_3 satisfies the ordinary differential equation

$$
\frac{dD^{\delta,i}_{3,jkm}}{dt} = F^\delta_{3,ilnp} D^{\delta,p}_m D^{\delta,n}_k D^{\delta,l}_j + F^\delta_{2,iln} D^{\delta,n}_{2,km} D^{\delta,l}_j + F^\delta_{2,iln} D^{\delta,n}_k D^{\delta,l}_{2,jm} + F^\delta_{2,iln} D^{\delta,n}_m D^{\delta,l}_{2,jk} + F^\delta_{il} D^{\delta,l}_{3,jkm}
$$

so that along each characteristic

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|D^\delta_3\|^2 = F^\delta_{3,ilnp} D^{\delta,p}_m D^{\delta,n}_k D^{\delta,l}_j D^{\delta,i}_{3,jkm} + F^\delta_{2,iln} D^{\delta,n}_{2,km} D^{\delta,l}_j D^{\delta,i}_{3,jkm} \\
&+ F^\delta_{2,iln} D^{\delta,n}_k D^{\delta,l}_{2,jm} D^{\delta,i}_{3,jkm} + F^\delta_{2,iln} D^{\delta,n}_m D^{\delta,l}_{2,jk} D^{\delta,i}_{3,jkm} + F^\delta_{il} D^{\delta,l}_{3,jkm} D^{\delta,i}_{3,jkm} \\
&\le \|F^\delta_3\|\|D^\delta\|^3\|D^\delta_3\| + 3\|F^\delta_2\|\|D^\delta\|\|D^\delta_2\|\|D^\delta_3\| + |F^\delta|\|D^\delta_3\|^2,
\end{aligned}
$$

where $F^\delta_{3,ijkn} = \partial F^\delta_{2,ijk}/\partial z_n$, and at $t=0$ we have $D_3(0) = 0$. Therefore we obtain

$$
\|D^\delta_3(t)\|_\infty \le \left(\frac{\|F^\delta_3\|_\infty}{|F^\delta|_\infty} + \frac{3\|F^\delta_2\|^2_\infty}{|F^\delta|^2_\infty}\right)\exp(3|F^\delta|_\infty t)
$$

and thus

$$
\begin{aligned}
\|u^\delta_q(t)\|_{\dot H^3} &\le \left[\left(\frac{\|F^\delta_3\|_\infty}{|F^\delta|_\infty} + \frac{3\|F^\delta_2\|^2_\infty}{|F^\delta|^2_\infty}\right)\|u^0_q\|_{\dot H^1} + 3\frac{\|F^\delta_2\|_\infty}{|F^\delta|_\infty}\|u^0_q\|_{\dot H^2} + \|u^0_q\|_{\dot H^3}\right]\exp(3|F^\delta|_\infty t) \\
&\le 6\left(\frac{\|F^\delta_3\|_\infty}{|F^\delta|_\infty} + \frac{\|F^\delta_2\|^2_\infty}{|F^\delta|^2_\infty} + 1\right)\|u^0_q\|_{H^3}\exp(3|F^\delta|_\infty t).
\end{aligned}
$$

This completes the proof of Lemma 3.6 because $\bar\gamma_\delta = |F^\delta|_\infty$.

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