Basic Discrete Probability

1. What is the probability that a five-card poker hand contains a royal flush, that is, the 10, jack, queen, king, and ace of one suit?

   There are four possibilities for an unordered royal flush, one for each suit. Therefore there are $\frac{4}{\binom{52}{5}}$ possibilities.

2. What is the probability that Abby, Barry, and Syliva win the first, second, and third prizes, respectively, in a drawing if 200 people enter a contest and

   (a) no one can win more than one prize.

   In this case, our sample space is unordered sets of prizes. There is only one potential solution, so our answer is
   $$\frac{1}{P(200,3)}$$

   (b) winning more than one prize is allowed.

   Here the only difference is that our sample space is larger, making our answer
   $$\frac{1}{200^3} = \frac{1}{8,000,000}$$

3. What is the probability of these events when we randomly select a permutation of $\{1, 2, 3\}$?

   (a) 1 precedes 3.

   There are three permutations where 1 proceeds 3, so the answer is 1/2.

   (b) 3 precedes 1.

   This is the complement of the previous question, so 1-1/2=1/2

   (c) 3 precedes 1 and 3 precedes 2.
   There are two permutations where this happens, 312 and 321, so 1/3.

4. Assume that the probability a child is a boy is 0.51 and that sexes of children born into a family are independent. What is the probability a family of five children has

   (a) exactly three boys?

   The probability of having three boys is, by the binomial distribution $\binom{5}{3}(.51)^3(.49)^2$

   (b) at least one boy?

   This is the complement of having all girls, so is probability $1 - (.51)^5$

(c) at least one girl?

$1 - (.49)^5$

(d) exactly two boys, conditional on there being at least two girls?
The probability that there are at least two girls is $1 - (.51)^5 - 5(.51)^4 .49$. The probability that there are exactly two boys is $(.51)^2(.49)^3$. Since this entirely lies within the condition that there are at least two girls, the conditional probability is

$$\frac{(.51)^2(.49)^3}{1 - (.51)^5 - 5(.51)^4 .49}$$

5. Assume that the probability of a 0 is 0.8 and a 1 is 0.2 for a randomly generated bit string of length six. What is the probability that there are

(a) at least 3 zeros?

We can sum up probabilities

$$\binom{6}{3}(.8)^3(.2)^3 + \binom{6}{4}(.8)^4(.2)^2 + \binom{6}{5}(.8)^5(.2)^1 + \binom{6}{6}(.8)^6$$

(b) two ones, conditional on the first digit being a zero. Here we can treat this as a string of length five, meaning that we have

$$\binom{5}{2}(.8)^3(.2)^2$$

6. (a) What is the probability that two people chosen at random were born during the same month of the year? Assume that it is equally likely that a person is born during any month.

Given the first person's birth month, there is a $\frac{1}{12}$ chance the second person has the same birth month, so $\frac{1}{12}$

(b) What is the probability that in a group of n people, two are born during the same month of the year?

We calculate the complement, namely if everyone is born in a different month. In this case there are 12 options for the first person, 11 for the second, etc. meaning there are $P(12, n)$ options total. As there are 12^n possible birth month sets in our state space, our answer is

$$1 - \frac{P(12, n)}{12^n}$$

(c) How many people chosen at random are needed such that the probability that two are born during the same month is at least 1/2?
By calculating this value for various n, we see we need 5 people.

7. What is the conditional probability that exactly 4 heads appears when a fair coin is flipped five times, given that the first coin comes up heads.

   We can treat this as if we have 4 coins of which 3 are heads. Using ELOP we see that the probability of this is $\frac{4}{16} = \frac{1}{4}$

8. For the Monty Hall problem, assume there are n doors, behind k of which are prizes. What is the probability of success now if we stick with our initial choice, versus switching to another door?

   If we don't switch doors, there is a $\frac{k}{n}$ chance that we choose a door with a prize behind it.

   If we do switch doors, we split into cases of whether we initially choose a prize door or a non prize door. If we start with a prize door, then there are $k - 1$ prize doors to choose from if we switch. If we start with a non prize door, there are k prize doors to choose from if we switch. Therefore the total probability of choosing a prize while switching is

$$\frac{k(k-1)}{n(n-2)} + \frac{(n-k)k}{n(n-2)} = \frac{(n-1)k}{n(n-2)}$$

   So even in this general case, it is always good to switch.

Source: Rosen's *Discrete Mathematics and its Applications.*