Worksheet 9

1. A weather forecaster in Brooklyn reports the following: The probability it will snow today is .7. The probability it will snow tomorrow is .5. The probability it will snow today and tomorrow is .3. The probability it will snow today or tomorrow is .9. Define an appropriate sample space, write these expressions in terms of unions and intersections of subsets of the sample space, and assess if this is a reasonable statement to make.

Solution: The sample space is the four possibilities of snow; it snows today and tomorrow, it snows today but not tomorrow, it snows tomorrow but not today, or it snows neither day. Call A the event that it snows today, and B the event that it snows tomorrow. These can be expressed as

$$P(A) = .7, P(B) = .5, P(A \cap B) = .3, P(A \cup B) = .9$$

We need $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for this statement to be reasonable. Here this is the case.

2. **original** A family has three children. Assume that the probability of each child being a boy or girl is 50% each.

   (a) What is the probability the family has exactly two girls if there is at least one girl?

   Solution: Using conditional probability, the probability of there being at least one girl (A) is $1 - \frac{1}{2^3} = \frac{7}{8}$. The probability of there being exactly two girls B is $\binom{3}{2}\frac{1}{2^3} = \frac{3}{8}$. $B \subset A$, so

   $$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)} = \frac{3}{7}$$

   (b) What is the probability the family has exactly two girls if the oldest child is a girl?

   Solution: $P(B|A)$ is now the probability that exactly one of the next two children is a girl, or $\binom{2}{1}\frac{1}{4} = \frac{1}{2}$

3. **7.2.29** A group of 6 people are playing a game of "odd person out" to determine who will buy refreshments. Each person flips a fair coin. If there is a person whose outcome is not the same as that of any other member of the group, this person has to buy the refreshments. What is the probability that there is an odd person out after the coins are flipped once?

> **Solution:** This is the probability that there is exactly one heads or exactly one tails. As these are disjoint, we sum the probabilities
> $$\binom{6}{1}\frac{1}{2^6} + \binom{6}{5}\frac{1}{2^6} = \frac{12}{64} = \frac{3}{16}$$

4. **original** Suppose that 20% of the population of Berkeley eats Smarties candy and 30% of those people get stomach aches after eating them. Suppose that the rate of stomach aches is 10% of the Berkeley population. What is the probability that someone with a stomach ache eats Smarties?

> **Solution:** Call A the probability someone eats Smarties and B the probability someone has a stomach ache. Then
> $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.06}{.1} = 60\%$$

5. **7.2.11** Suppose that E and F are events such that $p(E) = 0.7$ and $p(F) = 0.5$. Show that $p(E \cup F) \geq 0.7$ and $p(E \cap F) \geq 0.2$.

> **Solution:** $p(E \cup F) \geq .7$, as it must be at least as large as $p(E)$ and $p(F)$.
> $$p(E \cup F) = p(E) + p(F) - p(E \cap F)$$
> $p(E \cap F) \geq 0.2$ as $p(E \cup F)$ is at most 1.

6. **original** Suppose you roll a die 4 times

   (a) Describe the sample space Ω and calculate $|\Omega|$.

   > **Solution:** The sample space is set of 4 die rolls, which has size $6^4 = 1296$

   (b) What is the probability that the sum is less than 6?

   > **Solution:** There is 1 scenario where the sum is 4, and 4 scenarios where the sum is 5, so $\frac{5}{1296}$.

   (c) What is the probability that you roll at least one 2?

> **Solution:** The complement is rolling no 2's, which happens with probability $(\frac{5}{6})^4$, so our answer is $1 - \frac{5^4}{6^4}$.

(d) What is the probability that at least one of these two events happens; namely rolling less than 6 or rolling at least one 2?

> **Solution:** The first condition is a subset of the second condition unless we roll all 1's, so the answer is $1 - \frac{5^4}{6^4} + \frac{1}{6^4}$

7. **7.2.38** A pair of dice is rolled in a remote location and when you ask an honest observer whether at least one die came up six, this honest observer answers in the affirmative.

(a) What is the probability that the sum of the numbers that came up on the two dice is seven, given the information provided by the honest observer?

> **Solution:** There are 2 scenarios where the sum is 7 and at least one die is a 6. There are 11 scenarios where at least one die is a 6. So $\frac{2}{11}$.

(b) Suppose the honest observer tells us that at least one die came up five. What is the probability the sum of the numbers that came up on the dice is seven?

> **Solution:** By the same method as part a, the answer is also $\frac{2}{11}$

8. **original challenge**

(a) How can I replicate a probability of 1/3 by flipping a fair coin repeatedly?

> **Solution:** Flip the coin twice. If it is heads twice (HH) then this is success. If HT or TH, then failure. If TT, then start the process again.

(b) **mega challenge** How can I replicate any decimal probability between zero and one by flipping a fair coin repeatedly?

> **Solution:** Expand any decimal probability in binary. Consider the heads as 1's in the decimal binary expansion, and the tails as 0's. By flipping the coin over and over, we obtain some decimal expansion. If we end up with a number larger than our probability, this is failure. If we end up with a number smaller than our probability, this is success. Our process is uniform on all decimals, meaning that our probability of success is the given probability.

Source: Rosen's *Discrete Mathematics and its Applications.*